MATH 463: HOMEWORK ASSIGNMENT # 9: SOLUTIONS

51.6 Since $\cos z$ is an entire function, $\dfrac{\cos z}{z^2 - (\frac{\pi}{2})^2}$ is analytic everywhere except, possibly, at $\pm\frac{\pi}{2}$. Since $\cos z = 0$ at $z = \pm\frac{\pi}{2}$, the Taylor series for $\dfrac{\cos z}{z + \frac{\pi}{2}}$ in powers of $z - \frac{\pi}{2}$ has no constant term and is identically divisible by $z - \frac{\pi}{2}$. The quotient is a power series that converges to $\dfrac{\cos z}{z^2 - (\frac{\pi}{2})^2}$, which is therefore analytic at $\frac{\pi}{2}$. The argument for analyticity at $-\frac{\pi}{2}$ is analogous.

51.7 We begin by observing that

$$\frac{1}{w} = \frac{1}{1 + (w - 1)} = \sum_{n=0}^{\infty} (-1)^n (w - 1)^n,$$

for $|w - 1| < 1$. Integrating term by term (along contours contained in the disk of convergence, *e.g.* line segments originating at 1 and terminating at z with $|z - 1| < 1$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1} (w - 1)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1} (w - 1)^n,$$

where the two series are equivalent by shifting the summation index and observing that only the parity of the exponent of (-1) matters. Since the integrated series must be an anti-derivative of $\frac{1}{z}$, it must be $\operatorname{Log} z$ plus a constant. Evaluation at $z = 1$ establishes that the constant is 0.

51.8 We already know that $\operatorname{Log} z$, the principal logarithm with imaginary part between $-\pi$ and π is analytic on the complement of the negative real axis. It follows that the same is true of $\dfrac{\operatorname{Log} z}{z - 1} = f(z)$, except possibly at $z = 1$. However, by the results of problem 7,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1} (w - 1)^n$$

converges to $f(z)$ in a neighborhood of $z = 1$. It follows that $f(z)$ is analytic at $z = 1$.

51.16 $\cosh z$ is an entire function whose Maclaurin series contains only even powers of z. $\dfrac{1}{\cosh z}$ is analytic wherever $\cosh z \neq 0$. The nearest zero of $\cosh z$ to the origin is at $z = \frac{\pi}{2}i$. It follows that the Maclaurin series for $\dfrac{1}{\cosh z}$ has radius of convergence $\dfrac{\pi}{2}$. Since $\dfrac{1}{\cosh z}$, like $\cosh z$ is an even function, its Maclaurin series can contain only even powers of z. It follows that $E_n = 0$ for n odd. Finally, to obtain the first four non-vanishing Euler numbers, compute the first four terms of

$$\frac{1}{1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720}}.$$

by polynomial long division or, alternatively, compute the first six derivatives of $\dfrac{1}{\cosh z}$ at $z = 0$.