INFINITE MEASURE MIXING FOR SOME MECHANICAL SYSTEMS

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ABSTRACT. We show that if an infinite measure preserving system is well approximated on most of the phase space by a system satisfying the local limit theorem, then the original system enjoys mixing with respect to the global observables, that is, the observables which admit an infinite volume average. The systems satisfying our conditions include Lorentz gas with Coulomb potential, the Galton board and piecewise smooth Fermi-Ulam pingpongs.

1. Introduction

Mixing plays a central role in the study of stochastic properties of finite measure transformations. Recently there has been a surge of interest in studying mixing properties of infinite measure systems ([27, 35, 36, 34, 46, 3, 8, 33, 47, 39, 2, 37, 23, 38, 40]). In contrast with finite measure case there are several different notions of mixing for infinite measure preserving transformations. Recently, motivated by statistical mechanics considerations, Marco Lenci [29] introduced several notions of mixing for spatially extended observables (see also [30, 6, 31]). Let $T$ be a map of a space $X$ preserving an infinite measure $\mu$. The idea of [29] is to introduce two spaces: the space of local functions $L^1$ and the space of global functions $\mathcal{G} \subset L^\infty$. The functions from $\mathcal{G}$ are supposed to admit an average value

$$\bar{\Phi} = \lim_{\mu(V) \to \infty} \frac{1}{\mu(V)} \int_V \Phi d\mu$$

where the limit has to be understood in an appropriate sense. The map $T$ is called local global mixing (of type 1) if for each $\phi \in L^1$ and each $\Phi \in \mathcal{G}$ we have

$$\lim_{n \to \infty} \int \phi(x) \Phi(T^n x) d\mu = \left( \int \phi d\mu \right) \bar{\Phi}.$$  

(1.1)

$T$ is called global global mixing if for each $\Phi_1, \Phi_2 \in \mathcal{G}$ for large $n$ and large $V$

$$\frac{1}{\mu(V)} \int_V \Phi_1(x) \Phi_2(T^n x) d\mu \approx \Phi_1 \Phi_2.$$
The goal of this paper is to illustrate those notions on several examples of piecewise smooth hyperbolic dynamical systems preserving a smooth invariant measure.

2. Results.

Definition 2.1. $T$ is global-global mixing (of type 3) if for each $\Phi_1, \Phi_2 \in \mathbb{G}$

$$\lim_{n \to \infty} \limsup_{\mu(V) \to \infty} \frac{1}{\mu(V)} \int_V \Phi_1(x)\Phi_2(T^nx) d\mu = \lim_{n \to \infty} \liminf_{\mu(V) \to \infty} \frac{1}{\mu(V)} \int_V \Phi_1(x)\Phi_2(T^nx) d\mu = \bar{\Phi}_1 \bar{\Phi}_2.$$

We begin with systems having a lot of symmetry.

Let $X = M \times \mathbb{Z}^d, x = (y,z) \in X$ and $T(y,z) = (f(y), z + \tau(y))$ where $M$ is a locally compact metric space and $f$ preserves a probability measure $\nu$. We equip $X$ with the measure $\mu$ which is the product of $\nu$ and the counting measure on $\mathbb{Z}^d$. We write $\tau_n(y) = \sum_{j=0}^{n-1} \tau(f^j(y))$.

Definition 2.2. $T$ satisfies a mixing local limit theorem (MLLT) at scale $L_n$ if there is a continuous function $p$ such that

$$\int p(z) d\beta = 1$$

and for each $\phi_1, \phi_2 \in C(M)$ for each $\mathbb{Z}^d$-valued sequence $z_n^0$ such that $z_n^0 / L_n \to 0$ and for each $K < \infty$,

$$\lim_{n \to \infty} \sup_{z \in \mathbb{R}^d, |z| < K} \left| L_n^d \int \phi_1(y)\phi_2(f^n(y))1_{\tau_n = z_n^0 + [zL_n]} d\nu - \nu(\phi_1)\nu(\phi_2)p(z) \right| = 0$$

where $[\cdot]$ means taking lower integer part coordinate-wise.

$T$ satisfies a shifted mixing local limit theorem at scale $L_n$ if there is a sequence $D_n \in \mathbb{R}^d$ and a continuous function $p$ satisfying (2.1), such that for each $\phi_1, \phi_2 \in C(M)$ for each $\mathbb{Z}^d$-valued sequence $z_n^0$ such that $z_n^0 - D_n / L_n \to 0$, and for each $K < \infty$, (2.2) holds.

We remark that the MLLT implies the following useful a priori bound: if $\phi_1, \phi_2$ are bounded functions then the absolute value of the integral in (2.2) is bounded by $C \|\phi_1\|_{\infty} \|\phi_2\|_{\infty}$. Now a standard approximation argument shows that the convergence in (2.2) is uniform for $\phi_1, \phi_2$ in a compact subset of $C(M)$. The same remark applies to all variants of the MLLT considered in this paper, i.e. to the shifted
MLLT, the AMLLT and to condition (M4) (the last two are to be defined later).

We now let \( G_O \) be the space of bounded uniformly continuous functions such that for each \( a_1, a_2, \ldots, a_d, \pm \in \mathbb{R} \) with \( a_i, - < a_i, + \),

\[
\lim_{N \to \infty} \frac{1}{N^d} \prod_j (a_{j, +} - a_{j, -}) \int_{[a_{j, -}N, a_{j, +}N]} \Phi(x) d\mu(x) = \Phi
\]

and \( G_U \) be the space of bounded uniformly continuous functions such that for each \( \varepsilon \) there exists \( N_0 \) such that for each cube \( V \) of size greater than \( N_0 \) we have

\[
\left| \frac{1}{\mu(V)} \int_V \Phi(x) d\mu(x) - \Phi \right| \leq \varepsilon.
\]

**Theorem 2.3.** Suppose that \( T \) satisfies MLLT. Then

(a) \( T \) is local global mixing with respect to \( G_O \);
(b) \( T \) is global global mixing with respect to \( G_O \).

For random walks this result is proven in [7]. The proof in the general case follows the arguments of [7], however, we will provide the proof in §3.1 since our setting is quite different from that of [7].

**Theorem 2.4.** Suppose that \( T \) satisfies a shifted MLLT. Then

(a) \( T \) is local global mixing with respect to \( G_U \);
(b) \( T \) is global global mixing with respect to \( G_U \).

In fact, Theorem 2.3 holds under weaker conditions, to wit that (2.2) holds outside of a bounded subset of \( \mathbb{R}^{d_1 + d_2} \), whose closure has zero Lebesgue measure.

Namely we consider a map \( T \) defined on \( X = D \cup (M \times \mathbb{Z}_{d_1}^+ \times \mathbb{Z}^{d_2}) \).

Let

\[
(y(x), z(x)) = \begin{cases} (n, 3) & \text{if } x = (n, 3) \in M \times \mathbb{Z}_{d_1}^+ \times \mathbb{Z}^{d_2} \\ (\infty, \infty) & \text{if } x \in D. \end{cases}
\]

**Definition 2.5.** \( T \) satisfies the almost mixing LLT (AMLLT) if there is a bounded non-negative continuous function \( p : \mathbb{R}_{d_1}^+ \times \mathbb{R}_{d_2} \) such that (2.1) holds and a set \( B \subset \mathbb{R}_{d_1 + d_2} \) such that for each \( R, \varepsilon > 0 \) there is a set \( B_{\varepsilon, R} \subset \mathbb{R}_{d_1 + d_2} \) which is a finite union of boxes such that

\[
B \cap \{|z| \leq R\} \subset B_{\varepsilon, R} \subset \{|z| \leq R\},
\]

(2.3) \[ \text{mes}(B_{\varepsilon, R}) \leq \varepsilon \]
and if $x$ is distributed according to a measure $\tilde{\nu}$ which has compactly supported Lipschitz density $\phi$ with respect to $\mu$ then for each continuous function $\psi : M \to \mathbb{R}$ for any sequence $z_n^0$ such that $z_n^0/L_n \to 0$,

\begin{equation}
\lim_{n \to \infty} \sup_{z \in \mathbb{R}^{d_1+d_2}|z| \in \{||z|| \leq R\}\setminus B_z,R} |L_n^{d_1+d_2}\tilde{\nu}\left(\psi(y(T_n^nx))1_{z(T_n^nx)=z_n^0+|z|L_n}\right) - p(z)\nu(\psi)| = 0
\end{equation}

In equation (2.3), and also in the sequel, mes stands for the Lebesgue measure.

**Theorem 2.6.** Suppose that $T$ satisfies the AMLLT. Then $T$ enjoys local global mixing with respect to $\mathcal{G}_O$ and global global mixing with respect to $\mathcal{G}_O$.

Next we provide applications of Theorem 2.3 to maps which are asymptotically periodic at infinity.

**Proposition 2.7.** If $T$ is a map of a space $X$ preserving an infinite measure $\mu$ which is global global mixing and if $\tilde{T}$ equals to $T$ away from a finite measure set then $\tilde{T}$ is global global mixing.

We now allow more drastic perturbations.

**Definition 2.8.** We say that $\tilde{T}$ is very well approximated by $T$ at infinity if $\tilde{T}$ preserves $\mu$ and one of the following holds:

(i) For each $\varepsilon > 0$ there exists $R :$ for each $|z| > R$ there is a set $A_{z,\varepsilon} \subset M$ such that $\mu(A_{z,\varepsilon}) < \varepsilon$ and for $y \notin A_z \ d(\tilde{T}(y,z),T(y,z)) < \varepsilon$.

(ii) $d = 1$, $\tilde{T} : \tilde{X} \to \tilde{X}$ where $\tilde{X} = (M \times N) \cup D$ (or $\tilde{X} = (M \times Z) \cup D$) where $D$ is a finite measure set and the above estimates hold for $z > R$ (respectively for $|z| > R$).

We say that $\tilde{T}$ is well approximated by $T$ at infinity if either (i) or (ii) holds and $\tilde{T}$ preserves a measure $\tilde{\mu}$ such that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ which satisfies the following. If $V$ is a box such that $\sup_V z \leq (1 + \delta) \inf_V z$ then

\begin{equation}
\sup_V \frac{d\tilde{\mu}}{d\mu} \leq (1 + \varepsilon) \inf_V \frac{d\tilde{\mu}}{d\mu}.
\end{equation}

**Theorem 2.9.** Suppose that $\tau$ is bounded and both $\tau$ and $T$ are almost everywhere continuous.

(a) If $\tilde{T}$ is very well approximated by $T$ at infinity and $T$ is global global mixing with respect to either $\mathcal{G}_O$ or $\mathcal{G}_U$, then $\tilde{T}$ is global global mixing with respect to the same space.

(b) If $\tilde{T}$ is well approximated by $T$ at infinity and $T$ is global global mixing with respect to $\mathcal{G}_U$, then so is $\tilde{T}$. 
Next we provide conditions for local global mixing. We assume that there is a class \( \mathcal{M} \) of probability measures on \( X \) and for each \( \varepsilon > 0 \) there is a class \( \mathcal{M}_\varepsilon \) of probability measures on \( M \) such that

(M1) \( \tilde{T} \) preserves \( \mathcal{M} \).

(M2) For each compactly supported Lipschitz function \( \phi \) for each \( \varepsilon > 0 \) there is a finite set of functions \( \phi_1, \ldots, \phi_k \in L^\infty(X) \cap L^1(\mu) \) supported on a unit neighborhood of the support of \( \phi \) and constants \( c_1, \ldots, c_k \) such that

\[
\left\| \phi - \left( \sum_{j=1}^{k} c_j \phi_j \right) \right\|_\infty \leq \varepsilon \quad \text{and} \quad \phi_j \mu \in \mathcal{M}.
\]

(M3) For each \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) there exists \( R > 0 \) such that for each \( m \in \mathcal{M}_\varepsilon \)

\[
m(x) = R \quad \text{and} \quad d(T^n x, \tilde{T}^n x) \geq \varepsilon \) \leq \varepsilon.
\]

(M4) Measures from \( \mathcal{M}_\varepsilon \) satisfy uniform LLT in the sense that for each \( \phi \in C(M) \) for each \( K \) for each \( z \)

\[
L_n^d \mu_n \left( \phi(f^n x) 1_{z(T^n x) = z_n} \right) - p(z_n / L_n) \nu(\phi) \to 0
\]

and the convergence is uniform for \( m \in \mathcal{M}_\varepsilon \) and \( |z_n| / L_n \leq K \).

(M5) There is a constant \( C < \infty \) such that for each \( m \in \mathcal{M} \) and each \( \varepsilon > 0 \) there exists \( n_0 = n_0(m, \varepsilon) \) such that for \( n \geq n_0 \) there is a decomposition \( T^n \sum m = \sum_j (c'_j m'_j + c''_j m''_j) \) where \( m'_j, m''_j \) are supported on \( \{ z = j \} \), \( m'_j \in \mathcal{M}_\varepsilon \) and \( \sum_j c''_j \leq C \varepsilon \).

(M6) For each \( m \in \mathcal{M} \) for each \( R > 0 \) \( m(|z(T^n x)| \leq R) \to 0 \) as \( n \to \infty \).

We note that (M6) is satisfied if there is a random variable \( Z \) which has no atoms at 0 and such that for each \( m \in \mathcal{M} \), if \( x \) is distributed according to \( m \), then \( z(T^n x) / L_n \Rightarrow Z \).

**Theorem 2.10.** If (M1)-(M6) are satisfied then \( \tilde{T} \) is local global mixing with respect to \( \mathcal{G}_U \).

3. Proofs

Let \( \mathbb{L} \) be the space of compactly supported Lipschitz functions on \( X \). Note that \( \mathbb{L} \) is dense in \( L^1(\mu) \) so a standard approximation argument shows that it suffices to prove (1.1) for \( \phi \in \mathbb{L} \). So henceforth we will suppose that all local functions are in \( \mathbb{L} \).

3.1. Infinite Volume Mixing for cocycles.

**Proof of Theorem 2.3.** Let \( \phi \in \mathbb{L}, \Phi \in \mathcal{G}_O \). Since \( \phi \) is compactly supported, we have \( \phi(y, z) = \sum_k \phi_k(y) 1_{z = z_k} \) with a finite sum. Thus it suffices to prove the statement for \( \phi_k \), which for brevity is denoted by
\( \phi \) in the sequel. By the definition of \( G_O \), given \( \varepsilon > 0, R \) and \( \delta \), there exists \( K_0 \) such that for \( K \geq K_0 \) and for any cube \( V \) of size \( \delta K \) whose center is within \( RK \) from the origin, we have

\[
\left| \frac{1}{\mu(V)} \int_V \Phi d\mu - \bar{\Phi} \right| \leq \varepsilon.
\]

Choose \( R \) such that

\[
\int_{|z| \geq R} p(z)dz < \varepsilon.
\]

Then for large \( n \)

\[
\nu(y : |\tau_n(y)| \geq L_nR) < 2\varepsilon.
\]

Thus

\[
\left| \int \phi(x)\Phi(T^n x)d\mu - \int \phi(x)\hat{\Phi}(T^n x)d\mu \right| \leq 2||\phi||_\infty ||\Phi||_\infty \varepsilon
\]

where \( \hat{\Phi} = \Phi 1_{|z| \leq R L_n} \). Let \( \Phi_m = \Phi 1_{z=m} \) for \( m \in \mathbb{Z}^d \). By the foregoing discussion,

\[
\left| \int \phi(x)\Phi(T^n x)d\mu - \sum_{|m| \leq RL_n} \int \phi\Phi_m(T^n x)d\mu \right| \leq 2||\phi||_\infty ||\Phi||_\infty \varepsilon.
\]

By the MLLT, we can replace individual terms in the above sum by

\[
L_n^{-d} \mu(\phi)\mu(\Phi_m)p(m/L_n)
\]

with small relative error.

It remains to estimate

\[
\sum_{|m| \leq RL_n} \mu(\Phi_m)p(m/L_n).
\]

Divide \( \{ z \in \mathbb{Z}^d : |z| \leq RL_n \} \) into cubes \( C_j \) of size \( \delta L_n \). Let \( z_j \) be the center of \( C_j \). Assuming, as we can, that \( L_n > K \), we find

\[
\sum_{m \in C_j} \mu(\Phi_m)p(m/L_n) = \left[ p(z_j)\bar{\Phi} + e_j \right] \mu(M \times C_j),
\]

where \( e_j \) is an error term. By the continuity of \( p \), for fixed \( \varepsilon \) and \( R \) we can choose \( \delta \) so that \( |e_j| < 2\varepsilon \) for all \( j \). Summing over \( j \) and using (3.2) we obtain part (a). The proof of part (b) is similar but now we need to decompose both \( \Phi_1 \) and \( \Phi_2 \) as the sum of local functions. \( \square \)

The proof of Theorem 2.4 is similar except that we need to consider boxes around \( D_n \) rather than around the origin.
The proof of Theorem 2.6 is also similar. Namely we use (2.4) to control 
\[ m \in \{ |z| \leq L_n R : z/L_n \not\in B_{R, \varepsilon} \} \]. The points where
\[ z(T^n x) \in \{ |z| > L_n R \} \cup L_n B_{R, \varepsilon} \] or \( T^n x \in D \)
give small contribution due to (2.1) and (2.3).

3.2. Global mixing for approximations.

Proof of Proposition 2.7: Let \( A = \{ x : T x \neq \tilde{T} x \} \). Then
\[
\left| \int_V \Phi_1(x)[\Phi_2(T^n x) - \Phi_2(\tilde{T}^n x)]d\mu \right| 
\leq 2||\Phi_1||_\infty||\Phi_2||_\infty \nu(x : \exists 0 \leq k < n : T^k x \neq \tilde{T}^k x) \leq 2||\Phi_1||_\infty||\Phi_2||_\infty n \mu(A).
\]
Since the last expression does not grow as \( \mu(V) \to \infty \) we obtain the result. \( \square \)

Proof of Theorem 2.9. (a) We will show that for each \( n \)
\[
\lim_{\mu(V) \to \infty} \frac{1}{\mu(V)} \left[ \int_V \Phi_1(x)\Phi_2(\tilde{T}^n x)d\mu - \int_V \Phi_1(x)\Phi_2(T^n x)d\mu \right] = 0.
\]
Note that for each \( n \), \( T^n \) is continuous almost everywhere. Fix an arbitrary \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). An induction on \( n \) shows that for \( \nu \) a.e. \( y \) there exists \( \delta = \delta(y, \varepsilon) \) such that if \( \{ y_k' \}_{k=0}^n \) is a sequence such that
\[ d(y_0', y) < \delta \] and \( d(f(y_k'), y_{k+1}') \leq \delta \), then
\[ d(f^n(y), y_n') \leq \varepsilon \] and \( \tau_n(y) = \sum_{k=0}^{n-1} \tau(y_k') \).

We will say that \( y \) is \((\delta, \varepsilon)\)-good. Choose \( \delta = \delta(\varepsilon) \) so small that the measure of \( B_{n, \delta, \varepsilon} \), the set of not \((\delta, \varepsilon)\)-good points is less than \( \varepsilon \). Next, choose \( R = R(\varepsilon) \) such that for \( |z| > R \) we have \( \mu(A_{z, \delta}) \leq \varepsilon \).

We are now ready to establish (3.4). To fix ideas let us suppose that \( V \) is a cube of size \( L \). We split \( V \) into two parts. Let \( V_1 \) be the set of points \( x = (y, z) \in V \) for which
\bullet there is some \( k \leq n \) so that the absolute value of the \( z \)-coordinate of \( \tilde{T}^k x \) is less than \( R \), or
\bullet there is some \( k \leq n \) so that \( \tilde{T}^k x \in \cup_z A_{z, \delta} \), or
\bullet \( y \in B_{n, \delta, \varepsilon} \).

Denote \( V_2 = V - V_1 \). Assume \( |\tau| \leq r \). Then the orbit of points from \( V \) are within distance \( nr \) from \( V \). It follows that
\[ \mu(V_1) \leq (R + r)^d + 2(L + nr)^d n \varepsilon. \]
Thus the contribution of \( V_1 \) to (3.4) is less than
\[ [(R + nr)^d + 2(L + nr)^d n \varepsilon] ||\Phi_1||_\infty ||\Phi_2||_\infty. \]
On the other hand if \((x,z) \in V_2\) then \(d(T^n(x,z), \tilde{T}^n(x,z)) \leq \varepsilon\) and so the contribution of \(V_2\) is less \(\mu(V)||\Phi_1||_\infty \text{Osc}(\Phi_2, \varepsilon)\) where
\[
\text{Osc}(\Phi, \varepsilon) = \sup_{\delta(x',x'') \leq \varepsilon} |\Phi(x') - \Phi(x'')|.
\]

It follows that for large \(L\)
\[
\frac{1}{\mu(V)} \left| \int_V \Phi_1(x) \left[ \Phi_2(\tilde{T}^n x) - \Phi_2(T^n x) \right] d\mu \right| \\
\leq 3n\varepsilon||\Phi_1||_\infty||\Phi_2||_\infty + ||\Phi_1||_\infty \text{Osc}(\Phi_2, \varepsilon).
\]

Since \(\varepsilon\) is arbitrary, we can take the limit \(\varepsilon \to 0\) obtaining (3.4). This completes the proof of part (a).

To prove part (b) we may assume that \(V\) is such that \(\sup_V z \leq (1 + \delta(\varepsilon)) \inf_V z\). If this does not hold, we subdivide \(V\) into smaller boxes and remove the central part (which has small relative measure). Next we use (2.5) to replace
\[
\frac{1}{\mu(V)} \left[ \int_V \Phi_1(x)\Phi_2(\tilde{T}^n x) d\tilde{\mu} \right] \text{ by } \frac{1}{\mu(V)} \left[ \int_V \Phi_1(x)\Phi_2(T^n x) d\mu \right]
\]
and then conclude as before using (3.4).

\[\square\]

3.3. Local global mixing for approximations.

Proof of Theorem 2.10. Due to (M2) it suffices to show that for each \(m \in M\) and for each \(\Phi \in G_U\) we have \(m(\Phi(T^n x)) \to \Phi\).

In the proof, we will choose small parameters \(\varepsilon > 0, \delta = \delta(\varepsilon) > 0\) and large numbers \(\bar{n} = \bar{n}(\varepsilon), R = R(\varepsilon, \delta, \bar{n}), n = n(\varepsilon, \delta, \bar{n}, R)\).

Using (M4) and precompactness of the set \(\{\Phi_1\}\) (where \(\Phi_1(x) = \Phi(x, l)\)), we conclude as in the proof of Theorem 2.3 that for each \(\varepsilon\) there is a number \(\bar{n}\) such that for all \(m \in M_\varepsilon\) and all \(z \in \mathbb{Z}^d\)
\[
|m(\Phi(f^n y, z + \tau_n(y))) - \Phi| \leq \varepsilon.
\]

By equicontinuity of \(\{\Phi_1\}\) for each \(\varepsilon\) there exists \(\delta \leq \varepsilon\) such that if \(d(x', x'') < \delta\) then \(|\Phi(x') - \Phi(x'')| < \varepsilon\). Let \(\tilde{m} = \tilde{T}^{n-\bar{n}} m\). We claim that if \(n\) is large enough, then
\[
|\tilde{m}(\Phi(T^n x)) - m(\Phi(\tilde{T}^n x))| \leq \varepsilon.
\]

Indeed we can split the last integral into two parts corresponding to \(|z(x)| > R\) and \(|z(x)| \leq R\) where \(R\) is such that
\[
\tilde{m}(x : |z(x)| > R \text{ and } d(\tilde{T}^n x, T^n x) > \delta) < \delta
\]
(such \(R\) exists by (M3)).

The contribution of the set \(|z(x)| > R\} can be estimated by
\[
2||\Phi||_\infty \tilde{m}(x : |z(x)| > R, d(\tilde{T}^n x, T^n x) > \delta) + \varepsilon \leq 2||\Phi||_\infty \delta + \varepsilon
\]
while the contribution of \( \{|z(x)| < R\} \) can be made as small as we wish by (M6).

It remains to handle \( \tilde{m}(\Phi(T^n x)) \). By (M5) we can split

\[
\tilde{m}(\Phi(T^n x)) = \sum_j (c'_j m'_j(\Phi(T^n x)) + c''_j m''(\Phi(T^n x)))
\]

\[
= \sum_j c'_j m'_j(\Phi(T^n x)) \pm 2\varepsilon ||\Phi||_{\infty}.
\]

By the choice of \( \bar{n} \) for each \( j \)

\[
|m'_j(\Phi(T^n x)) - \bar{\Phi}| \leq \varepsilon.
\]

Since

\[
1 \geq \sum_j c'_j = 1 - \sum_j c''_j \geq 1 - \varepsilon
\]

the result follows.

\[\square\]


The results of Section 2 can be extended to flows in a straightforward way. Here, we briefly summarize the necessary changes in the definitions and theorems.

Let \( X = M \times \mathbb{Z}^d, x = (y, z) \in X \) and \( G^t(y, z) = (g^t(y), z + \tau^t(y)) \) for \( t \geq 0 \) (or for \( t \in \mathbb{R} \)) where \( X \) is as before, and \( g^t \) preserves a probability measure \( \kappa \). We equip \( X \) with the measure \( \lambda \) which is the product of \( \kappa \) and the counting measure on \( \mathbb{Z}^d \). We define the spaces \( L, G_O, G_U \) as before.

The definition of local-global and global-global mixing is analogous, we just need to replace \( T^n \) by \( G^t \) and let \( t \to \infty \) instead of \( n \to \infty \). Noting that the second coordinate of \( X \) is still discrete, we can extend the definition of MLLT and shifted MLLT by simply replacing \( f^n, \tau_n, z_n \in \mathbb{Z}^d, L_n, D_n \) and \( n \to \infty \) by \( g^t, \tau_t, z_t \in \mathbb{Z}^d, L_t, D_t \) and \( t \to \infty \) respectively. Similarly, we define AMLLT by replacing \( T^m, z_n, L_N \) and \( \lim_n \) by \( G^t, z_t, L_t \) and \( \lim_t \) respectively. With these adjustments, one can extend Theorems 2.3–2.6 as well as their proofs to the case of flows.

In the remaining results, the map \( \tilde{T} \) was approximated by a periodic map \( T \). In case of flows, we can define similar approximations by, say, comparing the two flows up to time 1. First, the following analogue of Proposition 2.7 holds:

**Proposition 4.1.** If \( G^t \) is a flow on a space \( X \) preserving an infinite measure \( \kappa \) which is global global mixing and if \( \tilde{G}^t \) equals to \( G^t \) for \( t \in [0, 1] \) away from a finite measure set, then \( \tilde{G} \) is global global mixing.
We can obtain a proof of Proposition 4.1 from the proof of Proposition 2.7 by replacing $A = \{x : Tx \neq \tilde{T}x\}$ by $A = \{x : \exists t \in [0, 1] : G_t(x) \neq \tilde{G}_t(x)\}$, and $n$ by $t$ in (3.3).

Similarly, in the definition of good and very good approximation, besides the obvious changes, we require that for all $y \notin A_z$ and for all $t \in [0, 1]$, $d(\tilde{G}_t(y, z), G_t(y, z)) < \varepsilon$. Then we have

**Theorem 4.2.** Suppose that $\{\tau_t(y) : y \in M, t \in [0, 1]\}$ is bounded and the set

$$\{y \in M : g_t(y) \text{ and } \tau_t(y) \text{ are continuous at } y\}$$

has full measure for any fixed $t$.

(a) If $\tilde{G}$ is very well approximated by $G$ at infinity and $G$ is global global mixing with respect to either $G_O$ or $G_U$, then $\tilde{G}$ is global global mixing with respect to the same space.

(b) If $\tilde{G}$ is well approximated by $G$ at infinity and $G$ is global global mixing with respect to $G_U$, then so is $\tilde{G}$.

The proof of Theorem 4.2 is similar to that of Theorem 2.9 with minor changes as before. We leave the details to the reader.

Finally, the assumptions (M1)–(M6) can analogously be formulated for flows. Namely, (M1) claims that $\tilde{G}_t$ preserves $\mathfrak{M}$ for every $t$, (M2) is unchanged and all changes in (M3)–(M6) amount to replacing $T, \tilde{T}$ by $G, \tilde{G}$ are as before. With these changes, and with a similar proof, we can derive the analogue of Theorem 2.10.

5. **Preliminaries on Lorentz gas and related systems.**

In the remaining part of the paper, we give several examples of systems satisfying the assumptions of Section 2. In those example we have a system moving in $\mathbb{R}^d$ with a number of scatterers removed and having elastic reflections from the boundary. The motion between the collisions will be either free (such as in case of Lorentz gas) or subject to a field. In this case the most interesting question from physical point of view is to study mixing properties of the continuous time system, however, mathematically one could also study the mixing properties of the collision map. We will also use natural examples above to illustrate several subtleties associated to the notions of local global and global global mixing.

In our examples, the system having approximate symmetry will be denoted by $\tilde{T}$ while its symmetric approximation will be denoted by $T$. In the continuous time setting, the corresponding systems will be denoted by $G^t$ and $\tilde{G}^t$ respectively.
For the reader’s convenience, we summarize some basic facts about Lorentz gas in this section. We will focus on the notions and results that are most important for studying global mixing properties. Everything in this section (as well as many other important results) can be found in [15]. Thus we do not give more references. Much of the theory presented in this section has been extended to billiards subject to external fields (see [10, 11, 16]). Additional reference will be given later when we discussing specific examples.

Let \( O_1, \ldots, O_J \) be disjoint convex subsets of the 2-torus \( \mathbb{T}^2 \) with \( C^3 \) boundary with non-vanishing curvature. These sets are also called scatterers. Let us consider a point particle that flies freely (with speed 1) in the interior of \( D_0 = \mathbb{T}^2 \setminus \bigcup O_j \), and upon reaching the boundary, it undergoes specular reflection (angle of incidence equals angle of reflection). This dynamics is called the Sinai billiard flow \( (g^t) \). It preserves the Lebesgue measure on \( D_0 \times S^1 \) (position and velocity). Let \( \kappa \) be the invariant Lebesgue measure normalized so as it is a probability measure. Identifying the torus with \([0, 1]^2\) and extending the scatterer configuration periodically to the plane, we define the billiard flow on \( D = \mathbb{R}^2 \setminus \bigcup_{\ell \in \mathbb{Z}^2} \bigcup_{j=1}^J (O_j + \ell) \) as before. We call the billiard flow in this infinite domain Lorentz gas. It is denoted by \( G^t \) and preserves \( \lambda \), the product of \( \kappa \) and the counting measure on \( \mathbb{Z}^2 \). We assume unless it is explicitly stated otherwise that the scatterer configuration is such that the free flight is bounded (aka finite horizon condition).

The billiard flow induces a billiard map (or collision map) by the Poincarè section taken at collisions. Namely, the phase space of the billiard map is

\[
M = \{(q,v) \in \partial D_0 \times S^1, \langle v,n \rangle \geq 0\},
\]

where \( n \) is the inward normal vector of \( \partial D \) at \( q \) (that is, \( q \) is the point of collision and \( v \) is the post-collisional velocity). The standard coordinates on \( M \) are \( r \) : arc length parameter for \( q \) and \( \phi \): the angle between \( n \) and \( v \) (\( \phi \in [-\pi/2, \pi/2] \) with clockwise orientation). The billiard map is denoted by \( f : M \to M \). It preserves the invariant measure \( \nu = c \cos \phi dr d\phi \), where \( c \) is a normalizing constant. Similarly, the billiard map of the Lorentz gas is \( T : X \to X \), where \( X = M \times \mathbb{Z}^2 \), \( T(y,z) = (f(y), z + \tau(y)) \) and \( \tau \in \mathbb{Z}^2 \) is the vector connecting the center of the cells where two consecutive collisions take place. It preserves the invariant measure \( \mu = \nu \times \text{counting} \).

The map \( f \) is hyperbolic: there is an (un)stable conefield, \( \mathcal{C}^{\text{u/s}}_y \subset T_y M \) so that \( Df(\mathcal{C}^{\text{u/s}}_y) \subset \mathcal{C}^{\text{u/s}}_{f(y)}, Df^{-1}(\mathcal{C}^{\text{u/s}}_y) \subset \mathcal{C}^{\text{u/s}}_{f^{-1}(y)} \). The cones are transversal, that is the angle between any stable vector (an element of...
for some $y$) and any unstable vector is uniformly bounded below by a positive number. (In fact there exist some constants $0 < c_1 < c_2$ so that $C^u$ can be defined as $c_1 \leq d\phi/dr \leq c_2$ and $C^s$ can be defined as $-c_2 \leq d\phi/dr \leq -c_1$ for all $y \in M$.)

The map $f$ is piecewise smooth with singularities at grazing collisions. Furthermore, as the expansion the and distortion are unbounded near grazing collisions, it is common to introduce artificial singularities

$$H_k = \{(r\phi), \phi = \pm \pi/2 \mp k^{-2}\}$$

for $k \geq k_0$. We call a smooth curve (un)stable if at each point its tangent vector belongs to the (un)stable cone. An (un)stable curve is homogeneous if it does not cross any singularity genuine or artificial. We call $W$ a local stable (unstable) manifold if $f^n(W)$ is a stable (unstable) curve for any $n \geq 0$ ($n \leq 0$, respectively).

For any unstable curve $W$ and point $y \in W$, we define the Jacobian of $f^n$ on $W$ at $y$ by $J_W f^n(y) = \|D_x f^n(dy)\|/\|dy\|$ with $dy \in T_y W$. The uniform hyperbolicity implies that there are constants $\Lambda > 1$ and $C$ so that $J_W f^n(y) \geq C\Lambda^n$ for $n > 0$ (and similarly for stable curves and $n < 0$). Furthermore, after the above extra partitioning of the phase space, one has the following distortion bounds. Let $W$ be a homogeneous unstable curve, such that $f^{-n}(W)$ is also homogeneous unstable for $n = 1, ..., N - 1$. Then for any $y_1, y_2 \in W$ and $n = 1, ..., N - 1$ we have

$$e^{-C|W|^{1/3}} \leq \frac{J_W f^{-n}(y_1)}{J_W f^{-n}(y_2)} \leq e^{C|W|^{1/3}}.$$  

(5.1)

Here, as well as in the sequel, $C$ denotes some finite number that only depend on the dynamical system (and not on the curve $W$ or $n$). Furthermore, the value of $C$ is not important and may change from line to line.

For any point $y \in M$, we denote by $r_u(y)$ ($r_s(y)$) the distance between $y$ and the singularity set, measured along the unstable (stable) manifold. More generally, given an unstable curve $W$ and $y \in W$, there is a homogeneous unstable curve $W' \subset f^n(W)$ that contains $f^n(y)$. Then $W'$ is cut into two pieces by $f^n(y)$, the length of the shorter one is denoted by $r_n(y)$.

The measure of points $y$ such that $r_u(y) = 0$ or $r_s(y) = 0$ is zero. It is also true that the measure of points having short (un)stable manifolds is small, namely

$$\nu(y : \min\{r_u(y), r_s(y)\} < \varepsilon) \leq C\varepsilon.$$  

(5.2)
A pair \( \ell = (W, \rho) \) is called a standard pair, if \( W \) is a homogeneous unstable curve and \( \rho \) is a probability measure on \( W \) satisfying

\[
\left| \log \frac{d\rho}{d\text{mes}}(y_1) - \log \frac{d\rho}{d\text{mes}}(y_2) \right| \leq C \frac{|W(y_1, y_2)|}{|W|^{2/3}},
\]

where \( |W(y_1, y_2)| \) is the length of the segment of \( W \) bounded by \( y_1 \) and \( y_2 \).

The image of a standard pair by the dynamics is a weighted sum of standard pairs (the image of a homogeneous unstable curve is a family of homogeneous unstable curves and the regularity of the density of \( \rho \) is preserved). A weighted average of standard pairs is called a standard family. Namely, a standard family is a (possibly uncountable) collection of standard pairs \( \mathcal{G} = \{(W_a, \nu_a)\}_{a \in \mathcal{A}} \) and a probability measure \( \eta = \eta_\mathcal{G} \) on \( \mathcal{A} \). Such a standard family \( \mathcal{G} \) induces a measure on \( M \) by

\[
\nu_\mathcal{G}(\cdot) = \int_\mathcal{A} \nu_a(\cdot \cap W_a) d\eta_\mathcal{G}(a).
\]

For standard families, the \( Z \)-function is defined as

\[
Z_\mathcal{G} = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_\mathcal{A} \nu_a(r_0 < \varepsilon) d\eta_\mathcal{G}(a).
\]

Important special cases are standard pair (\( \mathcal{A} \) has a single element) or the invariant measure \( \nu \). In particular, the conditional measures satisfy the required regularity and the \( Z \)-function of this family is finite.

Standard pairs are stretched by the dynamics due to expansion and are cut by singularities. The next result tells us that the "expansion is winning over fragmentation", that is the most of the weight is carried by long curves.

**Lemma 5.1** (Growth Lemma). There are constants \( \theta < 1, C_1, C_2 \) such that for a standard family \( \mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathcal{A} \), and \( \mathcal{G}_n = f^n(\mathcal{G}) \), we have

\[
Z_\mathcal{G}_n < C_1 \theta^n Z_\mathcal{G} + C_2.
\]

We also consider standard pairs on the phase space of the Lorentz gas, by shifting \( W \) with a vector \( m \in \mathbb{Z}^2 \), where \( \ell = (W, \rho) \) is a standard pair for the Smai billiard. In this case, we write \([\ell] = m\).

The Growth Lemma implies a local version of (5.2), namely, for any unstable curve \( W \) and for any \( n \geq 0 \),

\[
\text{mes}(y \in W : r_n(y) < \varepsilon) < C \varepsilon,
\]

where \( \text{mes} \) denotes the Lebesgue measure on \( W \).

We will also use the following important consequence of the Growth Lemma (see [15, §5.12] as well as the a proof of (7.11) below). Given
a curve $\gamma$ and a positive number $\delta$ let $\gamma_\delta = \{ x \in \gamma : r_s(x) \geq \delta \}$. Then there is a constant $K^*$ such that
\begin{equation}
\mes(\gamma - \gamma_\delta) \leq K^* \delta.
\end{equation}

Another application of the Growth Lemma requires an extra definition. Fix a large constant $\bar{Z}$. In particular we require that $\bar{Z} \geq 2C_2$ where $C_2$ is the constant from the Growth Lemma. In practice it is convenient to choose $\bar{Z}$ so large that there is a standard family $G$ with $Z_G < \bar{Z}$ such that $\nu_G$ is the invariant measure $\nu$. We say that a standard family $G$ is proper if $Z_G \leq \bar{Z}$. Then the Growth Lemma implies that there exists $n_0$ such that for any $n \geq n_0$ and for any measure $\check{\nu}$ defined by a proper standard family $G$, the measure $\check{\nu}_n(\phi) = \check{\nu}(\phi \circ f^n)$ also corresponds to a proper standard family (namely $f^nG$).

Another crucial property of partition of $(M, \nu)$ into stable (unstable) manifolds is absolute continuity. We refer the reader to [5, §8.6] for a comprehensive overview of absolute continuity of stable and unstable laminations. Here we just summarize the results for dispersive billiards we are going to use. Let $W_1$ and $W_2$ be two unstable curves which are close to each other. Let
\[ W_j = \{ x \in W_j : W^s(x) \cap \Gamma_{3-j} \} \]
and let $\pi_s : \tilde{W}_1 \to \tilde{W}_2$ be the stable holonomy $\pi_s(x) = W^s(x) \cap W_2$. Then $\pi_s$ is absolutely continuous and its Jacobian equals to $J(x, \pi_s x)$ where ([15, Equation (5.23)])
\begin{equation}
J(x, \pi_s x) = \prod_{n=0}^{\infty} \frac{J_{f^nW_1}(f^n x)}{J_{f^nW_2}(f^n \pi_s x)}.
\end{equation}

Next, [15, Theorem 5.42] tells us that there is a constant $C$ such that
\begin{equation}
e^{-C(d^{1/3}(x,\pi_s x)+\beta)} \leq J(x, \pi_s x) \leq e^{C(d^{1/3}(x,\pi_s x)+\beta)},
\end{equation}
where $\beta$ is the angle between the tangent vector to $W_1$ at $x$ and the tangent vector to $W_2$ at $\pi_s x$.

A similar statements hold for unstable holonomy.

Let us list several standard consequences of this fact ([5]).

Given an unstable curve $\gamma$ and a positive number $\delta$, consider the Hopf brush $\Lambda_\delta = \bigcup_{x \in \gamma_\delta} W^s(x)$. Consider the measure $\check{\nu}$ defined by
\[ \check{\nu}(A) = \int_{\gamma_\delta} \mes_{W^s}W^s(x) \cap A \mes_{\gamma}(x). \]
Let $\nu_{A_\delta}$ denote the restriction of $\nu$ to $A_\delta$. If $|\gamma| \geq 2K^*\delta$ so that (5.3) implies that $A_\delta \neq \emptyset$. Then there is a constant $\kappa_1 = \kappa_1(\delta)$ such that

$$(5.6) \quad \kappa_1 \leq \frac{d\nu}{d\nu_{A_\delta}} \leq \kappa_1^{-1}.$$ 

From the foregoing discussion it is not difficult to see that there is a constant $\kappa_2 = \kappa_2(\delta)$ such that for each $\gamma$ of length at least $2K^*\delta$,

$$(5.7) \quad \nu(A_\delta(\gamma)) \geq \kappa_2.$$ 

Another consequence of (5.6) is that if $A$ is a set of zero measure, then

$$(5.8) \quad \text{for } \nu \text{ almost every } x, \; \text{mes}(W^s(x) \cap A) = \text{mes}(W^u(x) \cap A) = 0.$$ 

6. Examples

6.1. Lorentz gas. The mixing local limit theorem holds for Lorentz gas with finite horizon in both discrete [44] and continuous setting [20]. Accordingly Theorem 2.3 applies to both Lorentz collision map and Lorentz flow, and so, both systems enjoy both local global and global global mixing with respect to $G_O$. In the case the horizon is infinite, to the best of our knowledge, only the MLLT for the collision map is available [45]. Therefore the discrete time system enjoys both local global and global global mixing with respect to $G_O$. It is quite likely that the MLLT holds also in the continuous time system and so Theorem 2.3 applies in that case as well.

One can also consider a Lorentz tube, where instead of motion on the plane the particle moves on the strip with a periodic configuration of convex scatterers removed. As before [44, 20] give MLLT in both discrete and continuous setting and so the system enjoys both local global and global global mixing with respect to $G_O$.

6.2. Local Perturbations of Lorentz gas. Next consider local perturbations of the Lorentz gas.

6.2.1. Local perturbations. Consider a billiard in a domain which is periodic outside of some ball. If the limiting periodic configuration has finite horizon (or equivalently, the perturbed configuration has finite horizon) then the conditions of Propositions 2.7 and 4.1 are satisfied and so the system enjoys global global mixing. On the other hand, local perturbations of the Lorentz gas do not have to be local global mixing. Indeed, we can trap particles in a bounded part of the phase space. For example, by allowing non-convex scatterers, one can arrange that the system has a stable elliptic orbit, so that the set $B$ of bounded orbits has positive measure. Let $B_L$ be the set of orbits which always
stay within distance $L$ from the origin. Take $\phi$ such that $\int_{B_L} \phi d\mu > 0$. Take two functions $\Phi_1, \Phi_2 \in G$ such that
(i) $\Phi_2 > \Phi_1$ and moreover
(ii) $\Phi_2 - \Phi_1 \geq 1$ inside the ball of radius $L$;
(iii) $\bar{\Phi}_2 = \bar{\Phi}_1$.
In this case
\[ \int \phi[(\Phi_2 - \Phi_1) \circ \tilde{T}^n] d\mu \geq \int_{B_L} \phi d\mu \]
does not tend to 0, so it is impossible that both
\[ \int \phi(x)\Phi_2(\tilde{T}^n x) d\mu(x) \to \mu(\phi)\Phi_2 \text{ and } \int \phi(x)\Phi_1(\tilde{T}^n x) d\mu(x) \to \mu(\phi)\Phi_1. \]

However, the system remains local global mixing if the configuration is a finite perturbation (i.e. finitely many scatterers discarded, finitely many new ones included) of a periodic Lorentz gas such that the scatterers in the entire configuration (including the perturbed part) are strictly convex, disjoint and have $C^3$ boundary. We call such a perturbation a mild perturbation. Mildly perturbed Lorentz gases are local global mixing as implied by Theorem 2.6 and

**Theorem 6.1.** The mildly perturbed periodic Lorentz gas satisfies the AMLLT with exceptional set $B = \{0\} \subset \mathbb{R}^2$ ($d_1 = 0, d_2 = 2$).

**Proof.** The proof is similar to (but easier than) the proof of Proposition 3.8 in [20] so we provide only a sketch of the argument.

We begin with discrete time. In the proof we will use letters with tildes to denote the objects associated to the mildly perturbed Lorentz gas, and the same letter without tildes will refer to periodic (unperturbed) system.

Let $\tilde{\nu}$ be as in the definition of AMLLT. The global central limit theorem for mildly perturbed periodic Lorentz gas is proved in [25, Theorem 1]. Thus there is a matrix $D$ such that
\[ \tilde{\nu} \left( \frac{\tilde{\tau}_n}{\sqrt{n}} \in \Omega \right) \to \int\int \mathcal{g}(u) du \]
where $\mathcal{g}$ is the density of the centered Gaussian distribution with covariance matrix $D$ and $\Omega \subset \mathbb{R}^2$ is a set whose boundary has zero Lebesgue measure.

Given a sequence $z_n$ such that $\frac{\tilde{\tau}_n}{\sqrt{n}} \to z \neq 0$ and a Lipshitz function $\psi$ we need to evaluate
\[ I_n = \tilde{\nu} (\psi(\tilde{x}_n) 1_{\tilde{\tau}_n = z_n}). \]
Take $\delta_1 \ll 1$ and denote $n_2 = \delta_1 n$, $n_1 = n - n_2$. 

Let the measure \( \nu^\tilde{x} \) be the normalized version of the restriction of \( \hat{T}^{n_1} \nu \) to the cell \( \tilde{z} \). That is, if \( p_{n_1}(\tilde{z}) = \nu(\hat{z} \circ \hat{T}^{n_1} = \tilde{z}) \), and \( A \subset M \), then
\[
\nu^\tilde{x}(A) = \frac{1}{p_{n_1}(\tilde{z})} \nu\left( \tilde{x} : \hat{T}^{n_1}(\tilde{x}) \in (A \times \{ z = \tilde{z} \}) \right).
\]
Then we have the decomposition
\[
I_n = \sum_{\tilde{z} \in \mathbb{Z}^2 - \{0\}} p_{n_1}(\tilde{z}) \nu^\tilde{x}(\psi(x_{n_2}) 1_{\tau_{n_2} = z_n - \tilde{z}}) + \hat{\epsilon}_1 \nu^\tilde{x}(\psi) \delta_t \sum_j \sum_{\tilde{z} \in B_j} p_{n_1}(\tilde{z}) g\left( \frac{\tilde{z} - z_n}{\sqrt{n_2}} \right),
\]
where \( \hat{\epsilon}_1 \) is an error term corresponding to the set of points \( \tilde{x} \) so that \( z \circ \hat{T}^{n_1}(\tilde{x}) = 0 \) and we assume without loss of generality that all perturbations are in the zeroth cell.

Choose \( K \gg 1 \) and consider the following approximation
\[
I_n = \sum_{|\tilde{z} - z_n| \leq K \sqrt{n_2}} p_{n_1}(\tilde{z}) \nu^\tilde{x}(\psi(x_{n_2}) 1_{\tau_{n_2} = z_n - \tilde{z}}) + \hat{\epsilon}_1 + \hat{\epsilon}_2
\]
where \( \hat{\epsilon}_2 \) is an error term. Note that there are no tildes inside \( \nu^\tilde{x}(\cdot) \). That is we pretend that the particle moves in the unperturbed environment for the last \( n_2 \) collisions. The error \( \hat{\epsilon} = \hat{\epsilon}_1 + \hat{\epsilon}_2 \) comes from two sources:

(A) There is a contributions from the cells with \( |\tilde{z} - z_n| > K \sqrt{n_2} \) and

(B) the particle may visit the perturbed region for some \( k \in [n_1, n] \).

Given \( \varepsilon \) we can choose \( \delta_t \) so small and \( K \) so large that both (A) and (B) have contributions which is less than \( \frac{\varepsilon}{n} \) similarly to [20, \S 6.2]. Note that [20, Lemma 2.8(b)], which is extensively used in this step, is formulated for the Lorentz tube and thus is not directly applicable here. However, we can replace it by [22, Lemma 4.8(b)], which is valid in a much more general setting, including the Lorentz gas.

Returning to main term in (6.1) we can use the MLLT for the periodic Lorentz gas to conclude that
\[
\nu^\tilde{x}(\psi(x_{n_2}) 1_{\tau_{n_2} = z_n - \tilde{z}}) \approx \frac{1}{n_2} g\left( \frac{z_n - \tilde{z}}{\sqrt{n_2}} \right) \nu(\psi).
\]
Let us divide the set \( \{ z : |z - z_n| \leq K \sqrt{n_2} \} \) into boxes \( B_j \) of size \( \delta_s \sqrt{n} \) where \( \delta_s \ll \delta_t \). Then
\[
\sum_{|\tilde{z} - z_n| \leq \sqrt{n_2}} p_{n_1}(\tilde{z}) \nu^\tilde{x}(\psi(x_{n_2}) 1_{\tau_{n_2} = z_n - \tilde{z}}) \approx \frac{\nu(\psi)}{\delta_t n} \sum_j \sum_{\tilde{z} \in B_j} p_{n_1}(\tilde{z}) g\left( \frac{\tilde{z} - z_n}{\sqrt{n_2}} \right),
\]
Since the oscillation of $g\left(\frac{z - z_n}{\sqrt{n_2}}\right)$ on $B_j$ is small, we can replace it by $g\left(\frac{z^{(j)} - z_n}{\sqrt{n_2}}\right)$ where $z^{(j)}$ is the center of $B_j$. Accordingly

$$\sum_{\bar{z} \in B_j} p_{n_1}(\bar{z}) g\left(\frac{\bar{z} - z_n}{\sqrt{n_2}}\right) \approx g\left(\frac{z^{(j)} - z_n}{\sqrt{n_2}}\right) \sum_{\bar{z} \in B_j} p_{n_1}(\bar{z}) =$$

(6.4) $g\left(\frac{z^{(j)} - z_n}{\sqrt{n_2}}\right) \bar{\nu}(\tilde{\tau}_n \in B_j)$.

The CLT for the mildly perturbed Lorentz gas and the fact that $z^{(j)}$ are close to $z_n$ for all $j$ imply that

(6.5) $\bar{\nu}(\tilde{\tau}_n \in B_j) \approx \delta_s^2 g(z)$

Combining (6.1)–(6.5) we obtain

$$I_n = \frac{g(z) \nu(\psi)}{n} \sum_j \frac{\delta_s^2}{\delta_t} g\left(\frac{z^{(j)} - z_n}{\sqrt{n_2}}\right).$$

The last sum is the Riemann sum of the integral of a Gaussian density over the set $\{|z| < K\}$. Accordingly taking $K$ large and choosing $\delta_s$ small to make the mesh sufficiently fine, we can make the last sum as close to 1 as we wish. This completes the sketch of proof of the AMLLT in the discrete time case.

The continuous time case is similar but we need to use the MLLT for flows proven in [22]. □

6.2.2. Lorenz gas in a half strip. Consider a Lorentz gas in a half strip, i.e. in $\mathbb{R}^+ \times [0, 1]$ with a periodic configuration of convex scatterers removed. (By periodicity we mean that if $S$ is a scatterer in our configuration and $S_{\pm} := S \pm (1, 0)$, then $S_+$ is in the scatterer configuration and if $S_- \subset \mathbb{R}^+ \times [0, 1]$, then $S_-$ also belongs to the configuration).

Using [25, Theorem 2] and proceeding as in the proof of Theorem 6.1, we have

**Theorem 6.2.** Lorentz gases in half strips satisfy the AMLLT with exceptional set $\mathcal{B} = \{0\} \subset \mathbb{R}_+$. Hence the Lorentz gas in a half strip satisfies both local global and global global mixing with respect to $G_O$.

6.3. Lorenz gas with external fields.
6.3.1. Lorentz gas in asymptotically vanishing potential fields. In this example we consider the same configuration of scatterers as in Example 6.1 but assume that the motion between collisions is subject to the potential

$$\ddot{q} = -\nabla U.$$ 

We suppose that the first three derivatives of $U$ are uniformly bounded and that

$$(6.6) \quad \lim_{|q| \to \infty} U(q) = 0, \quad \lim_{|q| \to \infty} \nabla U(q) = 0.$$ 

An example of such system is given by the Coulomb potential

$$(6.7) \quad U(q) = \frac{e}{|q|}.$$ 

For the Coulomb potential it is natural to assume that the origin is contained in the center of one of the scatterers. In this case $U$ is bounded.

In any case our system is Hamiltonian preserving the energy $H = v^2 + U(q)$. In particular under assumption (6.6) both the collision map $\tilde{T}$ and the continuous time system $\tilde{G}^t$ is well approximated by the Lorentz gas. Accordingly, Theorems 2.9 and 4.2 imply that both $\tilde{T}$ and $\tilde{G}^t$ enjoy global global mixing with respect to $\mathbb{G}_O$.

Also, as before, the assumption (6.6) is insufficient to ensure hyperbolicity close to the origin. In particular the system could have elliptic islands in the bounded part of the space (cf. [43]) and so it may fail to be local global mixing. On the other hand, we will show below that if $||U||_{C^3}$ is sufficiently small (e.g. in the Coulomb potential case the charge $e$ is small) then the system is local global mixing. To see this it suffices to check conditions (M1)-(M6).

We begin with the discrete time system. Let $\mathcal{M}$ to be the set of all compactly supported standard families such that for $m \in \mathcal{M}$ we have

$$(6.8) \quad m(x : r(x) < \varepsilon) \leq K \varepsilon$$

where $K$ is a sufficiently large constant. Then (M1) is checked in [11] (see also [15]). To check (M2) let $\phi$ be a Lipschitz function supported on a single scatterer $\Omega$. (Note that is suffices to check the local global mixing for Lipschitz functions $\phi$ as the set of Lipschitz functions is dense in $L$. The condition that $\phi$ is supported on a single scatter is also not restrictive since a function supported on a finite set of scatterers is a finite linear combination of the functions supported on a single scatterer.) We first observe that for each $\delta$ there exists $K(\delta)$ such that if $\phi$ has the following properties:

$$(6.9) \quad \delta \leq \phi \leq \delta^{-1}, \quad \mu(\phi) = 1, \quad \text{Lip}(\phi) \leq 2,$$

...
then $\phi \mu \in \mathcal{M}$ where $\mathcal{M}$ is defined by (6.8) with $K = K(\delta)$, see e.g. [15, Theorem 5.17]. Pick a large $R$. We have the following decomposition:

$$\phi = R||\phi||_1 - (R||\phi|| - \phi)1_{\Omega}.$$ 

Thus $\phi = c_1 \phi_1 - c_2 \phi_2$ where $c_1$ and $c_2$ are constants and

$$\phi_1 = \frac{1_{\Omega}}{\mu(\Omega)}, \quad \phi_2 = \frac{1_{\Omega} - \phi/R}{\mu(\Omega) - \frac{1}{R}\int_{\Omega} \phi d\mu}.$$ 

Note that as $R \to \infty$, $\phi_2 \to 1_{\Omega}/\mu(\Omega)$ in the space of Lipschitz functions, so if $R$ is sufficiently large then $\phi_1, \phi_2$ satisfy (6.9) with constant $\delta$ depending only on the minimal perimeter of the scatterers in our configuration. By the foregoing discussion, $\phi_1 \mu, \phi_2 \mu \in \mathcal{M}$.

(M3) follows from the transversality of the standard curves to singularities of the system, see [12, Section 4.5]. Next, let $\mathcal{M}_\varepsilon$ be the set of standard families on $\mathcal{M}$ such that all standard pairs in $\mathcal{m}$ is longer than $\varepsilon$. The local limit theorem for standard families follows from mixing LLT for $T$ as is explained in [20]. Thus (M4) holds. Next for $\mathcal{m}$ in $\mathcal{M}$ let $\mathcal{m}'_j$ is the measure corresponding to the standard pairs from $\mathcal{T}_n \mathcal{m}$ which belong to $\{z = j\}$ and have length greater than $\varepsilon$. Then (M5) follows from invariance of $\mathcal{M}$ (recall (M1)). Since checking (M6) requires more effort, we postpone it to Section 7.

The continuous time case can be handled similarly. We refer the reader to [21, 4] for the Growth Lemma and related results in the continuous time setting.

6.3.2. Lorentz gas in external field and Gaussian thermostat. Suppose that the system moves in the same domain as the Lorentz gas but the motion between the collisions is not free but rather satisfies

$$\ddot{q} = E(q) - \frac{\langle \dot{q}, E(q) \rangle}{||\dot{q}||^2}$$

where $E(q)$ is a periodic field and the second term models energy dissipation. This system is a $\mathbb{Z}^2$-cover of Sinai billiard in external field which we will denote by $f$. By [10] $f$ has unique SRB measure $\nu$ if $||E||_{C^1}$ is sufficiently small. Furthermore, a Young tower can be constructed by the results of [10, 11] (see also [9]). Thus (shifted) MLLT holds for $(f, \nu)$. The shifted MLLT for continuous time system follows from [22]. We note that for typical $E$ (including the constant field) the drift in the CLT is not equal to zero ([14]). Accordingly by Theorem 2.4 we have local global and global global mixing with respect to $(L, G_U)$. We also note that in the presence of the drift the system is dissipative in the sense of ergodic theory, that is, almost every particle tends to infinity. This gives a physical example of a system which enjoys both local global and global global mixing but is not ergodic.
6.4. Galton board. This model is similar to Example 6.3.1, however, we do not assume that the potential is vanishing at infinity. Namely we consider a particle moving in a half plane $q_1 > 0$ with a periodic configuration of convex scatterers removed (we confine the particle to the half plane by adding the vertical axis $q_1 = 0$ to the boundary of our domain). The motion between collisions is subject to a constant force field which corresponds to a linear potential $U = -gq_1$. This system preserves the energy

$$H = v^2/2 - gq_1.$$  

It is convenient to use the following coordinates: $q \in \mathbb{R}^2$ is the position of the particle and $\theta$ is the polar angle of the velocity vector $\tan \theta = \dot{q}_1/\dot{q}_2$. Then the speed could be recovered using the equation $|v| = \sqrt{2(H + gq_1)}$. Accordingly we consider the following space of global functions:

$G_O = \{ \Phi : \Phi \text{ is uniformly continuous in } (q, \theta) \text{ variables and for each } \varepsilon \text{ there is } R_0 \text{ such that if } \Omega_{q,R} = \{(q, \theta) : |q - q|_\infty \leq R, q_1 > 0 \} \text{ then for each } R \geq R_0 \text{ and any } q$

$$\left| \frac{1}{\mu(\Omega_{q,R})} \int_{\Omega_{q,R}} \Phi(q, \theta) d\mu - \bar{\Phi} \right| \leq \varepsilon \right\}.$$

Consider first the collision map $\tilde{T}$. Suppose that the kinetic energy of the particle, $K_n$, after $n$ collisions is large. In order to compute the next collision point it is convenient to make a time change $s = t/\sqrt{2K_n}$ so that with the new time units the particle moves with the unit speed. The time change is equivalent with replacing the field by a weaker one, thus the motion before the $(n + 1)$-st collision is governed by

$$\frac{d^2 q}{ds^2} = \frac{ge_1}{K_n}.$$  

Thus $\tilde{T}$ is well approximated at infinity by the Lorentz gas and hence it enjoys global global mixing with respect to $G_O$. The dynamics for small speed is more complicated, so we do not know if $\tilde{T}$ is local global mixing. We will see however, that a reasonable theory could be obtained if we assume that the energy $H$ is sufficiently large, so the system is a small perturbation of the Lorentz gas even for small $q_1$.

**Theorem 6.3.** There exists $H_0$ such that if $H \geq H_0$ then both the collision map $\tilde{T}$ and the continuous flow $\tilde{G}^t$ enjoy both global global mixing and local global mixing.

**Proof.** Global global mixing was proven above. To prove that $\tilde{T}$ is local global mixing we check conditions (M1)–(M6). We choose $\mathcal{M}$ and $\mathcal{M}_\varepsilon$
in the same way as in Example 6.3.1. (M2) and (M4) are checked in the same way as in that example. (M1) and (M5) follow from [13, Lemma 2.1] while (M3) is checked in [13, Section 3]. To check (M6) consider the following process

\[ K_n(t) = \frac{1}{\sqrt{n}} K_{\sqrt{n} t} \]

We claim that as \( t \to \infty \)

\[ K_n(t) \Rightarrow K(t) \]

where \( K(t) \) is the solution to the following stochastic differential equation:

\[ dK = \frac{\tilde{\sigma}^2}{4K} dt + \tilde{\sigma} dW, \quad K(0) = 0 \]

and \( \tilde{\sigma} \) is a positive constant (an explicit expression for \( \tilde{\sigma} \) is given in the first display on page 839 of [13]). Note that the equation (6.12) is well posed despite the singular coefficient as discussed in [13]. In case we start away from 0 and the process \( K_n \) is stopped when it reaches too high or too low values, (6.11) is proven in [13, Theorem 4]. The removal of this cutoff can be done in the same way as in the continuous time case, see the proof of Theorem 3 in [13] (note that this theorem assumes that the total energy \( H \) is large enough).

(6.11) implies that \( \frac{K_n}{\sqrt{n}} \Rightarrow Z := K(1) \). We note that \( K(t) \) is a power of the square Bessel process, so its density could be computed explicitly (cf. [19]). In particular, \( \mathbb{P}(Z = 0) = 0 \) proving (M6).

The proof in the continuous time case is similar. However, we need to modify (M1)–(M6) as explained below. Note that if \( q(t) \sim Q \gg 1 \) then \( v(t) \sim \sqrt{Q} \) so the particle will travel distance of order \( \sqrt{Q} \) during a unit time interval. This distance is too large for Lorentz particle to serve as a good approximation to the Galton particle. The good news is that a much shorter time is sufficient to observe the LLT on Galton board.

Accordingly we replace \( \mathcal{M}_\varepsilon \) by the family \( \mathcal{M}_{\varepsilon,t} \) consisting of the measures satisfying (6.8) and supported on the set \( \{ \hat{\varepsilon} \leq q_1/t^{2/3} < 1/\hat{\varepsilon} \} \) where \( \hat{\varepsilon} \) is chosen so that

\[ m\left( \hat{\varepsilon} < \frac{q_1(u)}{t^{2/3}} < \frac{1}{\hat{\varepsilon}} \text{ for all } u \in [t/2, t] \right) \geq 1 - \frac{\varepsilon}{100}. \]

Such \( \hat{\varepsilon} \) exists due to [13, Theorem 3].

Next we replace (M3) by

\[ \widetilde{(M3)} \forall m \in \mathcal{M} \forall s \exists T : \forall t \geq T : m\left( x : \hat{\varepsilon} < \frac{q_1(x)}{t^{2/3}} < \frac{1}{\hat{\varepsilon}} \text{ but } d(G^s(x), G^s(x)) > \varepsilon \right) \leq \varepsilon. \]

and replace by (M5) by
For each $m \in \mathcal{M}$ and each $\varepsilon > 0$ for each $s$ there exists $T$ and $\tau$ such that for $t \geq T$ we can decompose $\tilde{G}_{\tau}^{e-s/t^{1/3}} m = \sum_j (c_j m_j' + c_j'' m_j'')$ where $m_j'$, $m_j''$ are supported on $\{z = j\}$, $m_j' \in \mathcal{M}_{\varepsilon,t}$ and $\sum_j c_j'' \leq \varepsilon$.

The verification of (M1), (M2), (M3), (M4), (M5), (M6) is similar to the verification of (M1)–(M6) for the collision map $\tilde{T}$. This proves local global mixing. Also (M3) and the LLT for $G^t$ gives global global mixing similarly to the proof of Theorems 2.9 and 4.2. □

6.5. Fermi-Ulam pingpong. Consider the following one-dimensional system: a unit point mass moves horizontally between two infinite mass walls. Between collisions, the motion is free so that the kinetic energy is conserved, collisions between the particle and the walls are elastic. The left wall moves periodically, while the right one is fixed. The distance between the two walls at time $t$ is denoted by $\ell(t)$. We assume that $\ell$ is strictly positive, continuous and periodic of period 1. Moreover we assume that the restriction of $\ell$ to the open interval $(0, 1)$ is $C^5$ but $\dot{\ell}(1-) \neq \dot{\ell}(1+)$, where $\dot{\ell}(1+) = \lim_{t \downarrow 0} \dot{\ell}(t)$ and $\dot{\ell}(1-) = \lim_{t \uparrow 0} \dot{\ell}(t)$. Thus $\ell$ is piecewise smooth with singularities only at integers. Let $\tilde{T}$ be the map defined as follows. Let the particle move until the the next integer moment of time and then stop it after the first collision with the moving wall. Note that $\tilde{T}$ is conjugated to $G$-the time 1 map of the system. Namely for $\tilde{T}$ it is natural to use the following coordinates: the time of collision (taken modulo $\mathbb{Z}$) and the post collisional velocity at the moment of collision. For $G$ it is natural to use velocity and height. To pass from the first coordinate set to the second one, we replace the post collisional velocity with the precollisional one and then let the particle move backward until the first time it becomes an integer.

It is shown in [17] that $\tilde{T}$ is well approximated at infinity by the following map of the cylinder $\mathbb{T} \times \mathbb{R}$:

$$T(\tau, I) = (\tau - I, I + \Delta(\tau - I))$$ (6.13)

where

$$\Delta = \ell(0) \sigma \int_0^1 \ell^{-2}(s) \, ds, \quad \sigma = \dot{\ell}(1+) - \dot{\ell}(1-) .$$

$T$ covers a map $f$ of $\mathbb{T}^2$ which is defined by formula (6.13) with $I$ taken mod 1. If $\Delta \notin (0, 4)$ then the map $f$ is piecewise hyperbolic and according to [48, Section 7], it admits a Young tower and hence, satisfies the MLLT (see e.g. [26]). Therefore in this case $\tilde{T}$ and, hence, $G$ are global global mixing with respect to $G_U$.

We note that while the dynamics for large energies is described by a single parameter $\Delta$, the dynamics for low energies is far from universal.
In particular, it is easy to construct an example where $T$ has elliptic fixed points and so it is not ergodic. Thus we get another natural example where the map is global global mixing but is not ergodic.

On the other hand it is shown in [18] that if $\ell$ is piecewise convex, then $\tilde{T}$ is ergodic for most values of the parameter $\Delta$ (with at most a countable set of exceptions). One could expect that in that case $\tilde{T}$ is local global mixing, but this question requires a further investigation.

6.6. **Bouncing ball in a gravity field.** In this model the particle moves on $\mathbb{R}_+$ in a linear potential $U(x) = gx$ and collides elastically with an infinitely heavy wall whose position at time $t$ equals to $h(t)$. We assume that $h$ is 1-periodic and piecewise $C^2$ but not $C^2$. Let $\tilde{T}$ be the collision map in this model. It is shown in [49] that $\tilde{T}$ is well approximated at infinity by the map $T$ of the cylinder $\mathbb{T} \times \mathbb{R}$ given by

$$T(t, v) = (t + 2v/g, v + 2\tilde{h}(t + 2v/g)).$$

(6.14)

$T$ is a $\mathbb{Z}$ cover of the map $f$ of $\mathbb{T}^2$ defined by (6.14) with $t$ taken mod 1 and $v$ taken mod $\frac{g}{2}$. Moreover, it is proven in [49] that if either

$$\ddot{h} > 0 \text{ or } |\dot{h} + a| \leq \varepsilon$$

(6.15)

where $a > g$ and $\varepsilon = \varepsilon(a)$ is a small constant, then $f$ satisfies the conditions of [9]. Consequently it admits a Young tower with exponential tail and hence satisfies the MLLT. It follows that if (6.15) is satisfied, then $\tilde{T}$ enjoys global global mixing with respect to $\mathbb{G}_U$.

As in the previous example, the dynamics for small energies is not universal and the question about local global mixing may depend on the law energy dynamics of the system. Finally we note that the continuous time system is not global global mixing since on most of the phase space the motion is integrable. Namely let $\Phi$ be a non negative continuous functions which depends only on velocity, is 1-periodic and is supported on $\{v : d(v, \mathbb{Z}) \leq 0.01\}$. Then $\Phi = \int_{0}^{1} \Phi(v) dv > 0$. On the other hand for each $T$, on most of the set $\{v \leq V\}$ with $V \gg T$, velocity remains large on the time interval $[0,T]$. For such orbits $v(t) = v(0) - gt$ for $t \in [0,T]$ and so if $d(gT, \mathbb{Z}) > 0.04$ then $\Phi \cdot (\Phi \circ \tilde{G}^T) = 0$. Accordingly the large volume limit for such $T$’s is

$$\Phi \cdot (\Phi \circ \tilde{G}^T) = 0$$

precluding global global mixing. As in the discrete time case the question of local global mixing is more subtle and deserves a further investigation.
7. Checking (M6)

Here we check the condition (M6) for Lorentz gas with vanishing potential. We hope that a similar argument will apply to other hyperbolic systems with singularities, including the examples of §6.5 and §6.6 once their dynamics in the low energy regime is better understood.

7.1. Recurrence-transience dichotomy. For sets \( A, B \) we shall write \( A \equiv B \) if their symmetric difference satisfies \( \mu(A \triangle B) = 0 \).

In this section we prove an auxiliary result of independent interest. Let \( R^{\pm} = \{ x : |z(\tilde{T}^nx)| \not\to \infty \text{ as } n \to \pm\infty \} \).

Then, (see e.g. [1, §1.1]), \( R^- \equiv R^+ \). Let \( R = R^- \cap R^+ \) be the set of recurrent orbits. Then \( R \equiv R^+ \equiv R^- \).

Lemma 7.1. Either \( \tilde{\mu}(R) = 0 \) or \( \tilde{\mu}(R^c) = 0 \). In the second case \( \tilde{T} \) is ergodic.

Proof. Let \( R_0 = R, R_0^{\pm} = R^{\pm} \), and for \( n > 0 \) define inductively \( R_n = R_n^+ \cap R_n^- \) where

\[
R_n^+ = \{ x \in R_{n-1}^+ : \text{mes}(W_{loc}^s(x) \cap R_{n-1}^-) = 0 \},
\]

\[
R_n^- = \{ x \in R_{n-1}^- : \text{mes}(W_{loc}^u(x) \cap R_{n-1}^+) = 0 \}.
\]

We shall show inductively that

\[
(7.1) \quad R_n \equiv R_n^+ \equiv R_n^- = R_{n-1}.
\]

For \( n = 0 \) this follows from the foregoing discussion. Assuming that (7.1) holds for \( n-1 \) we obtain, using the absolute continuity of the stable lamination (namely, (5.8)) and the relation \( R_{n-1} \equiv R_{n-1}^+ \equiv R_{n-1}^- \), that

\[
R_n^+ \equiv \{ x \in R_{n-1}^+ : \text{mes}(W^s(x) \cap (R_{n-1}^+)^c) = 0 \} \equiv R_{n-1}^+
\]

where the last step use that, by construction, \( W^s(x) \cap (R_{n-1}^+)^c = 0 \) for \( x \in R_{n-1}^+ \). Thus \( R_n^+ \equiv R_{n-1} \). Likewise \( R_n^- \equiv R_{n-1}^- \), proving (7.1).

(7.1) shows that

\[
(7.2) \quad R_\infty := \bigcap_n R_n \equiv R.
\]

Let \( E_0 = E = E^+ \cap E^- \) where

\[
E^{\pm} = \{ x : |z(\tilde{T}^nx)| \not\to \infty \text{ as } n \to \pm\infty \}.
\]

and define \( E_n \) and \( E_\infty \) similarly to \( R_n \) and \( R_\infty \) respectively. Similarly to (7.2) we obtain that

\[
E_\infty \equiv E \equiv E^+ \equiv E^-.
\]
Denote $G = E_\infty \cup R_\infty$. By the foregoing discussion
$$G \equiv E \cup R \equiv E^+ \cup R^+.$$Since the last set equals to the whole phase space we conclude that
$$\mu(G^c) = 0.$$
Suppose for a moment that $R_\infty \neq \emptyset$. Pick $x' \in R_\infty$. Using an argument in [15, §6.4] we get that for every $x'' \in G$ where exists a Hopf chain, that is, a chain
$$x' = y_0, y_1, \ldots, y_n = x''$$ such that $y_j \in G$ and $y_{j+1} \in W^s_{loc}(y_j) \cup W^u_{loc}(y_j)$.
By construction since $y_0 = x' \in R_\infty$ then $y_j \in R_\infty$ for all $j$. Thus $x'' \in R_\infty$ and hence $\mu(R^c) = 0$.
On the other hand if $R_\infty = \emptyset$ then $\mu(R) = 0$. This proves the first claim of the lemma. The fact that recurrence implies ergodicity follows from [28]. □

**Corollary 7.2.** For any set $A$ of finite measure and for any $\varepsilon, R > 0$ there exists $n$ such that
\[(7.3) \quad \tilde{\mu}(x \in A : \tilde{T}^n x \in B_R) < \varepsilon.\]
where $B_R = \{x : |z(x)| \leq R\}$.

**Proof.** If $\tilde{\mu}(R) = 0$ then $\tilde{T}$ is dissipative ([1, §1.1]), that is, for a.e. $x$
$$\lim_{n \to +\infty} |z(\tilde{T}^n x)| = +\infty,$$
so (7.3) is obvious.

On the other hand if $\tilde{\mu}(R^c) = 0$ then $\tilde{T}$ is ergodic, so the Ratio Ergodic Theorem tells us that for each $z_1, z_2$ and for almost every $x$
$$\lim_{N \to \infty} \frac{\text{Card}(n \leq N : z(\tilde{T}^n x) = z_1)}{\text{Card}(n \leq N : z(\tilde{T}^n x) = z_2)} = \frac{\tilde{\mu}(x : z(x) = z_1)}{\tilde{\mu}(x : z(x) = z_2)}.$$Since the last expression is uniformly bounded away from 0 we have that for any $\bar{z}$ and almost every $x$
$$\lim_{N \to \infty} \frac{\text{Card}(n \leq N : z(\tilde{T}^n x) = \bar{z})}{N} = 0.$$By the Dominated Convergence Theorem
$$\frac{1}{N} \sum_{n=1}^{N} \tilde{\mu}(x \in A : z(\tilde{T}^n x) = \bar{z}) = \tilde{\mu}\left(\frac{\text{Card}(n \leq N : z(\tilde{T}^n x) = \bar{z})}{N}1_{\{x \in A\}}\right) \to 0.$$Summing over $\bar{z}$’s such that $|\bar{z}| \leq R$ we get
$$\frac{1}{N} \sum_{n=1}^{N} \tilde{\mu}(x \in A : \tilde{T}^n x \in B_R) \to 0.$$
Therefore the set of times $n$ when (7.3) is false has zero density. □

The preliminaries discussed in Section 5 extend to the case of billiards with small external fields by [10, 11]. In particular for an unstable curve $\gamma$, we write

$$\gamma_\delta = \{x \in \gamma: r_s(x) \geq \delta\}, \quad \Lambda_\delta(\gamma) = \bigcup_{x \in \gamma_\delta} W^s(x).$$

Then (5.3) holds and we have the analogue of (5.6):

$$\kappa_1 \leq \frac{d\hat{\mu}}{d\mu_\Lambda_\delta} \leq \kappa_1^{-1},$$

and the analogue of (5.7):

$$\tilde{\mu}(\Lambda_\delta(\gamma)) \geq \kappa_2.$$

**Corollary 7.3.** For any unstable curve $\gamma$ for any $\epsilon, R > 0$ there exists $n$ such that

$$\text{mes}(x \in \gamma: \tilde{T}^n x \in B_R) < \epsilon.$$  (7.6)

**Proof.** Since measure of $\gamma - \gamma_\delta$ tends to 0 as $\delta \to 0$ (see (5.3)), it suffices to prove that, for each fixed $\delta$, (7.6) holds with $\gamma$ replaced by $\gamma_\delta$. Combining Corollary 7.2 with (7.4) we obtain for each $\epsilon > 0$ there exists $n$ such that

$$\hat{\mu}(x \in \Lambda_\delta : |z(\tilde{T}^n x)| \leq R + 1) < \epsilon.$$  (7.7)

On the other hand the definition of $\hat{\mu}$ easily shows that

$$\hat{\mu}(x \in \Lambda_\delta : |z(\tilde{T}^n x)| \leq R + 1) \geq \delta \text{mes}(x \in \gamma_\delta : |z(\tilde{T}^n x)| \leq R)$$

proving the result. □

**7.2. Verifying (M6).** By our choice of $\mathfrak{M}$ it suffices to show that for each $\delta$, for each $\epsilon$ and $R$ there exists $n_0$ such that for $n \geq n_0$ for each curve $\Gamma$ of length at least $\delta$ we have

$$\text{mes}(x \in \Gamma: \tilde{T}^n x \in B_R) \leq \epsilon.$$  (7.7)

We first show this result under an additional assumption that

$$|z(\Gamma)| \geq \tilde{R}$$  (7.8)

provided $\tilde{R} = \tilde{R}(\epsilon, \delta, R)$ is sufficiently large and then use Corollary 7.3 to remove this restriction.

Before giving the formal proof let us describe the main idea. Given an unstable curve $\Gamma$ satisfying the conditions above and $\tilde{n} \in \mathbb{N}$ we consider the Hopf $\tilde{n}$-brush obtained by issuing the stable manifolds from all points of $\tilde{T}^n \Gamma$. We shall show that

(i) If $\tilde{n} = \tilde{n}(\epsilon, \delta, R)$ is large, then the brush has a large measure;
(ii) If at some time $n \geq \tilde{n}$ a significant proportion of $\Gamma$ came close to the origin then a significant portion of the $\tilde{n}$-brush would come close to the origin at time $n - \tilde{n}$. Since $\tilde{T}^{n-\tilde{n}}$ is measure preserving, there is not enough room in a fixed neighborhood of the origin, giving a contradiction.

To prove part (i) above we show that the image $\tilde{T}^\tilde{n}\Gamma$ stretches across a large number of cells. For $T$ this is true because of the LLT, while for $\tilde{T}$ this is true because it is well approximated by $T$ at infinity (at this step it is important that we take $\tilde{R} = \tilde{R}(\varepsilon, \delta, R, \tilde{n})$ sufficiently large). Next, the Growth Lemma implies that most of the components of $\tilde{T}^\tilde{n}\Gamma$ are not too short. Consequently, there are many cells whose intersection with $\tilde{T}^\tilde{n}\Gamma$ contains relatively long component. Now (7.5) implies that the brush has a significant measure in each such cell.

The proof of part (ii) uses the fact that if a point returns close to the origin then the same is true for its whole (homogeneous) stable manifold.

We now give a more detailed argument. Let $\delta_1 \ll \delta$ be a small constant. The precise requirements on $\delta_1$ will be given below. Here we require that for each curve $\Gamma$ of length at least $\delta$ and for each $n$,

\begin{equation}
\text{mes}(x \in \Gamma : x \text{ is not } (\delta_1, n) \text{-good}) \leq \varepsilon^2,
\end{equation}

where we call $x$ $(\delta_1, n)$-good if

$r_n(x) \geq \sqrt{\delta_1}$ and $r_s(\tilde{T}^n x) \geq \sqrt{\delta_1}$.

(the existence of $\delta_1$ with required properties follows from the Growth Lemma 5.1).

By transversality of stable and unstable directions there is a constant $K_1$ such that if $T$ is an unstable curve and $\pi$ is the projection to $T$ along the stable leaves, then

\begin{equation}
d(\pi x, x) \leq K_1 d(x, T)
\end{equation}

provided that $\pi$ is defined at $x$.

Let

\[ X_{\tilde{k}, \eta} = \{ x \in X : \forall y \in B(x, \eta) \forall 0 \leq j \leq \tilde{k} \quad \tilde{T} \text{ is continuous on } B(\tilde{T}^j y, \eta) \}, \]

and define $M_{\tilde{k}, \eta}$ similarly with $\tilde{T}$ replaced by $T$. Choose $\tilde{k}$ so large that for all sufficiently small $\delta_1$, if $x \in X_{\tilde{k}, 2K_1 \delta_1}$ and $T$ is an unstable curve of length $\delta_1$ through $x$, then

\begin{equation}
\text{mes}(t' \in T : r_s(t') \geq 2K_1 \delta_1) \geq \frac{\delta_1}{2}.
\end{equation}

The fact that (7.11) holds for large $\tilde{k}$ follows from [15]. For completeness we recall the argument.
Inequality (5.58) in [15] says that \( r_s(t') \geq \min_{n \geq 0} \Lambda^n d(\tilde{T}^n t', S) \) where \( \Lambda > 1 \) is the minimal expansion factor of \( \tilde{T} \) and \( S \) is the discontinuity set of \( \tilde{T} \). By the definition of \( X_{\tilde{\kappa}, 2K_1\delta_1} \), if the above minimum falls below \( 2K_1\delta_1 \), then also

\[
\min_{n \geq \tilde{k}} \Lambda^n d(\tilde{T}^n t', S) \leq 2K_1\delta_1.
\]

Applying the Growth Lemma to \( \ell = (T, \frac{1}{\delta_1} \text{mes}_T) \) and using \( Z_\ell = 2/\delta_1 \) we find that the Lebesgue measure of the set of \( t' \in T \) satisfying (7.12) is smaller than

\[
\delta_1 4K_1\delta_1 \left( \frac{C_1 \theta^k 2}{\delta_1} + C_2 \right).
\]

Thus if \( \tilde{k} \) is so large that

\[
\theta^k \leq \frac{1}{32CK_1}
\]

and

\[
\delta_1 < \frac{1}{16K_1C_2},
\]

then (7.11) follows.

Furthermore, we suppose that \( \delta_1 = \delta_1(\tilde{k}) \) is so small that

\[
\mu(M - M_{\tilde{k}, 2K_1\delta_1}) \leq \delta\varepsilon^2.
\]

Then for large \( \tilde{R} \) and for each cell \( C = \{ z = m \} \) which is at least \( \tilde{R} \) away from the origin,

\[
\tilde{\mu}((X - X_{\tilde{\kappa}, 2K_1\delta_1}) \cap C) < 2\delta\varepsilon^2.
\]

Next, pick an unstable curve \( \Gamma \) of length at least \( \delta \) satisfying (7.8). Divide \( X \) into squares of size \( \delta_1 \) and choose a curve transversal to the stable cone in each square. Let \( T \) be the union of all those transversals. Given \( \tilde{n} \in \mathbb{N} \) let \( \pi_{\tilde{n}} : \tilde{T}^{\tilde{n}} \Gamma \rightarrow T \) be the projection to the closest transversal along the stable leaves. Note that \( \pi_{\tilde{n}} \) is defined on \( \tilde{T}^{\tilde{n}} x \) if \( x \) is \((\delta_1, \tilde{n})\)-good. However, \( \pi_{\tilde{n}}(\tilde{T}^{\tilde{n}} x) \) may belong to a different square than \( \tilde{T}^{\tilde{n}} x \). Let \( J_{\tilde{n}} \) denote the Jacobian of \( \tilde{T}^{\tilde{n}} : \Gamma \rightarrow \tilde{T}^{\tilde{n}} \Gamma \).

For \( t \in T \) let

\[
J(t) = \sum_{x \text{ is } (\delta_1, \tilde{n})\text{-good}} J_{\tilde{n}}(x).
\]

Let \( L_{\tilde{n}} = \{ t \in T : 0 < J_{\tilde{n}}(t) < \frac{1}{\sqrt{\tilde{n}}} \} \). We claim that if \( \tilde{n} = \tilde{n}(\delta_1), \tilde{R} = \tilde{R}(\delta_1, \tilde{n}) \) are large enough, and \( t \in X_{\tilde{k}, 2K_1\delta_1} \) then \( t \in L_{\tilde{n}} \). To prove this
claim, first observe that by the definition of \( \tilde{\pi}_n \) and (7.10), if \( \pi(\tilde{T}_n x) = t \), then \( d(\tilde{T}_n x, t) \leq K_1 \delta_1 \). Take \( t' \) on the same transversal \( \Sigma \) as \( t \) with \( r_s(t') \geq 2K_1 \delta_1 \) (there are many such \( t' \) by (7.11)). Since \( x \) is \( \tilde{n} \) good, there is \( x' \in T_{\tilde{n}} x \) belongs to the same component as \( \tilde{T}_n x \) and \( \pi(\tilde{T}_n x') = t' \). By bounded distortion of \( \tilde{T}_n \) (see (5.1)), there exists a constant \( c \) such that if \( J_{\tilde{n}}(t) \geq \sqrt{\tilde{n}} \) then \( J_{\tilde{n}}(t') \geq c \sqrt{\tilde{n}} \).

Combining the absolute continuity of \( \pi_{\tilde{n}} \) (see (5.4) and (5.5)) with (7.11) we conclude that if there existed \( t \) such that \( J_{\tilde{n}}(t) \geq \sqrt{\tilde{n}} \), then we would have

\[
\text{(7.17)} \quad \text{mes}(x \in \Gamma : z(\tilde{T}_n x) = z(t)) \geq \frac{\bar{c}_1 \delta_1}{\sqrt{n}}.
\]

On the other hand the LLT for \( T \) shows that there is a constant \( \tilde{C} \) such that for each \( \tilde{n} \) there exists \( \tilde{R} \) such that if \( z(\Gamma) \geq \tilde{R} \), then

\[
\text{(7.18)} \quad \text{mes}(x \in \Gamma : z(\tilde{T}_n x) = z(t)) \leq \frac{\tilde{C}}{n}.
\]

If \( \tilde{n} \) is so large that \( \frac{\tilde{C}}{n} < \frac{\bar{c}_1 \delta_1}{\sqrt{n}} \), that is,

\[
\text{(7.19)} \quad \tilde{n} > \left( \frac{\tilde{C}}{\bar{c}_1 \delta_1} \right)^2,
\]

this gives a contradiction with (7.17) proving the claim.

By the foregoing discussion (see, in particular, (7.9) and (7.16)), if \( \delta_1 \) is small, then for appropriate \( \tilde{n}, \tilde{R} \) we have

\[
\text{(7.20)} \quad \text{mes}(\Gamma \setminus \Gamma^*) \leq 4\varepsilon^2
\]

where \( \Gamma^* \) is the set of points in \( \Gamma \) such that \( x \) is \((\delta_1, \tilde{n})\)-good and \( \pi_{\tilde{n}}(\tilde{T}_n x) \in L_{\tilde{n}} \).

By the definition of \( L_{\tilde{n}} \),

\[
\text{(7.21)} \quad \text{mes}(x \in \Gamma^* : T^N x \in B_{\tilde{R}}) \leq \frac{1}{\sqrt{\tilde{n}}} \text{mes}(y \in L_{\tilde{n}} : T^{N-\tilde{n}} y \in B_{\tilde{R}+1}).
\]

On the other hand combining the absolute continuity of the stable lamination (see (7.4)) with the fact that \( r_s \geq \delta_1 \) on \( L_{\tilde{n}} \), we obtain that there is a constant \( \tilde{C}' \) such that

\[
\text{(7.22)} \quad \text{mes}(y \in L_{\tilde{n}} : T^{N-\tilde{n}} y \in B_{\tilde{R}+1}) \leq \frac{\tilde{C}'}{\delta_1} \text{mes}(y \in \hat{L}_{\tilde{n}} : T^{N-\tilde{n}} y \in B_{\tilde{R}+2}),
\]

where \( \hat{L}_{\tilde{n}} = \bigcup_{z \in L_{\tilde{n}}} W^s(z) \).
Since \( \tilde{T} \) preserves \( \tilde{\mu} \), we have
\[
\tilde{\mu}(y \in \hat{L}_{\tilde{n}} : \tilde{T}^{N-n}y \in B_{R+2}) \leq D(R + 2)^2
\]
for some \( D > 0 \). Combining (7.21), (7.22), and (7.23), we see that
\[
\text{mes}(x \in \Gamma^* : T^N x \in B_R) \leq \frac{D\tilde{C}(R + 2)^2}{\delta_1 \sqrt{\tilde{n}}}
\]
Thus if
\[
\tilde{n} \geq \left[ \frac{D\tilde{C}(R + 2)^2}{2\delta_1(\varepsilon - 4\varepsilon^2)} \right]^2
\]
then
\[
\text{mes}(x \in \Gamma^* : T^N x \in B_R) \leq \varepsilon - 4\varepsilon^2
\]
Combining this with (7.20) we obtain (7.7) provided \( z(\Gamma) \) is large as required by (7.8).

Before completing the proof of (7.7) in the full generality, it makes sense to review the relations between the different parameters involved in the proof of (7.7) assuming (7.8). First, we take \( K_1 \) so that (7.10) holds. Then we select \( \tilde{k} \) so that (7.13) holds and hence (7.11) is satisfied. Next, we select \( \delta_1 \) so that (7.9) is valid, (7.14) is satisfied, and (7.16) holds for sufficiently large \( \tilde{R} \). After that we take \( \tilde{n} \) satisfying (7.19) and (7.24). Finally, we take \( \tilde{R} \) so large that (7.16) holds and (7.18) is satisfied.

It remains to obtain (7.7) without assuming (7.8). Fix \( \varepsilon > 0 \). Then take \( \delta_2 \) so small that for every unstable curve \( \Gamma \) and for all sufficiently large \( n \)
\[
\text{mes}(x \in \Gamma : r_n(x) \leq \delta_2) \leq \varepsilon^2
\]
Applying (7.7) with the assumption (7.8) and with \( \delta \) replaced by \( \delta_2 \varepsilon \), we find that there exists \( \tilde{R} \) so that for any curve \( \Gamma \) of length greater than \( \delta_2 \) such that \( |z(\Gamma)| \geq \tilde{R} \) we have
\[
\text{mes}(x \in \Gamma : z(\tilde{T}^n x) \leq R) \leq \varepsilon^2|\Gamma| \quad \text{for} \quad n \geq n_0(\tilde{R}, \varepsilon, \delta_2).
\]
Next for each \( \Gamma \) with \( |\Gamma| \geq \delta \), Corollary 7.3 shows that there is some time \( n_1 = n_1(\Gamma, \varepsilon) \) such that
\[
\text{mes}(x \in \Gamma : |z(\tilde{T}^{n_1} x)| \leq \tilde{R}) \leq \varepsilon^2.
\]
By compactness there exists \( N_1 \) such that for all curves \( \Gamma \) of length at least \( \delta \) one has \( n_1(\Gamma, \varepsilon) \leq N_1 \). Further increasing \( N_1 \) if necessary, we can
assume that (7.25) holds with \( n = N_1 \). Next, take \( n \geq N_1 + n_0(\tilde{R}, \varepsilon, \delta_2) \).
Divide the set of \( x \) such that \( |z(T^nx)| \leq R \) into three parts

\[
(i) : r_{N_1}(x) \leq \delta_2, \quad (ii) : |z(T^{N_1}x)| \leq \tilde{R},
\]

\[
(iii) : r_{N_1}(x) \geq \delta_2, |z(T^{N_1}x)| \geq \tilde{R} \text{ but } |z(T^nx)| \leq R.
\]

Inequalities (7.25), (7.26), and (7.27) show that contribution of each part to \( \text{mes}(x : |z(T^nx)| \leq R) \) is at most \( \varepsilon^2 \). This proves (7.7) for \( n \geq N_1 + n_0(\tilde{R}, \varepsilon, \delta_2) \).

8. Conclusions.

This paper deals with global mixing, that is, calculation of the expected value of an extended observable in a long time limit, for mechanical systems. The systems considered in this paper admit approximations at infinity, that is, when either the position or the velocity is large, by a periodic system. It turns out that if the map, obtained from the approximating system by factoring out the \( \mathbb{Z}^d \) extension, is chaotic (in our examples, the reduced systems are hyperbolic systems with singularities), then the original system enjoys global global mixing. To establish local global mixing, in addition to controlling the dynamics at infinity we also need to ensure the hyperbolicity in the whole phase space. In particular, we gave examples, where local modifications of the dynamics destroy local global mixing.

We note that notions of global mixing discussed in this paper are neither implied by nor imply the classical properties studied in infinite ergodic theory [1]. For example, Lorentz gas in a small external field is dissipative but it enjoys both local global and global global mixing. Non mild local perturbations of Lorentz gas are conservative but not ergodic and they enjoy global global mixing (even though under natural assumptions, ergodicity is a necessary prerequisite for local global mixing in the recurrent case, cf. discussion in §6.2.1). On the other hand, certain continuous time systems of bouncing balls in gravity field (i.e. special cases of the systems studied in §6.6) are likely to be ergodic and Krickeberg mixing but they are not global global mixing.

This logical independence between global mixing and other infinite ergodic theoretic properties is not surprising since those notions serve different purposes. Namely, classical ergodic theory strives to control the ergodic sum of localized \( (L^1) \) observables and the notions such as Krickeberg mixing are useful for that purpose (see e.g. [24, 41, 42]). The global mixing, on the other hand, is useful for studying ergodic sums of extended observables (cf. [6, 32]). In particular, it seems quite possible for us that the global mixing is more suitable for derivation of
macroscopic dynamics from microscopic laws, as statistical mechanics concerns itself with extended observables. In fact, in this paper we were able to prove

(A) global global mixing for systems where a good control on the dynamics in the bulk is already known and

(B) local global mixing for systems where full limit theorems are available due to a good control of the boundary conditions ([25, 17, 20]).

We also note that for mechanical systems there are more examples where the local global mixing is known than the examples where the Krickeberg mixing was proven. Intuitively, proving local global mixing is easier since it only requires control on most of the phase space, while Krickeberg mixing requires a good understanding of the dynamics in the localized regions of the phase space.

In summary global mixing is an interesting recent concept, which is relevant in physics and is easier to establish than other mixing properties. Our paper is a first step in studying global mixing for mechanical systems. We hope it will stimulate a further research in this area. Some of the natural questions motivated by our results include the multiple mixing, limit theorems for ergodic sums of global observables as well as quantitative aspects of global mixing.

References

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