

MATH 740 LECTURES OCTOBER 18-OCTOBER 27

1. CONNECTIONS AND SUBBUNDLES

We have a manifold M , a vector bundle E on M , a sub-bundle F and a connection ∇ on E . In general, ∇ will not respect F , that is, if η is a section of F , then $\nabla_X \eta$ is a section of E but not in general of F .

The extent to which ∇ does not respect F is measured by $A(x, \eta)$, which is the projection of $\nabla_X \eta$ on E/F . $A(X, \eta)$ is C^∞ -M linear in both X and η . If there is an inner product on E respected by ∇ , we can identify E/F with $F^\perp \subset E$, and can interpret $A(X, \eta)$ as taking values in F^\perp .

Special case: M is immersed in N , E is the pullback of $T(N)$ to a bundle over M , and F is $T(M)$. $\bar{\nabla}$ is the pullback to M of the Riemannian connection on N . $B(X, Y) \in T(M)^\perp = N(M)$. If ∇ is the Riemannian connection on M , $\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)$. It follows that $B(X, Y) = B(Y, X)$.

If we fix a unit vector field η normal to M , then $B_\eta(X, Y) = \langle B(X, Y), \eta \rangle$ becomes a bilinear form on $T(X)$.

Example 1: $N = \mathbb{R}^2$, M is a curve in \mathbb{R}^2 , η is a unit normal to the curve, and X is a unit tangent. Parametrizing the curve by arc length $X = \frac{\partial x}{\partial s} \frac{\partial}{\partial x} + \frac{\partial y}{\partial s} \frac{\partial}{\partial y}$. Then $B_\eta(X, X) = \langle \frac{\partial^2 x}{\partial s^2} \frac{\partial}{\partial x} + \frac{\partial^2 y}{\partial s^2} \frac{\partial}{\partial y}, \eta \rangle = k(s)$, the curvature.

Example 2: $N = \mathbb{R}^3$. M is a surface parametrized by $z = f(x, y)$.

$$\eta = \frac{\partial f}{\partial x} \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} - \frac{\partial}{\partial z}$$

$$X = \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial}{\partial z}$$

$$B_q(X, X) = \langle \bar{\nabla}_X X, \eta \rangle = \left\langle \frac{\partial^2 f}{\partial x^2} \frac{\partial}{\partial z}, \eta \right\rangle = \frac{-\frac{\partial^2 f}{\partial x^2}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

$$B_\eta = \frac{-H(f)}{\sqrt{f_x^2 + f_y^2 + 1}}$$

Sectional curvature is given by

$$\frac{f_{xx}f_{yy} - f_{xy}^2}{1 + f_x^2 + f_y^2}$$

If X, Y, Z are vector fields along M , and η, π, μ are vector fields normal to M . We write $\bar{\nabla}$ for the Riemannian connection on $T(N)$, ∇ for the Riemannian connection on $T(M)$, and $\nabla_X^\perp(\eta)$ for the projection of $\bar{\nabla}_X(\eta)$ on N .

$$\bar{\nabla}_X(Y) = \nabla_X(Y) + B(X, Y)$$

We define $S_\eta : T(M) \rightarrow T(M)$ by

$$\begin{aligned}\langle S_\eta X, Y \rangle &= \langle B(X, Y), \eta \rangle \\ \langle \bar{\nabla}_X \eta, Y \rangle &= -\langle \eta, \bar{\nabla}_X Y \rangle = -\langle \eta, B(X, Y) \rangle = -\langle S_\eta(X), Y \rangle \\ \bar{\nabla}_X(\eta) &= \nabla_X^\perp \eta - S_\eta(X) \\ \langle S_\eta(X), Y \rangle &= \langle B(X, Y), \eta \rangle\end{aligned}$$

2. THE GAUSS MAP

Let M^n be compact and immersed in \mathbb{R}^{n+1} with trivial normal bundle. Let η be a section of the normal bundle with unit length. The tangent bundle to the ambient manifold \mathbb{R}^{n+1} is trivial. Hence at every point m of M , we may identify $\eta(m)$ with a unit vector of \mathbb{R}^{n+1} or, equivalently, a point of S^n . This gives a map $M \rightarrow S^n$ which is called the **Gauss map**.

$$\langle \bar{\nabla}_x \eta, Y \rangle = - \langle B(X, Y), \eta \rangle$$

Choose an orthonormal basis e_0, \dots, e_n for $T(\mathbb{R}^{n+1})$.

$$\eta = \sum \alpha_i(m) e_i, \quad \sum \alpha_i^2(m) = 1$$

If X is a tangent vector along M , then

$$\bar{\nabla}_x \eta = \sum x(\alpha_i) e_i.$$

Choose a point m of M and choose a coordinate system so that $\eta(m) = e_0$. Set $X = \frac{\partial}{\partial x_i} = e_i, Y = \frac{\partial}{\partial x_j} = e_j$.

Then,

$$\bar{\nabla}_x \eta = \sum_{k=0}^n \frac{\partial \alpha_k}{\partial x_i} e_k$$

and

$$\langle \bar{\nabla}_x \eta, Y \rangle = \frac{\partial \alpha_j}{\partial x_i}.$$

As long as $\langle B(X, Y), \eta \rangle$ is nondegenerate as a bilinear form in X and Y , the the Gauss map is a local diffeomorphism. In particular, if $B(X, Y)$ is everywhere nondegenerate (i.e., has non-zero determinant), then the Gauss map is a diffeomorphism unless $n = 1$.

In general, let dV be the volume form on M induced by the metric induced by the immersion. Let dV_s be the volume form on S^n , the unit sphere, pulled back by the Gauss map, then

$$dV_s = \det(\langle \bar{\nabla} e_i \eta, e_j \rangle) dV_m = \det(\langle B(e_i, e_j), \eta \rangle) dV_m$$

So $\int_M dV_s = m V_0$ where V_0 is the volume of the unit sphere S^n , m an integer.

Special Case: Let $n = 2$. Then $\det B = K$ (where K is the curvature of the manifold). Hence we have $\int_M K dV_m = 4\pi m$.

In the case of the torus, $m = 0$, and in the case of the two-holed torus, $m = -1$. So $m = \frac{\chi(m)}{2}$ where χ is the Euler characteristic of M .

Definition: $M \subset \bar{M}$ is called **totally geodesic** if the second fundamental form vanishes identically. It follows that $\bar{\nabla}_M = \nabla_M$ and hence that geodesics in M are geodesics in \bar{M} .

Example: The equator of any sphere is totally geodesic.

3. COMPLETE RIEMANNIAN MANIFOLDS

A Riemannian manifold is called complete if the exponential map is defined on all of $T_p(M) \forall p \in M$. For any Riemannian manifold, the Riemannian metric induces a metric d given by defining $d(m, m')$ = the infimum of the lengths of all differentiable paths joining m to m' .

Theorem: The induced metric topology coincides with the manifold's underlying topology.

Proof: M can be covered by open sets that are diffeomorphic to open Euclidean balls by the exponential map. On these balls, the ratio between the Euclidean metric and the metric induced by the exponential map is bounded both above and below. It follows that the same is true for the geodesic distances.

We proceed now to show that the metric distance between two points on a complete Riemannian manifold is actually realized by a geodesic. Let $m, m' \in M$. Take $N_\epsilon(m)$ to be the normal ball around m of radius ϵ with boundary $S_\epsilon(m)$. Note that $d(c, m')$ is a continuous function on M and hence assumes a minimum on a compact set. In particular $\exists m_0 \in S_\epsilon(m)$ at which the minimum is assumed.

Claim: $d(m, m_0) + d(m_0, m') = d(m, m') = \epsilon + d(m_0, m')$

Proof: $d(m, m') \leq \epsilon + d(\hat{m}, m') \forall \hat{m} \in S_\epsilon(m)$

Now suppose $d(m, m') < \epsilon + d(m_0, m')$. Then $\exists \gamma \in C^\infty(M)$ from m to m' of length $< \epsilon + d(m_0, m')$

Since $S_\epsilon(m)$ disconnects M , this path must meet $S_\epsilon(m)$ at some point \hat{m} , which implies that

$$\begin{aligned} L(\gamma) &\geq d(m, \hat{m}) + d(\hat{m}, m') \\ &\geq \epsilon + d(m_0, m'). \end{aligned}$$

Let $c(t)$ be the geodesic originating at m and passing through m_0 at $t = \epsilon$. Let $r = d(m, m')$. We want to show that $c(r) = m'$. Consider the set $A = \{t \mid 0 \leq t \leq r \text{ and } d(c(t), m') = r - t\}$ Clearly $[0, \epsilon] \subset A$. Also A is closed in $[0, r]$ since d is continuous. To show that A is open, let $t > 0 \in A$ and look at the ball around $c(t)$ There is enough flexibility in our choice of geodesic to guarantee that $t + \delta \in A$ for suitably small δ so A is open. Thus we have established that the geodesic $c(t)$ realized the metric distance between m and m' .

Proposition: M is complete.

Let m_n be a Cauchy sequence in M . Pull m_n back to $T_{m_0}(M)$ via the exponential map. Since m_n is Cauchy, $d(m_0, m_n)$ is bounded. This implies that $g(\hat{m}_0, \hat{m}_n)$ is bounded. Now we have a sequence in a compact set, hence a subsequential limit in $T_{m_0}(M)$ whose image under exp_{m_0} must be the limit of the entire Cauchy sequence.

4. STANDARD MODEL FOR CONSTANT NEGATIVE CURVATURE

The standard model for constant negative curvature is $H^n = \{\mathbb{R}^n : x_n > 0\}$ with metric $g = \frac{\sum dx_i^2}{x_n^2}$.

How large can the isometry group of a Riemannian manifold be? Let $m \in M$, let G be the isometry group of M , and let G_m be the isotropy group of m .

Let $n = \dim M$. G_m is isomorphic to a subgroup of $O(n)$ because G_m acts isometrically on T_m in the flat metric fixing the origin.

We observe that $\dim O(n) = \binom{n}{2}$ where $O(n) = \{A \in GL_n(\mathbb{R}) : A^t = -A\}$, with Lie algebra $\mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) : A^t = -A\}$. Recall $\langle Av, Aw \rangle = \langle v, w \rangle$ if A is orthogonal. But for any orthogonal matrix B we also have $\langle Bv, w \rangle = \langle v, B^t w \rangle$, so $\langle v, w \rangle = \langle Av, Aw \rangle = \langle v, A^t Aw \rangle$, which implies $A^t A = I$, since v and w were general.

It follows that $\dim G$ is at most $n + \binom{n}{2}$ and if G acts transitively on M then M is diffeomorphic to G/G_m .

In order to force constant curvature, we need the isometry group to act transitively on M and G_m to act transitively on two dimensional subspaces of T_m .

By direct computation, each of the following is an isometry of H^n .

- Any isometry of \mathbb{R}^n that fixes H^n . This is because such an isometry fixes both the Euclidean metric and the plane $x_n = 0$; consequently it also fixes x_n^2 .
- Any rescaling, $x \rightarrow \lambda x$ of \mathbb{R}^n , since the scale factor obviously cancels out in the metric of H^n .
- Inversion in the unit sphere of \mathbb{R}^n . This has the form $x \rightarrow \frac{x}{\sum x_i^2}$ and fixes the metric of H^n by direct computation.

The group generated by rescaling and isometries that fix H^n is clearly transitive on H^n . Moreover, conjugation of the inversion in the unit sphere by elements of this group give inversion in every sphere centered on the hyperplane $x_n = 0$. Every point on such a sphere, is fixed by the inversion and the induced isometry of the tangent space is reflection in the tangent hyperplane to the sphere. For every point in H^n , every hyperplane in the tangent space is tangent to such a sphere. It follows that the isotropy group includes all hyperplane reflections, and thus is the entire orthogonal group. It now follows that H^n has constant sectional curvature. Moreover, it is clear that the submanifold obtained by setting all but the last two coordinates to zero is total geodesic and is isometric to H^2 , thus establishing that the curvature is negative.