MATH 463: HOMEWORK ASSIGNMENT # 2: SOLUTIONS

7.2(a) $\arg(-1) = \{\pi + 2n\pi\}$. The principal values of the cube roots of -1 are π and $\pm\frac{\pi}{3}$. Since $|-1| = 1$, the cube roots of -1 are -1 and $e^{\pm\frac{\pi}{3}i}$. The last two roots in rectangular coordinates are $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

7.4

(a) There is very little to prove here. Since $|a+i| = A$ and $\text{Arg}(a+i) = \alpha$, It follows that one of the square roots of $a+i$ has principal argument $\frac{\alpha}{2}$ and modulus $\sqrt{A}$. The result follows.

(b) Since $\alpha = \text{Arg}(a+i)$ it follows that $\cos(\alpha) = \frac{a}{A}$. Using the identity supplied, it follows that

$$\cos^2(\frac{\alpha}{2}) = \frac{1 + \frac{a}{A}}{2} = \frac{a+A}{2A}$$

and

$$\sin^2(\frac{\alpha}{2}) = \frac{A-a}{2A}.$$

Since $\Im(a+i)$ is positive, $0 < \alpha < \pi$, so that $\frac{\alpha}{2}$ is a first quadrant angle. It follows that

$$\pm\sqrt{A}e^{\frac{\alpha}{2}i} = \sqrt{A}(\cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2})i = \pm\frac{1}{\sqrt{2}}(\sqrt{A+a} + i\sqrt{A-a}).$$

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8.1

(a) This inequality defines a closed disk with center $2+i$ and radius 1. Since the disk is closed, the boundary is included and the set is not open, and therefore not a domain.

(b) This inequality defined the exterior of a disk of radius 2 and center $\frac{3}{2}$. Since the inequality is strict, the boundary is not included and this set is open. It is also connected, and is therefore a domain.

(c) This set consists of all points above the line $y = 1$. Again it is connected and, since the inequality is strict, the boundary line $y = 1$ is not included. Therefore the set is a domain.

(d) This set consists of the line $y = 1$. It does not contain any neighborhoods at all, and is therefore not open and not a domain.

(e) This set consists of all points interior to the angle formed by the positive x-axis and the ray $x = y > 0$. Although the origin is not included, the other points of the boundary rays are included so this set is not open and therefore not a domain.

(f) This set consists of the line $x = 2$ and all points to the left of that line. Since the boundary is included, the set is not open and not a domain.

8.5 The complement of S includes all z with $|z| = 1$. Any polygonal line connecting 2 with 0, both of which are in S must include a segment connecting some z_1 and z_2 with $|z_1| < 1$ and $|z_2| > 1$. Let $z(t) = z_1 + t(z_2 - z_1)$ for $0 \leq t \leq 1$. Then $z(t)$ parametrizes the line segment in question. Since $|z(t)|$ is a continuous function of $t$with $|z(0)| < 1$ and $|z(1) > 1$, by the intermediate value theorem $|z(t)|$ must take the value 1 for some t between 0 and 1. It follows that the line segment cannot be contained in S, so S is not connected.

10.1

(b) The only value of z for which $\text{Arg}(z)$ is undefined is 0. Thus $\text{Arg}(z)$ is defined on the set $z \neq 0$. However it is not continous on that set. If we remove the negative real axis, then $\text{Arg}(z)$ becomes a continous function on its complement.

(d) f is defined and continuous on the set $|z| \neq 1$.

10.8. Since the real and imaginary parts of z^2 are respectively $x^2 - y^2$ and $2xy$, the images of the given sets in the w plane are respectively $Re(w) = u = c_1$ and $\Im(w) = v = c_2$.

Since $c_1 < 0$, the two branches of the hyperbola $x^2 - y^2 = c_1$ cut the y-axis at $(0, 1)$ and $(0, -1)$ respectively. In the upper branch, since $v = 2xy$ and y is positive, increasing x corresponds to increasing v on the line $u = c_1$. On the other branch, where y is negative, decreasing x corresponds to increasing v.

Since $c_2 < 0$ the two branches of the hyperbola $2xy = c_2$ are in the second and fourth quadrants, where x and y have opposite signs. Since $u = x^2 - y^2$, on each branch, the direction of increasing $|x|$ and decreasing $|y|$ corresponds to increasing u.

supp 4.

If $b = 0$, then $\sqrt{a + bi} = \pm\sqrt{a}$ if a is non-negative and $\pm\sqrt{|a|}i$ if a is negative.

If b is positive, then, following problem 7.4, we have

$$\sqrt{a+bi} = \sqrt{b}\sqrt{\frac{a}{b}+i} =$$

$$\pm\frac{\sqrt{b}}{\sqrt{2}}(\sqrt{\sqrt{1+(\frac{a}{b})^2}+\frac{a}{b}}+i\sqrt{\sqrt{1+(\frac{a}{b})^2}-\frac{a}{b}}) =$$

$$\pm(\frac{1}{\sqrt{2}}(\sqrt{\sqrt{b^2+a^2}+a}+i\sqrt{\sqrt{b^2+a^2}-a}).$$

If b is negative, we must observe that $-\pi < \mathrm{Arg}(a+bi) < 0$, so that the square roots are in the second and fourth quadrants. After setting $a + bi = |b|(\frac{a}{|b|} - i$ and making the appropriate sign adjustment, we obtain

$$\sqrt{a+bi} = \pm(\frac{1}{\sqrt{2}}(\sqrt{\sqrt{b^2+a^2}+a}-i\sqrt{\sqrt{b^2+a^2}-a}).$$

supp 5.

$$z^3+z^2+1 = (x+iy)^3+(x+iy)^2+1 = x^3+3ix^2y-3xy^2-iy^3+x^2-y^2+2ixy+1 =$$

$$(x^3-3xy^2+x^2-y^2+1)+i(x^2y-y^3+x^2-y^2)$$

supp 6.

We simply set $x = \frac{z+\bar{z}}{2}$ and $y = fracz - \bar{z}2i$, make the appropriate substitutions and obtain

$$\frac{i}{4}((z+\bar{z})^2-(z-\bar{z})^2+(z+\bar{z})(z-\bar{z})),$$

or one of many equivalent expressions.