At each point $m \in M$, we have the *exponential map*, $\exp_m : B_\epsilon \to M$, where B_ϵ is a ball of radius ϵ in T_m. For $v \in B_\epsilon$, $\exp_m(v)$ is defined to be $\gamma(1)$, where γ is a geodesic with $\gamma(0) = m$ and $\gamma'(0) = v$. Since γ is a geodesic, $||\gamma'|| = ||v||$ for all t. So $\exp_m$ is an embedding of B_ϵ in M. T_m has a flat metric induced from the inner product structure. B_ϵ has two metrics: the flat one induced by the inclusion $B_\epsilon \hookrightarrow T_m$, and the one induced from the metric on M by the exponential map. We will write them as $< \, , \, >_f$ and $< \, , \, >_m$ respecitvely.

Theorem 1 (Gauss' Lemma) *Let $v \in B_\epsilon$ and also identify the tangent space of B_ϵ at v with T_m. Let w be such that $<w, v>_f = 0$, then $<w, v>_m (v) = 0$.*

Proof We will adopt polar coordinates in the two dimensional subspaces of T_m spanned by w and v. Then the vector fields on T_m induced by v and w can be identitfied respectively at the point v with multiples of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$. Indeed the vector v is a constant multiple of $\frac{\partial}{\partial r}$ along the radial line in question, and we can extend w as a constant multiple of $\frac{\partial}{\partial \theta}$, except at the origin. Thus for purposes of the following computation we have $v = a\frac{\partial}{\partial r}$ and $w = b\frac{\partial}{\partial \theta}$, and there is no loss of generality in setting $b = 1$.

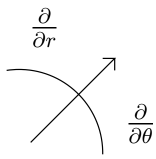

Figure 1:

We compute $<v, w>_m(tv)$. In deed, $\frac{\partial}{\partial t} <v, w>_m (tv) = <\nabla_v(v), \frac{\partial}{\partial \theta}>_m + < v, \nabla_v(\frac{\partial}{\partial \theta}) >_m = < v, \nabla_\theta(v) >_m = \frac{1}{2}\frac{\partial}{\partial \theta} < v, v >_m = 0$ which implies that $< v, \frac{\partial}{\partial \theta} >_m = a < \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} >_m$ is constant along the vertical line in the direction v. Moreover, the flat metric in terms of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ is $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$. The result now follows from the fact that the metric induced from M is asymptotically equal to the flat metric at the origin. □

Theorem 2 *Let $v \in B_\epsilon(m)$ and let c be any path from m to $\exp(v)$, then either c is a reparametrization of the radial geodesic or the length of c is longer than $||v||$; which is the length of the radial geodesic. And so we have geodesics are locally length minimizing.*

Proof We choose cooordinates on B_ϵ of the form r, $\{\theta_i\}$, where θ_i is a coordinate on the unit sphere in T_m. Then $c(t) = \exp(r(t)\theta(t))$ and $\frac{\partial c}{\partial t} = d\exp(\frac{\partial r}{\partial t}\theta + r\frac{\partial \theta}{\partial t})$ implying $||\frac{\partial c}{\partial t}|| = \sqrt{(\frac{\partial r}{\partial t})^2 + r^2||\frac{\partial \theta}{\partial t}||^2}$. □

Conversely, it can be shown any minimizing global curve is a geodesic.

Definition 1 *A Lie group is simultaneously a real analytic manifold and a group G so multiplication and inversion are real analytic operations.*

Definition 2 $L_g : G \to G$ *defined by* $L_g(h) = gh$ *is then real analytic and since* $L_{g^{-1}} = L_g^{-1}$, L_g *is a diffeomorphism.*

Proposition 1 *Let* $\phi : M \to N$ *be a diffeomorphism and* X *a vector field on* M*. Define* X_ϕ *on* N *by* $X_\phi(\phi(m)) = D\phi X(m)$*. Then* $[X_\phi, Y_\phi] = [X, Y]_\phi$*.*

Definition 3 *A vector field* X *is called left invariant if* $X(g) = DL_g(X(e))$ *or equivalently* $X_{L_g} = X$*.*

Proposition 2 *If* $e \in G$ *is the identity element, each* $X \in T_e(G)$ *determines a unique left invariant vector field and the Lie bracket of these are again left invariant.*

Definition 4 *The left invariant vector fields,* $\mathfrak{g}$*, with the Lie bracket operation form what is called the Lie algebra of* G*.*

Riemannian Geomery
Lecture notes 10/9/02

Let G be a Lie-group with Lie-algebra g identified both with $T_e(G)$ and with the Lie-algebra of left-invariant vectorfields on G. Then any positive inner product on G imidietly induces a metric on G[1]. Such a metric $\langle , \rangle$ is called *bi-invariant* if for any $x, y, z \in g$

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0$$

Proposition 1 *If* $\langle , \rangle$ *is a bi-invariant metric on* G*, the Riemannian Connection is given by*

$$\begin{aligned}
\nabla_x y &= \frac{1}{2}[x, y] \text{ for } x, y \in g \\
\nabla_x y - \nabla_y x &= \frac{1}{2}[x, y] - \frac{1}{2}[y, x] = [x, y] \\
0 &= x \langle y, z \rangle \\
&= \langle [x, y], z \rangle + \langle y, [x, z] \rangle \\
&= \langle \nabla_x y, z \rangle + \langle y, \nabla_x z \rangle
\end{aligned}$$

And

$$\begin{aligned}
R(x, y) z &= -\nabla_x \nabla_y z + \nabla_y \nabla_x z + \nabla_{[x,y]} z \\
&= -\frac{1}{4}[x, [y, z]] + \frac{1}{4}[y, [x, z]] + \frac{1}{2}[[x, y], z] \\
&= -\frac{1}{4}[z, [x, y]]
\end{aligned}$$

[1]which is left invariant in the sense that $x \langle y, z \rangle = 0$ for $\forall x, y, z \in g$

$$
\begin{aligned}
R(x,y,z,w) &= -\frac{1}{4}\langle [z,[x,y]],w\rangle \\
&= \frac{1}{4}\langle [x,y],[z,w]\rangle
\end{aligned}
$$

If $x \in g$ then $\nabla_x x = \frac{1}{2}[x,x] = 0$ and so Geodesics are a 1-parameter subgroup.

One g_t is a one parameter subgroup of G if

$$
\begin{aligned}
g_{t_1-t_2} &= g_{t_1} g_{t_2}^{-1} \\
g_s g_t &= g_t g_s \\
&= g_{s+t}
\end{aligned}
$$

G acts on itself both on the left and the right and the two actions commute. $(G \times G)$ acts on G by $(g_1, g_2)\, G = g_1^{-1} G g_2$. For a one-parameter subgroup g_t let

$$
X_g = \left.\frac{d}{dt} g\right|_{g_t=0}
$$

Then X is left-invariant because the left and right actions commute. It also follows that the right action of G on itself induces an action on the left invariant vector fields. This representation of G on its own Lie algebra is called the adjoint representation.

Lecture 17, October 11, 2002

Properties of the trace function, tr

1. $tr(AB)$ is invariant under the full general linear group. Moreover, $tr(AB) = tr(BA) = \sum_{i,j} a_{ij} b_{ji}$.
2. $tr(A^m B^m) = tr(M^{-1}AMM^{-1}BM) = tr(M^{-1}ABM) = tr(M^{-1}MAB) = tr(AB)$.
3. $tr(ad_x ad_y)$ is invariant on g but is not in general definite. If G is compact and semi-simple, then $tr(ad_x ad_y)$ is negative-definite. If G is semi-simple, then $tr(ad_x ad_y)$ is non-degenerate but, in general, indefinite.

If G admits a finite dimensional orthogonal representation, then G has a compact bi-invariant metric.

proof: If G is represented on V with inner product $<,>$ so that $< gv, gw > = < v, w >$, then g, the Lie Algebra of G, has the property that for $X \in g$, $< Xv, w > + < v, Xw > = 0$. With repsect to any orthonormal basis of V, G is represented by orthogonal matrices satisfying $M^T = M^{-1}$ and g is represented by anti-symmetric matrices satisfying $M^T = -M$. In that case $tr(MN) = -tr(MN^T)$ and is negative definite.

Curvature and Separation of Geodesics (from chapter 5)

We choose a plane in the tangent space at $p \in M$, and set $v(\theta) = r_0(\cos(\theta), \sin(\theta))$ $f(t,\theta) = exp_p(tv(\theta))$ gives is the restriction of the exponential map at p to our plane. We set $X = df(\frac{\partial}{\partial t}$ and $J = df(\frac{\partial}{\partial \theta}$. Both vector fields are defined near p on the exponential image of our plane, but not at p. Moreover, $[X, J] = 0$ and $\nabla_X X = 0$ since the radial exponential segments are geodesics, and X and J are the exponential images of commuting vector fields and the exponential map is a local diffeomorphism.

It now follows that

$$\nabla_X \nabla_X J = \nabla_X \nabla_J X = -R(X,J)X.$$

The first equality follows from the symmetry of the connection and the commutation of X and J, and the second follows again from the commutation and from the fact that $\nabla_X X = 0$. Strictly speaking, these equations should be interpreted on the pull back of the tangent bundle of M with its Riemannian connection to the two-dimensional submanifold that is the image of f

A vector field J along a geodesic is called a **Jacobi Field** if it satisifes the equation $= \nabla_x \nabla_x J + R(x,J)x = 0$, where x is the tangent to the geodesic.

We notice two important things about this equation. The first is that it is linear in J. The second is that, although we needed to differentiate along a two-dimensional submanifold in order to derive it in the case we have considered, we only need to differentiate along the geodesic in order to have it defined. We will think of J as a function of a parameter t proportional to arclength along the geodesic. If we choose an orthonormal basis of $T(M)$ at the origin of the geodesic and parallel transport it along the geodesic, then the components of $J(t)$ in that basis satisfy a second order linear ODE which has a unique solution along the geodesic for any initial conditions $J(0)$ and $J'(0)$.

Here, we want to study $< J(t), J(t) >$ as a function of t with $J(0) = 0$ and $< J'(0), X >= 0$.

We observe that, since $J'' = -R(X,J)X$, it follows from the general properties of the curvature that $< J'', X >= 0$ and hence that $< J', X >\equiv 0$. It also follows that $J''(0) = 0$.

We will also need to compute $J'''(0)$. Let W be covariantly constant along our geodesic, so that $W' \equiv 0$. Then

$$< J'''(t), W >= \frac{d}{dt} < J''(t), W >= -\frac{d}{dt} R(X,J,X,W) =$$

$$= -\frac{d}{dt} R(X,W,X,J) = -\frac{d}{dt} < R(X,W,X), J >=$$

$$= - < \nabla_X R(X,W)X, J > - R(X,W)X, J' > .$$

Since $J(0) = 0$, this yields

$$< J'''(0), W >= -R(X, W, X, J'(0)) = -R(X, J'(0), X, W) = - < R(X, J'(0))X, W >,$$

and it follows that $J'''(0) = -R(X, J'(0))X$.

Formally differentiating $< J, J >$, we obtain

$$\frac{d}{dt} < J, J >= 2 < J, J' >$$

$$\frac{d^2}{dt^2} < J, J >= 2 < J', J' > +2 < J, J'' >$$

$$\frac{d^3}{dt^3} < J, J >= 6 < J', J'' > +2 < J, J''' >$$

$$\frac{d^4}{dt^4} < J.J >= 8 < J', J''' > +6 < J'', J'' > +2 < J, J'''' >$$

At $t = 0$, the only terms that survive give

$$\frac{d^2}{dt^2} < J, J > |_{t=0} = 2 < J'(0), J'(0) >$$

and

$$\frac{d^4}{dt^4} < J, J > |_{t=0} = 8 < J'(0), J'''(0) >= -8R(X, J'(0), X, J'(0)),$$

where the curvature tensor is evaluated at the origin of the geodesic. This gives the Taylor expansion of $< J, J >$ to fourth order in t as

$$< J(t), J(t) >\approx< J'(0), J'(0) > t^2 - \frac{1}{3}R(X, J'(0), X, J'(0))t^4.$$

Let γ be a geodesic and consider the vector space V of Jacobi fields J along γ with $J(t_0) = 0$. Then $d\exp_{\gamma(t_0)}$ maps V linearly to $T_{\gamma(t)}M$ for each t. The point $\gamma(t_1)$ is called conjugate to $\gamma(t_0)$ along the geodesic γ if $d\exp_{\gamma(t_0)} \to T_{\gamma(t_1)}M$ is not a monomorphism, or equivalently, if there is a Jacobi field J along γ not identically zero, for which $J(t_0) = J(t_1) = 0$.

Big Theorem: $\gamma(t_1)$ is conjugate to $\gamma(t_0)$ along γ iff $d\exp_{t_0}$ is not a monomorphism at $\exp_{t_0}^{-1}\gamma(t_1)$.

Theorem (do Carmo p. xxx problem number yy) There are no conjugate points on a Riemannian manifold whose sectional curvatures are identically non-positive.

Compute: $\frac{\partial}{\partial t}\langle J, J'\rangle = \langle J', J'\rangle + \langle J, J''\rangle = \langle J', J'\rangle - R(\gamma', J, \gamma', J) \geq 0$.
If there is a conjugate point, then $\langle J, J'\rangle = 0$ at two distinct points. (Since $J = 0$ at two distict points.)
Hence, in between these two points, $\frac{\partial}{\partial t}\langle J, J'\rangle \equiv 0$. It follows that$2 < J, J' >= \frac{d}{dt} < J, J >\equiv 0$, and hence that $< J, J >$ is constant. But $J(0) = 0$; hence $< J, J >\equiv 0$ and $J \equiv 0$.