AMSC/CMSC 666 Dr. Wolfe SAMPLE FINAL EXAM Spring 2003

1. (10 points) Let $f(x) = \frac{1}{x}$. Let $x_0, x_1, \cdots, x_k$ be distinct non-zero numbers. Show that

$$f[x_0, x_1, \cdots, x_k] = (-1)^k \prod_{i=0}^{k} \frac{1}{x_i}$$

(Hint: Use induction.)

2. (10 points) Construct a polynomial $p(x)$ of degree at most 2 for which

$$p(x_0) = y_0, \quad p'(x_1) = z_0, \quad p(x_2) = y_2,$$

where $x_0, x_1, x_2, y_0, z_0, y_2$ are given numbers with $x_0 \neq x_2$. Under what conditions will $p(x)$ exist and be unique ?

3. (12 points) Let $f \in C[a, b]$. Let S_n be the result of applying the n panel Simpson's rule to f (of course n must be even). Show that

$$\lim_{n \to \infty} S_n = \int_a^b f(x)\, dx.$$

(Hint: Show S_n is a Riemann Sum.)

4. (13 points) Let

$$\int_a^b f(x)w(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

be an n-point Gaussian integration rule.
(a) What are the x_i ?
(b) For which polynomials is the formula exact ?
(c) Prove $w_i > 0,\ i = 1, \ldots, n$.

5. (15 points) Let $A \in R^{m \times n}$.
(a) What is the Singular Value Decomposition for A ?
(b) Define

$$\|A\|_2 = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\|\mathbf{x}\|_2 = 1} \|A\mathbf{x}\|_2$$

Show that $\|A\|_2 = \sigma_1$, the largest singular value of A.

6. (15 points) Let A be an invertible matrix. Let C_0 be given. Define $R_0 = I - AC_0$ and assume $\|R_0\| < 1$ for some matrix norm. Define the iteration method

$$C_{m+1} = C_m(I + R_m), \quad R_{m+1} = I - AC_{m+1}, \quad m \geq 0$$

(a) Show that C_m converges to A^{-1} by first relating the error $A^{-1} - C_m$ to the residual R_m. Then examine the behavior of the residual R_m by showing $R_{m+1} = R_m^2,\ m \geq 0$.

(b) Relate C_m to the expansion

$$A^{-1} = C_0(I - R_0)^{-1} = C_0 \sum_{j=0}^{\infty} R_0^j.$$

7. (25 points) Let $A \in R^{n \times n}$ be nonsingular and $\mathbf{b} \in \mathbf{R^n}$. Consider the following iteration for the solution of $A\mathbf{x} = \mathbf{b}$.

$$\mathbf{x_{k+1}} = \mathbf{x_k} + \alpha(\mathbf{b} - A\mathbf{x_k}).$$

(a) Show that if all the eigenvalues of A have positive real part then there will be some real α such that the iterates converge for any starting vector $\mathbf{x_0}$. Discuss how to choose α optimally in case A is symmetric and determine the rate of convergence.

(b) Show that if some eigenvalues of A have negative real part and some have positive real part, then there is no real α for which the iterations converge.

(c) Let $\rho = \|I - \alpha A\| < 1$ for a matrix norm associated to a vector norm. Show that the error can be expressed in terms of the difference between consecutive iterates, namely

$$\|\mathbf{x} - \mathbf{x_{k+1}}\| \leq \frac{\rho}{1-\rho} \|\mathbf{x_k} - \mathbf{x_{k+1}}\|$$

(The proof of this is short but a little tricky.)

(d) Let A be the tridiagonal matrix

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 3 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 3 & 1 \\ 0 & \cdots & \cdots & \cdots & -1 & 3 \end{pmatrix}$$

Find a value of α that guarantees convergence.