Change in Notation

If $f : M \to N$ is a smooth map, then $df : T(M) \to T(N)$ is the induced map on the tangent bundles.

In particular, if $N = \mathbb{R}$, then $df(X) = Xf$.

0.1 Induced Length

Because a metric induces a volume on every orientable manifold, then the induced metric induces a volume on every submanifold.

The most important example of this is the length of a curve. If g is a Riemannian metric on M and $c : [t_0, t_1]$ is a smooth curve, then the length of c is given by

$$\int_{t_0}^{t_1} \sqrt{g\left(dc\frac{d}{dt}, dc\frac{d}{dt}\right)}$$

or in coordinate notation

$$\int_{t_0}^{t_1} \sqrt{\sum_{i,j=1}^{n} g_{ij}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial t}}$$

The length-minimizing curve is called a **geodesic**. This will be covered later.

1 Differentiating a Section of the Vector Bundle

Let $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ be a smooth vector bundle, with the n being the dimension of M and m being the fiber dimension of E - the dimension of the vector space $\pi^{-1}(x)$ $\forall x \in M$.

A section η of E is a map $M \to E$ with $\pi \circ \eta = \textit{identity}$.

It seems like we should be able to differentiate locally, though perhaps not globally (different vector spaces). But it is not obvious how to do it.

Suppose we want to differentiate cross-sections of E.

- Let X be a vector field on M
- Let η be a section of E
- Let $\nabla_X \eta$ be the result of "differentiating" η along X

Then $\nabla_X \eta$ should have the following properties.

1. $\nabla_X \eta$ should be $\mathbb{R}$-linear in both η and X
2. $\nabla_X \eta$ should depend on the germ of η at x, but only on the value of $X(x)$ at x. As a result it follows that $\nabla_X \eta$ is $C^\infty(M)$-linear in X.

3. $\nabla_X \eta$ should satisfy Leibniz' rule with respect to η, i.e.

$$\nabla_X f\eta = f\nabla_X \eta + Xf\eta$$

where $f \in C^\infty(M)$

A rule $X, \eta \to \nabla_X \eta$ that satisfies 1-3 above is called an **affine connection** on the smooth vector bundle E.

Theorem Affine connections always exist
Proof

1. Let ∇^1, ∇^2 be affine connections on E; and let f_1, f_2 be smooth functions with $f_1 + f_2 \equiv 1$. Then $f_1\nabla^1 + f_2\nabla^2$ is an affine connection.
2. If $\eta_1, \cdots, \eta_m$ are a local basis of E over U, then we can define a local connection by setting $\nabla_X \eta_i = 0 \ \forall x, i$
3. We can now define a connection as $\sum \varphi_\alpha \nabla^\alpha$ where ∇^α are local connections and $\sum \varphi_\alpha$ is an appropriate, locally finite, partition of unity.

Proposition Let ∇^1, ∇^2 be two connections on E, then $\nabla^1_X \eta - \nabla^2_X \eta$ is **tensorial**. It is $C^\infty(M)$-linear in both X, η. To compute the values at any given point, you only need the values of X, η at that point.

Proposition Let $\varphi : M \to N$ be a smooth map; E be a smooth vector bundle over N, with affine connection ∇. E' will be the pull back of E to N, so that we have the following commutative diagram.

$$\begin{tikzcd}
E' \arrow[r, "d\varphi"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
M \arrow[r, "\varphi"] & N
\end{tikzcd}$$

Then $\exists!\ \nabla'$ on E' such that if η is a section of E and X is a vector field on M, then[1]

$$\varphi^{-1}\left(\nabla_{d\varphi(X)}\eta\right) = \nabla'_X \varphi^{-1}(\eta \circ \varphi)$$

Proof We restrict to an open set $U \subset N$ for which E is locally trivial and an open set U' for which $T(M)$ is trivial and $\varphi(U') \subset U$.

Let $\eta_1, \cdots, \eta_n$ be a local basis for E over N, and let $X_1, \cdots, X_m$ be a local basis for $T(M)$. Then $\{\varphi^{-1}(\eta_i \circ \varphi)\}$ is a local basis of E' over U'. Now set

$$\nabla'_{X_i} \varphi^{-1} \circ \eta_j \circ \varphi \equiv \varphi^{-1} \nabla_{d\varphi(X_i)} \eta_j$$

and extend using linearity and Leibniz' rule

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[1] this is a generalization of proposition 2.2 on page 50 where $M = [t_0, t_1]$ and $E = T(N)$.

$$\begin{aligned}
\nabla'_{X_i} \sum g_j \varphi^{-1} \eta_j \varphi &= \sum g_j \varphi^{-1} \nabla_{d\varphi(X_i)} \eta_j + \sum X_i g_j \varphi^{-1} \eta_j \varphi \\
\nabla_{X_i} \left(h_j \circ \varphi\right) \varphi^{-1} \eta_j \varphi &= \nabla_{X_i} \varphi^{-1} \left(h_i \eta_j \varphi\right) \\
&= \varphi^{-1} \nabla_{d\varphi(X_i)} h_i \eta_j
\end{aligned}$$

As so often in this sort of context, global existence is a consequence of local uniqueness.

A connection Δ on a vector bundle E is called flat if M is covered by open subsets U on which E admits a basis over $C^\infty(U)$ consisting of sections η_i for which $\Delta_X \eta_i = 0$ for all vector fields X. Such sections are called covariantly constant. Every trivial vector bundle admits a flat connection (Take any basis of global sections and declare them to be covariantly constant. Then Δ is defined by $\Delta_X \sum f_i \eta_i$.) A vector bundle that admits a flat connection is not necessarily trivial. Suppose $\{\eta_i\}$ and $\{\nu_i\}$ are bases of the covariantly constant sections over U. Let $A_{ij}(x)$x for $x \in U$ be such that $\nu_i = \sum A_{ij} \eta_j$. Then for any vector field X, $0 = \Delta_X \nu_i = \sum X A_{ij} \eta_j$. Since the η_j are linearly independent over $C^\infty(U)$ it follows that $X A_{ij} = 0$, so $A_{ij}(x)$ does not depend on X.

Let $C : [0,1] \to M$ be a curve and let η_0 be in $\pi^{-1}(C(0))$. The connection on E (whether flat or not) pulls back to a connection in the induced bundle E' over $[0,1]$. Let us assume that $C([0,1])$ is in some U with $E|_U$ trivial. Then E' is trivial, and we choose a trivial section. Let $\frac{d}{dt}$ be the flat connection induced by that trivialization. Let Δ' be the pullback connection on E'. Then, $\Delta'_{\frac{d}{dt}} \chi = \frac{d\chi}{dt} + A(t)\chi$, where $A(t)$ is an n by n matrix. Such a system has a unique solution by a standard theorem on linear ODE's. The system $\eta(0) = \eta_0$, $\frac{d\eta}{dt} + A(t)\eta = 0$ has a unique solution $\eta(t)$, which is by definition covariantly constant along $[0,1]$.

If we also write $C : E' \to E$ for the lift that makes the diagram

$$\begin{tikzcd}
E' \arrow[r, "C"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] \\
{[0,1]} \arrow[r, "C"] & M
\end{tikzcd}$$

commutative, $C(\eta_i)$ is called the parallel transport of $C(\eta_0)$ along the curve η_0. It is a standard ODE theorem that the parallel transport is independent of the parametrization.

Let x_0, x_1 be points of M. We can now define parallel transport along any curve from x_0 to x_1 by considering subcurves contained in open sets for which the restriction of E is trivial.

Theorem. If Δ is flat and c_1, c_2 are homotopic rel x_0, x_1 curves with $c_1(0) = c_2(0) = x_0$ and $c_1(1) = c_2(1) = x_1$ then the parallel transport along either curve produces the same result.

Proof. Let H be the homotopy from c_1 to c_2. Let

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & E \arrow[d] \\
H \arrow[r] & M
\end{tikzcd}$$

be the pullback bundle. Assume, for the moment, that the image of H is contained in some U for which $E|_U$ admits a covariantly constant basis. Let $\eta_0 \in \pi^{-1}(x_0)$ with $\eta_0 = \sum a_i \eta_i(x_0)$ where $\{\eta_i\}$ is the covariantly constant basis. Then $\sum a_i \eta_i$ is the unique covariantly constant section over U for which $\eta(x_0) = \eta_0$.

If E, a bundle over M, is a bundle with a flat connection Δ, then for any $x_0 \in M$, $\pi_1(M, x_0)$ acts on $\pi^{-1}(x_0)$, and the nontriviality of that action is the only obstruction to the triviality of E. In particular, if M is simply connected, then any bundle admitting a flat connection is trivial.

Suppose Δ is a flat connection on E. Let $\eta_1, ..., \eta_n$ be a local basis of covariantly constant sections. Then, a general section has the form $\sum f_i \eta_i$. Let X and Y be vector fields. We want to compute $\Delta_X \Delta_Y \eta - \Delta_Y \Delta_X \eta$.

$$\Delta_Y \eta = \sum (Y f_i) \eta_i;$$

$$\Delta_X \Delta_Y \eta_i = \sum (XY f_i) \eta_i;$$

$$\Delta_Y \Delta_X \eta = \sum (YX f_i) \eta_i;$$

$$\Delta_X \Delta_Y \eta - \Delta_Y \Delta_X \eta = \sum [X, Y] f_i \eta_i = \Delta_{[X,Y]} \eta.$$

Definition 1 *A connection ∇ is called flat, if E admits local bases of covariantly constant sections.*

Proposition 1 *If ∇ is flat, then $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$.*

Definition 2 *For every connection ∇ on E, we define the curvature, R_∇, by $R_\nabla(X, Y, \eta) = \nabla_X \nabla_Y \eta - \nabla_Y \nabla_X \eta - \nabla_{[X,Y]} \eta$, where X and Y are vector fields and η is a section of E.*

Proposition 2 *$R_\nabla(X, Y, \eta)$ is anti-symmetric in X, Y and is $C^\infty(M)$-linear in all three arguments.*

Proof Anti-symmetry follows from the defintion and the Lie bracket being anti-symmetric. $R_\nabla(fX, Y, \eta) = \nabla_{fX} \nabla_Y \eta - \nabla_Y \nabla_{fX} \eta - \nabla_{[fX,Y]} \eta = f \nabla_X \nabla_Y \eta - f \nabla_Y \nabla_X \eta - Y f \nabla_X \eta - f \nabla_{[X,Y]} \eta + \nabla_{Y f X} \eta = f R(X, Y, \eta)$, and so by anti-symmetry the same holds for Y. Lastly, $R(X, Y, f\eta) = \nabla_X \nabla_Y f\eta - \nabla_Y \nabla_X f\eta - \nabla_{[X,Y]} f\eta = f \nabla_X \nabla_Y \eta + X f \nabla_Y \eta + \nabla_X Y f \eta - \{\text{anti}-\text{symmetric terms}\} = f R(X, Y, \eta)$, since the anti-symmetric terms cancel out all unwanted terms. □

Definition 3 *Let $< , >$ be a metric on E. A connection ∇ is said to respect the metric on E if $X < \eta_1, \eta_2 > = < \nabla_X \eta_1, \eta_2 > + < \eta_1, \nabla_X \eta_2 >$, for all vector fields X.*

Theorem 1 *For every metric on E, there exists a connection, ∇, that respects the metric.*

Outline of Proof

1. Apply Gram-Schmidt to a local basis, to obtain an orthonormal basis.
2. Declare the local orthonomal basis to be covariantly constant to obtain a local connection that respects the metric.
3. Observe if ∇_i respects the metric $\forall i$ and $\Sigma\phi_i = 1$, then $\Sigma\phi_i\nabla_i$ respects the metric.
4. But there exists partitions of unity, so choose one. $\square$

Definition 4 *If ∇ respects a metric on E, write $R_\nabla(X, Y, \eta_1, \eta_2) =< R_\nabla(X, Y, \eta_1), \eta_2 >$.*

Proposition 3 *$R_\nabla(X, Y, \eta_1, \eta_2)$ is anti-symmetric in η_1, η_2 and is C^∞-linear in all four arguments.*

Definition 5 (SPECIAL CASE) *$E = T(M)$ and $< X, Y >= g(X, Y)$ is Riemannian on M. We define $T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.*

Proposition 4 *$T_\nabla(X, Y)$ is $C^\infty(M)$-linear and anti-symmetric in both arguments.*

Proof It is enough to show linearity in one variable, as anti-symmetric is immediate. $T_\nabla(fX, Y) = \nabla_{fX}Y - \nabla_Y fX - [fX, Y] = f\nabla_X Y - f\nabla_Y X - YfX - f[X, Y] + YfX = fT_\nabla(X, Y)$. $\square$

Theorem 2 *Let g be a Riemannian metric on M, then there exista a unique connection, ∇, on $T(M)$, which repects g and has vanishing torsion.*