MATH 463: FIRST HOUR EXAMINATION

1. Find the real and imaginary parts of $\frac{1+2i}{3-i}$.

Solution: Multiplying numerator and denominator by the complex conjugate of the denominator, we obtain $\frac{1}{10}(1+7i)$ so that the real and imaginary parts are respectively .1 and .7.

2. Find all values of $(8i)^{\frac{4}{3}}$ in the form $a+bi$, where a and b are expressions with exact values formulated in terms of numbers and radicals.

Solution: This can be rewritten as $16i^{\frac{4}{3}}$. Since $i^4 = 1$, the values of $i^{\frac{4}{3}}$ are the cube roots of 1, which are 1, $e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}i}{2}$ and $e^{\frac{2\pi i}{3}} = \frac{-1-\sqrt{3}i}{2}$. Thus the numbers in question are 16, $-8+8\sqrt{3}i$, and $-8-\sqrt{3}i$.

3. Evaluate those of the following limits that exist, and explain why the remaining one(s) do not exist.

Note: Each part is worth five points.

(a) $\lim_{z\to 0} e^{\frac{1}{z^2}}$

Solution: As z approaches 0 through real values, the limit in the real sense is infinite; as z approaches 0 through imaginary values, the limit in the real sense is 0. Hence the limit in the complex sense does not exist.

(b) $\lim_{z\to\infty} e^{\frac{1}{z^2}}$

Solution: $\lim_{z\to\infty} \frac{1}{z^2} = 0$. The exponential function is analytic at 0 and takes the value 1 there; consequently the limit exists and has the value 1.

(c) $\lim_{z\to i} \mathrm{Log}(z)$

Solution: The prinicipal branch of the logarithm function is analytic at $z = i$ and takes the value $\frac{\pi}{2}i$, which is therefore the value of the limit.

(d) $\lim_{z\to -1} \mathrm{Log}(z)$

Solution: The limit in this case does not exist, since the limit is πi when 1 is approached from above the real axis and $-\pi i$ when 1 is approached from below the real axis.

(e) $\lim_{z\to 0} \dfrac{\bar{z}^3}{z^3}$

Solution: $\dfrac{\bar{z}^3}{z^3}$ can be rewritten as $e^{-6\mathrm{Arg}(z)i}$, which is constant on lines through the origin, and takes all unimodular values arbitrarily close to the origin. Hence there is no limit.

(f) $\lim_{z\to 0} \dfrac{\bar{z}^3}{z^2}$

Solution: $\dfrac{\bar{z}^3}{z^2}$ can be rewritten as $|z|e^{-5\mathrm{Arg}(z)i}$, which has modulus $|z|$. Hence the limit is 0.

4. Let $u(x, y) = x^3 - 3xy^2 + 4xy + y$.

(a) [five points] Verify by direct differentiation that u is harmonic.

Solution: $u_x = 3x^2 - 3y^2 + 4y$, $u_{xx} = 6x$, $u_y = -6xy + 4x + 1$, and $u_{yy} = -6x$. Since $u_{xx} = -u_{yy}$, u is harmonic.

(b) [ten points] Find a harmonic function v conjugate to u by any method you choose.

Solution: $v_x = -u_y = 6xy - 4x - 1$. It follows that $v(x, y) = 3x^2y - 2x^2 - x + h(y)$ Differentiating with respect to y, we obtain $u_x = v_y = 3x^2 - 3y^2 + 4y = 3x^2 + h'(y)$. It follows that $h'(y) = 4y - 3y^2$ so that $h(y) = 2y^2 - y^3$ plus a constant that we may set to 0 since we are only looking for one harmonic conjugate of u. It follows that $v(x, y) = 3x^2y - y^3 + 2y^2 - 2x^2 - x$.

(c) [five points] Let $f(x + iy) = u(x, y) + iv(x, y)$. Identify f as an explicit polynomial in z.

Solution: This can be done in at least three ways. The least efficient is to make the substitution $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, and expand. All the terms involving $\bar{z}$ will cancel. A more efficient method is to anticipate the cancellation by taking only those terms in each expansion that are powers of z. Best of all is to separate the terms in $u + iv$ by total degree and recognize each term as a monomial in z. This quickly gives the answer $z^3 - 2z^2 - iz$.

5. Let $f(z) = \dfrac{z^2 - 9}{z^2}$.

Note: Each part is worth six points.

(a) Locate the zeros and singularities of f.

Solution: f has zeros at $z = \pm 3$ and a singularity at $z = 0$.

(b) Evaluate $\lim_{z\to\infty} f(z)$.

Solution: Since the numerator and denominator have the same degree and leading coefficient, the limit at ∞ is 1.

(c) Determine for what values of z $f(z)$ takes real non-positive values.

Solution: We set $f(z) = -t$ with t non-negative. This gives $z^2-9 = -tz^2$ or $z^2 = \frac{9}{1+t}$. This means that z^2 is a positive number less than or equal to 9, so z is between -3 and 3. (Actually the singular point at 0 is not included, a point which I overlooked along with all the students who took the test. In what follows, one must remove the entire interval $[-3, 3]$ on the real axis.)

(d) Show that $\sqrt{f(z)}$ has a single-valued analytic branch that is defined on the complement of the set you identified in part (c) and has a positive limit at ∞.

Solution: Since the square root function has an analytic single-valued branch that is defined on the complement of the negative real axis and takes positive values on the positive real axis, and f is analytic on the complement of $[-3, 3]$ and takes values in the complement of the negative real axis, $\sqrt{(f)}$ has a single valued branch that is the composition of analytic functions and therefore analytic. It is a little trickier than I realized to prove the existence of a limit at infinity; it involves the observation that since f has the limit 1 at infinity, $\sqrt{f}$ maps the exteriors of large circles into small neighborhoods of ± 1. Since the exterior of a large circle is connected, the image must be entirely contained in a neighborhood of 1 or of -1, and it must be 1 since f takes positive values on the positive real axis.

(e) Deduce from the foregoing that $\sqrt{z^2 - 9}$ has an analytic single-valued branch defined on the complement of the set you found in part (c). Evaluate such a branch at $z = 5$ and at $z = -5$.

Solution: Since $\sqrt{f(z)}$ is analytic on the set in question, so is $z\sqrt{f(z)}$, which gives the desired branch. Since we have seen that $\sqrt{f}$ is positive for both large positive and large negative real z. $zf(z)$ must take opposite signs on the two parts of the real axis.