MATH 463: HOMEWORK ASSIGNMENT # 5: SOLUTIONS

24.2 The point of this problem is that e^z was defined to make this identity true for real z, but it then needs to be verified for general complex z. However the proof is very simple. We observe that $e^{iz} - \cos(z) - i\sin(z)$ is analytic, in fact entire, since the complex exponential, sin and cosine functions are all entire. We also know that $e^{iz} - \cos(z) - i\sin(z)$ is identically 0 for z real. It follows that it is identically 0 for all complex z.

24.8 Both parts follow from the general identity for real numbers $|a| \leq \sqrt{a^2 + b^2}$.

25.9 The zeros of the hyperbolic tangent function are the zeros of the hyperbolic sine. These occur when $e^z = e^{-z}$ Since $|e^z| = e^x$, this requires $e^x = e^{-x}$ which occurs only at $x = 0$. Thus the zeroes of $sinh(z)$ must occur when $z = iy$ and $e^{iy} = e^{-iy}$. This is true if and only if y is a multiple of π. A similar analysis will show that the zeros of the hyperbolic cosine also occur along the imaginary axis where y is an odd multiple of $\frac{\pi}{2}$. Thus we have shown that the zeros and singularities of the hyperbolic tangent occur respectively at even and odd multiples of $\frac{\pi}{2}i$.

27.4 $\log i = \{i(\frac{\pi}{2} + 2n\pi)\}$; $\log(i^2) = \log(-1) = \{(2n+1)\pi\}$. Parts (a) and (b) correspond to two different ways of choosing single values branches of the logarithm function. In the case of part (a), the values of $\log(i)$ and $\log(-1)$ are, respectively, $\frac{\pi}{2}i$ and πi, so the equation is true. In the case of part (b), the values of $\log(i)$ and $\log(-1)$ selected by the single-valued branch are, respectively, $\frac{5\pi}{2}i$ and πi, so the equation is false.

27.5

(a) $i = e^{\frac{\pi}{2}i}$ so that $i^{\frac{1}{2}} = \pm e^{\frac{\pi}{4}i)}$. Then

$$\log(i^{\frac{1}{2}}) = \log(e^{\frac{\pi}{4}i}) \cup \log -e^{\frac{\pi}{4}i} =$$

$$= \{\frac{\pi}{4}i + 2n\pi i\} \cup \{\frac{\pi}{4}i + \pi i + 2n\pi i\} = \{\frac{\pi}{4}i + n\pi i\}.$$

On the other hand, we have

$$\frac{1}{2}\log(i) = \frac{1}{2}\{\frac{\pi}{2}i + 2n\pi i\} = \{\frac{\pi}{4}i + n\pi i\}.$$

(b)

$$\log(i^2) = \log(-1) = \{\pi i + 2n\pi i\}.$$
$$2\log(i) = 2\{\frac{\pi}{2}i + 2n\pi i\} = \{\pi i + 4n\pi i\}.$$

supp 11.

(a) $\sin^{-1}(\frac{1}{2}) = -i\log(\frac{i}{2} + (\frac{3}{4})^{\frac{1}{2}}) = -i\log\frac{i\pm\sqrt{3}}{2})$. Since $\frac{i\pm\sqrt{3}}{2}$ has modulus 1 for both values of the sign, $\mathrm{Arg}(\frac{i+\sqrt{3}}{2}) = \frac{\pi}{6}$, and $\mathrm{Arg}(\frac{i-\sqrt{3}}{2}) = \frac{5\pi}{6}$, the values in question are

$$\{\frac{\pi}{6} + 2n\pi\} \cup \{\frac{5\pi}{6} + 2n\pi\}$$

.

(b) We have $\cosh^{-1}(i) = \log(i \pm \sqrt{2}i = \log((1+\sqrt{2})i) \cup \log((1-\sqrt{2})i)$. The first of these is $\ln(1+\sqrt{2}) + i\arg(i)$, while the second is $\ln(\sqrt{(2)} - 1) + i\arg(-i)$.

(c) If $\log(z) = 2 + \frac{\pi}{3}i$, then $z = e^{2+\frac{\pi}{3}i} = \frac{e^2}{2} + \frac{\sqrt{3}e^2}{2}i$.