SOLUTIONS: HOMEWORK ASSIGNMENT #4

19.1 In each case the Cauchy-Riemann equations are violated:

(a) $u_x = 1;\ \ v_y = -1 \neq u_x$
(b) $u_x = 0;\ \ v_y = 2.$
(c) $u_x = 2;\ \ v_y = 2xy.$
(d) $u_x = e^x \cos(y);\ \ v_y = -e^x \cos(y).$

19.6 $f(z)$ can be rewritten as

$$\frac{\bar{z}^3}{z\bar{z}} = \frac{x^3 - 3xy^2 - i(x^2y - y^2)}{x^2 + y^2}.$$

To compute the partial derivatives with respect to x at the origin, we first set $y = 0$, then divide by x and take a limit as $x \to 0$. This gives $u_x(0,0) = 1;\ v_x(0,0) = 0$. To compute the partial derivaties with respect to y, we do the corresponding thing. This gives $u_y(0,0) = 0;\ v_y(0,0) = 1$. It follows that the partial derivatives at the origin (although nowhere else) satisfy the Cauchy-Riemann equations.

22.2 This is another exercise in the use of the Cauchy Riemann equations.

(a) $u_x = y,\ v_y = 1,\ u_y = x,\ v_x = 0$. This function satisfies the Cauchy-Riemann equations only at the point $(0,1)$ and is not analytic anywhere, since analyticity requires the existence of a derivative on a neighborhood.
(b) $u_x = 2y,\ v_y = -2y,\ u_y = 2x,\ v_x = 2x$. This function satisfies the Cauchy-Riemann equations only at the origin, and is therefore nowhere analytic.
(c) $u_x = -e^y \sin(x),\ v_y = e^y \sin(x),\ u_y = e^y \cos(x),\ v_x = e^y \cos(x)$. The Cauchy-Riemann equations are satisfied nowhere; consequently the function is nowhere analytic.

22.7

(a) If $f(z)$ is identically real, then $v = \Im(z)$ vanishes identically together with its partial derivatives. It then follows from the Cauchy-Riemann equations that the partial derivatives of $u = \Re(z)$ vanish as well, so that f is constant.

(b) If both $f(z)$ and $\overline{f(z)}$ are analytic, so is their sum, which is identically real. Hence The sum is constant. Similarly, $i(f(z) - \overline{f(z)}$ is analytic and identically real, and therefore constant. It follows that both f and its conjugate are constant.

(c) Suppose $|f(z)| = c$. If $c = 0$, then $f(z)$ is identically 0 and therefore constant. Otherwise,

$$\overline{(f(z)} = \frac{f(z)\overline{f(z)}}{f(z)} = \frac{c^2}{f(z)};$$

moreover, $f(z)$ is never 0. It follows that $\overline{f(z)}$ is analytic, and f is constant by part (b)

23.5 In general, we have $|\exp(z)| = e^x$. This gives $|\exp(2z+i)| = e^{2x}$ and $|\exp(iz^2| = \exp(\Re(iz^2)) = e^{-2xy}$. Then we have

$$|\exp(2z+i) + \exp(iz^2)| \leq |\exp(2z+i)| + |\exp(iz^2)| = e^{2x} + e^{-2xy}.$$

23.10 If e^z is real, then $\sin(\Im(z)) = 0$, from which it follows that $\Im(z)$ is a multiple of π. Similarly, if e^z is imaginary, then $\cos(\Im(z)) = 0$ from which it follows that $\Im(z)$ is equal to $\frac{\pi}{2} + n\pi$.

23.11

(a) As x tends to $-\infty$, $exp(x+iy)$ tends to 0 for any y.

(b) As y tends to ∞ for fixed x, the values of $exp(x+iy)$ wind around the circle of radius e^x and do not approach any limit.

supp 9.

(a) f has singular points at $z = \pm i$. $u = \dfrac{x^3 + xy^2 + x}{(x^2+y^2)^2 + 2x^2 - 2y^2 + 1}$, and $y = \dfrac{-y^3 - x^2y + y}{(x^2+y^2)^2 + 2x^2 - 2y^2 + 1}$.

(b) f has singular points at $z = \pm 2$. $u = \dfrac{(x^2+y^2)^2 - 2(x^2-y^2) - 8}{(x^2+y^2)^2 - 8(x^2-y^2) + 16}$ and $v = \dfrac{-12xy}{(x^2+y^2)^2 - 8(x^2-y^2) + 16}$.

supp 10.

(a) f is the exponential function, which is entire.
(b) f is not analytic since it does not satisfy the Cauchy-Riemann equations.
(c) f is not entire since it has a singular point at $z = -2$, although f is analytic everywhere else.