Riemannian Geometry
9/25/02 Lecture notes

Existence and uniqueness of a connection that respects the metric and has vanishing tension:

We consider a coordinate neighbourhood U and write
$X_i = \frac{\partial}{\partial x_i}$ and $g_{ij} = \langle X_i, X_j \rangle$ and $\Gamma_{ijk} = \langle \nabla_{X_i} X_j, X_k \rangle$.
The torsion free condition sats that
$\nabla_{X_i} X_j = \nabla_{X_j} X_i$ or, equivalently $\Gamma_{ijk} = \Gamma_{jik}$
The condition of respecting the metric sats that
$X_i \langle X_j, X_k \rangle = \langle \nabla_{X_i} X_j, X_k \rangle + \langle X_j, \nabla_{X_i} X_k \rangle$
or, equivalently

$$\Gamma_{ijk} + \Gamma_{ikj} = \frac{\partial}{\partial x_i} g_{jk} \qquad (1)$$
$$\Gamma_{jki} + \Gamma_{jik} = \frac{\partial}{\partial x_j} g_{ik} \qquad (2)$$
$$\Gamma_{kji} + \Gamma_{kij} = \frac{\partial}{\partial xk} g_{ij} \qquad (3)$$
$$\implies (1) + (2) + (3) : \qquad 2\Gamma_{ijk} = \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}$$

Suppose, we have a connection, then

$$\nabla_{X_i} X_j = \sum_l \Gamma^l_{ij} X_l \qquad (a)$$

and $\quad \Gamma_{ijk} = \sum_l g_{lk} \Gamma^l_{ij} \qquad (b)$

thus $\quad \Gamma^m_{ij} = \sum_k g^{mk} \Gamma_{ijk} \qquad (c)$

Do (a), (b), (c) in the other direction and see that Γ_{ijk} determines the connection.

Curvature:

$$\begin{aligned} R_{ijkl} &= \langle \nabla_{X_i} \nabla_{X_j} X_k, X_l \rangle - \langle \nabla_{X_j} \nabla_{X_i} X_k, X_l \rangle \\ &= \langle \nabla_{X_i} \sum_m \Gamma^m_{jk} X_m, X_l \rangle \\ &= \langle \sum_m \sum_k \Gamma^k_{im} \Gamma^m_{jk} X_m, X_l \rangle + \langle \sum_m \frac{\partial}{\partial X_i} \Gamma^m_{jk} X_m, X_l \rangle \\ &= \sum_m \sum_k \Gamma^n_{im} \Gamma^m_{jk} + \sum_m \frac{\partial}{\partial X_i} \Gamma^m_{jk} g_{ml} \\ &= \sum_m \Gamma_{iml} \Gamma^m_{ik} + \sum_m \frac{\partial}{\partial X_i} \Gamma^m_{jk} g_{ml} \end{aligned}$$

Example to compute curvature:

The Hyperbolic plane is covered by a single coordinate patch with coordinates

$$\{(x,y) \,|y > 0\}, \qquad g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

An orthonormal basis is given by $\quad X := y\frac{\partial}{\partial x} \quad Y := y\frac{\partial}{\partial y}$ It follows that $[X, Y] = -X$.

Differentiating $< Y, Y >= 1$, we learn that $\nabla_X Y$ and $\nabla_Y Y$ are both perpendicular to Y and therefore multiples of X. We write accordingly, $\nabla_X Y = \alpha X$ and $\nabla_Y Y = \beta X$.

Similarly, $\nabla_X X$ and $\nabla_Y X$ are both multiples of X. Differentiating $< X, Y >= 0$, we obtain $\nabla_X X = -\alpha Y$ and $\nabla_Y X = -\beta Y$. From the symmetry, it now follows that

$$\nabla_X Y - \nabla_Y X = \alpha X + \beta Y = [X, Y] = -X,$$

which yields $\alpha = -1$ and $\beta = 0$. This now allows a direct computation of the sectional curvature $R(X, Y, X, Y)$. We have

$$R(X, Y, X) = \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X,Y]} X.$$

The first two terms give 0, the last is $\nabla_{-X} X = -\nabla_X X = -Y$. It now follows that $R(X, Y, X, Y) = -1$.

Tensors and Indices

Some definitions:

- $T(M)$ is the tangent bundle of M.
- $T^*(M)$ is the dual of the cotangent bundle of M.
- Sections of $T(M)$ are vector fields and are generated locally by $\frac{\partial}{\partial x_i}$.
- Sections of $T^*(M)$ are differentials and are generated locally by dx_i.
- A **tensor field** on M is a section of $(T^*(M))^m \otimes (T(M))^n$.
- A section of $T^*(M)^m$ is called **covariant**.
- A section of $T(M)^n$ is called **contravariant**.

The **metric** g is a section of $(T^*)^2$ and is generally written as $g_{ij} dx_i dx_j$.

The **curvature tensor** R of (x, y, z, w) is a section of $(T^*)^4$.

The metric g allows us to identify $T(M)$ with $T^*(M)$ as follows: If ω is a 1-form and x is a vector, then ω is identified with x provided $g(x, y) = \omega(y)$ for all y. And we have:

$$x = \sum a^i \frac{\partial}{\partial x_i}$$

$$\omega = \sum b_j dx_j$$

$$y = \sum c^i \frac{\partial}{\partial x_i}$$

$$\langle x, y \rangle = \sum_{ij} g_{ij} c^i a^j$$

$$\omega(y) = \sum c^i b_j$$

which implies

$$b_i = \sum g_{ij} a^j$$

If we denote the inverse matrix to g_{ij} by g^{ij} and transport the metric on T to a metric on T^*, then g_{ij} turns out to be the coefficient matrix for that metric.

$$\langle X, Y \rangle >= \sum g_{ij} a^i c^j$$

Let η correspond to Y so that $\eta = \sum d_i dx_i = \sum g_{ij} c^j$. Then,

$$\sum_{ij} g^{ij} b_i d_j = \sum \sum \sum g^{ij} g_{jk} a^k g_{il} c^l$$

If we have a tensor which is a section of $T^* \bigotimes T^* \bigotimes \ldots \bigotimes T \bigotimes T$, we can obtain a new tensor by **contraction** by letting a factor of T^* and a factor of T act on each other to produce a number or by using the metric to produce a number from two factors of the same type.

Definition: The **curvature tensor** R is given by $R = \sum R_{ijkl} dx_i \otimes dx_j \otimes dx_k \otimes dx_l$.

In the case of the Riemannian connection,

$$R_{ijkl} + R_{kijl} + R_{jkil} = 0$$

Consequently, $R_{ijkl} = R_{klij}$.

Definition: The **Ricci tensor** is given by

$$R_{ij} = \frac{1}{n} \sum_{kl} R_{ikjl} g^{kl}$$

where n is the dimension of the manifold.

Definition: The **scalar curvature** K is given by

$$K = \frac{1}{n(n-1)} \sum_{ijkl} g^{ik} g^{jl} R_{ijkl}.$$

We can restrict the metric and the curvature tensor to any two-dimensional subspace of the tangent space. The complete contraction of the restricted curvature tensor is called the **sectional curvature** for that subspace.

Geodesics

A geodesic is a curve $c : [t_0, t_1] \to M$ in a Riemannian manifold such that $Dc(\frac{\partial}{\partial t}$ is parallel along c. In coordinates,

$$\tfrac{d^2 x_k}{dt^2} = -\textstyle\sum_{ij} \Gamma^k_{ij} \frac{\partial x_j}{\partial t} \frac{\partial x_i}{\partial t}$$

if $c(t) = (x_1(t), \ldots, x_n(t))$. The push forward of $\frac{d}{dt}$ is $\sum \frac{dx_i}{dt} \frac{\partial x_i}{\partial t}$.

If $\eta = \sum a_i \eta^i$ where the η^i are sections for some local trivialization, then η is covariantly constant means that

$$\sum_i \frac{dx_i}{dt} \nabla_{x_i} \eta = 0 = \sum_i \frac{dx_i}{dt} \sum_{jk} a_j \Gamma^i_{jk} \eta_k + \sum_{ij} \frac{dx_i}{dt} \frac{\partial a_j}{\partial x_i} \eta_j$$

Geodesics on the Hyperbolic Plane:

$$\begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} = metric$$

$\Gamma^x_{xy} = \Gamma^y_{yy} = -\frac{1}{y^2}$, $\Gamma^y_{xx} = \frac{1}{y}$.
These are nonzero because they have an even number of x's. The rest are zero.

Geodesics on the upper-half plane are vertical lines and circles centered on the plane. The length of a vector in the hyperbolic plane is the Euclidean length divided by y.

$$x = \cos\theta, y = \sin\theta$$
$$\tfrac{dx}{dt} = -\sin\theta \tfrac{d\theta}{dt}, \tfrac{dy}{dt} = -\cos\theta \tfrac{d\theta}{dt}$$

$\frac{d\theta}{dt}$ must be proportional to y. (in order to get constant length)

Then $\dfrac{dx}{dt} = -\sin\theta \dfrac{d\theta}{dt}$ and $\dfrac{dy}{dt} = \cos\theta \dfrac{d\theta}{dt}$.

By parallelism, we have $y = c\dfrac{d\theta}{dt}$ for some constant c. For computational ease, we may assume $c = 1$, and thus

$\dfrac{dx}{dt} = -y^2$ and $\dfrac{dy}{dt} = xy$. Then

$\dfrac{d^2x}{dt^2} = -2xy$

$= \dfrac{2}{y}(\dfrac{dy}{dt}, \dfrac{dx}{dt})$

$= -2\Gamma^x_{xy}\dfrac{dx}{dt}\dfrac{dy}{dt}$ and

$\dfrac{d^2y}{dt^2} = x^2y - y^3$

$= -\Gamma^y_{yy}(\dfrac{dy}{dt})^2 - \Gamma^y_{xx}(\dfrac{dx}{dt})^2$

$= \dfrac{1}{y}(\dfrac{dy}{dt})^2 - \dfrac{1}{y}(\dfrac{dx}{dt})^2$ which shows that $(\cos\theta, \sin\theta)$ is a geodesic in the hyperbolic plane.

For vertical lines, we will assume $x = 0$ and $t = \log y$. We want to show that

$\dfrac{d^2y}{dt^2} = \dfrac{1}{y}(\dfrac{dy}{dt})^2$. This is true since

$\dfrac{dy}{dt} = \dfrac{1}{y}$ and

$\dfrac{d^2y}{dt^2} = \dfrac{1}{y}(\dfrac{dy}{dt}) = -\dfrac{1}{y^3}$.

Theorem: Let M be immersed in N. Let g be a metric on N and the induced metric on M. Let $\bar{\nabla}$ be the Riemannian connection on T(N), and the induced connection on $T(N)|_M$. Let ∇ be the Riemannian connection on M. If X and Y are vector fields on M then $\nabla_X Y$ is the orthogonal projection of $\bar{\nabla}_X Y$ on T(M).

Lie Groups on Manifolds

Let G be a Lie group with Lie algebra g identified both with $T_e(G)$ and with the Lie algebra of left invariant vector fields on G. Left invariant in the sense that $x < y, z >= 0 \; \forall x, y, z \in g$.

Any positive definite inner product on g induces a metric on G. Such a metric, $<, >$ is called **bi-invariant** if $\forall x, y, z \in g \; \langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0$.

Proposition

If $<, >$ is a bi-invariant metric on G, the Riemannian connection is given by $\nabla_X Y = \frac{1}{2}[X, Y]$ $\forall X, Y \in g$.

$$
\begin{aligned}
\nabla_X Y - \nabla_Y X &= \frac{1}{2}[X,Y] - \frac{1}{2}[Y,X] \\
&= [X,Y] \\
0 &= X < Y, Z > \\
&= \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle \\
&= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle
\end{aligned}
$$

Now inspect the curvature.

$$
\begin{aligned}
R(X,Y)Z &= \nabla_{[X,Y]}Z - (\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z) \\
&= \frac{1}{2}\,[[X,Y],Z] - \frac{1}{4}\,[X,[Y,Z]] + \frac{1}{4}\,[Y,[X,Z]] \\
&= -\frac{1}{4}\,[Z,[X,Y]] \text{ using the Jacobi identity} \\
R(X,Y,Z,W) &= \frac{1}{4}\,\langle [X,Y],[Z,W]\rangle
\end{aligned}
$$

If $X \in g$ then $\nabla_X X = \frac{1}{2}[X,X] = 0$.

One-parameter Subgroups

g_t is a **one-parameter subgroup** of G if $g_{t_1-t_2} = g_{t_1} g_{t_2}^{-1}$.

G acts on itself both on the left and on the right and manifestly the two actions commute. $(G \times G)$ acts on G by $(g_1, g_2)\, G = g_1^{-1} G g_2$.

For a one-parameter subgroup g_t, let $X_g = \frac{d}{dt} g g_t|_{t=0}$, then X is left invariant, because left and right actions commute.

Let $H \subset G$ be a Lie subgroup, then the coset space G/H is a manifold and is called a homogeneous space.

$$
\begin{array}{ccc}
gH & \to & G \\
 & & \downarrow \quad G \text{ acts on } G/H \text{ on the left.} \\
 & & G/H
\end{array}
$$

A bi-invariant metric on G induce a metric on G/H as roughly the orthogonal compliment to the tangent bundle along the fibers. We will go further into this later.