3.7a

$X_1, \dots, X_n$ iid $f(x) = \lambda^{-1} \exp(-\lambda^{-1}(x-\theta))\, I\{x \geq \theta\}$ $\quad H_0: \theta \leq 0$ $\quad H_1: \theta > 0$

$\cancel{\log L = -n\log\lambda - \frac{1}{\lambda}\sum X_i + \frac{n\theta}{\lambda}\, I\{X_{min} \geq \theta\}}$

$\cancel{\leq -n\log\lambda - \frac{1}{\lambda}\sum X_i - \frac{1}{\lambda}X_{min}}$

$$L = \lambda^{-n} \exp\left(-\frac{1}{\lambda}\sum X_i + \frac{n\theta}{\lambda}\right) I\{X_{min} \geq \theta\} \leq \lambda^{-n} \exp(-n\bar{X}/\lambda + nX_{min}/\lambda)$$

$$\therefore \hat{\theta} = X_{min}$$

$\cancel{-2\log L}$

$$\log L(\hat{\theta}, \lambda) = -n\log\lambda - \frac{1}{\lambda}\sum(X_i - X_{min})$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{min}) = \bar{X} - X_{min} \qquad L(\hat{\theta}, \hat{\lambda}) = (\bar{X} - X_{min})^{-n} e^{-n}$$

Under H_0 $\quad \hat{\theta}_0 = \min\{X_{min}, 0\}$

If $\hat{\theta}_0 = X_{min} < 0$ $\quad \hat{\lambda}_0 = \hat{\lambda}$ and $\Lambda = 1$

If $\hat{\theta}_0 = 0$ and $X_{min} > 0$, $\quad \hat{\lambda} = \bar{X}$ and $L(\hat{\theta}_0, \hat{\lambda}_0) = \bar{X}^{-n} e^{-n}$

$$\Lambda = \begin{cases} 1 & \text{if } X_{min} < 0 \\ \left(\frac{\bar{X} - X_{min}}{\bar{X}}\right)^n & \text{if } X_{min} \geq 0 \end{cases} \qquad \text{Reject if } \Lambda < c$$

$$\Lambda^{-1/n} = \frac{\sum_1^n X_{(i)}}{\sum_2^n X_{(i)}} = \frac{nX_{(1)} + \sum(X_i - X_{(1)})}{\sum_2^n (X_{(i)} - X_{(1)})} = 1 + \frac{nX_{(1)}}{\sum_2^n (X_{(i)} - X_{(1)})}$$

Make change of variable:

$$[X_{(1)}, \dots, X_{(n)}] \longrightarrow [Y_1 = X_{(1)},\ Y_2 = X_{(2)} - X_{(1)},\ \dots,\ Y_n = X_{(n)} - X_{(1)}]$$

The Y's have density $\quad n!\, f_{\underset{\sim}{X}}(\underset{\sim}{x}(\underset{\sim}{y}))\, |\underset{\sim}{J}|^{-1} = \frac{n!}{\lambda^n} \exp\left[-\frac{n}{\lambda}(y_1 - \theta) - \frac{1}{\lambda}\sum_2^n y_j\right]$

$\cancel{= \frac{n}{\lambda}\exp(-\frac{1}{\lambda}y}$ $\quad \times I\{\theta < y_1,\ 0 < y_2 < y_3 < \dots$

$\underset{\sim}{Y}$ has density

$$\frac{n}{\lambda}\exp\left[-\frac{n}{\lambda}(y_1 - \theta)\right] \frac{(n-1)!}{\lambda^{n-1}} \exp\left[-\frac{1}{\lambda}\sum_2^n y_j\right] I\{y_1 > \theta\}\, I\{0 < y_2 < \dots < y_n\}$$

$\therefore$ Y_1 is independent of $(Y_2, \cdots, Y_n)$ and $(Y_2, \cdots, Y_n)$ have distribution of order statistics from $(n-1)$ iid exponentials with density $\lambda^{-1} \exp(-\lambda^{-1} y)$

$\therefore$ ~~2Σ~~

The LR test rejects if $\Lambda < c$ iff $X_{(1)} > 0$ and $\frac{n X_{(1)}}{\sum_2^n (X_{(j)} - X_{(1)})}$ is large. If $\theta = 0$, $\frac{n X_{(1)}}{\sum_2^n (X_{(j)} - X_{(1)})}$ has same dist as $\frac{n Y_1}{\sum_2^n Y_j}$

which is $\frac{2}{2(n-1)} F_{2, 2n-2}$ under $H_0 : \theta = 0$

$$\therefore \text{ Reject } H_0 \text{ iff } T = \frac{n(n-1) X_{(1)}}{\sum [X_{(j)} - X_{(1)}]} > F_{2, 2n-2, \alpha}$$

The test has size α if $\theta = 0$ for any λ. If $\theta < 0$,

$$T = \frac{n(n-1) X_{(1)}}{\sum (X_{(j)} - X_{(1)})} = \frac{n(n-1)(X_{(1)} - \theta) + n(n-1)\theta}{\sum_2^n (X_{(j)} - X_{(1)})}$$

and $P_{\theta\lambda}\left[T > F_{2, 2n-2, \alpha} \right] \le P_{\theta\lambda}\left[\frac{n(n-1)(X_{(1)} - \theta)}{\sum_2^n (X_{(j)} - X_{(1)})} > F_{2, 2n-2, \alpha} \right] = \alpha$

Therefore the test has level α for all $\theta \le 0$.

$X_1, \dots X_n$ iid $\quad f_{\beta\gamma}(x) = \beta^{-\gamma} x^{\gamma-1} \exp[-(x/\beta)^\gamma]$ $\qquad H_0: \gamma = 1 \qquad H_1: \gamma \neq 1$, $\gamma > 0$

$\boxed{\text{8.7b}}$

$$\log L = -n\gamma \log\beta + (\gamma-1)\Sigma \log X_i - \Sigma (X_i/\beta)^\gamma$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{n\gamma}{\beta} + \gamma\beta^{-\gamma-1}\Sigma X_i^\gamma = 0 \Rightarrow \hat\beta = \left[\frac{1}{n}\Sigma X_i^\gamma\right]^{1/\gamma} \quad [\text{Under } H_0\ \hat\beta_0 = \bar X]$$

$$\left.\frac{\partial \log L}{\partial \gamma}\right|_{\hat\beta} = -n\log\hat\beta + \Sigma\log X_i - \Sigma(X_i/\hat\beta)^\gamma \log(X_i/\hat\beta)$$

$$= -\frac{n}{\gamma}\log\left(\frac{1}{n}\Sigma X_i^\gamma\right) + \Sigma\log X_i - \left(\frac{1}{n}\Sigma X_i^\gamma\right)^{-1}\Sigma X_i^\gamma \log X_i$$

$$+ \left(\frac{1}{n}\Sigma X_i^\gamma\right)^{-1}\frac{1}{\gamma}\log\left(\frac{1}{n}\Sigma X_i\right)\Sigma X_i^\gamma$$

$$= -\frac{n}{\gamma}\log\left(\Sigma X_i^\gamma\right) + \frac{n}{\gamma}\log n + \Sigma\log X_i - \frac{n}{\Sigma X_i^\gamma}\Sigma X_i^\gamma \log X_i$$

$$+ \frac{n}{\Sigma X_i^\gamma}\left(\Sigma X_i^\gamma\right)\frac{1}{\gamma}\left(\log\Sigma X_i - \log n\right)$$

$$\Rightarrow \left(\Sigma\log X_i\right)\left(\Sigma X_i^\gamma\right) - n\Sigma X_i^\gamma \log X_i = 0 \quad \text{Choose pos. sol'n}$$

Under H_0: $\max\limits_{\gamma=1} \log L = -n\log\bar X - \frac{\Sigma_i X_i}{\beta} = -n\log\bar X - n$

$$\max_{\gamma,\beta} \log L = -n\log\frac{1}{n}\Sigma X_i^{\hat\gamma} + (\hat\gamma-1)\Sigma\log X_i - \frac{\Sigma X_i^\gamma}{\frac{1}{n}\Sigma X_i^\gamma}$$

$$= +n\log n - n\log\left(\Sigma X_i^{\hat\gamma}\right) + (\hat\gamma-1)\Sigma\log X_i - n$$

$$-2\log\Lambda = +2n\log n - 2n\log\left(\Sigma X_i^{\hat\gamma}\right) + 2(\hat\gamma-1)\Sigma\log X_i - 2n$$

$$+2n\log\bar X + 2n$$

$$= -2n\log\left(\Sigma X_i^{\hat\gamma}/\Sigma X_i\right) + 2(\hat\gamma-1)\Sigma\log X_i$$

Reject if $-2\log\Lambda > \chi^2_{1,\alpha}$ [approx. test for large n]

8.12 (a) $H_0 : \mu \le \mu_0$ $\quad H_1 : \mu > 0$

Reject H_0 if $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha$

$$\text{Power} = P_\mu[\text{reject}] = P_\mu\left[\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha\right]$$

$$= P_\mu\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > z_\alpha - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right]$$

$$= 1 - \Phi\left(z_\alpha - \sqrt{n}(\mu-\mu_0)/\sigma\right)$$

Power depends on n and $(\mu-\mu_0)/\sigma$

(b) $H_0 : \mu = 0$ vs $H_1 : \mu \ne 0$

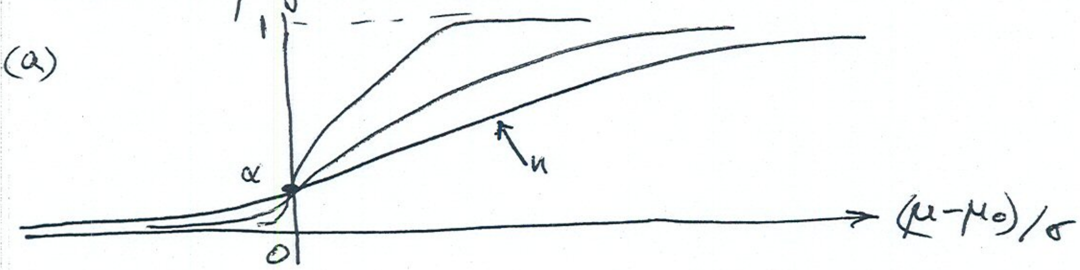
Reject H_0 if $\dfrac{\bar{X}}{\sigma/\sqrt{n}} > z_{\alpha/2}$ or if $\dfrac{\bar{X}}{\sigma/\sqrt{n}} < -z_{\alpha/2}$

$$\text{Power} = P_\mu\left[\frac{\bar{X}}{\sigma/\sqrt{n}} > z_{\alpha/2}\right] + P_\mu\left[\frac{\bar{X}}{\sigma/\sqrt{n}} < -z_{\alpha/2}\right]$$

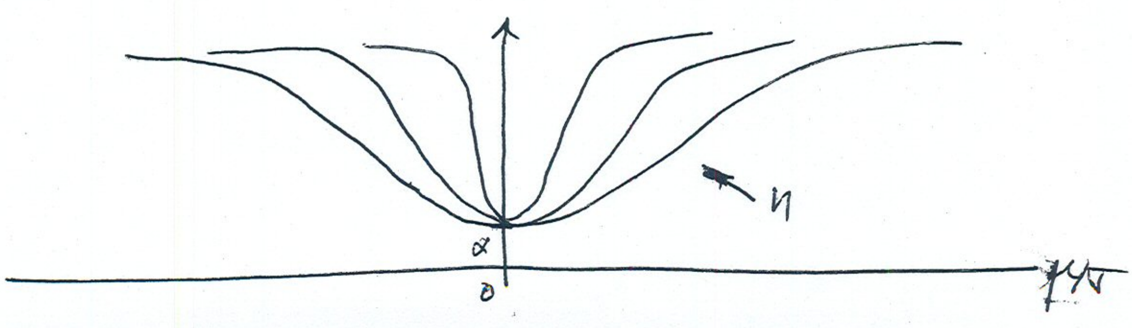
$$= 1 - \Phi\left(z_{\alpha/2} - \sqrt{n}\,\mu/\sigma\right) + \Phi\left(-z_{\alpha/2} - \sqrt{n}\,\mu/\sigma\right)$$

Sketch of power functions

(a)



(b)



8.26 (a) We prove a stronger result: Let $\psi(X)$ be a nondecreasing function of X. Then if $\{f_\theta(x) \mid \theta \in \Theta\}$ is MLR, then $E_\theta[\psi(X)]$ is nonincreasing in θ.

Proof: Let $\theta_1 < \theta_2$. Then,

$$E_{\theta_2}[\psi(X)] - E_{\theta_1}[\psi(X)] = \int \psi(x)[f_{\theta_2}(x) - f_{\theta_1}(x)]\,dx$$

$$= \int_{f_{\theta_2}(x) > f_{\theta_1}(x)} \psi(x)[f_{\theta_2}(x) - f_{\theta_1}(x)]\,dx + \int_{f_{\theta_2}(x) < f_{\theta_1}(x)} \psi(x)[f_{\theta_2}(x) - f_{\theta_1}(x)]\,d$$

$$\geq \inf\{\psi(x) \mid f_{\theta_2}(x) > f_{\theta_1}(x)\} \int_{f_{\theta_2} > f_{\theta_1}} [f_{\theta_2}(x) - f_{\theta_1}(x)]\,dx + \sup\{\psi(x) \mid f_{\theta_2}(x) < f_{\theta_1}(x)\} \int_{f_{\theta_2} < f_{\theta_1}} [f_{\theta_2}(x) - f_{\theta_1}(x)]\,d$$

$$= (b-a)\int [f_{\theta_2}(x) - f_{\theta_1}(x)]\,dx + a\int_{-\infty}^{\infty} [f_{\theta_2}(x) - f_{\theta_1}(x)]\,d$$

$$\geq 0 \quad \text{since } b \geq a.$$

To see that $b-a \geq 0$, note that $f_{\theta_2}(x)/f_{\theta_1}(x)$ is monotone in x, so $\{x \mid f_{\theta_2}(x) < f_{\theta_1}(x)\}$ is an interval $(-\infty, c)$ and $\{x \mid f_{\theta_2}(x) > f_{\theta_1}(x)\}$ is an interval $(-\infty, d)$ with $c \leq d$. Because ψ is monotone, $a = \sup_{x<c} \psi(x) \leq \inf_{y>d} \psi(y) = b$

Now let $\psi(x) = I\{X \leq x\}$

(b) $f_\theta(x) = \dfrac{1}{\pi[1+(x-\theta)^2]}$; $F_\theta(x) = \frac{1}{2} + \frac{\pi}{2}\tan^{-1}(x-\theta)$

Obviously $F_{\theta_1}(x) \geq F_{\theta_2}(x)$ but $\dfrac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \dfrac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2}$ is not monot

8.33 $X_1, \ldots, X_n$ iid unif $(\theta, \theta+1)$ $\quad H_0: \theta = 0 \quad H_1: \theta > 0$

(a) Reject H_0 if $X_{(1)} > k$ or if $X_{(n)} > 1$

$$P_0\left[\{X_{(1)} > k\} \cup \{X_{(n)} > 1\}\right] = P\left[X_{(1)} > k\right] = (1-k)^n = \alpha$$

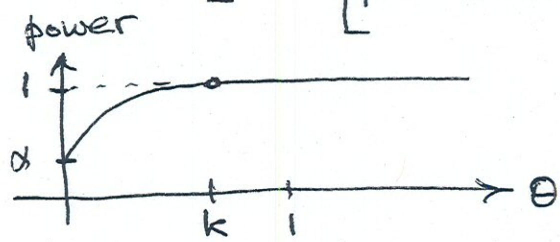
$$\therefore \; k = 1 - \alpha^{1/n}$$

(b) Power $= P\left[\{X_{(1)} > k\} \cup \{X_{(n)} > 1\}\right]$

$$= P_\theta\left[\{k < X_1, \ldots, X_n \le 1\} \cup \{X_{(n)} > 1\}\right]$$

$$= \prod_1^n P_\theta\left[k < X_i \le 1\right] + P_\theta\left[X_{(n)} > 1\right]$$

$$= \left[1 - \max\{k, \theta\}\right]^n + 1 - (1-\theta)^n$$



Power $= 1$ if $\theta \ge k$

(c) If $\theta < k$, use Neyman-Pearson Lemma.

$H_0: \theta = 0$ vs $H_1: \theta = \theta_1 < k$

$$f_0(X_1, \ldots, X_n) = I\{0 \le X_{(1)} < X_{(n)} \le 1\}$$

$$f_{\theta_1}(X_1, \ldots, X_n) = I\{\theta \le X_{(1)} < X_{(n)} \le \theta + 1\}$$

- If $X_{(n)} > 1$, then $f_0(\underline{X}) = 0$, $f_1(\underline{X}) = 1$, so reject
- If $X_{(1)} > k$ and $X_{(n)} \le 1$, then $f_0(\underline{X}) = f_1(\underline{X})$ and we reject. Neyman Pearson Lemma does not specify decision in this case.
- If $\theta_1 \le X_{(1)} \le k$, $f_0(\underline{X}) = f_{\theta_1}(\underline{X})$. Our test ~~rejects~~ accepts and Neyman Pearson Lemma does not specify a decision

- If $X_{(1)} < \theta_1$ ~~and if~~ and $X_{(n)} \le 1$, then $f_0(\underset{\sim}{X}) = 1$ and ~~then~~ $f_{\theta_1}(\underset{\sim}{X}) = 0$. Our test rejects, as does the Neyman Pearson test.

Our test agrees with the Neyman-Pearson specifications for $\theta < k$. For $\theta \ge k$ it has power 1, so no test is more powerful. Furthermore, our test has size α. Therefore it is UMP for $H_0: \theta = 0$ vs $H_1: \theta > 0$.

<u>Note</u> If we were testing $H_0: \theta = 0$ vs $H_1: \theta \ge k$, a test with power 1 and smaller size would be available. But that test would not be UMP against $H_0: \theta > 0$.

8.35 (a) $T' = \dfrac{Z+\mu}{\sqrt{Y/\nu}}$ where Y, Z independent, $Z \sim N(0,1)$

and $Y \sim \chi^2_\nu$

(a) $E[T'] = E[Z]\, E[(Y/\nu)^{-1/2}] + \mu E[(Y/\nu)^{-1/2}]$

$$E[(Y/\nu)^{-1/2}] = \sqrt{\nu} \int_0^\infty y^{-1/2} \frac{y^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)} e^{-y/2}\, dy$$

$$= \sqrt{\nu/2}\, \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\nu/2)} \int_0^\infty \frac{y^{\frac{\nu-1}{2}-1}}{2^{\frac{\nu-1}{2}}\Gamma(\frac{\nu-1}{2})} e^{-y/2}\, dy$$

$$= \sqrt{\frac{\nu}{2}}\, \Gamma(\tfrac{\nu-1}{2}) / \Gamma(\nu/2)$$

<u>Note:</u> if $\nu = 1$, the expected value doesn't exist

$$E[T'^2] = E\left[\frac{(Z+\mu)^2}{Y/\nu}\right] = E\left[\frac{Z^2 + 2\mu Z + \mu^2}{Y/\nu}\right]$$

$$= (1+\mu^2)\, E(\nu/Y)$$

$$E(\nu/Y) = \nu \int_0^\infty y^{-1} \frac{y^{\nu/2-1} e^{-y/2}}{2^{\nu/2}\Gamma(\nu/2)}\, dy$$

$$= \frac{\nu}{2} \Gamma(\tfrac{\nu-2}{2}) / \Gamma(\nu/2) \quad \text{if } \nu > 2$$

$$\therefore E(T') = (1+\mu^2) \frac{\nu}{2} \Gamma(\tfrac{\nu-2}{2}) / \Gamma(\nu/2) - \frac{\mu^2 \nu}{2} \Gamma(\tfrac{\nu-1}{2})^2 / \Gamma(\nu/2)^2$$

$$= \frac{\nu}{2} \frac{2}{\nu-2} + \frac{\mu^2 \nu}{2\Gamma(\nu/2)^2} \left[\Gamma(\nu/2)\Gamma(\tfrac{\nu-2}{2}) - \Gamma(\tfrac{\nu-1}{2})^2 \right]$$

(b) $f(t;\delta)$

$$= \frac{e^{-\delta^2/2}}{\Gamma(1/2)\Gamma(\nu/2)\sqrt{\nu}} \sum_0^\infty \left(\frac{2}{\nu}\right)^{k/2} \frac{(\delta t)^k}{k!} \frac{\Gamma(\frac{\nu+k+1}{2})}{[1+t^2/\nu]^{(\nu+k+1)/2}}$$

If $\delta = 0$ $\quad f(t;0) = \dfrac{\Gamma(\frac{\nu+1}{2})}{\Gamma(1/2)\Gamma(\nu/2)\sqrt{\nu}} \dfrac{1}{(1+t^2/\nu)^{(\nu+1)/2}}$

Since $\Gamma(1/2) = \sqrt{\pi}$, this is formula (5.3.6)

$$\log f = h_1(\delta) + h_2(t) + \log\left[\sum_0^\infty \left(\frac{2}{\nu}\right)^{k/2} \frac{(\delta t)^k}{k!} \frac{\Gamma\left(\frac{\nu+k+1}{2}\right)}{(1+t^2/\nu)^{k/2}}\right]$$

$$= h_1(\delta) + h_2(t) + \log\left[\sum_0^\infty \frac{1}{k!}\Gamma\left(\frac{\nu+k+1}{2}\right)\left(\frac{2\delta^2 t^2}{\nu(1+t^2/\nu)}\right)^{k/2}\right]$$

$$\frac{\partial^2 \log f}{\partial t \partial \delta} = \frac{\partial}{\partial t}\left\{\frac{\sum_1^\infty \frac{1}{k!}\Gamma\left(\frac{\nu+k+1}{2}\right)\frac{k}{2}\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{\frac{k}{2}-1}\frac{4\delta t^2}{\nu+t^2}}{\sum_0^\infty \frac{1}{k!}\Gamma\left(\frac{\nu+k+1}{2}\right)\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{k/2}}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{\sum_1^\infty \frac{k}{k!}\Gamma\left(\frac{\nu+k+1}{2}\right)\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{k/2}}{\delta \sum_0^\infty \frac{1}{k!}\Gamma\left(\frac{\nu+k+1}{2}\right)\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{k/2}}\right]$$

$$= \frac{\sum_1^\infty \frac{1}{(k-1)!}\Gamma\left(\frac{\nu+k+1}{2}\right)\frac{k}{2}\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{\frac{k}{2}-1}\frac{(\nu+t^2)4\delta^2 t - 2\delta^2 t^2(2t)}{(\nu+t^2)^2}}{\delta\left[\sum_0^\infty \frac{1}{k!}\Gamma\left(\frac{\nu+k+1}{2}\right)\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{k/2}\right]^2}$$

$$= \frac{\sum_1^\infty \frac{1}{(k-1)!}\Gamma\left(\frac{\nu+k+1}{2}\right)\frac{k}{2}\left(\frac{2\delta^2 t^2}{\nu+t^2}\right)^{\frac{k}{2}-1}\frac{4\delta^2\nu t + 4\delta^2 t^3 - 4\delta^2 t^3}{(\nu+t^2)^2}}{\delta\left[\sum - - - -\right]^2}$$

$$> 0$$

Note that if $f(t;\delta_2)/f(t;\delta_1) \uparrow$ as $t\uparrow$, for $\delta_1 < \delta_2$

$$\frac{f(t;\delta+\Delta\delta) - f(t;\delta)}{f(t;\delta)\,\Delta\delta} \uparrow \quad \text{fa } \Delta\delta > 0$$

$$\Rightarrow \quad \frac{\partial}{\partial\delta}\log f(t;\delta) \uparrow$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t\partial\delta}\log f(t;\delta) \geq 0$$

Conversely, if $\frac{\partial^2}{\partial t\partial\theta} g(t;\theta) \geq 0$, then $g(t;\theta_2)/g(t;\theta_1)$ is increasing in t if $\theta_1 < \theta_2$.

8.39 (X_i, Y_i), $i=1, \dots, n$ iid bivariate normal
$H_0: \mu_X = \mu_Y \qquad H_1: \mu_X \neq \mu_Y$

(a) $X_i - Y_i \sim N(\mu_X - \mu_Y, \operatorname{Var}(X_i - Y_i))$

Define $\mu_X - \mu_Y \equiv \mu_W$

$$\operatorname{Var}(X_i - Y_i) = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2 \equiv \sigma_W^2$$

Normality holds because any linear combination of multivariate normals is normal.

(b) Reduce data to $W_i = X_i - Y_i$, $i=1, \dots, n$.

The likelihood of the W_i is $(2\pi)^{-n/2}\sigma_W^{-n}\exp\left[-\frac{1}{2\sigma_W^2}\Sigma(W_i-\mu_W\right.$

$$\frac{\partial \log L}{\partial \mu_W} = -\frac{1}{2\sigma_W^2}\sum_1^n (W_i - \mu)(-2) = 0 \Rightarrow \overline{W} = \hat{\mu}_W = \overline{X} - \overline{Y}$$

$$\left.\frac{\partial \log L}{\partial \sigma_W}\right|_{\mu_W = \overline{W}} = -\frac{n}{\sigma_W} + \frac{2}{\sigma_W^3}\sum_1^n (W_i - \overline{W})^2 = 0$$

$$\Longrightarrow \hat{\sigma}_W^2 = \frac{1}{n}\sum_1^n (W_i - \overline{W})^2$$

$$\sup L = (2\pi)^{-\frac{n}{2}}\left[\frac{1}{n}\Sigma(W_i - \overline{W})^2\right]^{-\frac{n}{2}} e^{-n/2}$$

Under $H_0: \mu_W = 0$, $L = (2\pi)^{-n/2}\sigma_W^{-n}\exp\left[-\frac{1}{2\sigma_W^2}\Sigma W_i^2\right]$

$$\frac{\partial \log L}{\partial \sigma_W} = -\frac{n}{\sigma_W} + \frac{2}{2\sigma_W^2}\Sigma W_i^2 = 0$$

$$\Longrightarrow \hat{\sigma}_{W0}^2 = \frac{1}{n}\Sigma W_i^2 = \frac{1}{n}\Sigma(W_i - \overline{W})^2 + \overline{W}^2$$

$$\sup_{H_0} L = (2\pi)^{-n}(\hat{\sigma}^2 + \overline{W}^2)^{-\frac{n}{2}} e^{-n/2}$$

$$\Lambda = \left[\frac{\hat{\sigma}^2 + \overline{W}^2}{\hat{\sigma}^2}\right]^{-n/2} \qquad \Lambda < c \Rightarrow \left|\frac{\overline{W}}{\hat{\sigma}_W}\right| > k$$

$$\Rightarrow \left|\frac{\overline{W}}{S_W/\sqrt{n}}\right| > t_{n-1, \alpha/2}$$

Note that under H_0,

$$T = \frac{\overline{W}}{S_W/\sqrt{n}} = \left[\frac{\overline{W} - 0}{\sigma_W/\sqrt{n}}\right] \Big/ \left[\frac{\sum (W_i - \overline{W})^2/\sigma_W^2}{n-1}\right]^{1/2}$$

The numerator of the RHS is $N(0,1)$, the denominator is $\chi^2_{n-1}/(n-1)$ and the numerator and denominator are independent. $\therefore$ T has the Student t distribution with $(n-1)$ degrees of freedom.

<u>8.47</u> $H_0^-: \mu_X - \mu_Y \le -\delta$ vs $H_1^-: \mu_X - \mu_Y > -\delta$

$$\log L = -(m+n)\log\sigma - (m+n)\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\left\{\sum_1^m (X_i - \mu_X)^2 + \sum (Y_j - \mu_Y)^2\right\}$$

<u>Unrestricted</u>: $\dfrac{\partial \log L}{\partial \mu_X} = -\dfrac{1}{2\sigma^2}\sum (X_i - \mu)(-2) = 0 \Rightarrow \hat{\mu}_X = \bar{X}$

Similarly $\hat{\mu}_Y = \bar{Y}$

$$\frac{\partial}{\partial\sigma}\log L(\bar{X}, \bar{Y}, \sigma) = -\frac{m+n}{\sigma} + \frac{2}{2\sigma^3}\left\{\sum_1^m (X_i - \bar{X})^2 + \sum_1^n (Y_j - \bar{Y})^2\right\} = 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{(m+n-2)S_p^2}{m+n}$$

where $S_p^2 = \dfrac{\sum_1^m (X_i - \bar{X})^2 + \sum_1^n (Y_j - \bar{Y})^2}{m+n-2}$

$$\sup L = (2\pi)^{-\frac{m+n}{2}}\left[\frac{(m+n-2)S_p^2}{m+n}\right]^{-\frac{m+n}{2}} e^{-\frac{m+n}{2}}$$

Under H_0^-: If $\bar{X} - \bar{Y} \le -\delta$, the MLE's are the same as above and $\Lambda = 1$.

If $\bar{X} - \bar{Y} > -\delta$, look for maximum on the boundary of the H_0^- region: $\mu_X - \mu_Y = -\delta$, $\sigma \ge 0$

$$\log L(\mu_X, \mu_X + \delta, \sigma)$$

$$= -(m+n)\log\sigma - (m+n)\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\left\{\sum_1^m (X_i - \bar{X})^2 + m(\bar{X} - \mu_X)^2 + \sum_1^n (Y_j - \bar{Y})^2 + n(\bar{Y} - \mu_X - \delta)^2\right\}$$

$$\frac{\partial \log L}{\partial \mu_X} = -\frac{m}{2\sigma^2}(\bar{X} - \mu_X)(-2) - \frac{n}{2\sigma^2}(\bar{Y} - \mu_X - \delta)(-2) = 0$$

$$\Rightarrow \hat{\mu}_{X0} = \frac{m\bar{X} + n(\bar{Y} - \delta)}{m+n}$$

$$0 = \frac{\partial \log L}{\partial \sigma} = -\frac{m+n}{\sigma} + \frac{1}{\sigma^3}\left\{(m+n-2)S_p^2 + mn^2\left(\frac{\bar{X} - (\bar{Y} - \delta)}{m+n}\right)^2 + nm^2\left(\frac{\bar{X} - \bar{Y} - \delta}{m+n}\right)^2\right\}$$

$$\Rightarrow \hat{\sigma}^2 = (m+n)^{-1}\left\{(m+n-2)S_p^2 + (1/m + 1/n)^{-1}(\bar{X} - \bar{Y} + \delta)^2\right\}$$

$$\sup_{H_0} L = \left[\frac{1}{m+n}\left\{\cancel{}(m+n-2)S_p^2 + \left(\frac{1}{m}+\frac{1}{n}\right)^{-1}(\bar{X}-\bar{Y}+\delta)^2\right\}\right]^{-\frac{m+n}{2}} e^{-\frac{m+n}{2}}$$

If $\bar{X}-\bar{Y} > \delta$,

$$\Lambda = \left[\frac{(m+n-2)S_p^2}{(m+n-2)S_p^2 + (1/m+1/n)^{-1}(\bar{X}-\bar{Y}+\delta)^2}\right]^{\frac{m+n}{2}}$$

$$= \left[1 + \frac{1}{m+n-2}(T^-)^2\right]^{-\frac{m+n}{2}}$$

$$\Lambda < c \Rightarrow (T^-)^2 > k \text{ and } \bar{X}-\bar{Y} > -\delta$$

$$\Rightarrow T^- > k' = t_{m+n-2,\alpha} \quad \text{to get size } \alpha$$

(b) Use one sided test which rejects if

$$T^+ = \frac{\bar{Y}-\bar{X}-(-\delta)}{\sqrt{S_p^2(1/m+1/n)}} > t_{m+n-2,\alpha}$$

Derive by interchanging X and Y in (a)

(c) $H_0: \mu_X - \mu_Y \le -\delta$ or $\mu_X - \mu_Y \ge +\delta$ } equivalence problem

$H_1: -\delta < \mu_X - \mu_Y < \delta$

Reject if both H_0^- and H_0^+ are rejected, using T^- and T^+ as in (a) and (b). This is an <u>intersection – union</u> test.

(d) Apply Theorem 8.3.24. Let $\theta_\ell = \{\mu_X, \mu_X+\delta, \sigma_\ell\}$ where μ_X is fixed and $\sigma_\ell \to 0$. Each θ_ℓ satisfies H_0^- and the T^- test has size α for each θ_ℓ.

$$P_{\theta_\ell}\left[T^+ > t_{m+n-2,\alpha}\right] = P_{\theta_\ell}\left[\frac{\bar{Y}-\bar{X}-(+\delta)+2\delta}{S_p\sqrt{1/m+1/n}} > t\right]$$

$$= P\left[T^+ + \frac{2\delta/\sigma}{(S_p/\sigma)\sqrt{1/m+1/n}} > t\right] \to 1$$