1 Manifolds, coordinates, vector fields, and vector bundles

1.1 Manifolds

1.1.1 Topological Manifolds

A **topological manifold** of dimension n is a topological space that is the union of open subsets, each homeomorphic to an open subset of $\mathbb{R}^n$.

Invariance of Domain: The dimension is set in stone. The subset can only be homeomorphic to a unique dimensional Euclidean space.[1]

1.1.2 Differentiable Manifolds

A **differentiable manifold** is the union of open subsets U_α such that

1. $\forall U_\alpha$ $\exists$ homeomorphism $\varphi_\alpha : U_\alpha \to$ open subsets of $\mathbb{R}^n$
2. $U_\alpha \cap U_\beta \neq \emptyset \implies \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta\left(U_\alpha \cap U_\beta\right) \to \varphi_\alpha\left(U_\alpha \cap U_\beta\right)$ is a **diffeomorphism**: a differentiable homeomorphism with a differentiable inverse.

To be more explicit about the definition of a diffeomorphism, let $\mathbf{x} = (x_1, \cdots, x_n)$ be coordinates in $\varphi_\beta\left(U_\alpha \cap U_\beta\right)$, and let $\mathbf{y} = (y_1, \cdots, y_n)$be coordinates in $\varphi_\alpha\left(U_\alpha \cap U_\beta\right)$. Then

$$\varphi_\alpha \circ \varphi_\beta^{-1}(\mathbf{x}) = \mathbf{y}$$

is a diffeomorphism provided

1. It is bijective: one-to-one, onto
2. each y_i is a differentiable function of $\mathbf{x}$
3. $det\left|\left[\frac{\partial y_i}{\partial x_j}\right]\right|$ is nowhere vanishing.

It is a consequence of 1 above that there is an inverse mapping; it is a consequence of 3 that the inverse mapping also satisfies 2 and 3.

1.2 Examples of Manifolds

1.2.1 The Sphere

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$$

The complement of a point is homeomorphic to $\mathbb{R}^n$. The intersection of the complements of two distinct points is homeomorphic to $\mathbb{R}^n - \{\mathbf{0}\}$

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[1] The Theorem of Invariance of Domain states that no open subset of R^n is homeomorphic to any open subset of R^m for $m \neq n$.

Let $\mathbf{n} = (0, \cdots, 0, 1) \in S^n$ be the north pole.
Then the associated steriographic projection is

$$\pi_{\mathbf{n}} : S^n - \{\mathbf{n}\} \to \mathbb{R}^n$$

$$\pi_{\mathbf{n}} (x_1, \cdots, x_{n+1}) = \frac{(x_1, \cdots, x_n)}{1 - x_{n+1}} = \mathbf{y}$$

Let $\mathbf{s} = (0, \cdots, 0, -1) \in S^n$ be the south pole.
Then the associated steriographic projection is

$$\pi_{\mathbf{s}} : S^n - \{\mathbf{s}\} \to \mathbb{R}^n$$

$$\pi_{\mathbf{s}} (x_1, \cdots, x_{n+1}) = \frac{(x_1, \cdots, x_n)}{1 + x_{n+1}} = \mathbf{y}'$$

The change of coordinates is

$$\mathbf{y}' = \frac{\mathbf{y}}{\sum_{i=1}^{n} y_i^2}$$

1.2.2 Real Projective Space

$P^n(\mathbb{R})$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Typically a point of $P^n(\mathbb{R})$ is described by homogeneous coordinates $(x_1, \cdots, x_{n+1})$ where not all the coordinates are zero and the equivalence relation $\forall \lambda \neq 0 \ \lambda \mathbf{x} \sim \mathbf{x}$.

We choose open sets $U_i = \{\mathbf{x} \in P^n(\mathbb{R}) | x_i \neq 0\}$. Then we set $x_i = 1$ and all other coordinates arbitrarily given a homeomorphism $U_i \to \mathbb{R}^n$

$$U_1 \cap U_2 = \{\mathbf{x} \in P^n(\mathbb{R}) | x_1, x_2 \neq 0\}$$

$$\begin{aligned}
\varphi_1^{-1}(y_1, \cdots, y_n) &= (1, y_1, \cdots, y_n) \\
&= \left(\frac{1}{y_1}, 1, \frac{y_2}{y_1}, \cdots, \frac{y_n}{y_1}\right) \\
&= (z_1, 1, z_2, \cdots, z_n)
\end{aligned}$$

1.3 Differentiability

If U, V are open subsets of $\mathbb{R}^n$ and $\varphi : U \to V$ is diffeomorphism, then $f : V \to \mathbb{R}$ is differentiable iff $\varphi \circ f : U \to \mathbb{R}$ is differentiable.

We know what it means for a function to be differentiable on open sets of a manifold. Coordinates are differentiable functions.

Let M, N be differentiable manifolds and $f : M \to N$ is a continuous mapping.

f is **differentiable** if $\forall p \in M$

- U an open subset of M containing p and $\varphi : U \to \mathbb{R}^n$ a coordinate mapping
- V an open subset of N containing $f(p)$ and $\psi : V \to \mathbb{R}^n$ a coordinate mapping

then $\varphi\left(U \cap f^{-1}(V)\right) \xrightarrow{\varphi^{-1}} U \cap f^{-1}(V) \xrightarrow{f} f(U) \cap V \xrightarrow{\psi} \mathbb{R}^m$ is differentiable at p [2].

A more useful definition of a differentiable map is given by

Proposition: If M, N are smooth manifolds, a continuous map $\phi : M \to N$ is differentiable iff for every open set $U \subset N$, and differentiable function $f : U \to R$, $f \circ \phi$ is a differentiable function on $\phi^{-1}(U)$.

Proof: (Basically coordinates are smooth functions)

1.3.1 Example

$T : \mathbf{x} \to -\mathbf{x}$,the *antipodal map*, is a diffeomorphism of $S^n \to S^n$.

$T^2 = I$ and $\forall p \in S^n \; T(p) \neq p$

If we identify $T(p)$ with p, then we obtain $P^n(\mathbb{R})$. For $U \subseteq P^n$ sufficiently small that its inverse image under the identification map consists of two disjoint copies, then U inherits coordinates from either copy. Because the antipodal map is a diffeomorphism, it doesn't matter which copy we choose.

In a similar manner, the quotient of a differentiable manifold by a free action of any finite group is again a differentiable manifold.[3]

1.4 Germs, tangent vectors, and vector fields

Let p be a point of the differentiable manifold M, let f, g be functions defined on neighborhoods of p. Then f and g have the same germ at p if there is an open neighborhood of p such that $f = g$ on this neighborhood. Note that the set of germs at p preserves addition, multiplication, etc.

A tangent vector at p is a linear function D from the ring of germs at p to the real numbers such that $D(fg) = f(p)Dg + g(p)Df(p)$. Note that the set of tangent vectors at $x \in R^n$ is a vector space over R with basis $\{\frac{\partial}{\partial x_1}|_x, ..., \frac{\partial}{\partial x_n}|_x\}$.

A vector field x on an open subset of U of M is a derivation in the ring of differentiable functions on U: $f \mapsto xf$ when $x(fg) = fxg + gxf$. For each $p \in U$, $f \to xf(p)$ is a tangent vector. In particular, a vector field on an open subset of R^n has the form $\sum f_i \frac{\partial}{\partial x_i}$ where the f_i are differentiable functions.

If x and y are vector fields on $U \subset M$ then so is $[x, y]$, defined by $[x, y]f = xyf - yxf$. Just as the space of tangent vectors at p has dimension $n = \mathrm{dim} M$, the space of vector fields on U (contained in one coordinate patch) also has dimension n over the ring of differentiable functions on U. A basis $\{x_i\}_{i=1}^n$ of the Lie algebra of differentiable vector fields on U is called commuting if $[x_i, x_j] = 0$ for all i, j.

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[2] The coordinate maps in this definition are going backwards relative to the book.

[3] Removing the restriction of finiteness involves defining what it means for a group action to be properly discontinuous; we will not involve ourselves in that for the present.

1.5 Vector bundles

Let B be a topological space (almost invariably a C^∞-manifold for our purposes). A vector bundle over B consists of a topological space E and a projection $\pi : E \to B$ such that:

1. For every $b \in B$, $\pi^{-1}(b)$ has the structure of an n-dimensional vector space over R.
2. For any open subset $U \subset B$, a continuous function $\Xi : B \to E$ such that $\pi \circ \Xi = id$ is a cross section. The vector space structures on $\pi^{-1}(b)$ endow a set of cross-sections over U with the structure of a $C^0(U)$ module.
3. The $C^0(U)$ module structure is locally trivial; that is, B is covered by open sets for which the module of cross-sections has a basis.

Assume bundles are C^∞ vector bundles over C^∞ manifolds. If $E \xrightarrow{\pi} M$ is a C^∞ vector bundle of dimension n, then $\forall$ open subset U of M, the sections of E over U form a $C^\infty(U)$-module. We assume M is covered by open subsets U for which $\exists$ a base; that is n sections where values form a base for each fibre corresponding to a point of U.

If we have two different bases, they are related by an $n \times n$ matrix whose entries are C^∞ functions on U and whose determinants are nowhere vanishing on U. Such a matrix can be reinterpreted as a differentiable function $U \to GL_n(\mathbb{R})$. These are called the transition functions, and the bundle can be reconstrunted from them.

Example Suppose M is an n-dimensional manifold. The tangent bundle of M has as its fibres over $m \in M$, the n-dimensional vector spaces of tangent vectors of M. The C^∞ sections of the tangent bundle are C^∞ vector fields. If U is a coordinate neighborhood in M, then $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ gives a base for the vector field over U. If V, with coordinates $\{y_1, \ldots, y_n\}$ is another such neighborhood, then the transition matrix over $U \cap V$ is exactly the matrix whose ij^{th} entry is $\frac{\partial y_i}{\partial x_j}$.

Example A one-dimensional vector bundle is called a line bundle. There is a line bundle over $\mathbb{R}\mathrm{P}^n$ whose fibre over a point in $\mathbb{R}\mathrm{P}^n$ is the line through the orgin in $\mathbb{R}^{n+1}$ to which it corresponds. The fibre over a given point is the set of all possible homgeneous coordinates fo rthat point tgether with 0. If U is the subset of $\mathbb{R}\mathrm{P}^n$ with $x_i \neq 0$, then there is a section over U, given by setting x_i to 1. If we have a point of $U_i \cap U_j$ with coordinates $(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$ and $x_i, x_j \neq 0$, then the transition function is $\frac{x_i}{x_j}$.

A vector bundle over M is *trivial* if it admits n sections over M whose values are everywhere linearly dependant. In particular, a line bundle is trivial if it admits a non-vanishing section over M. We have:

Theorem 1 *A line bundle over a connected manifold is trivial* $\iff$ *the complement of the zero section is not connected.*

If we have a smooth map $M \xrightarrow{f} N$ and a smooth vector bundle $E \to N$, then $f^!(E)$ is a smooth vector bundle over M whose total space is the subset of $M+E$ consisting of points (m,e) with $f(M) = \pi(e)$. If E has a basis over $U \subset N$, then $f^!(E)$ has a basis over $f^{-1}(U) \subset M$, defined by $f^!(\chi_M(V)) = \chi(f(M))$ for $f(V) \in U$, and χ a section of the base over U. We have the following commutative diagram:

$$\begin{tikzcd} f^!(E) \arrow[r] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

If $f : M \to N$ is smooth, $T(M)$ is the tangent bundle of M and $T(N)$ is the tangent bundle of N. Then there exists a map $T(f) : T(M) \to T(N)$. If v is a tangent vector at m and g is a germ of a smooth function at $f(m)$, $Tf(v)g = v(g \circ f)$. If $dim(M) \leq dim(N)$, f is called an *immersion*, if $T(f)$ is injective on every fibre of $T(M)$. If $dim(M) \geq dim(N)$, f is called a *submersion*, if $T(f)$ is surjective on every fibre of $T(M)$. Note: *a covering map is both an immersion and submersion, but not a diffeomorphism.*

Functors on Vector Spaces Give Functors on Vector Bundles

Example Let E be a vector bundle over M. E^*, the dual vector bundle to E, is the bundle for which E^*_m, the fibre of E^* over m, is the real dual of E_m. A section χ of $E^*|U$ is smooth $\iff$ (χ, ξ) is a smooth function on U, $\forall$ smooth sections ξ of $E|U$ where $(,)$ denotes the pairing between a vector space and its dual, taken fibre-wisely.

Similarly, we may define sums and tensor products for $E^1 \to M$ and $E^2 \to M$: $E^1 \oplus E^2$ is the vector bundle whose fibres over $m \in M$ are $E^1_m \oplus E^2_m$; and $E^1 \otimes E^2$ is the vector bundle whose fibres over $m \in M$ are $E^1_m \otimes E^2_m$. The sections are anologous to the above.

Exterior Algebra: Let V be any real vector space. Then the full tensor algebra over V is $\sum_{i=0}^{\infty} V^i$, where $V^0 = \mathbb{R}$, $V^1 = V$, and V^i is the i^{th} tensor power of V for $i > 1$. We have quotients: $S(V)$ and $\Lambda(V)$, the symmetric and anti-symmetric tensor algebras respectively.