ADDENDUM TO "GLUING FORMULAS FOR DETERMINANTS OF DOLBEAULT LAPLACIANS ON RIEMANN SURFACES"

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The purpose of this note is two-fold. The first is to explain in a little more detail the discussion of page 458 of [8], and in particular, how to derive the result in Corollary 1.1 from Theorem 1.2 of that paper. The second is to compare this result with an earlier derivation of the Deligne constant by Jay Jorgenson [6]. In [7, Remark 1.6], I incorrectly implied that the two constants were slightly different. In fact, they agree exactly. Moreover, once differences in normalizations are taken into account, these results also agree with the prior work of Gillet-Soulé [4]. I would like to thank Changwei Zhao, Jay Jorgenson, and Gerard Freixas i Montplet for discussions of these issues.

1. Fay's constants

Let's begin with a precise summary of the constants appearing in the bosonization formulas of [3], and especially Corollary 5.12 of that work. Let M be a closed Riemann surface of genus $g \geq 1$, and with a fixed symplectic homology basis A_i, B_i. We let $\{\omega_i\}_{i=1}^{g}$ denote the normalized basis of abelian differentials:

$$\int_{A_i} \omega_j = \delta_{ij} \quad , \quad \int_{B_i} \omega_j = \Omega_{ij} \ ,$$

where (Ω_{ij}) is the period matrix. The associated Riemann divisor Δ satisfies $2\Delta = K$, where K is the canonical divisor of M. Set

$$\lambda(M) = \frac{\operatorname{Det}^* \Delta_M}{A(M) \det \operatorname{Im} \Omega} \tag{1}$$

where Δ_M is the (positive) Laplace-Beltrami operator for a conformal metric on M, and $A(M)$ is its total area. For convenience of this exposition, I assume here that M has the Arakelov metric. Let $\delta(M)$ denote Faltings' delta invariant. Then we have

$$\delta(M) = c_g - 6 \log(\lambda(M) \det \operatorname{Im} \Omega) \tag{2}$$

for a constant c_g depending only on the genus. In fact, in [7, Theorem 1.3] it was shown that

$$\begin{aligned} c_g &= (1-g)c_0 + gc_1 \\ c_1 &= -8\log(2\pi) \\ c_0 &= -24\zeta'(-1) + 1 - 6\log(2\pi) - 2\log 2 \end{aligned} \tag{3}$$

where $\zeta(s)$ is the Riemann zeta function.

Below I summarize Fay's constants and their interrelationships.

(i) $c_0(M)$: This is the constant first appearing in [3, Theorem 4.9] (see also (14) below). In general, $c_0(M)$ depends on the choice of conformal metric, but here we have fixed the Arakelov metric once and for all.

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(ii) $c_1(M)$: This is related to the Faltings invariant by

$$c_1(M) = e^{\delta(M)/12}(\det \operatorname{Im}\Omega)^{1/2} \tag{4}$$

(see [3, eq. (1.43)]).

(iii) δ_g : This appears in [3, Theorem 5.9] via the relation

$$c_0(M) = \delta_g c_1(M) \tag{5}$$

(iv) $\kappa_{g,n}$: This appears in [3, Theorem 5.8] and is related to the Mumford isometry [3, eq. (5.21)']. The result states that $\kappa_{g,n}$ is a constant depending only on the genus and the tensor powere $\Delta^{\otimes n}$ of the Riemann divisor taken in the Mumford isomorphism. By [3, eq. (5.22)],

$$\kappa_{g,2} = 4\pi^2 c_0^2(M)\lambda(M) \ . \tag{6}$$

(v) $\varepsilon_{g,d}$: This constant first appears in [3, Theorem 5.11]. We record here the key relation

$$\kappa_{g,n} = \varepsilon_{g,n(g-1)}\delta_g^{3(n-1)^2} \ . \tag{7}$$

Warning! Please do not confuse Fay's functions $c_0(M)$ and $c_1(M)$ with the values c_0 and c_1 of the constants c_g appearing in (3); nor should Fay's constant δ_g be confused with Faltings' invariant $\delta(M)$.

It will be convenient to express $c_0(M)$ and $c_1(M)$ in terms of $\lambda(M)$. First, using (2) and (4), we get

$$c_1(M) = e^{c_g/12}(\lambda(M))^{-1/2} \tag{8}$$

Next, using (5), (6), (7), and (8), we have

$$c_0^2(M) = \delta_g^2 c_1^2(M) = \delta_g^2 e^{c_g/6}(\lambda(M))^{-1} = (2\pi)^{-2}\kappa_{g,2}(\lambda(M))^{-1} = (2\pi)^{-2}\varepsilon_{g,2g-2}\delta_g^3(\lambda(M))^{-1} \ ;$$

or,

$$c_0(M) = \frac{1}{2\pi}(\varepsilon_{g,2g-2}\delta_g^3)^{1/2}(\lambda(M))^{1/2} \tag{9}$$

2. The evaluation of Fay's constants

By (5) and (8),

$$\varepsilon_{g,d}c_0(M) = \varepsilon_{g,d}\delta_g e^{c_g/12}(\lambda(M))^{-1/2}$$

With this change of constants, the bosonization formula in [3, Theorem 5.11] then becomse the expression appearing in [8, eq. (1.1)].

Next, using [8, Theorem 1.2], we claim that

$$\varepsilon_{g,d} = (2\pi)^{g-1-d} \tag{10}$$

(see [8, Corollary 1.1]). This is verified as follows: for a hermitian holomorphic line bundle $L \to M$ with admissible metric, let $T(L)$ denote the zeta-regularized determinant $\operatorname{Det}^* \square_L$ of the Dolbeault laplacian for L (recall that we always assume M has the Arakelov metric). Choose $p \in M$. For a flat line bundle $\chi \to M$, generic in the sense that $h^0(\chi \otimes \Delta) = h^1(\chi \otimes \Delta) = 0$, we have by [3, Theorem 5.11]:

$$\frac{T(\chi \otimes \Delta(p))}{\|\omega\|_{L^2}^2} = \frac{\varepsilon_{g,g}c_0(M)\|\vartheta\|^2(\chi + p - x)}{\|\omega\|^2(x)} \ , \tag{11}$$

where we have made a choice ω of nonzero holomorphic section of $\chi \otimes \Delta(p)$, and general point x. Note that we regard χ as a point in the Jacobian, and we pass freely between additive and multiplicative notation for line bundles and their divisors. On the other hand, by [8, Theorem 1.2],

$$T(\chi\otimes\Delta)=2\pi\|\omega\|^2(p)\frac{T(\chi\otimes\Delta(p))}{\|\omega\|^2_{L^2}} \tag{12}$$

Combine (11) and (12) to get

$$T(\chi\otimes\Delta)=2\pi\varepsilon_{g,g}c_0(M)\frac{\|\omega\|^2(p)}{\|\omega\|^2(x)}\|\vartheta\|^2(\chi+p-x)\ . \tag{13}$$

The invariant $c_0(M)$ is define in [3, eq. (4.37)] by the equation

$$T(\chi\otimes\Delta)=c_0(M)\|\vartheta\|^2(\chi)\ . \tag{14}$$

Hence, if we let $x\to p$ in (13) we get $1=2\pi\varepsilon_{g,g}$, or $\varepsilon_{g,g}=(2\pi)^{-1}$. A similar argument using [8, eq. (1.1)] and [8, Theorem 1.2] shows that $\varepsilon_{g,d+1}=(2\pi)^{-1}\varepsilon_{g,d}$ for all $d\geq g$, and the claim follows.

Combining this with (9) implies

$$\delta_g=\frac{(2\pi)^2}{\varepsilon_{g,2g-2}}e^{c_g/6}=(2\pi)^{g+1}e^{c_g/6}\ . \tag{15}$$

This is the result that appears in [8, Corollary 1.1].

3. Deligne's constant

We begin with a preliminary result. The insertion formula [8, Theorem 1.1] for line bundles $L\to M$ assumes that $h^1(L)=0$. We wish to extend this to the sequence

$$0\longrightarrow K\longrightarrow K(q)\longrightarrow K(q)\big|_q\longrightarrow 0$$

In the following, $G(z,w)$ denotes the Arakelov-Green's function (see [2, p. 393]), and recall from that paper the notion of *admissible* metric.

Proposition 3.1. *Let 1_q be a nonzero section of $\mathcal{O}_M(q)$ vanishing at q, and use this to define an admissible metric $\|1_q\|(z)=G(z,q)$. Let $\{\omega_i\}_{i=1}^g$ be normalized abelian differentials giving a basis of $H^0(M,K)$, and set $\widehat{\omega}_i=\omega_i\otimes 1_q$, so that $\{\widehat{\omega}_i\}_{i=1}^g$ is a basis for $H^0(M,K(q))$. Then with respect to the Arakelov metric on M and K and the tensor product metric on $K(q)$, we have*

$$\frac{T(K(q))}{\det\langle\widehat{\omega}_i,\widehat{\omega}_j\rangle}=2\pi\cdot\lambda(M)\ .$$

Proof. Let $\chi(s)$, $s\in D\subset\mathbb{C}$, be a generic family of $\mathsf{U}(1)$ characters on $\pi_1(M)$, with $\chi(0)=1$, and $\chi(s)\neq 1$ for $s\neq 0$. We first choose an L^2-orthonormal basis $\{\nu_i\}_{i=1}^g$ of $H^0(M,K)$. As in Fay [3], we may find independent meromorphic sections $\eta_i(z,s)$ of $\chi(s)\otimes K$ with at most a simple pole at q, and such that $\eta_i(z,0)=\nu_i(z)$, for all $i=1,\ldots,g$. With respect to a fixed trivialization of $\chi(s)$ at q, let $r_i(s)$ be the residue of $\eta_i(z,s)$ at q, $\mathbf{r}(s)=(r_1(s),\ldots,r_g(s))$. Note that by the assumption on $\chi(s)$ we must have $\mathbf{r}(s)\neq 0$ for $s\neq 0$. For generic choices we may furthermore assume $\mathbf{r}'(0)\neq 0$. Then there is a unitary martix $A=A(0)$ such that $A\mathbf{r}'(0)=(\widetilde{r}_1'(0),0,\ldots,0)$. Extend $A(s)$ holomorphically so that $A(s)\mathbf{r}(s)=(\widetilde{r}_1(s),0,\ldots,0)$, and set $\widetilde{\eta}_i(z,s)=A_{ij}(s)\eta_j(z,s)$. In this way (we henceforth omit the tildes), we reduce to the case where $\{\eta_i(z,s)\}_{i=1}^g$ is a basis of $H^0(M,\chi(s)\otimes K)$ for $s\neq 0$, and $\eta_1(z,s)$ has a simple pole at q with residue $r_1(s)$, and $r_1(s)$ has a simple zero at $s=0$.

Let $\widehat{\eta}_i(z,s) = \eta_i(z,s) \otimes 1_q$. Then by [8, Theorem 1.2] we have for $s \neq 0$,

$$2\pi \cdot \|\widehat{\eta}_1\|^2(q,s) \frac{T(\chi(s) \otimes K(q))}{\det\langle \widehat{\eta}_i, \widehat{\eta}_j \rangle} = \frac{T(\chi(s) \otimes K)}{\det\langle \eta_i, \eta_j \rangle} \tag{16}$$

By [3, Theorem 4.2],

$$T(\chi(s) \otimes K) = \frac{4\pi^2}{A(M)} |r_1'(0)|^2 s^2 T(K) + O(|s|^3) \tag{17}$$

On the other hand, in a local coordinate z centered at q we may write

$$\widehat{\eta}_1(z,s) = r_1(s) \frac{dz}{z} \otimes 1_q(z) + \text{ regular}$$

$$\|\widehat{\eta}_1\|^2(z,s) = |r_1(s)|^2 \|dz\|^2 \frac{G^2(z,q)}{|z|^2} + O(|z|)$$

$$\|\widehat{\eta}_1\|^2(q,s) = |r_1(s)|^2 \lim_{z \to 0} \|dz\|^2 \frac{G^2(z,q)}{|z|^2} = |r_1'(0)|^2 s^2 + O(|s|^3) \tag{18}$$

because $\mathcal{O}_M(q)$ has an admissible metric and M has the Arakelov metric. Using (17) on the right hand side of (16), and (18) on the left, and taking the limit as $s \to 0$, we have

$$\frac{T(K(q))}{\det\langle \widehat{\nu}_i, \widehat{\nu}_j \rangle} = \frac{2\pi}{A(M)} \frac{T(K)}{\det\langle \nu_i, \nu_j \rangle}$$

The result now follows by changing the basis $\{\nu_i\}$ to $\{\omega_i\}$. □

For the following, see [6, Theorem 1.2] (and note that there is a different sign convention).

Definition 3.2. Deligne's constant $a(g)$ is defined by the expression

$$-\frac{a(g)}{4} = \log T(\chi \otimes \Delta) + \frac{1}{2} \log \lambda(M) - \log \|\vartheta\|^2(\chi)$$

where Δ is the Riemann divisor and χ is chosen so that $h^0(\chi \otimes \Delta) = h^1(\chi \otimes \Delta) = 0$.

Using the insertion method of Faltings, we now derive

Proposition 3.3. *The following equality holds for all $g \geq 1$:*

$$a(g) + c_g + 4\pi(g+1)\log(2\pi) = 0 \ .$$

Proof. Choose generic points $p_1, \ldots, p_g, q$, and let $L = K(-p_1 - \cdots - p_g + q)$. The genericity condition is chosen so that $h^0(L) = h^1(L) = 0$. Notice that $L = \chi \otimes \Delta$, where the divisor of χ is $\Delta - p_1 - \cdots - p_g + q$. Choose a basis $\{\eta_i\}_{i=1}^g$ of $H^0(K)$ so that $\eta_i(p_j) \neq 0 \Leftrightarrow i = j$. For $k = 0, \ldots, g-1$, and $i > k$ let

$$\eta_i^{(k)} = \eta_i \otimes 1_q \otimes 1_{p_1}^{-1} \otimes \cdots \otimes 1_{p_k}^{-1} \ .$$

Then $\{\eta_i^{(k)}\}_{i=k+1}^g$ is a basis for $H^0(K(q - p_1 - \cdots - p_k)$, and $\eta_i^{(k+1)} \otimes 1_{p_{k+1}} = \eta_i^{(k)}$. Put admissible metrics on the bundles $K(q - p_1 - \cdots - p_k)$. We have the exact sequence

$$0 \longrightarrow K(q - p_1 - \cdots - p_{k+1}) \longrightarrow K(q - p_1 - \cdots - p_k) \longrightarrow K(q - p_1 - \cdots - p_k)\big|_{p_{k+1}} \longrightarrow 0$$

given by $\eta_{k+1}^{(k)}$. Hence, by [8, Theorem 1.2],

$$2\pi \cdot \|\eta_{k+1}^{(k)}\|^2(p_{k+1}) \frac{T(K(q - p_1 - \cdots - p_k)}{\det\langle \eta_i^{(k)}, \eta_j^{(k)} \rangle} = \frac{T(K(q - p_1 - \cdots - p_{k+1})}{\det\langle \eta_i^{(k+1)}, \eta_j^{(k+1)} \rangle} \ .$$

We also have

$$\|\eta_{k+1}^{(k)}\|^2(p_{k+1}) = \frac{\|\eta_{k+1}\|^2 G^2(q, p_{k+1})}{\prod_{i\leq k} G^2(p_i, p_{k+1})}$$

Iterating this for $k = 0, \ldots, g-1$, we obtain

$$\frac{(2\pi)^g \prod_{i=1}^g \|\eta_i\|^2(p_i) G^2(q, p_i)}{\prod_{i<j} G^2(p_i, p_j)} \frac{T(K(q)}{\det\langle \eta_i^{(0)}, \eta_j^{(0)}\rangle} = T(L)$$

Replacing the basis $\{\eta_i\}$ with $\{\omega_i\}$, and using Proposition 3.1 and Definition 3.2, we arrive at

$$\frac{(2\pi)^{g+1}\|\det \omega_i(p_j)\|^2 G^2(q, p_i)}{\prod_{i<j} G^2(p_i, p_j)} \lambda(M) = \lambda(M)^{-1/2} e^{-a(g)/4} \|\vartheta\|^2(p_1 + \cdots + p_g - q - \Delta) \ .$$

Using the definition of the Faltings invariant [2, p. 402], this becomes

$$(2\pi)^{g+1}\lambda(M)^{3/2} e^{a(g)/4} = e^{-\delta(M)/4} (\det \mathrm{Im}\, \Omega)^{-3/2} \ ,$$

and so

$$a(g) + 4(g+1)\log(2\pi) + \delta(M) + 6\log(\lambda(M)\det \mathrm{Im}\,\Omega) = 0 \ .$$

Now use the definition (2) of the constant c_g. This completes the proof. □

Remark 3.4. This result also follows from the expressions in the previous section. Namely, using Definition 3.2 and (14), we have

$$\begin{aligned}
-\frac{a(g)}{4} &= \log c_0(M) + \frac{1}{2}\log\lambda(M) \\
&= -\log(2\pi) + \frac{1}{2}\log(\varepsilon_{g,2g-2}\delta_g^3) \qquad \text{(by (9))} \\
&= -\log(2\pi) - \frac{1}{2}(g-1)\log(2\pi) + \frac{3}{2}(g+1)\log(2\pi) + \frac{c_g}{4} \qquad \text{(by (10) and (15))} \\
&= (g+1)\log(2\pi) + \frac{c_g}{4} \\
a(g) &= -4(g+1)\log(2\pi) - c_g
\end{aligned}$$

The proof given on the previous page is, however, direct and only uses the insertion formula and the expression for the Faltings invariant.

Finally, by Proposition 3.1 and the expression (3), we obtain

$$\begin{aligned} a(g) &= (1-g)a(0) \\ a(0) &= 24\zeta'(-1) - 1 + 2\log 2 + 2\log(2\pi) \end{aligned} \tag{19}$$

This agrees with the result in [6, Theorems 3.13 and 6.3], where we again note the difference in sign.

4. Comparison with the result of Gillet-Soulé

The first derivation of Deligne's constant is due to Gillet-Soulé [4] (see also [1] and [5]), and their formulation of the constant gives the value:

$$\begin{aligned} \widetilde{a}(g) &= (1-g)\widetilde{a}(0) \\ \widetilde{a}(0) &= 24\zeta'(-1) - 1 \end{aligned} \tag{20}$$

The goal of this section is to show that once normalizations are accounted for, the result for $a(g)$ in the previous section is consistent with the value of $\widetilde{a}(g)$ above. To this end, we first state a well-known result on the scaling of determinants.

Lemma 4.1. *Let $L \to M$ be a hermitian holomorphic line bundle of degree d, and set $\widetilde{\Box}_L = c\Box_L$ for some $c > 0$. Then*

$$\operatorname{Det}^* \widetilde{\Box}_L = c^{\frac{1}{3}(1-g)+d/2-h^0(L)} \operatorname{Det}^* \Box_L \ .$$

Proof. The zeta functions for $\widetilde{\Box}_L$ and $\Box_L$ are related by: $\zeta_{\widetilde{\Box}_L}(s) = c^{-s}\zeta_{\Box_L}(s)$, and so

$$\zeta'_{\widetilde{\Box}_L}(s) = -(\log c)\zeta'_{\widetilde{\Box}_L}(s) + c^{-s}\zeta'_{\Box_L}(s) \ .$$

Hence,

$$\log \operatorname{Det}^* \widetilde{\Box}_L := -\zeta'_{\widetilde{\Box}_L}(0) = (\log c)\zeta_{\Box_L}(0) + \log \operatorname{Det}^* \Box_L \ ,$$

and the result follows from the fact that

$$\zeta_{\Box_L}(0) = \frac{1}{3}(1-g) + \frac{d}{2} - h^0(L)$$

(see [3, eq. (2.38)]). □

There are *three* differences in the normalizations that go into the definitions of $a(g)$ and $\widetilde{a}(g)$. The normalizations for $a(g)$ are the ones used in [3], [7], and [8]. The normalizations for $\widetilde{a}(g)$ are used by Bismut-Gillet-Soulé (BGS). Here are the differences:

(i) BGS define the Dolbeault laplacian as

$$\Box_{L,\mathsf{BGS}} = \bar\partial_L^*\bar\partial_L = (1/2)\Box_L \ .$$

Thus, if L has degree $g-1$ with $h^1(L) = 0$, then by Lemma 4.1,

$$\begin{aligned} \operatorname{Det} \Box_{L,\mathsf{BGS}} &= 2^{-\frac{1}{6}(g-1)} \operatorname{Det} \Box_L \ ; \\ \operatorname{Det}^* \Delta_{M,\mathsf{BGS}} &= 2^{\frac{1}{3}(g-1)+1} \operatorname{Det}^* \Delta_M \ . \end{aligned} \tag{21}$$

(ii) Following Deligne, BGS introduce a factor of $1/2\pi$ into the L^2-norms. This changes the normalization of the Quillen metric by a factor of $(2\pi)^{-\chi(L)}$.

(iii) The metric we use on the canonical bundle is twice that of BGS. Hence, for example, for the normalized abelian differentials,

$$\frac{i}{2}\int_M \omega_i \wedge \overline{\omega}_j = \frac{1}{2}\int_M \langle\omega_i, \omega_j\rangle_{\mathsf{BGS}}\, dA$$

$$\det \operatorname{Im} \Omega = \frac{1}{2^g} \det\langle\omega_i, \omega_j\rangle_{\mathsf{BGS}}$$

Because of the changes in (ii) and (iii), we have by (21) that

$$\lambda(M)_{\mathsf{BGS}} = 2^{\frac{1}{3}(g-1)+1-g}(2\pi)^{-(g-1)}\lambda(M) = 2^{-\frac{2}{3}(g-1)}(2\pi)^{-(g-1)}\lambda(M) \ . \tag{22}$$

Using Definition 3.2, along with (21) and (22), we see that

$$\begin{aligned} \frac{\widetilde{a}(g)}{4} &= \frac{a(g)}{4} + \frac{1}{6}(g-1)\log 2 + \frac{1}{3}(g-1)\log 2 + \frac{1}{2}(g-1)\log(2\pi) \\ \widetilde{a}(g) &= a(g) - 2(1-g)\log 2 - 2(1-g)\log(2\pi) \ . \end{aligned}$$

The claim then follows from the expressions in (19) and (20).

References

[1] Jean-Michel Bismut and Gilles Lebeau. Complex immersions and Quillen metrics. *Inst. Hautes Études Sci. Publ. Math.*, (74):ii+298 pp. (1992), 1991.

[2] Gerd Faltings. Calculus on arithmetic surfaces. *Ann. of Math. (2)*, 119(2):387–424, 1984.

[3] John Fay. Kernel functions, analytic torsion, and moduli spaces. *Mem. Amer. Math. Soc.*, 96(464):vi+123, 1992.

[4] H. Gillet and C. Soulé. Analytic torsion and the arithmetic Todd genus. *Topology*, 30(1):21–54, 1991. With an appendix by D. Zagier.

[5] Henri Gillet and Christophe Soulé. An arithmetic Riemann-Roch theorem. *Invent. Math.*, 110(3):473–543, 1992.

[6] Jay Jorgenson. Degenerating hyperbolic Riemann surfaces and an evaluation of the constant in Deligne's arithmetic Riemann-Roch thoerem. *Unpublished manuscript*, 1991.

[7] Richard A. Wentworth. Precise constants in bosonization formulas on Riemann surfaces. I. *Comm. Math. Phys.*, 282(2):339–355, 2008.

[8] Richard A. Wentworth. Gluing formulas for determinants of Dolbeault Laplacians on Riemann surfaces. *Comm. Anal. Geom.*, 20(3):455–499, 2012.

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