1. MATH 740 NOTES, SEPTEMBER 11, 2002

If we have a bundle E of dimension n over M, then $E \otimes E$ is a bundle of dimension n^2. $E \otimes E$ has a subbundle generated by sections of the form $\psi \otimes \eta + \eta \otimes \psi$ that can be identified with $S^2(E)$, and the quotient bundle can be identified with $\Lambda^2(E)$.

When $\dim(E) = n$, $S^k(E)$ has dimension $\binom{n+k}{k}$ and $\Lambda^k(E)$ has dimension $\binom{n}{k}$ for $k \leq n$. In particular, $\Lambda^n(E)$ is a line bundle.

Definition. M is *orientable* if $\Lambda^n(TM)$ is trivial.

On a coordinate neighborhood, TM has basis $\{\frac{\partial}{\partial x_i}\}$. To each smooth function f we associate the *section* df of $T^*(M)$, where $T^*(M)$ is the cotangent bundle of M defined by $(X, df) = Xf$ for any vector field X. Clearly, $dx_1, dx_2, \ldots, dx_n$ form a basis of $T^*(M)$ over any coordinate neighborhood. A general section of $T^*(M)$, called a 1-*form*, has the form $\Sigma f_i dx_i$, where the f_i are smooth functions. A 1-form df is called *exact.*

$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ gives a local section of $\Lambda^n T^*(m)$, and the transition functions are given by $\det \frac{\partial x_i}{\partial x_j}$. If E has dimension n, then $\Lambda^n(E)$ is sometimes written $\det(E)$.

Theorem. *If A is a subbundle of B, then* $\det(B)$ *is isomorphic as a line bundle to* $\det(A) \otimes \det(B/A)$.

Proof. If $a_1, a_2, \ldots, a_k$ is a basis of A which is completed to a basis of B by $c_{k+1}, c_{k+2}, \ldots, c_n$, then then $a = a_1 \wedge a_2 \wedge \cdots \wedge a_k$ is a generator of $\det(A)$, and $c_{k+1} \wedge c_{k+2} \wedge c_n$ descends to a generator c of $\det(B/A)$. Moreover the generator $a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge c_{k+1} \wedge c_{k+2} \wedge c_n$ of $\det(B)$ depends only on a and c and is bilinear in a and c. This establishes the desired isomorphism.

$\square$

Definition. Suppose $\phi : M \to N$ is an immersion. Then $T(M)$ is a subbundle of $\phi^!(TN)$. The quotient bundle $\phi^!(TN)/TM$ is called the *normal bundle* of the immersion.

Definition. An *embedding* of manifolds is an immersion that is also a *topological* embedding.

Example. Consider the natural embedding of S^n in $\mathbb{R}^{n+1}$. The tangent bundle $T(\mathbb{R}^{n+1})$ is trivial, with global basis $\frac{\partial}{\partial x_i}$. The normal bundle of S^n in $\mathbb{R}^{n+1}$ is a line bundle with canonical section given by $\Sigma x_i \frac{\partial}{\partial x_i}$. Thus, $\det(T(\mathbb{R}^n)) = \det(T(S^n)) \otimes \{\text{trivial line bundle}\}$, and thus is also a trivial line bundle, because tensoring with a trivial line bundle gives back an isomorphic bundle.

Definition. A *homogeneous* function on $\mathbb{R}^{n+1}$ is a function of the form $x = (x_1, x_2, \ldots, x_n)$ with $f(\lambda x) = \lambda^k f(x)$. k is called the *degree of homogeneity.*

Homogeneous functions of degree 0 are just functions on $\mathbb{R}P^n$. Products of homogeneous functions are homogeneous, and the degree of the product is the sum of the degrees. Thus, the homogeneous functions of degree k are a locally trivial module over the homogeneous functions of degree 0, and in fact constitute the sections of a line bundle. These bundles are the *tensor powers* of the canonical bundle H. So, if L is a line bundle, $L^{-k} = (L^*)^k$. H^2 is trivial, with nowhere vanishing section given by Σx_i^2.

Theorem: $\Re\mathrm{P}^{\mathrm{n}}$ is orientable if and only if n is odd.

Proof: We can represent differentiable functions on open subsets of $\Re\mathrm{P}^{\mathrm{n}}$ as differentiable functions of homogeneity zero on the corresponding subsets of $\Re^{n+1}$.

Claim: If f has homogeneity zero, so does $x_i\frac{\partial f}{\partial x_j}$.

$$\frac{\partial f}{\partial x_j}(\lambda x) = lim_{\triangle x_j \to 0}\frac{f(\lambda x_1,\ldots,\lambda x_j+\lambda\triangle x_j,\lambda x_{j+1},\ldots)-f(\lambda x)}{\lambda\triangle x_j}$$

$$= \frac{1}{\lambda} lim_{\triangle x_j \to 0}\frac{f(x_1,\ldots x_j+\triangle x_j,x_{j+1},\ldots)-f(x)}{\triangle x_j}$$

$$= \frac{1}{\lambda}\frac{\partial f}{\partial x_j}(x)$$

Hence, $\frac{\partial f}{\partial x_j}$ has homogeneity (-1) and since x_i has homogeneity 1, $x_i\frac{\partial f}{\partial x_j}$ has homogeneity zero. Consequently, $x_i\frac{\partial f}{\partial x_j}$ induces a vector field on $\Re\mathrm{P}^{\mathrm{n}}$.

Now, consider the trivial bundle E over $\Re\mathrm{P}^{\mathrm{n}}$, generated by global sections that we call $\frac{\partial}{\partial x_i}$, for $i = 1, \ldots, n+1$. We can now interpret $x_i\frac{\partial}{\partial x_j}$, a section of the canonical line bundle H, as a section of E⊗H.

Observation: If f has homogeneity zero, then $\sum_i x_i\frac{\partial f}{\partial x_i} = 0$.

$$\frac{df}{d\lambda}(\lambda x) = \sum_i x_i\frac{\partial f}{\partial x_i}(\lambda x) = 0$$

So we have a map from sections of E⊗H to vector fields which:

1. is linear over the functions of homogeneity zero
2. annihilates $\sum_i x_i \otimes \frac{\partial}{\partial x_i}$ which is a nowhere vanishing section of E⊗H

(3) generates the tangent bundle of $\Re$P^n, T($\Re$P^n) at every x (specifically, if $x_i \neq 0$, then $\{x_i \frac{\partial}{\partial x_j}\}_{i \neq j}$ generates the tangent space at the point x) so $x_i \frac{\partial}{\partial x_j}(\frac{\partial x_k}{\partial x_i}) = \delta_{jk}$.
In other words, there is a trivial line bundle L$\subset$ E$\otimes$H with E$\otimes$H/L $\approx$ T($\Re$P^n).

It follows that det$\{$T($\Re$P^n)$\}$ = det(E $\otimes$ H) = H^{n+1}.

Since even powers of H are trivial and odd powers are not, this gives the theorem. ∎

A differentiable map $\varphi(M \times R) \to M$ is called an **action** of R on M if

(1) $\varphi(m, 0) = m$
(2) $\varphi(\varphi(m, s), t) = \varphi(m, s + t)$

In other words, φ(M × R) for fixed m is called a **trajectory** or **orbit** of φ. Associated to such an action is a vector field χ on M defined by

$$\chi f(m) = \frac{d}{dt} f(\varphi(m, t))|_{t=0}$$

or $\chi = \mathrm{T}\varphi(\frac{d}{dt}) \mid_{t=0}$.

$\varphi : U \times (-\delta, \delta) \to M$ is called a **local action** if $\varphi(m, 0) = m$ and $\varphi(\varphi(m, s), t) = \varphi(m, s + t)$ whenever both sides are defined.

From a standard theorem in ODE's we know the following:

Let X be a vector field on M, let $m \in M$. Then there is a real number δ and an open neighborhood U of M for which $X \mid_U$ is the vector field of a local action $\varphi(U \times (-\delta, \delta)) \to M$.

Some "unraveling" of the terminology:

Let $\varphi(x.t)$ be a local action on an open subset of $\Re^n$. Then, $\chi(x) = \sum \frac{\partial x_i}{\partial t} \frac{\partial}{\partial x_i}$. Now, as the theorem states, let X be the vector field $X = y_i \frac{\partial}{\partial x_i}$. Then the initial value problem $\chi(0) = X$, $\frac{\partial x}{\partial t} = y(x)$ has a local solution (just use Euler's method).

Begin Differential Geometry

A metric on any vector bundle is a smooth bilinear form g on each fibre. In other words, if χ and η are local sections, then $g(\chi, \eta)$ is bilinear and symmetric over $C^\infty(M \; or \; U)$ and takes real values; also

$g(\chi, \chi) > 0$.

Let E be a line bundle over M and let g_1 and g_2 be metrics on $E \mid_U$. Let f_1 and f_2 be non-negative real valued functions on U with $f_1 + f_2 > 0$. Then $f_1 g_1 + f_2 g_2$ is also a metric on $E \mid_U$.

Observation: If $E \mid_U$ is trivial, then E admits a metric over U.
proof: let $x_1, \ldots, x_n$ be a basis of sections. Set $g(x_i, x_j) = \delta_{ij}$ and extend by bilinearity over $C^\infty(U)$.

Observation: Any vector bundle over a **paracompact** manifold admits a metric (use partitions of unity to show):
proof: Let U be a cover of M by open sets for which $E \mid_U$ is trivial. Then, there are non-negative functions ϕ_α such that $\phi_\alpha = 0$ outside of U_α and $\sum \phi_\alpha = 1$ since there are only finitely many non-vanishing summands for each $m \in M$. Now let g_α be a metric on $E \mid_{U_\alpha}$. Then $\sum \phi_\alpha g_\alpha$ is a metric on E (again only finitely many non-zero terms in the sum).

Definition: A **Riemannian metric** on M is by definition a metric on $T(M)$.

2. INDUCED METRICS

Let $\varphi : M \rightarrow N$ be an immersion and let g be a Riemannian metric on N. Then there is an induced metric g_φ on M edfined by $g_\varphi(x_1, x_2) = g(T\varphi(x_1), T\varphi(x_2))$. It is immediate that g_φ is a metric on M.

Notation: On a coordinate neighborhood, we write $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and we may also write $g = \sum g_{ij} dx_i dx_j$

Recall: We can parametrize a 2-sphere of radius ρ by

$$\begin{aligned} x &= \rho \sin(\varphi)\cos(\theta) \\ y &= \rho \sin(\varphi)\sin(\theta) \\ z &= \rho \cos(\theta) \end{aligned}$$

The Euclidean metric in $\mathbf{R}^3$ is simply $(dx)^2 + (dy)^2 + (dz)^2$
We can write down the metric:

$$\begin{aligned} dx &= \rho \cos(\varphi)\cos(\theta) d\varphi - \rho \sin(\varphi)\sin(\theta) d\theta \\ dy &= \rho \cos(\varphi)\sin(\theta) d\varphi - \rho \sin(\varphi)\cos(\theta) d\theta \\ dz &= -\rho \sin(\varphi) d\varphi \end{aligned}$$

Thus, $(dx)^2 + (dy)^2 + (dz)^2 = \rho^2 (d\varphi)^2 + \rho^2 \sin^2(\varphi)(d\theta)^2$

We can do the same for the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
$$x = a\sin(\varphi)\cos(\theta)$$
$$x = b\sin(\varphi)\cos(\theta)$$
$$x = c\cos(\varphi),$$

but the expression for the metric will be more complicated. In particular it will have off-diagonal terms.

Volume of an orientable Riemannian manifold:
Recall: M is orientable means that $\Lambda^n T(M)$ is trivial, which means, in turn, that M admits a nowhere vanishing n-form written locally as

$$f(x_1, \ldots, x_n) dx_1 \wedge \cdots \wedge dx_n$$

with f identically positive.

Suppose we look at a parallel piped in $T\mathbf{R}^n$ with edges $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. At a point P, we consider a basis $e_1, \ldots, e_n$ with $g(e_i, e_j) = \delta_{ij}$. Then we have $\frac{\partial}{\partial x_i} = \sum a_{ij} e_j$. Notice: It follows that $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle =$

$$\sum_k a_{ik} a_{jk}$$

The volume of a parallel piped is $det(a_{ij})$. We can write $a = (a_{ij})$, where (a_{ij}) denotes the matrix with entries equal to a_{ij}. Then, $(g_{ij}) = aa^t$. Moreover, $det(g_{ij}) = (det(a_{ij}))^2$. The volume form is $\sqrt{det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n$.

The matrix that represents the metric on the 2-sphere is:

$$\begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \sin^2(\varphi) \end{pmatrix}$$

The determinant is thus $\rho^4 \sin^2(\phi)$ and the volume form is the familar $\rho^2 \sin(\phi)\, d\phi\, d\theta$ from undergraduate multivariate calculus.

The n-sphere is defined by

$$\sum_{i=1}^{n+1} x_i^2 = \rho^2$$

There are ways of defining the analogue of spherical coordinates on a sphere of any dimension. However, we can also coordinatize any open hemisphere by writing: $x_{n+1} = \sqrt{\rho^2 - \sum_{i=1}^{n} x_i^2}$

Then

$$dx_{n+1} = \frac{-\sum_{i=1}^{n} x_i dx_i}{\sum_{i=1}^{n} x_i^2},$$

and we can use this to write the metric $\sum dx_i^2$ in terms of $\{x_i\}_{i \leq n}$ taken as coordinates on the sphere.

More General Phenomenon: Suppose $dimN < dimM$ and $\varphi : M \to N$ is a smooth map. We say φ is regular at $p \in M$ if $T\varphi : TM_p \to TN_{\varphi(p)}$ is surjective. If not, then $\varphi(p)$ is called a singular value of φ. In other words, $n \in N$ is a regular value of φ if $T\varphi$ is surjective at every point of $\varphi^{-1}(n)$.

Theorem: The inverse image of a regular value of φ is an embedded submanifold of M.

Proof: Let $y_1, \ldots, y_n$ be the coordinates on a neighborhood V of n. Let U be a coordinate neighborhood of $p \in \varphi^{-1}(n)$. Then $d(y_1 \circ \varphi), \ldots, d(y_n \circ \varphi)$ are linearly independent because of regularity. Then, assuming $dimM = n + k$, we can choose k linear combinations of $dx_1, \ldots, dx_{n+k}$ to form a basis of $T^*(M)_p$. Then the corresponding linear combinations of the x_i together with $y_i \circ \varphi$ form a coordinate system $U \cap \varphi^{-1}(V)$