MATH 463: HOMEWORK ASSIGNMENT # 6: SOLUTIONS

31.2 If m =n, then the integrand is identically equal to 1, so the value of the integral is 2π. Otherwise, the integrand has the form $\cos((m-n)\theta) + i\sin((m-n)\theta$, and the integral over a full period is 0.

31.8 We have

$$\int_0^{2\pi} e^{it}\,dt = \int_0^{2\pi} \cos(t)\,dt + i\int_0^{2\pi} \sin(t)\,dt = 0.$$

On the other hand, the integrand never takes the value 0, so there can be no c such that the value of the integral is $e^{ic}(2\pi - 0)$.

33.7 We parametrize the circle by setting $z = e^{it}, /quad0 \le t \le 2\pi$. Then $\bar{z} = e^{-it}$ and $dz = ie^{it}\,dt$. Putting all this together, the integral becomes $\int_0^{2\pi} ie^{(m-n+1)it}\,dt$, which takes the value 0 unless $m-n+1 = 0$, in which case it takes the value $2\pi i$.

33.14 The modulus of the integral is bounded by the product of the length of the contour and the maximum modulus of the integrand along the contour. In this case, the contour is a semicircle of radius R, and so has length πR. Along the contour, $|z| = R$, so that the modulus of the numerator is at most $R^2 + 1$. The denominator can be factored as $(z^2+1)(z^2+4)$ so, for $R > 2$, the modulus of the denominator is at least $(R^2-1)(R^2-4)$. Putting all this together, we obtain the bound claimed in the problem. Since the denominator has higher degree than the numerator, the limit of this bound as $R \to \infty$ is 0 as claimed.

supp 12. Setting $f(z) = \dfrac{z^2+2z-5}{z^2}$, $z = 2e^{i\theta}$, so that $dz = 2ie^{i\theta}\,d\theta$, the integrand becomes, in terms of θ, $i(2e^{i\theta} + 2 - \frac{5}{2}e^{-i\theta})\,d\theta$, so that the formal anti-derivative with respect to θ is $2e^{i\theta} + 2i\theta + \frac{5}{2}e^{-i\theta}$. This gives the answers

(a) $2\pi i - 9$
(b) $2\pi i + 9$
(c) $4\pi i$.

supp 13. We set $f(z) = \exp(3\pi\bar{z} = e^{3\pi x}e^{-3\pi iy}$. Taking the sides of the square in counter-clockwise order, starting with the side on the x-axis, we evaluate the line integral along each side in order.

- $y = 0; \quad z = x, \quad 0 \leq x \leq 1.$
$$\int_0^1 e^{3\pi x}\,dx = \frac{e^{3\pi} - 1}{3\pi}.$$
- $x = 1; \quad z = iy, \quad 0 \leq y \leq 1.$
$$ie^{3\pi}\int_0^1 e^{-3\pi iy}\,dy = \frac{2}{3\pi}e^{3\pi}.$$
- $y = 1; \quad z = x + i$. This is the same as the first edge with two differences. We pick of a factor of $e^{-3\pi i} = -1$, and we must take the edge from right to left instead of from left to right. These two effects cancel and we get the same answer as for the first edge.
- The integral along the last edge is similarly computed as $\dfrac{-2}{3\pi}$.

We obtain a final answer of $\dfrac{(3e^{3\pi} - 3)}{3\pi} = \dfrac{e^{3\pi} - 1}{\pi}$.