MATH 463: HOMEWORK ASSIGNMENT # 7: SOLUTIONS

38.2 The respective anti-derivatives, with their values at the endpoints are

(a) $\dfrac{e^{\pi z}}{\pi}$, $\quad \dfrac{i}{\pi}, \dfrac{-1}{\pi}$

(b) $2\sin(\frac{z}{2})$, $\quad \cosh(1)$, $\quad 0$

(c) $\dfrac{(z-2)^4}{4}$, $\quad \frac{1}{4}$, $\quad \frac{1}{4}$

38.5 The singularities in each case are the zeros of the denominator. In each case these line either inside the square or outside the circle or both, but never on either curve or between them. Specifically, in part (a) the singularities are at $\pm\frac{i}{\sqrt{3}}$, inside the square. In part (b) the singularities are at multiples of 2π, of which 0 is inside the square, while all others are outside the circle. In part (c) the singularities are at multiples of $2\pi i$, excluding 0. All of these are outside the circle.

38.6 The point $2+i$ is interior to the given rectangle. It follows that the value of the line integral is the same as if the contour were a circle centered at $2+i$.

38.9 The Cauchy-Goursat theorem does not apply because f is not analytic at 0. However, we can evaluate the integral directly. On the semi-circle, we set $z = e^{i\theta}$, $dz = ie^{i\theta}$ and $f(z) = e^{i\frac{\theta}{2}}$. The integral then takes the value

$$\int_0^\pi ie^{i\frac{3\theta}{2}}\, d\theta = \frac{2}{3}(e^{i\frac{3\theta}{2}} - 1) = \frac{-2i-2}{3}.$$

Evaluating over the radius along the positive axis, we have $z = x$ and $f(z) = \sqrt{x}$, so the integral takes the value

$$\int_0^1 \sqrt{x}dx = \frac{2}{3}.$$

Finally, along the radius on the negative axis, we have $z = -x$, $dz = -dx$ and $f(z) = \sqrt{x}i$. The minus sign is cancelled by the change of direction and the integral therefore takes the value $\frac{2}{3}i$. The result now follows.

40.2 $2i$ is interior to the circle $|z-i| = 2$, while $-2i$ is exterior to the circle. Consequently we can set $f(z) = \dfrac{1}{z+2i}$, and by the Cauchy integral formula, the value of the first integral is $2\pi i f(2i) = \frac{\pi}{2}$, and the value of the second integral is $2\pi i$ times the derivative of $f(z)^2$ at $2i$, which is $4\pi i f(2i)f'(2i) = \frac{\pi}{16}$.

40.9 Set $f(z) = e^{az}$. Then by the Cauchy integral formula, the integral around the unit circle takes the value $2\pi i f(0) = 2\pi i$. Now, making the substitution $z = e^{i\theta}$ so that $dz = ie^{i\theta}d\theta$, we obtain $\frac{f(z)}{z}\,dz = ie^{a\cos(\theta)}eai\sin(\theta)\,d\theta$. Since the value of the integral is purely imaginary, we can equate imaginary parts and obtain

$$\int_{-\pi}^{\pi} e(a\cos(\theta)cos(a\sin(\theta))\,d\theta = 2\pi.$$

Finally, we observe that the integrand is even and the result follows.

42.1 By the Cauchy integral formula,

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(w)}{(w-z)^3}\,dw,$$

Where C is any closed contour surrounding z. If we take C to be a circle of radius R surrounding z, the length of the contour is $2\pi R$, while the integrand is bounded by $A\frac{|z|+R}{R^3}$. Consequently, $|f''(z)| \le 2A\frac{|z|+R}{R^2}$. Since ths must be true for all positive R, it follows that $f''(z) = 0$ and hence that f is a linear function of z.

42.2 If $f(z)$ is entire, so is $e^{f(z)}$. But $|e^{f(z)}| = e^{u(z)}$. Consequently, if $u(z)$ is bounded, then so is $e^{f(z)}$. It then follows from Liouville's theorem that $e^{f(z)}$, and therefore $f(z)$, is constant.

42.6 We note first that $|f|$ must take its maximum region on the boundary and the minimum must either be 0 or also occur on the boundary. Since -1 is not included in the triangle, both maximum and minimum occur on the boundary at the nearest and furthest points from -1. The nearest point is $z = 0$ with $|f(0)| = 1$. The furthest point is $z = 2$ with $|f(2)| = 9$.

42.8 We have $u = e^x\cos(y)$. Since e^x is identically positive and and takes its maximum on the region at $x = 1$, u takes its maximum at 1, where $y = 0$ and $\cos(y) = 1$ and its minimum at $1 + i\pi$, where $\cos(y) = -1$.