

**MATH 463: HOMEWORK ASSIGNMENT # 10:
SOLUTIONS**

55.1

- (a) In this case the residue is the value of $\frac{1}{z+1}$ at $z = 0$, which gives the value 1.
- (b) The easiest way to do this is to look at the Laurent series, which begins $z - \frac{1}{2z}$, so that the residue is $-\frac{1}{2}$.
- (c) In this case, the residue is the value of $z - \sin z$ at $z = 0$, which is 0.
- (d) In this case, we need to compute the first three non-vanishing terms in the Laurent series about the origin for $\cot z$, which gives

$$\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45}.$$

Dividing by z^4 , we obtain the coefficient of z^{-1} for the series representing $\frac{\cot z}{z^4}$, which is $-\frac{1}{45}$.

- (e) Setting $\sinh z = z + \frac{z^3}{6} + \dots$, $\frac{1}{1+z^2} = 1 - z^2 + \dots$, we obtain for the product of these two, $z + \frac{7}{6}z^3 + \dots$, so that, dividing by z^4 , we obtain $\frac{7}{6}$ as the residue.

55.2

- (a) This represents $2\pi i$ times the derivative of e^{-z} at $z = 0$, giving $-2\pi i$.
- (b) This represents $2\pi i$ times the derivative of e^{-z} at $z = 1$, giving $-\frac{2\pi i}{e}$.
- (c) The residue at 0 is the coefficient of z^{-1} in the Laurent expansion of $z^2 e^{\frac{1}{z}}$ about $z = 0$, which is $\frac{1}{3!}$. It follows that the value of the integrand is $\frac{\pi i}{3}$.

- (d) In this case there are singular points at both $z = 0$ and $z = 2$. The residue at 0 is $-\frac{1}{2}$, and the residue at 2 is $\frac{3}{2}$. Consequently the sum of the residues is 1 and the value of the integral is $2\pi i$.

55.3

- (a) This is equivalent to $2\pi i$ times the residue at $w = 0$ of $\frac{1}{w^7 - w^4}$. This can be evaluated by taking the coefficient of w^{-1} in the product of w^4 with the power series converging to $\frac{1}{w^3 - 1}$, which gives -1 for the residue and $-2\pi i$ for the integral.
- (b) This works out to $2\pi i$ times the residue of $\frac{1}{1+w^2}$ at $w = 0$, which is 0, since that function is analytic at the origin.
- (c) This works out to $2\pi i$ times the residue of $\frac{1}{w}$ at $w = 0$. The residue is 1, and the value of the integral is $2\pi i$.

55.4

- (a) Since $e^{\frac{1}{z}}$ has an essential singularity at the origin, so does $ze^{\frac{1}{z}}$. The principal part, which is the sum of all the terms of strictly negative degree is $\sum_{n=1}^{\infty} \frac{1}{(n+1)!z^{-n}}$.
- (b) This has a simple pole at $z = -1$, and the principal part is $\frac{1}{z+1}$.
- (c) The singularity at the origin is removable.
- (d) There is a simple pole at the origin, with principal part $\frac{1}{z}$.
- (e) This function has a pole of order 3 at $z = 2$ and the principal part is the same as the function.

55.6 Since f is analytic, f has a Taylor series about $z = z_0$. We can obtain a Laurent series for g by dividing the Taylor series for f by $(z - z_0)$. If $f(z_0) = 0$, then the Laurent series for g will contain no negative power of $(z - z_0)$ and will provide a Taylor series for g , which is therefore analytic at z_0 . If $f(z_0) \neq 0$, then the principal part of the Laurent series for g is $\frac{f(z_0)}{z - z_0}$, and the residue of $g(z)$ at z_0 is therefore $f(z_0)$.