$\boxed{2.14}$

(a) Use integration by parts: $U = x$, $dV = f(x)dx$, $V = -[1-F(x)]$

$$\mu = \int_0^\infty x f(x)\,dx = \Big[x\,(F(x)-1)\Big]_0^\infty - \int_0^\infty (F(x)-1)\,dx$$

$$= \int_0^\infty [1-F(x)]\,dx$$

This calculation is only valid if $\lim_{x\to\infty} x[1-F(x)] = 0$, but this holds iff $E(X) < \infty$.

Proof: If $\mu < \infty$, then $\int_0^\infty x f(x)\,dx$ is convergent

$\therefore$ $\forall \varepsilon$, $\exists M > 0$ such that $\int_M^\infty x f(x)\,dx < \varepsilon$.

But $\int_M^\infty x f(x)\,dx \geq M \int_M^\infty f(x)\,dx = M[1-F(M)]$.

Therefore $M[1-F(M)] \to 0$ as $M \to \infty$. To see the converse, suppose ~~lim inf x[1-F(x)]~~

$\liminf_{x\to\infty} x[1-F(x)] > 0$. Then

$\int_M^\infty x f(x)\,dx > M[1-F(M)]$ can not converge to zero, so $E(X) = \infty$.

(b) $\mu = \sum_{k=0}^\infty k\,P(X=k) = 0$

$+ P(X=1)$

$+ P(X=2) + P(X=2)$

$+ P(X=3) + P(X=3) + P(X=3)$

$+ \cdots$

Sum down the columns of this array to obtain

$\mu = P(X>0) + P(X>1) + \cdots$

$= \sum_{k=0}^\infty [1-F(k)]$

The existence of μ means the terms of this array of positive quantities can be summed in any order.

$\boxed{4.20}$ $\quad Y_1 = X_1^2 + X_2^2 \qquad Y_2 = X_1/\sqrt{Y_1} = X_1/\sqrt{X_1^2+X_2^2}$

(a) The transformation is not 1-1 because we can not identify the sign of X_2 from Y_1 and Y_2. However, it is 1-1 on the sets $\{(x_1, x_2) \mid x_2 > 0\}$ and $\{(x_1, x_2) \mid x_2 < 0\}$. [We neglect $\{(x_1, x_2) \mid x_2 = 0\}$ because $P[X_2 = 0] = 0$.]

On $\{(x_1, x_2) \mid x_2 > 0\}$ we have

$$X_1 = Y_1^{1/2} Y_2 \qquad X_2 = Y_1^{1/2}(1-Y_2^2)^{1/2}$$

$$J = \det \begin{bmatrix} \frac{1}{2} y_1^{-1/2} y_2 & y_1^{1/2} \\ \frac{1}{2} y_1^{-1/2}(1-y_2^2)^{1/2} & y_1^{1/2}\left(\frac{1}{2}\right)(1-y_2^2)^{-1/2}(-2y_2) \end{bmatrix}$$

$$= -\frac{1}{2}\,\frac{1}{(1-y_2^2)^{1/2}} \qquad |J| = \frac{1}{2}(1-y_2^2)^{-1/2}$$

On $\{(x_1, x_2) \mid x_2 < 0\}$ we have

$$X_1 = Y_1^{1/2} Y_2 \qquad X_2 = -Y_1^{1/2}(1-Y_2^2)^{1/2}$$

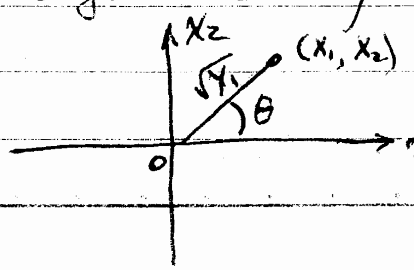
and $|J| = \frac{1}{2}(1-y_2^2)^{-1/2}$

$$\therefore f_Y(y_1, y_2) = \frac{1}{2\pi} e^{-y_1/2}\left[\frac{1}{2}(1-y_2^2) + \frac{1}{2}(1-y_2^2)\right]$$

$$= \frac{1}{2} e^{-y_1/y_2} \cdot \frac{1}{\pi}(1-y_2^2)^{-1/2}$$

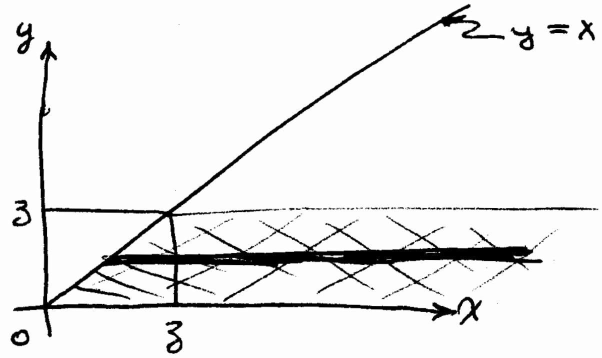
(b) The joint density factors, so Y_1 and Y_2 are independent.

(c)



The point (X_1, X_2) is represented by $(\sqrt{Y_1}, \theta)$ in polar coordinates, and $\cos\theta = Y_2$. We see that the length and direction of (X_1, X_2) are independent.

$\boxed{4.26}$



(a) We need $P[Z \le z \text{ and } W=0]$ and $P[Z \le z \text{ and } W=1]$

$$P[Z \le z \text{ and } W=0] = P[\min\{X,Y\} \le z \text{ and } Y \le X]$$

$$= \int_{y=0}^{z} \int_{x=y}^{\infty} \frac{1}{\lambda\mu} e^{-x/\lambda} e^{-y/\mu}\, dx\, dy$$

[see diagram]

$$= \int_{y=0}^{z} \frac{1}{\lambda} e^{-y/\lambda} e^{-y/\mu}\, dy$$

$$= \frac{1/\lambda}{1/\lambda + 1/\mu}\left[1 - \exp\left(-z\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\right)\right]$$

$$= \frac{\mu}{\lambda+\mu}\left[1 - \exp\left(-z\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\right)\right]$$

Similarly $P[Z \le z \text{ and } W=1] = \frac{\lambda}{\lambda+\mu}\left[1 - \exp\left(-z\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\right)\right]$

$\therefore P[Z \le z] = 1 - \exp\left(-z\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\right)$

and $P[W=1] = \frac{\lambda}{\lambda+\mu}$ (let $z \to \infty$)

(b) Clearly $P[Z \le z \text{ and } W=w] = P[Z \le z]\,P[W=w]$ for all $z \ge 0$ and all w. Hence Z and W are independent.

$\boxed{4.36}$ (a) We assume $X_i \mid P_i \sim \text{Bernoulli}(P_i)$

~~and~~ $\qquad P_i \sim \text{beta}(\alpha, \beta)$.

We also assume the P_i are independent to make sure the trials are independent.

$$Y = \sum_1^n X_i$$

$$E(Y) = \sum_1^n E(X_i) = \sum_1^n E\left(E(Y_i \mid P_i)\right)$$

$$= \sum E(P_i) = \frac{n\alpha}{\alpha+\beta}$$

The last equality comes from properties of the beta distribution

(b) Since Y is a sum of independent terms, $E(Y) = nE(X_i)$ and $\operatorname{Var} Y = n \operatorname{Var} X_i$.

$$\operatorname{Var} X_i = E[\operatorname{Var}(X_i \mid P_i)] + \operatorname{Var}[E(X_i \mid P_i)]$$

$$= E[P_i(1-P_i)] + \operatorname{Var} P_i$$

$$= E(P_i) - E(P_i^2) + E(P_i^2) - [E(P_i)]^2$$

$$= \frac{\alpha}{\alpha+\beta}\left[1 - \frac{\alpha}{\alpha+\beta}\right] = \frac{\alpha\beta}{(\alpha+\beta)^2}$$

$$\therefore \operatorname{Var} Y = \frac{n\alpha\beta}{(\alpha+\beta)^2}$$

In fact, since the X_i are iid with values $0, 1$, $Y = \sum X_i$ is binomial with parameters n and $\frac{\alpha}{\alpha+\beta}$.

(c) $E(Y) = \sum_{i=1}^{k} E[E(X_i \mid P_i)] = \sum_{i=1}^{k} n_i \alpha/(\alpha+\beta)$

Because the X_i are still independent $\operatorname{Var} Y = \sum_1^k \operatorname{Var} X_i$ as in (b).

$$\operatorname{Var} X_i = E[\operatorname{Var}(X_i \mid P_i)] + \operatorname{Var}[E(X_i \mid P_i)]$$

$$= E[n_i P_i (1-P_i)] + \operatorname{Var}[n_i P_i]$$

$$= n_i E(P_i) - n_i E(P_i^2) + n_i^2 \operatorname{Var} P_i$$

$$= n_i \frac{\alpha}{\alpha+\beta} - n_i \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} + n_i^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{n_i \alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\therefore \operatorname{Var} Y = \sum_{i=1}^{n} \frac{n_i \alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$\boxed{4.56}$

(a) $P(\text{positive in pooled sample}) = 1 - P(\text{all subjects negative})$
$= 1 - (1-p)^k$

(b) $X = \#$ tests for $N = mk$ subjects

Let $Z_j = \#$ tests for $k = \begin{cases} k+1 \text{ with prob } 1-(1-p)^k \\ 1 \text{ with prob } (1-p)^k \end{cases}$

$$X = \sum_{j=1}^{m} Z_j$$

$$E(X) = \sum_{1}^{m} E(Z_j) = m\left[(1-p)^k + (k+1)\left[1-(1-p)^k\right]\right]$$

$$= m\left[k+1-k(1-p)^k\right]$$
$$= mk + m - mk(1-p)^k$$

If $p \to 0$, $E(X) \to m$ so Plan (ii) is preferred if p is near 0

4.58

(a) $\mathrm{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$

$= E[E\{(X-\mu_x)(Y-\mu_y)|X\}] = E[(X-\mu_x)E\{Y-\mu_y \mid X\}]$

$= E[(X-\mu_x)(E(Y|X)-\mu_Y)] = \mathrm{Cov}(X, E(Y|X))$

The last equality follows from the fact that

$\mu_Y = E(Y) = E[E(Y|X)]$.

(b) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, Y-E(Y|X)+E(Y|X))$

$= \mathrm{Cov}(X, Y-E(Y|X)) + \mathrm{Cov}(X, E(Y|X))$

$= \mathrm{Cov}(X, Y-E(Y|X)) + \mathrm{Cov}(X,Y)$ from (a)

$\therefore \mathrm{Cov}(X, Y-E(Y|X)) = 0$

(c) $\mathrm{Var}\, Y = \mathrm{Var}[E(Y|X)] + E[\mathrm{Var}(Y|X)]$ (Thm 4.4.7)

$= \mathrm{Var}[Y - E(Y|X) + E(Y|X)]$

$= \mathrm{Var}[E(Y|X)] + \mathrm{Var}[Y-E(Y|X)]$

$+ 2\,\mathrm{Cov}[Y-E(Y|X), E(Y|X)]$

But the covariance is

$E\{E[(Y-E(Y|X))E(Y|X)|X]\} = 0$

$\therefore E[\mathrm{Var}(Y|X)] = \mathrm{Var}[Y-E(Y|X)]$