AMSC/CMSC 666 Dr. Wolfe SAMPLE MIDTERM EXAM Spring 2003

1. (20 points) Let Π_n be the vector space of all (real) polynomials of degree $\leq n$. Let $x_0, x_1, \ldots, x_n$ be distinct real numbers. For $p, q \in \Pi_n$ define

$$\langle p, q \rangle = \sum_{i=0}^{n} p(x_i) q(x_i). \tag{1}$$

   (a) Show that (1) defines an inner product on Π_n.
   (b) Π_n with the inner product (1) has a 'natural' orthonormal basis. Write down this basis.

2. (15 points) Use divided differences to show that the following data can be interpolated by a <u>cubic</u> polynomial.

x	-2	-1	0	1	2	3
y	1	4	11	16	13	-4

3. (25 points) The idea of this problem is to give a direct proof of the uniqueness of the natural cubic spline interpolating data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with $x_0 < x_1 < \cdots < x_n$. The only analytic tool you will need is Rolle's theorem.
So suppose $S_1(x), S_2(x)$ are natural cubic spline functions satisfying

$$S_1(x_i) = S_2(x_i) = y_i, \quad i = 0, 1, \ldots, n, \quad S_j''(x_0) = S_j''(x_n) = 0, \quad j = 1, 2.$$

   Let $S(x) = S_1(x) - S_2(x)$.
   (a) Show that $S(x)$ is a natural cubic spline function.
   (b) Show that $S''(x)$ has at least $n-1$ zeros on (x_0, x_n) and therefore at least $n+1$ zeros on $[x_0, x_n]$.
   (c) Show that there is an $i, 0 \leq i \leq n-1$ such that $S(x) \equiv 0$ on $[x_i, x_{i+1}]$.
   (d) Show $S(x) \equiv 0$ on $[x_0, x_n]$ and therefore $S_1(x) \equiv S_2(x)$ on $[x_0, x_n]$.

4. (10 points) How many panels are needed to approximate $I = \int_0^1 e^{-x^2} dx$ with error no more than 10^{-6} using the composite trapezoid rule ? (neglect roundoff error.)

5. (30 points)
   (a) Find $\{p_0, p_1, p_2\}$ such that p_i is a polynomial of degree i and this set is orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.
   (Note: $\int_0^\infty x^n e^{-x} dx = n!,\ 0! = 1$.)
   (b) Derive the 2 point Gaussian formula

$$\int_0^\infty f(x) e^{-x} dx \approx w_1 f(x_1) + w_2 f(x_2).$$

   i.e. find the weights and nodes.