DELIGNE PAIRINGS AND FAMILIES OF RANK ONE LOCAL SYSTEMS ON ALGEBRAIC CURVES

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ABSTRACT. For smooth families $\mathcal{X} \to S$ of projective algebraic curves and holomorphic line bundles $\mathcal{L}, \mathcal{M} \to \mathcal{X}$ equipped with flat relative connections, we prove the existence of a canonical and functorial "intersection" connection on the Deligne pairing $\langle \mathcal{L}, \mathcal{M} \rangle \to S$. This generalizes the construction of Deligne in the case of Chern connections of hermitian structures on $\mathcal{L}$ and $\mathcal{M}$. A relationship is found with the holomorphic extension of analytic torsion, and in the case of trivial fibrations we show that the Deligne isomorphism is flat with respect to the connections we construct. Finally, we give an application to the construction of a meromorphic connection on the hyperholomorphic line bundle over the twistor space of rank one flat connections on a Riemann surface.

1. INTRODUCTION

Let $\pi : \mathcal{X} \to S$ be a smooth proper morphism of smooth quasi-projective complex varieties with 1-dimensional connected fibers. Let $\mathcal{L}$ be a holomorphic line bundle on $\mathcal{X}$, and denote by $\omega_{\mathcal{X}/S}$ the relative dualizing sheaf of the family π. In his approach to understanding work of Quillen [31] on determinant bundles of families of $\overline{\partial}$-operators on a Riemann surface, Deligne [13] established a canonical (up to sign) functorial isomorphism of line bundles on S

$$\det R\pi_*(\mathcal{L})^{\otimes 12} \xrightarrow{\sim} \langle \omega_{\mathcal{X}/S}, \omega_{\mathcal{X}/S} \rangle \otimes \langle \mathcal{L}, \mathcal{L} \otimes \omega_{\mathcal{X}/S}^{-1} \rangle^{\otimes 6}. \qquad (1)$$

The isomorphism refines to the level of sheaves the Grothendieck-Riemann-Roch theorem in relative dimension 1. It relates the determinant of the relative cohomology of $\mathcal{L}$ (on the left hand side of (1)) to certain "intersection bundles" $\langle \mathcal{L}, \mathcal{M} \rangle \to S$ (on the right hand side of (1)), known as *Deligne pairings*, which associate line bundles on S to pairs of holomorphic bundles $\mathcal{L}, \mathcal{M} \to \mathcal{X}$. The relationship with Quillen's construction in Deligne's approach is in part inspired by Arakelov geometry, where metrized line bundles play a central role. Given smooth hermitian metrics on $\omega_{\mathcal{X}/S}$ and

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$\mathcal{L}$, there is an associated *Quillen metric* on $\det R\pi_*(\mathcal{L})$. The relevant input in the definition of this metric is the (holomorphic) analytic torsion of Ray-Singer: a spectral invariant obtained as a zeta regularized determinant of the positive self-adjoint $\overline{\partial}$-laplacians for $\mathcal{L}$ and the chosen metrics. We will call the associated Chern connection the *Quillen connection* on $\det R\pi_*(\mathcal{L})$. Also, the Deligne pairings in (1) inherit hermitian metrics, defined in the style of the archimedean contribution to Arakelov's arithmetic intersection pairing. Using the Chern connections associated to these hermitian metrics, the cohomological equality

$$c_1(\langle\mathcal{L},\mathcal{M}\rangle) = \pi_*\left(c_1(\mathcal{L}) \cup c_1(\mathcal{M})\right) \tag{2}$$

becomes an equality of forms for the Chern-Weil expressions of c_1 in terms of curvature. For these choices of metrics, the Deligne isomorphism (1) becomes an isometry, up to an overall topological constant. Consequently, the isomorphism is flat for the Chern-Weil connections, *i.e.* preserves these connections. This picture has been vastly generalized in several contributions by Bismut-Freed [5, 6], Bismut-Gillet-Soulé [4, 7, 8], Bismut-Lebeau [9], and others. They lead to the proof of the Grothendieck-Riemann-Roch theorem in Arakelov geometry, by Gillet-Soulé [20].

In another direction, Fay [17] studied the Ray-Singer torsion as a function on unitary characters of the fundamental group of a marked compact Riemann surface X with a hyperbolic metric. He showed that this function admits a unique holomorphic extension to the complex affine variety of complex characters of $\pi_1(X)$. He goes on to prove that the divisor of this function determines the marked Riemann surface structure. As for the classical Ray-Singer torsion, the holomorphic extension of the analytic torsion function to the complex character variety can be obtained by a zeta regularization procedure, this time for non-self-adjoint elliptic operators. Similar considerations appear in [27, 11, 30, 12], and in more recent work [23], where Hitchin uses these zeta regularized determinants of non-self-adjoint operators in the construction of a hyperholomorphic line bundle on the moduli space of Higgs bundles.

From a modern perspective, it is reasonable to seek a common conceptual framework for the results of Deligne, Fay and Hitchin, where the object of study is the determinant of cohomology of a line bundle endowed with a flat relative connection instead of a hermitian metric. Hence, on the left hand side of (1), one would like to define a connection on the determinant of the cohomology in terms of the spectrum of some natural non-self-adjoint elliptic operators, specializing to the Quillen connection in the unitary case. On the right hand side of (1), one would like to define natural connections on the Deligne pairings, specializing to Chern connections in the metric case. The aim would then be to show that the Deligne isomorphism is flat for these connections. This is the first motivation of the present article, where we achieve the core of this program. Specifically, we address the following points:

- we define an *intersection connection* on the Deligne pairing $\langle \mathcal{L}, \mathcal{M} \rangle \to S$ of line bundles $\mathcal{L}, \mathcal{M} \to \mathcal{X}$ equipped with *flat relative connections* (see Definition 2.3);
- in the case of trivial families $\mathcal{X} = X \times S$, we build a holomorphic connection on the determinant of cohomology by spectral methods. We then show that the Deligne isomorphism is flat with respect to this connection and intersection connections on Deligne pairings;
- we recover some of the results of Fay and Hitchin as applications of our results.

In a separate paper [18], the ideas developed here are used to construct an intersection theory for flat line bundles on arithmetic surfaces, and we establish an arithmetic Riemann-Roch in this formalism. The second *raison d'être* of the present article is thus providing the foundations that sustain this new arithmetic intersection formalism.

We now state the main results and outline of this paper more precisely. Let $\mathcal{L} \to \mathcal{X}$ be a holomorphic line bundle[1]. Let ∇ be a smooth connection on the underlying smooth bundle L that is *compatible* with the holomorphic structure on $\mathcal{L}$ in the sense that its $(0,1)$ part $\nabla^{0,1}$ coincides with the Dolbeault operator $\bar{\partial}_L$ induced by $\mathcal{L}$. Suppose in addition that the curvature F_∇ of ∇ vanishes on the fibers $\mathcal{X}_s$ of $\pi : \mathcal{X} \to S$, we wish to define an associated compatible connection on the Deligne pairing $\langle \mathcal{L}, \mathcal{M} \rangle$. The existence of the Deligne pairing, the construction of which we briefly review in Section 2.3, relies on the Weil reciprocity law for meromorphic functions on Riemann surfaces. Similarly, the construction of a connection on $\langle \mathcal{L}, \mathcal{M} \rangle$ requires a corresponding property of ∇ which we will call *Weil reciprocity for connections*, or (WR) for short (see Definition 3.1). It turns out that not every connection satisfies this condition! However, suppose $\nabla_{\mathcal{X}/S}$ is a compatible flat *relative* connection on $\mathcal{L}$; that is, a family of connections on the restricted line bundles $L|_{\mathcal{X}_s}$ to the fibers of $\pi : \mathcal{X} \to S$ which varies smoothly in s, and such that the connections on each fiber are flat and are compatible with the restricted holomorphic bundles $\mathcal{L}|_{\mathcal{X}_s}$ (see Definition 2.3). Then we shall show that $\nabla_{\mathcal{X}/S}$ can always be extended to a smooth connection ∇ on L that is compatible with $\mathcal{L}$ and which satisfies (WR). Moreover, this extension is functorial with respect to tensor products and base change (we shall simply say "functorial"). The extension is unique once the bundle is rigidified, *i.e.* trivialized along a given section; in general we characterize the space of all such extensions. It is important to stress that the extension is in general not a holomorphic connection: even if the initial flat relative connection varies holomorphically in s, the extension will in general only have a smooth (meaning C^∞) dependence on the base parameters of the family $\mathcal{X} \to S$.

The precise result may be formulated as follows.

[1]Throughout the paper, script notation such as $\mathcal{L}$, $\mathcal{M}$, etc., will be used for holomorphic bundles, whereas roman letters L, M denote underlying C^∞ bundles.

Theorem 1.1 (Trace connection)**.** *Let $\mathcal{L}, \mathcal{M}$ be holomorphic line bundles on $\pi : \mathcal{X} \to S$. Assume we are given:*

- *a section $\sigma : S \to \mathcal{X}$;*
- *a rigidification $\sigma^*\mathcal{L} \simeq \mathcal{O}_S$;*
- *a flat relative connection $\nabla_{\mathcal{X}/S}$ on $\mathcal{L}$, compatible with the holomorphic structure on $\mathcal{L}$ in the sense described above (see also Definition 2.3).*

Then the following hold:

(i) there is a unique extension of $\nabla_{\mathcal{X}/S}$ to a smooth connection on $\mathcal{L}$ that is compatible with the holomorphic structure, satisfies (WR) universally (i.e. *after any base change $T \to S$) and induces the trivial connection on $\sigma^*\mathcal{L}$;*

(ii) consequently, $\nabla_{\mathcal{X}/S}$ uniquely determines a functorial connection $\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}$ on the Deligne pairing $\langle\mathcal{L},\mathcal{M}\rangle$ that is compatible with the holomorphic structure;

(iii) in the case where $\nabla_{\mathcal{X}/S}$ is the fiberwise restriction of the Chern connection for a hermitian structure on $\mathcal{L}$, then $\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}$ coincides with the Chern connection for Deligne's metric on $\langle\mathcal{L},\mathcal{M}\rangle$ (and any metric on $\mathcal{M}$).

We shall use the term *trace connection* for the connections $\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}$ that arise from Theorem 1.1 (see Definition 3.3 for a precise definition). The extension result makes use of the moduli space of line bundles with flat relative connections and the infinitesimal deformations of such, which we call *Gauss-Manin invariants* $\nabla_{\mathrm{GM}}\nu$. These are 1-forms on S with values in the local system $H^1_{dR}(\mathcal{X}/S)$ that are canonically associated to a flat relative connection $\nabla_{\mathcal{X}/S}$ (see Section 2.2). In Section 3 we formalize the notion of Weil reciprocity and trace connection, and we formulate general existence and uniqueness theorems in terms of Poincaré bundles. This provides an explanation for why our constructions are canonical, and it demonstrates as well the importance of the functoriality conditions. In Section 4 we attack the proof of Theorem 1.1. The main result is Theorem 4.6, where we show that a certain *canonical extension* of $\nabla_{\mathcal{X}/S}$ satisfies all the necessary requirements. A closed expression for the curvature of a trace connection on $\langle\mathcal{L},\mathcal{M}\rangle$ is given in Proposition 4.15.

In the symmetric situation where both $\mathcal{L}$ and $\mathcal{M}$ are endowed with flat relative connections, $\nabla^L_{\mathcal{X}/S}$ and $\nabla^M_{\mathcal{X}/S}$, say, the construction in Theorem 4.6 can be modified to produce a connection on the Deligne pairing $\langle\mathcal{L},\mathcal{M}\rangle$ which takes into account both $\nabla^L_{\mathcal{X}/S}$ and $\nabla^M_{\mathcal{X}/S}$. We call these *intersection connections* (see Definition 3.13). In the special case where the $\nabla^L_{\mathcal{X}/S}$ and $\nabla^M_{\mathcal{X}/S}$ are the Chern connections of hermitian structures on $\mathcal{L}$ and $\mathcal{M}$, the intersection connection is simply the Chern connection on Deligne's metric on $\langle\mathcal{L},\mathcal{M}\rangle$. Thus, intersection connections give a generalization of Deligne's construction. We formulate this in the following

Theorem 1.2 (Intersection connection)**.** *Let $\mathcal{L}, \mathcal{M}$, be holomorphic line bundles on $\pi : \mathcal{X} \to S$ with flat relative connections $\nabla^L_{\mathcal{X}/S}$ and $\nabla^M_{\mathcal{X}/S}$ compatible with the* *holomorphic structures. Then there is a uniquely determined connection* $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ *on* $\langle\mathcal{L},\mathcal{M}\rangle$ *satisfying:*

(i) $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ *is functorial and compatible, and it is symmetric with respect to the isomorphism* $\langle\mathcal{L},\mathcal{M}\rangle \simeq \langle\mathcal{M},\mathcal{L}\rangle$*;*

(ii) the curvature of $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ *is given by*

$$F_{\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}} = \frac{1}{2\pi i}\,\pi_*\left(\nabla_{\mathrm{GM}}\nu_L \cup \nabla_{\mathrm{GM}}\nu_M\right) \tag{3}$$

where $\nabla_{\mathrm{GM}}\nu_L$ *and* $\nabla_{\mathrm{GM}}\nu_M$ *are the Gauss-Manin invariants of* $\mathcal{L}$ *and* $\mathcal{M}$*, respectively, and the cup product is defined in* (47)*;*

(iii) in the case where $\nabla^M_{\mathcal{X}/S}$ *is the fiberwise restriction of the Chern connection for a hermitian structure on* $\mathcal{M}$*, then* $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle} = \nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}$ *(where the trace connection is from Theorem 1.1 and exists under these hypotheses);*

(iv) in the case where both $\nabla^L_{\mathcal{X}/S}$ *and* $\nabla^M_{\mathcal{X}/S}$ *are the restrictions of Chern connections for hermitian structures, then* $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ *is the Chern connection for Deligne's metric on* $\langle\mathcal{L},\mathcal{M}\rangle$*.*

We call $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ the *intersection connection* of $\nabla^L_{\mathcal{X}/S}$ and $\nabla^M_{\mathcal{X}/S}$. In Section 5 we illustrate the construction in the case of a trivial family $\mathcal{X} = X \times S$, where the definition of the connections on $\langle\mathcal{L},\mathcal{M}\rangle$ described in Theorems 1.1 and 1.2 can be made very explicit. Given a holomorphic relative connection on $\mathcal{L} \to X \times S$, there is a classifying map $S \to \mathrm{Pic}^0(X)$, and $\det R\pi_*(\mathcal{L})$ is the pull-back to S of the corresponding determinant of cohomology. Viewing the jacobian $J(X) = \mathrm{Pic}^0(X)$ as the character variety of $\mathsf{U}(1)$-representations of $\pi_1(X)$, and choosing a conformal metric on X, the determinant of cohomology carries a natural Quillen metric and associated Chern connection. If we choose a theta characteristic κ on X, $\kappa^{\otimes 2} = \omega_X$, and consider instead the map $S \to \mathrm{Pic}^{g-1}(X)$ obtained from the family $\mathcal{L} \otimes \kappa \to \mathcal{X}$, then $\det R\pi_*(\mathcal{L} \otimes \kappa)$ is the pull back of $\mathcal{O}(-\Theta)$. Using a complex valued holomorphic version of the analytic torsion of Ray-Singer, $T(\chi \otimes \kappa)$, we show that the tensor product of the determinants of cohomology for X and $\overline{X}$ (the conjugate Riemann surface) admits a canonical holomorphic connection. On the other hand, in this situation the intersection connection on the tensor product of $\langle\mathcal{L},\mathcal{L}\rangle$ with its counterpart for $\overline{X}$ is also holomorphic. In Theorem 5.11 we show that the Deligne isomorphism, which relates these two bundles, is flat with respect to these connections. The importance of working with both X and $\overline{X}$ simultaneously appears as well in related constructions of Cappell-Miller [12]. The precise relationship of [12] to the present work will be explained in our second paper [18].

Finally, again in the case of a trivial fibration, we point out a link with some of the ideas in the recent paper [23]. The space $M_{dR}(X)$ of flat rank 1 connections on X has a hyperkähler structure. Its twistor space $\lambda : Z \to \mathbb{P}^1$ carries a holomorphic line bundle $\mathcal{L}_Z$, which may be interpreted as a determinant of cohomology via Deligne's characterization of Z as the space of λ*-connections* (see Definition 5.14). We will show how the connection obtained from the intersection connection on the Deligne pairing of the universal bundle on $M_{dR}(X)$ gives a proof of the following result (see Theorem 5.15 for a more precise statement).

Theorem 1.3 (Hitchin, *cf.* [24, Theorem 3])**.** *The line bundle $\mathcal{L}_Z$ admits a meromorphic connection with logarithmic singularities along the preimage $\lambda^{-1}\{0,\infty\}$. The curvature of this connection restricted to each fiber of λ over $\mathbb{C}^\times$ is a holomorphic symplectic form. The residue of the connection at $\lambda = 0$ (resp. $\lambda = \infty$) is the Liouville* 1*-form on $T^*J(X)$ (resp. $T^*J(\overline{X})$).*

Similar methods will potentially produce a higher rank version of this result; this will be the object of future research.

We end this introduction by noting that considerations similar to the central theme of this paper have been discussed previously by various authors. We mention here the work of Bloch-Esnault [10] on the determinant of deRham cohomology and Gauss-Manin connections in the algebraic setting, and of Beilinson-Schechtman [3]. Complex valued extensions of analytic torsion and reciprocity laws do not seem to play a role in these papers. Gillet-Soulé [19] also initiated a study of Arakelov geometry for bundles with holomorphic connections, but left as an open question the possibility of a Riemann-Roch type theorem.

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2. Relative Connections and Deligne Pairings

2.1. **Preliminary definitions.** Let $\pi : \mathcal{X} \to S$ be a submersion of smooth manifolds, whose fibers are compact and two (real) dimensional. We suppose that the relative complexified tangent bundle $T_{\pi,\mathbb{C}}$ comes equipped with a relative complex structure $J : T_{\pi,\mathbb{C}} \to T_{\pi,\mathbb{C}}$, so that the fibers of π have the structure of compact Riemann surfaces. It then makes sense to introduce sheaves of (p,q) relative differential forms $\mathcal{A}^{p,q}_{\mathcal{X}/S}$, $p,q \in \{0,1\}$. There is a relative Dolbeault operator $\overline{\partial} : \mathcal{A}^0_{\mathcal{X}} \to \mathcal{A}^{0,1}_{\mathcal{X}/S}$, which is just the projection of the exterior differential to $\mathcal{A}^{0,1}_{\mathcal{X}/S}$. The case most relevant in this paper and to which we shall soon restrict ourselves is, of course, when π is a holomorphic map of complex manifolds. Then, the relative Dolbeault operator is the projection to relative forms of the Dolbeault operator on $\mathcal{X}$.

Let $L \to \mathcal{X}$ be a C^∞ line bundle. We may consider several additional structures on L. The first one is relative holomorphicity.

Definition 2.1. A *relative holomorphic structure* on L is the choice of a *relative Dolbeault operator on* L: a $\mathbb{C}$-linear map $\overline{\partial}_L : \mathcal{A}^0_{\mathcal{X}}(L) \to \mathcal{A}^{0,1}_{\mathcal{X}/S}(L)$ that satisfies the Leibniz rule with respect to the relative $\overline{\partial}$-operator. We will write $\mathcal{L}$ for a pair $(L, \overline{\partial}_L)$, and call it a *relative holomorphic line bundle*.

Remark 2.2. In the holomorphic (or algebraic) category we shall always assume the relative Dolbeault operator is the fiberwise restriction of a global integrable operator $\overline{\partial}_L : \mathcal{A}^0_{\mathcal{X}}(L) \to \mathcal{A}^{0,1}_{\mathcal{X}}(L)$, so that $\mathcal{L} \to \mathcal{X}$ is a holomorphic bundle. In order to stress the distinction, we will sometimes refer to a *global* holomorphic line bundle on $\mathcal{X}$.

The second kinds of structure to be considered are various notions of connections.

Definition 2.3.

(i) A *connection on* $L \to \mathcal{X}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{A}^0_{\mathcal{X}}(L) \to \mathcal{A}^1_{\mathcal{X}}(L)$ satisfying the Leibniz rule: $\nabla(fe) = df \otimes e + f\nabla e$, for local C^∞ functions f and sections e of L. Its *curvature* is $F_\nabla := \nabla \wedge \nabla \in \mathcal{A}^2_{\mathcal{X}}$, and ∇ is called *flat* if $F_\nabla = 0$.

(ii) A *relative connection on* $L \to \mathcal{X}$ is a $\mathbb{C}$-linear map $\nabla_{\mathcal{X}/S} : \mathcal{A}^0_{\mathcal{X}}(L) \to \mathcal{A}^1_{\mathcal{X}/S}(L)$ satisfying the Leibniz rule with respect to the relative exterior differential $d : \mathcal{A}^0_{\mathcal{X}} \to \mathcal{A}^1_{\mathcal{X}/S}$.

(iii) A relative connection on $L \to \mathcal{X}$ is called *flat* if the induced connection on $L|_{\mathcal{X}_s}$ is flat for each $s \in S$.

(iv) If $\nabla_{\mathcal{X}/S}$ is a relative connection on a line bundle $L \to \mathcal{X}$, then a smooth connection ∇ on L is called *an extension of* $\nabla_{\mathcal{X}/S}$ if the projection to relative forms makes the following diagram commute:

$$L \xrightarrow{\nabla} \mathcal{A}^1_{\mathcal{X}}(L) \longrightarrow \mathcal{A}^1_{\mathcal{X}/S}(L), \qquad L \xrightarrow{\nabla_{\mathcal{X}/S}} \mathcal{A}^1_{\mathcal{X}/S}(L)$$

(v) If $\pi : \mathcal{X} \to S$ is a holomorphic map of complex manifolds, given a global holomorphic line bundle $\mathcal{L} = (L, \overline{\partial}_L)$ on $\mathcal{X}$, a connection ∇ on $\mathcal{L}$ is called *compatible* with the holomorphic structure if the $(0,1)$-part of the connection $\nabla^{0,1} = \bar{\partial}_L$.

(vi) Given the structure of a relative holomorphic line bundle $\mathcal{L} = (L, \overline{\partial}_L)$ (see Definition 2.1), a *relative connection on* $\mathcal{L}$ is a relative connection on the underlying C^∞ bundle L that is compatible with $\mathcal{L}$, in the sense that the vertical $(0,1)$ part satisfies: $(\nabla_{\mathcal{X}/S})'' = \bar{\partial}_L$ (relative operator).

(vii) If $\pi : \mathcal{X} \to S$ is a holomorphic map of complex manifolds and $\mathcal{L} \to \mathcal{X}$ is a global holomorphic line bundle on $\mathcal{X}$, a relative connection on $\mathcal{L}$ is called *holomorphic* if it induces a map $\nabla_{\mathcal{X}/S} : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{\mathcal{X}/S}$.

Remark 2.4.

(i) Note that a holomorphic connection in the sense of part (vii) above is automatically flat.
(ii) If $\mathcal{L}$ is a relative holomorphic line bundle and $\nabla_{\mathcal{X}/S}$ is a flat relative connection, then its restrictions to fibers are holomorphic connections.
(iii) The important special case (vii) above occurs, for example, when $\nabla_{\mathcal{X}/S}$ is the fiberwise restriction of a holomorphic connection on $\mathcal{L}$. This is perhaps the most natural situation from the algebraic point of view. However, the more general case of flat relative connections considered in this paper is far more flexible and is necessary for applications, as the next example illustrates (see also Remark 5.1 below).

Example 2.5. Let $\mathcal{L} \to \mathcal{X}$ be a holomorphic line bundle with relative degree zero. Then there is a smooth hermitian metric on $\mathcal{L}$ such that the restriction of the Chern connection ∇_{ch} to each fiber is flat, and for a rigidified bundle (*i.e.* the choice of a trivialization along a given section) this metric and connection can be uniquely normalized (by imposing triviality along the section). Abusing terminology slightly, we shall refer to the connection ∇_{ch} as *the Chern connection of* $\mathcal{L} \to \mathcal{X}$. The fiberwise restriction of ∇_{ch} then gives a flat relative connection $\nabla_{\mathcal{X}/S}$. Note that outside of some trivial situations it is essentially never the case that $\nabla_{\mathcal{X}/S}$ is holomorphic in the sense of Definition 2.3 (vii).

2.2. **Gauss-Manin invariant.** Let $\pi : \mathcal{X} \to S$ be as in the preceding discussion, and suppose it comes equipped with a fixed section $\sigma : S \to \mathcal{X}$. The problem of extending relative connections to global connections requires infinitesimal deformations of line bundles with relative connections. In our approach, it is convenient to introduce a moduli point of view. Let $L \to \mathcal{X}$ be a fixed C^∞ complex line bundle that is topologically trivial on the fibers and is endowed with a fixed trivialization along σ. We set

$$M_{dR}(\mathcal{X}/S) = \{\text{moduli of flat relative connections on } L\} \tag{4}$$

Consider the functor of points: $\{T \to S\} \mapsto M_{dR}(\mathcal{X}_T/T)$, where $T \to S$ is a morphism of smooth manifolds and $\mathcal{X}_T$ is the base change of $\mathcal{X}$ to T. This functor can be represented by a smooth fibration in Lie groups over S. To describe it, let us consider the relative deRham cohomology $H^1_{dR}(\mathcal{X}/S)$. This is a complex local system on S, whose total space may be regarded as a C^∞ complex vector bundle. The local system $R^1\pi_*(2\pi i\underline{\mathbb{Z}}) \to S$ is contained and is discrete in $H^1_{dR}(\mathcal{X}/S)$. We can thus form the quotient: $H^1_{dR}(\mathcal{X}/S)/R^1\pi_*(2\pi i\underline{\mathbb{Z}}) \to S$. This space represents $T \mapsto M_{dR}(\mathcal{X}_T/T)$ by the Riemann-Hilbert correspondence. Indeed, using the base point we have a well-defined logarithm of the holonomy map $\mathrm{Hom}(\pi_1(\mathcal{X}/S, \sigma), \mathbb{C}/2\pi i\mathbb{Z}) \to S$, and the assertion follows by duality. Therefore, given a pair $(\mathcal{L}, \nabla_{\mathcal{X}/S})$ (or more generally $(\mathcal{L}, \nabla_{\mathcal{X}_T/T})$) formed by a relative holomorphic line bundle together with a flat relative connection, there is a classifying C^∞ morphism

$$\nu : S \longrightarrow H^1_{dR}(\mathcal{X}/S)/R^1\pi_*(2\pi i\underline{\mathbb{Z}}).$$

Locally on S, this map lifts to $\widetilde{\nu} : U \to H^1_{dR}(\mathcal{X}/S)$. For future reference (*e.g.* Proposition 4.15), we note that since the quotient involves purely imaginary integral forms, $\mathrm{Re}\,\tilde{\nu}$ is well-defined independent of the lift. Applying the Gauss-Manin connection gives an element $\nabla_{\mathrm{GM}}\tilde{\nu} \in H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^1_U$. Now since $\nabla_{\mathrm{GM}} R^1\pi_*(2\pi i\underline{\mathbb{Z}}) = 0$, it follows that the above expression is actually well-defined globally, independent of the choice of lift (and we therefore henceforth omit the tilde from the notation). We define the *Gauss-Manin invariant* of $(\mathcal{L}, \nabla_{\mathcal{X}/S})$ by

$$\nabla_{\mathrm{GM}}\nu \in H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^1_S. \tag{5}$$

We mention an intermediate condition that is also natural:

Definition 2.6. A flat relative connection will be called *of type* $(1,0)$ if $\nabla_{\mathrm{GM}}\nu \in H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^{1,0}_S$.

It will be useful to recall the following, known as the Cartan-Lie formula (*cf.* [34, Section 9.2.2]). A local expression for $\nabla_{\mathrm{GM}}\nu$ is computed as follows: let s_i be local coordinates on $U \subset S$ and $\widetilde{\partial_{s_i}}$ a lifting to $\mathcal{X}_U$ of the vector field $\partial/\partial s_i$. Suppose ∇ is a connection with curvature F_∇ such that the restriction of ∇ to the fibers in U coincides with the relative connection $\nabla_{\mathcal{X}/S}$. Then

$$\nabla_{\mathrm{GM}}\nu = \sum_i \left[\mathrm{int}_{\widetilde{\partial_{s_i}}}(F_\nabla)\big|_{\mathrm{fiber}}\right] \otimes ds_i \in H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^1_U\,, \tag{6}$$

where "int" is the interior product of vector fields with forms. With this formula in hand, one easily checks that the Gauss-Manin invariant is compatible with base change. Let $\varphi : T \to S$ be a morphism of manifolds. Then there is a natural pull-back map $\varphi^* : H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^1_S \to H^1_{dR}(\mathcal{X}_T/T) \otimes \mathcal{A}^1_T$. Under this map, we have

$$\varphi^*(\nabla_{\mathrm{GM}}\nu) = \nabla_{\mathrm{GM}}(\varphi^*\nu), \tag{7}$$

where $\varphi^*\nu$ corresponds to the pull-back of $(\mathcal{L}, \nabla_{\mathcal{X}/S})$ to $\mathcal{X}_T$. Finally, we introduce the following notation. Let

$$(\nabla_{GM}\nu)' = \Pi'\nabla_{GM}\nu \in H^{1,0}_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^1_S \tag{8}$$

$$(\nabla_{GM}\nu)'' = \Pi''\nabla_{GM}\nu \in H^{0,1}_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^1_S \tag{9}$$

where Π', Π'' are the projections onto the $(1,0)$ and $(0,1)$ parts of $\nabla_{GM}\nu$ under the relative Hodge decomposition of C^∞ vector bundles

$$H^1_{dR}(\mathcal{X}/S) = H^{1,0}(\mathcal{X}/S) \oplus H^{0,1}(\mathcal{X}/S)\,.$$

2.3. **Deligne pairings, norm and trace.** Henceforth, we suppose that $\pi : \mathcal{X} \to S$ is a smooth proper morphism of smooth quasi-projective complex varieties, with connected fibers of relative dimension 1. Let $\mathcal{L}, \mathcal{M} \to \mathcal{X}$ be algebraic line bundles. The *Deligne pairing* $\langle \mathcal{L}, \mathcal{M} \rangle \to S$ is a line bundle on S defined as follows. As an $\mathcal{O}_S$-module, it can be described locally for the Zariski or étale topologies on S (at our convenience), in terms of generators and relations. In this description, we may thus localize S for any of these topologies, without any further comment:

- **Generators**: local generators of $\langle \mathcal{L}, \mathcal{M} \rangle \to S$ are given by symbols $\langle \ell, m \rangle$ where ℓ, m are rational sections of $\mathcal{L}$, $\mathcal{M}$, respectively, with disjoint divisors that are finite and flat over S. We say that ℓ and m are in general position.
- **Relations**: for $f \in \mathbb{C}(\mathcal{X})^{\times}$ and rational sections ℓ, m, such that $f\ell$, m and ℓ, m are in general position,

$$\langle f\ell, m \rangle = N_{\operatorname{div} m/S}(f) \langle \ell, m \rangle \tag{10}$$

  and similarly for $\langle \ell, fm \rangle$. Here $N_{\operatorname{div} m/S} : \mathcal{O}_{\operatorname{div} m} \to \mathcal{O}_S$ is the norm morphism.

The Deligne pairing has a series of properties (bi-multiplicativity, compatibility with base change, cohomological construction à la Koszul, etc.) that we will not recall here; instead, we refer to [14] for a careful and general discussion.

Remark 2.7. There is a holomorphic variant of Deligne's pairing in the analytic category, defined analogously, which we denote temporarily by $\langle \cdot, \cdot \rangle^{\mathrm{an}}$. If "an" denotes as well the analytification functor from algebraic coherent sheaves to analytic coherent sheaves, there is a canonical isomorphism, compatible with base change, $\langle \mathcal{L}, \mathcal{M} \rangle^{\mathrm{an}} \xrightarrow{\sim} \langle \mathcal{L}^{\mathrm{an}}, \mathcal{M}^{\mathrm{an}} \rangle^{\mathrm{an}}$. Actually, there is no real gain to working in the analytic as opposed to the algebraic category, since we assume our varieties to be quasi-projective. Indeed, the relative Picard scheme of degree d line bundles $\mathrm{Pic}^d(X/S)$ is quasi-projective as well. By use of a projective compactification $\overline{S}$ of S and $\overline{P}$ of $\mathrm{Pic}^d(X/S)$ and Chow's lemma, we see that holomorphic line bundles on $\mathcal{X}$ of relative degree d are algebraizable. For instance, if $\mathcal{L}$ is holomorphic on $\mathcal{X}$, after possibly replacing S by a connected component, it corresponds to a graph Γ in $S^{\mathrm{an}} \times \mathrm{Pic}^d(\mathcal{X}/S)^{\mathrm{an}}$. By taking the Zariski closure in $\overline{S}^{\mathrm{an}} \times \overline{P}^{\mathrm{an}}$, we see that Γ is an algebraic subvariety of $S^{\mathrm{an}} \times \mathrm{Pic}^d(X/S)^{\mathrm{an}}$, and then the projection isomorphism $\Gamma \to S^{\mathrm{an}}$ is necessarily algebraizable. Therefore, the classifying morphism of $\mathcal{L}$, $S^{\mathrm{an}} \to \mathrm{Pic}^d(X/S)^{\mathrm{an}}$, is algebraizable. For the rest of the paper we shall interchangeably speak of algebraic or holomorphic line bundles on $\mathcal{X}$ (or simply line bundles). Similarly, we suppress the index "an" from the notation.

The following is well-known (cf. [15, Thm. I.1.1]).

Lemma 2.8. *Let $\pi : \mathcal{X} \to S$, $\mathcal{L}$, $\mathcal{M}$ be as above. Locally Zariski over S, the Deligne pairing $\langle \mathcal{L}, \mathcal{M} \rangle$ is generated by symbols $\langle \ell, m \rangle$, with rational sections ℓ, m whose divisors are disjoint, finite and étale over S. In addition, if $\sigma : S \to \mathcal{X}$ is a given section, one can suppose that* $\operatorname{div} \ell$ *and* $\operatorname{div} m$ *avoid σ.*

The relevance of the lemma will be apparent later when we discuss connections on Deligne pairings. While the defining relations in the Deligne pairing make use of the norm morphism of rational functions, the construction of connections will require traces of differential forms. This is possible when our divisors are finite étale over the base: for a differential form ω defined on an open neighborhood of an irreducible divisor $D \hookrightarrow \mathcal{X}$ that is finite étale on S, the trace $\operatorname{tr}_{D/S}(\omega)$ is the map induced by inverting the map $\pi^* : \mathcal{A}^i_S \to \mathcal{A}^i_D$ (which is possible because $D \to S$ is finite étale). The trace is extended by linearity to Weil divisors whose irreducible components are finite étale over the base. The following is then clear:

Lemma 2.9. *If D is a Weil divisor in $\mathcal{X}$ whose irreducible components are finite étale over S, then $d \log N_{D/S}(f) = \operatorname{tr}_{D/S}(d \log f)$.*

2.4. **Metrics and connections.** We continue with the previous notation. Suppose now that $\mathcal{L}$, $\mathcal{M}$ are endowed with smooth hermitian metrics h, k, respectively. For both we shall denote the associated norms $\|\cdot\|$. Then Deligne [13] defines a metric on $\langle \mathcal{L}, \mathcal{M} \rangle$ via the following formula:

$$\log \|\langle \ell, m \rangle\| = \pi_* \left(\log \|m\| \, c_1(\mathcal{L}, h) + \log \|\ell\| \, \delta_{\operatorname{div} m}\right) \tag{11}$$

where $c_1(\mathcal{L}, h) = (i/2\pi) F_\nabla$ is the Chern-Weil form of the Chern connection ∇ of $(\mathcal{L}, h)$ and π_* denotes fiber integration. For the convenience of the reader, we recall the value of the Chern connection of $(\mathcal{L}, h)$ on a non-vanishing local holomorphic section e is: $\partial \log h(e, e)$. The curvature is the $(1,1)$-form locally given by $F_\nabla = d\partial \log h(e,e) = \bar{\partial}\partial \log h(e, e)$. The expression in parentheses in (11) is $\log \|\ell\| * \log \|m\|$ as defined in [20]. If ∇ is flat on the fibers of $\mathcal{X}$, namely F_∇ vanishes on fibers, then

$$\log \|\langle \ell, m \rangle\|^2 = \pi_* \left(\log \|\ell\|^2 \delta_{\operatorname{div} m}\right) = \operatorname{tr}_{\operatorname{div} m/S} \left(\log \|\ell\|^2\right)$$

and

$$\partial \log \|\langle \ell, m \rangle\|^2 = \operatorname{tr}_{\operatorname{div} m/S} \left(\partial \log \|\ell\|^2\right) = \operatorname{tr}_{\operatorname{div} m/S} \left(\frac{\nabla \ell}{\ell}\right). \tag{12}$$

Given a flat relative connection on $\mathcal{L}$, not necessarily unitary, we wish to take the right hand side of (12) as the definition of a *trace connection* on the pairing $\langle \mathcal{L}, \mathcal{M} \rangle$. In this case, we **define**

$$\nabla \langle \ell, m \rangle := \operatorname{tr}_{\operatorname{div} m/S} \left(\frac{\nabla \ell}{\ell}\right) \otimes \langle \ell, m \rangle \, . \tag{13}$$

We extend this definition to the free $C^\infty(S)$-module generated by the symbols, by enforcing the Leibniz rule:

$$\nabla(\varphi \langle \ell, m \rangle) := d\varphi \otimes \langle \ell, m \rangle + \varphi \nabla \langle \ell, m \rangle \tag{14}$$

for all $\varphi \in C^\infty(S)$. Later, in Section 3, we will see that this is the only sensible definition whenever we neglect the connection on $\mathcal{M}$. To show that (13) gives a well-defined connection on $\langle \mathcal{L}, \mathcal{M}\rangle$, we must verify compatibility with the relations defining the Deligne pairing. Because of the asymmetry of the pair, this amounts to two conditions: compatibility with the change of frame ℓ, which is always satisfied, and compatibility with the choice of section m, which is not.

Let us address the first issue. Consistency between (10) and (14) requires the following statement:

Lemma 2.10. *With ∇ defined as in (13) and (14), then*

$$\nabla\langle f\ell, m\rangle = dN_{\operatorname{div} m/S}(f)\otimes\langle \ell, m\rangle + N_{\operatorname{div} m/S}(f)\nabla\langle \ell, m\rangle$$

for all $f \in \mathbb{C}(\mathcal{X})^\times$ for which the Deligne symbols are defined.

Proof. By Lemma 2.9, the right hand side above is

$$N_{\operatorname{div} m/S}(f)\operatorname{tr}_{\operatorname{div} m/S}(d\log f)\otimes\langle \ell, m\rangle + N_{\operatorname{div} m/S}(f)\operatorname{tr}_{\operatorname{div} m/S}\left(\frac{\nabla\ell}{\ell}\right)\otimes\langle \ell, m\rangle$$
$$= \operatorname{tr}_{\operatorname{div} m/S}\left(\frac{df}{f}+\frac{\nabla\ell}{\ell}\right)\langle f\ell, m\rangle\ .$$

By (13), the left hand side is

$$\operatorname{tr}_{\operatorname{div} m/S}\left(\frac{\nabla(f\ell)}{f\ell}\right)\otimes\langle f\ell, m\rangle = \operatorname{tr}_{\operatorname{div} m/S}\left(\frac{df}{f}+\frac{\nabla\ell}{\ell}\right)\langle f\ell, m\rangle$$

by the Leibniz rule for the connection on $\mathcal{L}$. □

The second relation is consistency with the change of frame $m \mapsto fm$, $f \in \mathbb{C}(\mathcal{X})^\times$ (whenever all the symbols are defined). By Lemma 2.9 and (14), we require

$$\frac{\nabla\langle \ell, fm\rangle}{\langle \ell, fm\rangle} = \frac{\nabla\langle \ell, m\rangle}{\langle \ell, m\rangle} + \operatorname{tr}_{\operatorname{div} \ell/S}\left(\frac{df}{f}\right). \tag{15}$$

By (13),

$$\frac{\nabla\langle \ell, fm\rangle}{\langle \ell, fm\rangle} = \operatorname{tr}_{\operatorname{div}(fm)/S}\left(\frac{\nabla\ell}{\ell}\right) = \frac{\nabla\langle \ell, m\rangle}{\langle \ell, m\rangle} + \operatorname{tr}_{\operatorname{div} f/S}\left(\frac{\nabla\ell}{\ell}\right).$$

So (15) is satisfied if and only if

$$I(f,\ell,\nabla) := \operatorname{tr}_{\operatorname{div} f/S}\left(\frac{\nabla\ell}{\ell}\right) - \operatorname{tr}_{\operatorname{div} \ell/S}\left(\frac{df}{f}\right) = 0.$$

Note that under a change $\ell \mapsto g\ell$, with $\operatorname{div} g$ locally in general position (*i.e.* relative to some nonempty Zariski open subset of the base), we have

$$I(f, g\ell, \nabla) = I(f,\ell,\nabla) + \operatorname{tr}_{\operatorname{div} f/S}\left(\frac{dg}{g}\right) - \operatorname{tr}_{\operatorname{div} g/S}\left(\frac{df}{f}\right)$$

But taking the logarithmic derivative of the equation of Weil reciprocity $N_{\operatorname{div} f/S}(g) = N_{\operatorname{div} g/S}(f)$ we obtain $\operatorname{tr}_{\operatorname{div} f/S}(dg/g) = \operatorname{tr}_{\operatorname{div} g/S}(df/f)$, and so

$I(f,\ell,\nabla)$ is actually independent of ℓ and is defined for all f. In particular, $I(f,\nabla) := I(f,\ell,\nabla)$ depends only on the isomorphism class of ∇. Moreover, $I(fg,\nabla) = I(f,\nabla) + I(g,\nabla)$. Thus, extending the trace trivially on vertical divisors, $f \mapsto I(f,\nabla)$ gives

$$I(\nabla) : \mathbb{C}(\mathcal{X})^\times \longrightarrow \mathcal{A}^1_{\mathbb{C}(S)} := \varinjlim\{\Gamma(U,\mathcal{A}^1_S) : \text{ Zariski open } U \subset S\}\,.$$

We will say that a connection ∇ on $\mathcal{L} \to \mathcal{X}$ satisfies *Weil reciprocity* (WR) if $I(\nabla) = 0$. In the next section we elaborate on this notion as well as a functorial version, whose importance will be seen in the uniqueness issue. So far, we have shown the following

Proposition 2.11. *If ∇ as above satisfies (WR), then for any line bundle $\mathcal{M} \to \mathcal{X}$, ∇ induces a connection on $\langle \mathcal{L}, \mathcal{M}\rangle$.*

Example 2.12. The Chern connection ∇_{ch} on $\mathcal{L}$ from Example 2.5 induces a well-defined (Chern) connection $\langle \mathcal{L}, \mathcal{M}\rangle$ for all $\mathcal{M}$, by using Deligne's metric. Notice from (11) that this is independent of a choice of metric on $\mathcal{M}$. We then clearly have $I(\nabla_{ch}) = 0$. To see this explicitly, note that if h is the metric on $\mathcal{L}$ in the frame ℓ, then

$$\begin{aligned}
\mathrm{tr}_{\mathrm{div}\, f/S}\left(\frac{\nabla \ell}{\ell}\right) &= \mathrm{tr}_{\mathrm{div}\, f/S}(\partial \log h) = \partial(N_{\mathrm{div}\, f/S} \log h) = \partial \pi_*\left(\log \|\ell\|^2 \delta_{\mathrm{div}\, f}\right) \\
&= \partial \pi_*\left(\log \|\ell\|^2 \left(\frac{1}{2\pi i}\bar\partial\partial \log |f|^2\right)\right) && \text{(by Poincaré-Lelong)} \\
&= \partial \pi_*\left(\left(\frac{1}{2\pi i}\bar\partial\partial \log \|\ell\|^2\right) \log |f|^2\right) \\
&= \partial \pi_*\left(\log |f|^2 \delta_{\mathrm{div}\, \ell}\right) && \text{(since } \nabla_{\mathcal{X}/S} \text{ is relatively flat)} \\
&= \mathrm{tr}_{\mathrm{div}\, \ell/S}\left(\frac{df}{f}\right)
\end{aligned}$$

3. Trace Connections and Intersection Connections

3.1. **Weil reciprocity and trace connections.** Consider a smooth and proper morphism of smooth quasi-projective complex varieties $\pi : \mathcal{X} \to S$, with connected fibers of dimension 1. We suppose π is endowed with a section $\sigma : S \to \mathcal{X}$. Let $\mathcal{L} \to \mathcal{X}$ be a holomorphic line bundle. Assume also that $\mathcal{L}$ is rigidified along σ; that is, there is an isomorphism $\sigma^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_S$, fixed once for all.

Let $\nabla : \mathcal{A}^0_{\mathcal{X}}(L) \to \mathcal{A}^1_{\mathcal{X}}(L)$ be a connection on $\mathcal{L}$ (recall for holomorphic line bundles we usually assume compatibility with holomorphic structures). We say that ∇ is rigidified along σ if it pulls back to the trivial connection under the fixed isomorphism $\sigma^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_S$. For technical reasons, we require an enlargement of the notion of Weil reciprocity from the previous section. The precise definition is the following.

Definition 3.1. We say that the rigidified connection ∇ satisfies *Weil reciprocity* (WR) if for every meromorphic section ℓ of $\mathcal{L}$ and meromorphic function

$f \in \mathbb{C}(\mathcal{X})^{\times}$, whose divisors $\operatorname{div}\ell$ and $\operatorname{div} f$ are étale and disjoint over a Zariski open subset $U \subset S$ with $\operatorname{div}\ell$ disjoint from σ, the following identity of smooth differential forms on U holds:

$$\operatorname{tr}_{\operatorname{div} f/U}\left(\frac{\nabla\ell}{\ell}\right) = \operatorname{tr}_{\operatorname{div}\ell/U}\left(\frac{df}{f}\right). \tag{16}$$

We say that ∇ satisfies the (WR) *universally* if for every morphism of smooth quasi-projective complex varieties $p : T \to S$, the pull-back (rigidified) connection $p^*(\nabla)$ on $p^*(\mathcal{L})$ also satisfies Weil reciprocity.

Remark 3.2.

(i) In the definition above and in the sequel, we allow an abuse of notation such as writing $p^*(\mathcal{L})$. Indeed, the actual notation should be $p'^*(\mathcal{L})$, where $p' : X_T \to X$ is the natural projection induced from $p : T \to S$. Also, to simplify the presentation, we will write ∇ instead of $p^*(\nabla)$.

(ii) There is a nonrigidified version of this definition. It is also possible to dispense with the compatibility with the holomorphic structure of $\mathcal{L}$ (in this case, (WR) is much a stronger condition).

(iii) The assumption that $\operatorname{div}\ell$ be disjoint from σ is not essential, but it simplifies the proof of the theorem below.

(iv) Similarly, it may be possible to prove directly the compatibility with base change; again, the assumption of universal Weil reciprocity made here simplifies the arguments.

(v) Condition (16) is highly nontrivial: it relates a **smooth** $(1,0)$ differential 1-form on the left hand side to a **holomorphic** 1-form on the right.

Definition 3.3. A *trace connection* for $\mathcal{L}$ consists in giving, for every morphism of smooth quasi-projective complex varieties $p : T \to S$ and every holomorphic line bundle $\mathcal{M}$ on $\mathcal{X}_T$ of relative degree 0, a connection $D_{\mathcal{M}}$ on $\langle p^*(\mathcal{L}), \mathcal{M}\rangle$, compatible with the holomorphic structure on $\mathcal{L}$, subject to the following conditions:

- (Functoriality) If $q : T' \to T$ is a morphism of smooth quasi-projective complex varieties, the base changed connection $q^*(D_{\mathcal{M}})$ corresponds to $D_{q^*(\mathcal{M})}$ through the canonical isomorphism
$$q^*\langle p^*(\mathcal{L}), \mathcal{M}\rangle \xrightarrow{\sim} \langle q^*p^*(\mathcal{L}), q^*(\mathcal{M})\rangle.$$
- (Additivity) Given $D_{\mathcal{M}}$ and $D_{\mathcal{M}'}$ as above, the connection $D_{\mathcal{M}\otimes\mathcal{M}'}$ corresponds to the "tensor product connection" $D_{\mathcal{M}} \otimes \mathrm{id} + \mathrm{id} \otimes D_{\mathcal{M}'}$ through the canonical isomorphism
$$\langle p^*(\mathcal{L}), \mathcal{M}\otimes\mathcal{M}'\rangle \xrightarrow{\sim} \langle p^*(\mathcal{L}), \mathcal{M}\rangle \otimes \langle p^*(\mathcal{L}), \mathcal{M}'\rangle.$$
- (Compatibility with isomorphisms) Given a (holomorphic) isomorphism of line bundles (of relative degree 0) $\varphi : \mathcal{M} \to \mathcal{M}'$ on $\mathcal{X}_T$, the connections $D_{\mathcal{M}}$ and $D'_{\mathcal{M}}$ correspond through the induced isomorphism on Deligne pairings: $\langle \mathrm{id}, \varphi\rangle : \langle p^*(\mathcal{L}), \mathcal{M}\rangle \xrightarrow{\sim} \langle p^*(\mathcal{L}), \mathcal{M}'\rangle$.

We shall express a trace connection as an assignment $(p : T \to S, \mathcal{M}) \mapsto D_{\mathcal{M}}$, or just $\mathcal{M} \mapsto D_{\mathcal{M}}$.

Remark 3.4.

(i) It is easy to check that the additivity axiom implies that $D_{\mathcal{O}_{\mathcal{X}}}$ corresponds to the trivial connection through the canonical isomorphism $\langle \mathcal{L}, \mathcal{O}_{\mathcal{X}} \rangle \xrightarrow{\sim} \mathcal{O}_S$.

(ii) The compatibility with isomorphisms implies that $D_{\mathcal{M}}$ is invariant under the action of automorphisms of $\mathcal{M}$ on $\langle p^*\mathcal{L}, \mathcal{M} \rangle$.

In case that $\mathcal{L}$ is of relative degree 0, a trace connection for $\mathcal{L}$ automatically satisfies an extra property that we will need in the next section. In this situation, for a line bundle on $\mathcal{X}$ coming from the base $\pi^*(\mathcal{N})$, there is a canonical isomorphism: $\langle \mathcal{L}, \pi^*(\mathcal{N}) \rangle \xrightarrow{\sim} \mathcal{N}_S^{\otimes \deg \mathcal{L}} = \mathcal{O}_S$ (see the proof in the lemma below). With these preliminaries at hand, we can state:

Lemma 3.5. *Suppose that $\mathcal{L}$ is of relative degree 0. Then, for any $(p : T \to S, \mathcal{M})$ as above, with $\mathcal{M} = \pi_T^*(\mathcal{N})$, the connection $D_{\mathcal{M}}$ corresponds to the trivial connection through the canonical isomorphism $\langle p^*\mathcal{L}, \mathcal{M} \rangle \xrightarrow{\sim} \mathcal{O}_T$.*

Proof. The statement is local for the Zariski topology on T, so we can localize and suppose there is a trivialization $\varphi : \mathcal{N} \xrightarrow{\sim} \mathcal{O}_T$. This trivialization induces a trivialization $\widetilde{\varphi} : \pi_T^*(\mathcal{N}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_T}$. The isomorphism $\langle p^*\mathcal{L}, \mathcal{M} \rangle \xrightarrow{\sim} \mathcal{O}_T$ is such that there is commutative diagram:

$$\begin{array}{ccc} \langle p^*\mathcal{L}, \mathcal{M} \rangle & \xrightarrow{\sim} & \mathcal{O}_T \\ {\scriptstyle \langle \mathrm{id}, \widetilde{\varphi} \rangle} \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\ \langle p^*\mathcal{L}, \mathcal{O}_{\mathcal{X}_T} \rangle & \xrightarrow{\sim} & \mathcal{O}_T . \end{array}$$

Observe that if we change φ by a unit in $\mathcal{O}_T^{\times}$, then the degree 0 assumption on $\mathcal{L}$ ensures $\langle \mathrm{id}, \widetilde{\varphi} \rangle$ does not change! This is compatible with the rest of the diagram being independent of φ. Now we combine: a) the compatibility of trace connections with isomorphisms, b) the triviality of $D_{\mathcal{O}_{\mathcal{X}_T}}$ through the lower horizontal arrow, c) the commutative diagram. We conclude that $D_{\mathcal{M}}$ corresponds to the canonical connection through the upper horizontal arrow. ◻

Now for the characterization of connections satisfying (WR) universally in terms of trace connections. Let ∇ be a rigidified connection on $\mathcal{L}$ satisfying (WR) universally, $p : T \to S$ a morphism of smooth quasi-projective complex varieties, and $\mathcal{M}$ a line bundle on $\mathcal{X}_T$ of relative degree 0. If ℓ and m are rational sections of $p^*(\mathcal{L})$ and $\mathcal{M}$ whose divisors are étale and disjoint over an open Zariski subset $U \subset T$, and $\operatorname{div} \ell$ is disjoint with $p^*\sigma$, we define (*cf.* (13))

$$D_{\mathcal{M}}\langle \ell, m \rangle = \langle \ell, m \rangle \otimes \operatorname{tr}_{\operatorname{div} m/U}\left(\frac{\nabla \ell}{\ell}\right). \tag{17}$$

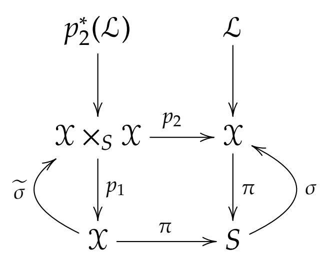
Theorem 3.6. *The definition* (17) *gives a bijection between the following types of data:*

- *A rigidified connection ∇ on $\mathcal{L}$ satisfying (WR) universally.*
- *A trace connection for $\mathcal{L}$.*

Remark 3.7. In Section 2.4 we guessed the formula (13) from the Chern connection of metrics on Deligne pairings. The theorem above shows that this is indeed the only possible construction of trace connections, once we impose some functoriality. The functoriality requirement is natural, since Deligne pairings behave well with respect to base change.

Proof of Theorem 3.6. Given a rigidified connection satisfying (WR) universally, we already know that the rule (17) defines a trace connection for $\mathcal{L}$. Indeed, the condition (WR) guarantees that this rule is compatible with both the Leibniz rule and the relations defining the Deligne pairing (Proposition 2.11). The compatibility with the holomorphic structure of the trace connection is direct from the definition.

Now, let us consider a trace connection for $\mathcal{L}$, *i.e.* the association $(p : T \to S, \mathcal{M}) \mapsto D_{\mathcal{M}}$, on $\langle p^*(\mathcal{L}), \mathcal{M}\rangle$, for $\mathcal{M}$ a line bundle on $\mathcal{X}_T$ of relative degree 0. Let us consider the particular base change $\pi : \mathcal{X} \to S$. The new family of curves is given by $p_1 : \mathcal{X} \times_S \mathcal{X} \to \mathcal{X}$, the projection onto the first factor. The base change of $\mathcal{L}$ to $\mathcal{X} \times_S \mathcal{X}$ is the pull-back $p_2^*(\mathcal{L})$. The family p_1 comes equipped with two sections. The first one is the diagonal section, that we denote δ. The second one, is the base change of the section σ, that we write $\widetilde{\sigma}$. Hence, at the level of points, $\widetilde{\sigma}(x) = (x, \sigma\pi(x))$. See (18).



(18)

The images of these sections are Cartier divisors in $\mathcal{X} \times_S \mathcal{X}$, so that they determine line bundles that we denote $\mathcal{O}(\delta)$, $\mathcal{O}(\widetilde{\sigma})$. Let us take for $\mathcal{M}$ the line bundle $\mathcal{O}(\delta - \widetilde{\sigma})$, namely $\mathcal{O}(\delta) \otimes \mathcal{O}(\widetilde{\sigma})^{-1}$. By the properties of the Deligne pairing, there is a canonical isomorphism $\langle p_2^*(\mathcal{L}), \mathcal{O}(\delta - \widetilde{\sigma})\rangle \xrightarrow{\sim} \delta^* p_2^*(\mathcal{L}) \otimes \widetilde{\sigma}^* p_2^* \mathcal{L}^{-1}$. But now, $p_2\delta = \mathrm{id}$, and $p_2\widetilde{\sigma} = \sigma\pi$. Using the rigidification $\sigma^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_S$, we obtain an isomorphism

$$\langle p_2^*(\mathcal{L}), \mathcal{O}(\delta - \widetilde{\sigma})\rangle \xrightarrow{\sim} \mathcal{L}. \tag{19}$$

Through this isomorphism, the connection $D_{\mathcal{O}(\delta - \tilde{\sigma})}$ corresponds to a connection on $\mathcal{L}$, that we temporarily write ∇_S. It is compatible with the holomorphic structure, as trace connections are by definition. More generally, given a morphism of smooth quasi-projective complex varieties $p : T \to S$, the same construction applied to the base changed family $\mathcal{X}_T \to T$

(with the base changed section σ_T) produces a connection ∇_T on $p^*(\mathcal{L})$, and it is clear that $\nabla_T = p^*(\nabla_S)$. We shall henceforth simply write ∇ for this compatible family of connections. It is important to stress the role of the rigidification in the construction of ∇.

First, we observe that the connection ∇ is rigidified along σ. Indeed, on the one hand, by the functoriality of the Deligne pairing with respect to base change, there is a canonical isomorphism

$$\sigma^*\langle p_2^*(\mathcal{L}), \mathcal{O}(\delta - \widetilde{\sigma})\rangle \xrightarrow{\sim} \langle \mathcal{L}, \sigma^*\mathcal{O}(\delta - \widetilde{\sigma})\rangle \simeq \langle \mathcal{L}, \mathcal{O}_S\rangle. \tag{20}$$

Here we have used the fact that the base change of δ and $\widetilde{\sigma}$ along σ both coincide with the section σ itself, so that $\sigma^*\mathcal{O}(\delta - \widetilde{\sigma}) \simeq \mathcal{O}_S$ (recall our abuse of notation for base change, cf. Remark 3.2). On the other hand, the functoriality assumption on trace connections and compatibility with isomorphisms ensure that through the isomorphism (20) we have an identification $\sigma^*(D_{\mathcal{O}(\delta-\tilde{\sigma})}) = D_{\mathcal{O}_S}$. But we already remarked that $D_{\mathcal{O}_S}$ corresponds to the trivial connection through the isomorphism

$$\langle \mathcal{L}, \mathcal{O}_S\rangle \xrightarrow{\sim} \mathcal{O}_S. \tag{21}$$

Now, the rigidification property for ∇ follows, since the composition of σ^*(19)–(21) gives back our fixed isomorphism $\sigma^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_S$.

Second, we show that ∇ satisfies (WR) universally. Actually, we will see that for $(p : T \to S, \mathcal{M})$, the connection $D_{\mathcal{M}}$ is given by the rule

$$D_{\mathcal{M}}\langle \ell, m\rangle = \langle \ell, m\rangle \otimes \mathrm{tr}_{\mathrm{div}\, m/U}\left(\frac{\nabla \ell}{\ell}\right),$$

for sections ℓ and m as in the statement. Using the fact that $D_{\mathcal{M}}$ is a connection (and hence satisfies the Leibniz rule) and imposing the relations defining the Deligne pairing, this ensures that (WR) for ∇ is satisfied.

To simplify the discussion, and because the new base T will be fixed from now on, we may just change the meaning of the notation and write S instead of T. Also, observe that the equality of two differential forms can be checked after étale base change (because étale base change induces isomorphisms on the level of differential forms). Therefore, after possibly localizing S for the étale topology, we can suppose that $\mathrm{div}\, m = \sum_i n_i D_i$, where the divisors D_i are given by sections σ_i, and the n_i are integers with $\sum_i n_i = 0$. Because $\mathcal{M} \simeq \mathcal{O}(\sum_i n_i D_i) \simeq \bigotimes_i \mathcal{O}(\sigma_i - \sigma)^{\otimes n_i}$, the additivity of the trace connection and the trace $\mathrm{tr}_{\mathrm{div}\, m/S}$ with respect to m, and the compatibility with isomorphisms, we reduce to the case where $\mathcal{M} = \mathcal{O}(\sigma_i - \sigma)$ and m is the canonical rational section $\mathbb{1}$ with divisor $\sigma_i - \sigma$. In order to trace back the definition of ∇, we effect the base change $\mathcal{X} \to S$. By construction of the connection ∇ on $\mathcal{L}$ (that, recall, involves the rigidification), we have

$$\frac{D_{\mathcal{O}(\delta-\tilde{\sigma})}\langle p_2^*\ell, \mathbb{1}\rangle}{\langle p_2^*\ell, \mathbb{1}\rangle} = \frac{\nabla \ell}{\ell} - \pi^*\left(\frac{d\sigma^*(\ell)}{\sigma^*\ell}\right), \tag{22}$$

where we identify $\sigma^*(\ell)$ with a rational function on S through the rigidification. We now pull-back the identity (22) by σ_i, and for this we remark that the base change of δ by σ_i becomes σ_i, while the base change of $\widetilde{\sigma}$ by σ_i becomes σ! Taking this into account, together with the functoriality of Deligne pairings and trace connections, we obtain

$$\frac{D_{\mathcal{O}(\sigma_i-\sigma)}\langle \ell, \mathbb{1}\rangle}{\langle \ell, \mathbb{1}\rangle} = \sigma_i^*\left(\frac{\nabla\ell}{\ell}\right) - \sigma_i^*\pi^*\left(\frac{d\sigma^*(\ell)}{\sigma^*\ell}\right) = \sigma_i^*\left(\frac{\nabla\ell}{\ell}\right) - \sigma^*\left(\frac{\nabla\ell}{\ell}\right)$$
$$= \mathrm{tr}_{\mathrm{div}\,\mathbb{1}/S}\left(\frac{\nabla\ell}{\ell}\right).$$

In the second inequality we used that ∇ is rigidified along σ, that we already showed above. This completes the proof of the theorem. □

Remark 3.8. Notice that the above notions do not require $\mathcal{L}$ to have relative degree 0. It may well be that the objects we introduce do not exist at such a level of generality. In the relative degree 0 case we have shown that connections satisfying (WR) universally on $\mathcal{L}$ do indeed exist and can be constructed from relative connections that are compatible with the holomorphic structure of $\mathcal{L}$. The latter, of course, always exist by taking the Chern connection of a hermitian metric. In the next section, we confirm the existence in relative degree 0 by other methods, and we classify them all.

Corollary 3.9. *Let $\mathcal{M} \mapsto D_{\mathcal{M}}$ be a trace connection for the rigidified line bundle $\mathcal{L}$. Then there is a unique extension of the trace connection to line bundles $\mathcal{M}$ of arbitrary relative degree, such that $D_{\mathcal{O}(\sigma)}$ corresponds to the trivial connection through the isomorphism $\langle \mathcal{L}, \mathcal{O}(\sigma)\rangle \simeq \sigma^*(\mathcal{L}) \simeq \mathcal{O}_S$. This extension satisfies the following properties:*

(i) if ∇ is the connection on $\mathcal{L}$ determined by Theorem 3.6, the extension is still given by the rule (17);

(ii) the list of axioms of Definition 3.3, i.e. *functoriality, additivity and compatibility with isomorphisms.*

Proof. Let ∇ be the rigidified connection on $\mathcal{L}$ corresponding to the trace connection $\mathcal{M} \mapsto D_{\mathcal{M}}$. Then we extend the trace connection to arbitrary $\mathcal{M}$ by the rule (17). The claims of the corollary are straightforward to check. □

3.2. **Reformulation in terms of Poincaré bundles.** In case $\mathcal{L}$ is of relative degree 0, the notion of trace connection can be rendered more compact by the introduction of a Poincaré bundle on the relative jacobian. Let $\pi : \mathcal{X} \to S$ be our smooth fibration in proper curves, with a fixed section $\sigma : S \to \mathcal{X}$. We write $p : J \to S$ for the relative jacobian $J = J(\mathcal{X}/S)$, and $\mathcal{P}$ for the Poincaré bundle on $\mathcal{X} \times_S J$, rigidified along the lift $\widetilde{\sigma} : J \to \mathcal{X} \times_S J$ of the section σ. This rigidified Poincaré bundle has a neat compatibility property with respect to the group scheme structure of J. Let us introduce the addition map $\mu : J \times_S J \longrightarrow J$. If $T \to S$ is a morphism of schemes, then at the level of T valued points the addition map is induced by the tensor product of line bundles on $\mathcal{X}_T$. If p_1, p_2 are the projections of $J \times_S J$ onto the first and second factors, then there is an isomorphism of line bundles on $\mathcal{X} \times_S J \times_S J$,

$$\mu^*\mathcal{P} \xrightarrow{\sim} (p_1^*\mathcal{P}) \otimes (p_2^*\mathcal{P}). \tag{23}$$

In particular, given a line bundle $\mathcal{L}$ on $\mathcal{X}$ and its pull-back $\widetilde{\mathcal{L}}$ to $\mathcal{X} \times_S J \times_S J$, there is an induced canonical isomorphism of Deligne pairings,

$$\langle \widetilde{\mathcal{L}}, \mu^*\mathcal{P}\rangle \xrightarrow{\sim} \langle \widetilde{\mathcal{L}}, p_1^*\mathcal{P}\rangle \otimes \langle \widetilde{\mathcal{L}}, p_1^*\mathcal{P}\rangle.$$

Let $\mathcal{M} \mapsto D_{\mathcal{M}}$ be a trace connection for $\mathcal{L}$. Then, we can evaluate it on the data $(p : J \to S, \mathcal{P})$, thus providing a connection $D_{\mathcal{P}}$ on $\langle p^*\mathcal{L}, \mathcal{P}\rangle$. By the functoriality of trace connections, we have $\mu^* D_{\mathcal{P}} = D_{\mu^*\mathcal{P}}$, $p_1^* D_{\mathcal{P}} = D_{p_1^*\mathcal{P}}$, $p_2^* D_{\mathcal{P}} = D_{p_2^*\mathcal{P}}$. Furthermore, by the compatibility with isomorphisms and additivity, there is an identification through (23): $\mu^* D_{\mathcal{P}} = (p_1^* D_{\mathcal{P}}) \otimes \mathrm{id} + \mathrm{id} \otimes (p_2^* D_{\mathcal{P}})$. We claim the data $\mathcal{M} \mapsto D_{\mathcal{M}}$ is determined by $D_{\mathcal{P}}$. For if $q : T \to S$ is a morphism of smooth quasi-projective complex varieties, and $\mathcal{M}$ a line bundle on $\mathcal{X}_T$ of relative degree 0, then we have a classifying morphism $\varphi : T \to J$ and an isomorphism $\mathcal{M} \xrightarrow{\sim} (\varphi^*\mathcal{P}) \otimes (\pi_T^*\sigma_T^*\mathcal{M})$. This induces an isomorphism on Deligne pairings $\langle q^*\mathcal{L}, \mathcal{M}\rangle \xrightarrow{\sim} \langle q^*\mathcal{L}, (\varphi^*\mathcal{P}) \otimes (\pi_T^*\sigma_T^*\mathcal{M})\rangle$. But now, because $\mathcal{L}$ is of relative degree 0, there is a canonical isomorphism $\langle q^*\mathcal{L}, \pi_T^*\sigma_T^*\mathcal{M}\rangle \xrightarrow{\sim} \mathcal{O}_T$, and the connection $D_{\pi_T^*\sigma_T^*\mathcal{M}}$ is trivial by Lemma 3.5. Hence, through the resulting isomorphism on Deligne pairings $\langle q^*\mathcal{L}, \mathcal{M}\rangle \xrightarrow{\sim} \varphi^*\langle p^*\mathcal{L}, \mathcal{P}\rangle$, we have an identification of connections $\varphi^* D_{\mathcal{P}} = D_{\mathcal{M}}$. Moreover this identification does not depend on the precise isomorphism $\mathcal{M} \xrightarrow{\sim} (\varphi^*\mathcal{P}) \otimes (\pi_T^*\sigma_T^*\mathcal{M})$, by the compatibility of trace connections with isomorphisms of line bundles (and hence with automorphisms of line bundles). The next statement is now clear.

Proposition 3.10. *Suppose that $\mathcal{L}$ is of relative degree 0. The following data are equivalent:*

- *A trace connection for $\mathcal{L}$.*
- *A connection $D_{\mathcal{P}}$ on the Deligne pairing $\langle p^*\mathcal{L}, \mathcal{P}\rangle$ (compatible with the holomorphic structure), satisfying the following compatibility with addition on J:*

$$\mu^* D_{\mathcal{P}} = (p_1^* D_{\mathcal{P}}) \otimes \mathrm{id} + \mathrm{id} \otimes (p_2^* D_{\mathcal{P}}). \tag{24}$$

The equivalence is given as follows. Let $D_{\mathcal{P}}$ as above, $q : T \to S$ a morphism of smooth quasi-projective complex varieties, and $\mathcal{M}$ a line bundle on $\mathcal{X}_T$ of relative degree 0 and $\varphi : T \to J$ its classifying map, so that there are isomorphisms $\mathcal{M} \xrightarrow{\sim} \varphi^*\mathcal{P} \otimes (\pi_T^*\sigma_T^*\mathcal{M})$, $\langle q^*\mathcal{L}, \mathcal{M}\rangle \xrightarrow{\sim} \varphi^*\langle p^*\mathcal{L}, \mathcal{P}\rangle$. Then, through these identifications, the rule: $\mathcal{M} \longmapsto \varphi^* D_{\mathcal{P}}$ on $\langle q^*\mathcal{L}, \mathcal{M}\rangle$, defines a trace connection for $\mathcal{L}$. The proposition justifies calling $D_{\mathcal{P}}$ a *universal trace connection*.

The formulation of trace connections in terms of Poincaré bundles makes it easy to deal with the uniqueness issue. Let $\mathcal{M} \mapsto D_{\mathcal{M}}$ and $\mathcal{M} \mapsto D'_{\mathcal{M}}$ be trace connections. These are determined by the respective "universal" connections $D_{\mathcal{P}}$ and $D'_{\mathcal{P}}$. Two connections on a given holomorphic line bundle, compatible with the holomorphic structure, differ by a smooth $(1,0)$ differential one form. Let θ be the smooth $(1,0)$ form on J given by $D_{\mathcal{P}} - D'_{\mathcal{P}}$. Then, the compatibility of universal trace connections with additivity imposes the restriction on θ: $\mu^*\theta = p_1^*\theta + p_2^*\theta$. We say that θ is a translation invariant form and denote the space of such differential forms by $\mathrm{Inv}(J)^{(1,0)} \subset \Gamma(J, A_J^{1,0})$.

It remains to consider the problem of existence. The line bundle $\langle p^*\mathcal{L}, \mathcal{P}\rangle$ on J is rigidified along the zero section and compatible with the relative addition law on J. In particular, $\langle p^*\mathcal{L}, \mathcal{P}\rangle$ lies in $J^\vee(S)$, the S-valued points of the dual abelian scheme to J. Equivalently, it is a line bundle on J of relative degree 0. Now $\langle p^*\mathcal{L}, \mathcal{P}\rangle$ admits a hermitian metric that is invariant under addition. The resulting Chern connection is invariant under addition and compatible with the holomorphic structure. We thus arrive at the following theorem.

Theorem 3.11 (Structure theorem)**.** *The space of trace connections for $\mathcal{L}$, and thus of rigidified connections on $\mathcal{L}$ satisfying (WR) universally, is a torsor under* $\mathrm{Inv}(J)^{(1,0)}$.

In Section 4 we will provide a constructive approach to Theorem 3.11.

3.3. **Intersection connections.** We continue with the notation of the previous sections. In particular, $\mathcal{L} \to \mathcal{X} \to S$ is a holomorphic line bundle of relative degree zero. In Theorem 3.6 we have related rigidified connections on $\mathcal{L}$ satisfying (WR) universally and trace connections for $\mathcal{L}$, as equivalent notions. We have also given a structure theorem for the space of such objects (Theorem 3.11). There is, of course, a lack of symmetry in the definition of a trace connection $\mathcal{M} \mapsto D_{\mathcal{M}}$, since the holomorphic line bundles $\mathcal{M}$ require no extra structure. In this section, we show that given a relatively flat connection on $\mathcal{L}$ satisfying (WR) universally, we can build an *intersection connection* "against" line bundles with connections $(\mathcal{M}, \nabla')$. Moreover, if ∇' is relatively flat and also satisfies (WR), then the resulting connection is symmetric with respect to the symmetry of the Deligne pairing. Recall from Definition 2.3 that F_∇ denotes the curvature of a connection ∇.

Theorem 3.12. *Let ∇ be a compatible connection on $\mathcal{L} \to \mathcal{X}$ satisfying (WR), not necessarily rigidified, and such that its vertical projection $\nabla_{\mathcal{X}/S}$ is flat. Let ∇' be a connection on another line bundle $\mathcal{M} \to \mathcal{X}$ of arbitrary relative degree. Then:*

(i) on the Deligne pairing $\langle \mathcal{L}, \mathcal{M}\rangle$, the following rule defines a connection compatible with the holomorphic structure:

$$\frac{D\langle \ell, m\rangle}{\langle \ell, m\rangle} = \frac{i}{2\pi}\pi_*\left(\frac{\nabla' m}{m} \wedge F_\nabla\right) + \mathrm{tr}_{\mathrm{div}\, m/S}\left(\frac{\nabla \ell}{\ell}\right), \tag{25}$$

where π_* is integration over the fiber;

(ii) if both connections ∇ and ∇' are unitary for some hermitian structures, then D is the Chern connection of the corresponding metrized Deligne pairing;

(iii) if ∇ satisfies (WR) universally, then the construction of D is compatible with base change and coincides with a trace connection when restricted to line bundles with unitary connections $(\mathcal{M}, \nabla')$.

Proof. To justify that the rule D defines a connection, it is enough to show the compatibility between Leibniz' rule and the relations defining the Deligne pairing. One readily checks the compatibility for the change $\ell \mapsto f\ell$ for f a rational function. For the change $m \mapsto fm$, the trace term in the definition of D already satisfies (WR). We thus have to show the invariance of the fiber integral under the change $m \mapsto fm$, or equivalently

$$\pi_*\left(\frac{df}{f} \wedge F_\nabla\right) = 0. \tag{26}$$

It will be useful to compare ∇ to the Chern connection ∇_{ch} on $\mathcal{L}$ from Example 2.5, which we assume is relatively flat (the rigidification is irrelevant for this discussion). The connection ∇_{ch} also satisfies (WR) (see Example 2.12). We write: $\nabla = \nabla_{ch} + \theta$, $F_\nabla = F_{\nabla_{ch}} + d\theta$. Then θ is of type $(1, 0)$ and has vanishing trace along divisors of rational functions. We exploit this fact, together with the observation that since $F_{\nabla_{ch}}$ is of type $(1, 1)$ it is ∂-closed. Write:

$$\begin{aligned}\pi_*\left(\frac{df}{f} \wedge F_\nabla\right) &= \pi_*\left(\frac{df}{f} \wedge F_{\nabla_{ch}}\right) + \pi_*\left(\frac{df}{f} \wedge d\theta\right)\\ &= \partial\pi_*(\log|f|^2 F_{\nabla_{ch}}) + \pi_*\left(\frac{df}{f} \wedge d\theta\right).\end{aligned} \tag{27}$$

Because $F_{\nabla_{ch}}$ is flat on fibers,

$$\pi_*(\log|f|^2 F_{\nabla_{ch}}) = 0. \tag{28}$$

Furthermore, by the Poincaré-Lelong formula for currents on $\mathcal{X}$, $\bar\partial\partial\log|f|^2 = 2\pi i \cdot \delta_{\operatorname{div} f}$. Hence, by type considerations,

$$\begin{aligned}\pi_*\left(\frac{df}{f} \wedge d\theta\right) &= \pi_*\left(\partial\log|f|^2 \wedge \bar\partial\theta\right)\\ &= \pi_*\left(\bar\partial\partial\log|f|^2 \wedge \theta\right) - \bar\partial\pi_*\left(\partial\log|f|^2 \wedge \theta\right)\\ &= 2\pi i\operatorname{tr}_{\operatorname{div} f/S}(\theta) - \bar\partial\pi_*\left(\partial\log|f|^2 \wedge \theta\right).\end{aligned} \tag{29}$$

The trace term vanishes by the Weil vanishing property of θ, and the second term vanishes because the integrand it is of type $(2, 0)$ and π reduces types by $(1, 1)$. Combining (27)–(29) we conclude with the desired (26).

The second assertion follows by construction of D, and the third item is immediate. □

Definition 3.13 (Intersection Connection)**.** Suppose that a relatively flat connection ∇ on $\mathcal{L}$ satisfies (WR) universally, and let ∇' be an arbitrary connection on $\mathcal{M}$, where $\mathcal{M}$ can have arbitrary relative degree. Both connections are assumed to be compatible with holomorphic structures. Then the connection $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle} := D$ on $\langle\mathcal{L},\mathcal{M}\rangle$ constructed in Theorem 3.12 is called *the intersection connection attached to* $(\mathcal{L}, \nabla)$ *and* $(\mathcal{M}, \nabla')$. We write $\langle(\mathcal{L}, \nabla), (\mathcal{M}, \nabla')\rangle$ for the Deligne pairing of $\mathcal{L}$ and $\mathcal{M}$ with the intersection connection.

The next result is a direct consequence of the Poincaré-Lelong formula.

Proposition 3.14. *The curvature of the intersection connection* $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ *attached to* $(\mathcal{L},\nabla)$ *and* $(\mathcal{M},\nabla')$ *is given by:* $F_{\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}} = \frac{i}{2\pi}\pi_*(F_\nabla\wedge F_{\nabla'})$.

Intersection connections satisfy the expected behavior with respect to tensor product and flat isomorphisms. Furthermore, if $\mathcal{L}$ and $\mathcal{M}$ are line bundles endowed with connections ∇,∇' that satisfy (WR), one might expect a symmetry of the intersection connections on $\langle\mathcal{L},\mathcal{M}\rangle$ and $\langle\mathcal{M},\mathcal{L}\rangle$, through the canonical isomorphism of Deligne pairings

$$\langle\mathcal{L},\mathcal{M}\rangle \xrightarrow{\sim} \langle\mathcal{M},\mathcal{L}\rangle. \tag{30}$$

This is indeed the case.

Proposition 3.15. *Suppose the connections* ∇,∇' *on* $\mathcal{L}$ *and* $\mathcal{M}$ *both satisfy (WR) and are relatively flat. Then, the symmetry isomorphism* (30) *is parallel (or* flat*) with respect to the intersection connections.*

Proof. Let ℓ, m be a couple of sections providing bases elements $\langle\ell,m\rangle$ and $\langle m,\ell\rangle$ of $\langle\mathcal{L},\mathcal{M}\rangle$ and $\langle\mathcal{M},\mathcal{L}\rangle$, respectively. We denote by T_ℓ and T_m the currents of integration against $\nabla\ell/\ell$ and $\nabla'm/m$, respectively. These currents have disjoint wave front sets. The same holds for the Dirac currents $\delta_{\mathrm{div}\,\ell}$ and $\delta_{\mathrm{div}\,m}$, as well as $\delta_{\mathrm{div}\,\ell}$ and $\nabla'm/m$, etc. For currents with disjoint wave front sets, the usual wedge product rules and Stokes' formulas for differential forms remain true. Applying the Poincaré-Lelong type equations for T_ℓ and T_m we find the chain of equalities

$$\begin{aligned}
\frac{\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}\langle\ell,m\rangle}{\langle\ell,m\rangle} &:= \frac{i}{2\pi}\pi_*\left(\frac{\nabla'm}{m}\wedge F_\nabla\right) + \mathrm{tr}_{\mathrm{div}\,m/S}\left(\frac{\nabla\ell}{\ell}\right)\\
&= \frac{i}{2\pi}\pi_*\left(\frac{\nabla'm}{m}\wedge dT_\ell\right) + \mathrm{tr}_{\mathrm{div}\,\ell/S}\left(\frac{\nabla'm}{m}\right) + \mathrm{tr}_{\mathrm{div}\,m/S}\left(\frac{\nabla\ell}{\ell}\right)\\
&= \frac{i}{2\pi}\pi_*\left(\frac{\nabla\ell}{\ell}\wedge dT_m\right) + \mathrm{tr}_{\mathrm{div}\,m/S}\left(\frac{\nabla\ell}{\ell}\right) + \mathrm{tr}_{\mathrm{div}\,\ell/S}\left(\frac{\nabla'm}{m}\right)\\
&= \frac{i}{2\pi}\pi_*\left(\frac{\nabla\ell}{\ell}\wedge F_{\nabla'}\right) + \mathrm{tr}_{\mathrm{div}\,\ell/S}\left(\frac{\nabla'm}{m}\right) =: \frac{\nabla^{int}_{\langle\mathcal{M},\mathcal{L}\rangle}\langle m,\ell\rangle}{\langle m,\ell\rangle}.
\end{aligned}$$

The proof is complete. □

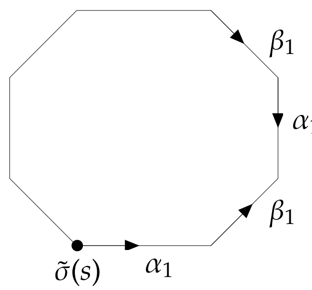
For later use, it will be useful to study the change of intersection connections under change of connection on $\mathcal{L}$.

Proposition 3.16. *Assume that* $\mathcal{M}$ *has relative degree 0. Let* $\theta\in\Gamma(S,\mathcal{A}^{1,0}_S)$. *Then* $\langle(\mathcal{L},\nabla+\pi^*\theta),(\mathcal{M},\nabla')\rangle = \langle(\mathcal{L},\nabla),(\mathcal{M},\nabla')\rangle$.

Proof. Let $\langle\ell,m\rangle$ be a local basis of the Deligne pairing. We observe that $\pi_*\left(\frac{\nabla'm}{m}\wedge d\left(\pi^*\theta\right)\right) = \pi_*\left(\frac{\nabla'm}{m}\right)\wedge d\theta = 0$. The vanishing of the last fiber integral is obtained by counting types: $\nabla'm/m$ is of type $(1,0)$ and π reduces types by $(1,1)$. Also, because $\mathrm{div}\,m$ is of degree 0, we have $\mathrm{tr}_{\mathrm{div}\,m/S}(\pi^*\theta) = (\deg\mathrm{div}\,m)\theta = 0$. These observations together imply the proposition. □

4. Proofs of the Main Theorems

4.1. **The canonical extension: local description and properties.** In this step, we work locally on S for the analytic topology. We replace S by a contractible open subset S°. Hence, local systems over S° are trivial. We write $\mathcal{X}^{\circ}$ for the restriction of $\mathcal{X}$ to S°, but to ease the notation, we still denote $\mathcal{L}$ for the restriction of $\mathcal{L}$. For later use (in the proof of Theorem 4.6), we fix a family of symplectic bases $\{\alpha_i, \beta_i\}_{i=1}^{g}$ for $H_1(\mathcal{X}_s)$, that is flat with respect to the Gauss-Manin connection. Observe that this trivially determines a symplectic basis after any base change $T \to S^{\circ}$ that is also flat. We may assume that these are given by closed curves based at $\sigma(s)$. We view these curves as the polygonal boundary of a fundamental domain $\mathcal{F}_s \subset \widetilde{\mathcal{X}}_s$ in the local relative universal cover $\widetilde{\mathcal{X}} \to \mathcal{X}^{\circ}$, in the usual way:



In the figure we have written $\tilde{\sigma}(s)$ for a lift of the section $\sigma(s)$ to a fundamental domain $\mathcal{F}_s$. Let $\nu : S \to E(\mathcal{X}/S)$ be the C^{∞} classifying map corresponding to $(\mathcal{L}, \nabla_{\mathcal{X}/S})$. By the choice of S°, $\nu\mid_{S^{\circ}}$ lifts to $\tilde{\nu} : S^{\circ} \to H^1_{dR}(\mathcal{X}/S)$. We identify the fundamental groups $\pi_1(\mathcal{X}_s, \sigma(s))$, $s \in S^{\circ}$, to a single Γ. Then to $\tilde{\nu}$ there is associated a smooth family of complex valued holonomy characters of Γ

$$\chi_s(\gamma) = \exp\left(-\int_{\gamma} \tilde{\nu}_s\right), \quad \gamma \in \Gamma, \quad s \in S^{\circ}.$$

where the integral is taken with respect to any de Rham representative of $\tilde{\nu}_s$. Observe that the definition of χ_s only depends on ν, and not the particular choice of the lift $\tilde{\nu}$. We can thus write ν in the integral. With this understood, local smooth sections ℓ of $\mathcal{L} \to \mathcal{X}^{\circ}$ are identified with smooth functions $\tilde{\ell}$ on $\widetilde{\mathcal{X}}$ satisfying the equivariance rule:

$$\tilde{\ell}(\gamma\tilde{z}, s) = \chi_s(\gamma)^{-1}\tilde{\ell}(\tilde{z}, s), \quad \gamma \in \Gamma, \quad s \in S^{\circ}. \tag{31}$$

Rational sections are meromorphic in $\tilde{z} \in \widetilde{\mathcal{X}}_s$, for fixed s. Notice that for every $s \in S^{\circ}$, we have a holomorphic structure $\overline{\partial}_s$ on $\widetilde{\mathcal{X}}_s$.

Remark 4.1.

(i) We clarify this important construction. Choose a lift $\tilde{\sigma}(s)$ to $\widetilde{\mathcal{X}}_s$ lying in $\mathcal{F}_s$. The rigidification $\sigma^*\mathcal{L} \simeq \mathcal{O}_S$ gives a nonzero element $\mathbf{e} \in \widetilde{\mathcal{L}}|_{\tilde{\sigma}(s)}$.

Using the relative flat connection $\nabla_{\mathcal{X}_s}$, $\mathbf{e}$ extends to a global frame $\widetilde{\mathbf{e}}$ of $\widetilde{\mathcal{L}}_s \to \widetilde{\mathcal{X}}_s$. Then the pullback of a section ℓ can be written $\tilde{\ell}(\tilde{z})\widetilde{\mathbf{e}}$.

(ii) If two lifts of $\sigma(s)$ are related by $\tilde{\sigma}_2(s) = \gamma \cdot \tilde{\sigma}_1(s)$, then $\widetilde{\mathbf{e}}_2 = \chi(\gamma)^{-1}\widetilde{\mathbf{e}}_1$, and therefore $\tilde{\ell}_2 = \chi(\gamma)\tilde{\ell}_1$.

(iii) With the identification above, the relative connection $\nabla_{\mathcal{X}/S}$ is given by $\ell \to \tilde{\ell} \,:\, \nabla_{\mathcal{X}/S}\ell/\ell \longleftrightarrow d\tilde{\ell}/\tilde{\ell}$, projected to $\mathcal{A}^{1,0}_{\widetilde{\mathcal{X}}/S^\circ}$.

To extend the relative connection we must differentiate $\tilde{\ell}$ with respect to s, as well as the factor $s \mapsto \exp\left(\int_\gamma \nu_s\right)$, all in a way which preserves the condition (31). Note that the dependence on γ factors through homology. Hence, we regard γ as a parallel section of $(R^1\pi_*\underline{\mathbb{Z}})^\vee$ on S°. Then, by the very definition of the Gauss-Manin connection,

$$d \exp\left(\int_\gamma \nu\right) = \exp\left(\int_\gamma \nu\right) d\int_\gamma \tilde{\nu} = \exp\left(\int_\gamma \nu\right) \int_\gamma \nabla_{\mathrm{GM}}\nu. \tag{32}$$

Here, the "integral" over $\gamma \subset \mathcal{X}_s$ means the integral of the part of $\nabla_{\mathrm{GM}}\nu$ in $H^1_{dR}(\mathcal{X}/S)$. Now choose any frame $\{[\eta_i]\}_{i=1}^{2g}$ of the local system $H^1_{dR}(\mathcal{X}/S)$ on S°. Then we may write: $\nabla_{\mathrm{GM}}\nu = \sum_{i=1}^{2g}[\eta_i] \otimes \theta_i$, $\theta_i \in \mathcal{A}^1_{S^\circ}$. For each $s \in S^\circ$ and $i = 1, \dots, 2g$, there is a unique harmonic representative $\eta_i(z,s)$ of $[\eta_i](s)$ on the fiber $\mathcal{X}_s$ (this is a relative form; recall that harmonic one forms on surfaces depend only on the conformal structure). For $\tilde{z} \in \widetilde{\mathcal{X}}_s$, we consider a path joining $\widetilde{\sigma}(s)$ to $\tilde{z}$ in $\widetilde{\mathcal{X}}_s$. Then we set

$$\int_{\tilde{\sigma}(s)}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu := \sum_{i=1}^{2g} \left\{ \int_{\tilde{\sigma}(s)}^{\tilde{z}} \eta_i(z,s) \right\} \theta_i, \tag{33}$$

where we have abused notation and wrote η_i for its lift to the universal cover. This expression varies smoothly in $s \in S^\circ$ and $\tilde{z}$. It is independent of the choice of local frame. Indeed, if $[\tilde{\eta}_i] = \sum_j A_{ij}[\eta_j]$ for a (constant) matrix (A_{ij}), then $\tilde{\eta}_i(z,s) = \sum_j A_{ij}\eta_j(z,s)$ (uniqueness of harmonic representatives), and hence $\sum_i A_{ij}\tilde{\theta}_i = \theta_j$ (since $\nabla_{\mathrm{GM}}\nu$ is intrinsically defined). It follows that, since (A_{ij}) is constant,

$$\begin{aligned} \sum_{i=1}^{2g} \left\{ \int_{\tilde{\sigma}(s)}^{\tilde{z}} \tilde{\eta}_i(z,s) \right\} \tilde{\theta}_i &= \sum_{i,j=1}^{2g} \left\{ \int_{\tilde{\sigma}(s)}^{\tilde{z}} A_{ij}\eta_j(z,s) \right\} \tilde{\theta}_i = \sum_{i,j=1}^{2g} \left\{ \int_{\tilde{\sigma}(s)}^{\tilde{z}} \eta_j(z,s) \right\} A_{ij}\tilde{\theta}_i \\ &= \sum_{j=1}^{2g} \left\{ \int_{\tilde{\sigma}(s)}^{\tilde{z}} \eta_j(z,s) \right\} \theta_j. \end{aligned}$$

With this understood, on $\widetilde{\mathcal{X}}$ we extend the relative connection $\nabla_{\mathcal{X}/S}$ by the following (see Remark 4.1): if $\tilde{z} \in \widetilde{\mathcal{X}}_s$,

$$\frac{\nabla\ell}{\ell}(\tilde{z},s) := \frac{d\tilde{\ell}}{\tilde{\ell}}(\tilde{z},s) - \int_{\tilde{\sigma}(s)}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu. \tag{34}$$

We claim that this expression descends to a 1-form on $\mathcal{X}^\circ$ and is independent of the choice of lift of $\sigma(s)$. Both facts follow from the same argument. Suppose, for example, that $\tilde{\sigma}_2(s) = \gamma\tilde{\sigma}_1(s)$ are two choices of local lifts. Then by Remark 4.1 (ii), it follows that $\tilde{\ell}_2(\tilde{z}) = \chi(\gamma)^{-1}\tilde{\ell}_1(\tilde{z})$, and therefore $d\tilde{\ell}_2/\tilde{\ell}_2 = d\tilde{\ell}_1/\tilde{\ell}_1 + d\log\chi(\gamma)$. On the other hand,

$$\int_{\tilde{\sigma}_2}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu = \int_{\gamma\tilde{\sigma}_1}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu = \int_{\tilde{\sigma}_1}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu - \int_{\tilde{\sigma}_1}^{\gamma\tilde{\sigma}_1} \nabla_{\mathrm{GM}}\nu = \int_{\tilde{\sigma}_1}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu - \int_{\gamma} \nabla_{\mathrm{GM}}\nu$$
$$= \int_{\tilde{\sigma}_1}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu + d\log\chi(\gamma)$$

where we have used (32). It follows that $\nabla\ell/\ell$ in (34) is independent of the lift. The fact that $\nabla\ell/\ell$ descends to a 1-form on $\mathcal{X}^\circ$ follows by the same argument. This proves the claim.

From the previous discussion it also follows that given overlapping contractible open subsets of S, say S_1° and S_2°, the corresponding extensions ∇_1 and ∇_2 agree over the intersection, so that they can be glued together. Therefore, there exists a smooth connection $\nabla : L \to L \otimes \mathcal{A}^1_{\mathcal{X}}$ that, locally on contractible open subsets of S, is of the form (34).

Definition 4.2. The extended connection ∇ on $L \to \mathcal{X}$ given by the procedure above will be called *the canonical extension* of $\nabla_{\mathcal{X}/S}$ to $\mathcal{X}$.

Remark 4.3.

(i) It is immediate that ∇ indeed satisfies the Leibniz rule and is a smooth connection. It is also trivially rigidified along σ.

(ii) It is, however, perhaps not so clear the ∇ is compatible with the holomorphic structure on $\mathcal{L}$, and this will be checked below in Theorem 4.6.

(iii) From now on, for notational convenience we confuse points on fibers of $\mathcal{X}^\circ \to S^\circ$ with their lifts to fundamental domains of universal covers. Therefore, we will write expressions such as $\int_{\sigma(s)}^{z} \nabla_{\mathrm{GM}}\nu$.

Lemma 4.4. *The construction of the canonical extension ∇ is compatible with base change.*

Proof. The lemma follows from the expression (34) and the compatibility of the Gauss-Manin invariant with base change (7). □

Because the line bundle $\mathcal{L} \to \mathcal{X}$ is of relative degree 0, we can endow it with the rigidified Chern connection ∇_{ch}, which is flat on fibers. We next show that the Chern connection is the canonical extension of its vertical projection, as given by the preceding construction.

Lemma 4.5. *Suppose that the flat relative connection $\nabla_{\mathcal{X}/S}$ is induced by a hermitian structure h on $\mathcal{L}$ whose Chern connection is flat on the fibers, i.e. by the Chern connection ∇_{ch} of $(\mathcal{L}, h)$ in the sense of Example 2.5. Then ∇_{ch} coincides with the canonical extension (34) of $\nabla_{\mathcal{X}/S}$.*

Proof. This essentially follows from the construction outlined in Remark 4.1. Choose a point $0 \in S$, and let $X = \mathcal{X}_0$. We assume a local C^∞ pointed trivialization of a restriction $\mathcal{X}^\circ$ over a contractible subset $0 \in S^\circ$ of S: hence, $\mathcal{X}^\circ \simeq X \times S^\circ$ with the section $\sigma(s)$ mapping to a fixed point $\sigma = \sigma(0) \in X$. We fix a hermitian metric on a bundle $L \to X$ with a fixed C^∞ trivialization, so that we have a nowhere vanishing smooth section $\mathbf{1}$ of L with $\|\mathbf{1}\| = 1$. The connection ∇_{ch} gives a family of fiberwise flat unitary connections on $\mathcal{L} \to \mathcal{X}^\circ$, and up to isomorphism we may assume that the pull-back of $\mathcal{L}$ to $X \times S^\circ$ is isometrically identified with the trivial extension of L to $X \times S^\circ$. Hence, on $L \to X$ we have a family ∇_s of flat unitary connections. Let $\widetilde{X}$ denote the universal cover of X, and $\widetilde{\sigma}$ a lift of the base point $\sigma = \sigma(0) \in X$. Let d denote the trivial connection on L with respect to its trivialization, i.e. $d(\mathbf{1}) = 0$. Then we may write $\nabla_s = d + A_s$. By again applying a unitary gauge transformation, we may assume that A_s is a harmonic 1-form on X for each s. Set $\mathbf{e}_s = \exp\left(-\int_\sigma^z A_s\right) \cdot \mathbf{1}$. Then the extension $\mathbf{e}$ of $\mathbf{e}_s$ to $X \times S^\circ$ is a vertically flat section, well-defined on $\widetilde{X} \times S^\circ$. Let A be the difference of the Chern and trivial connections on $X \times S^\circ$, and d_s the de Rham operator on the S° factor. Then on $X \times S^\circ$, $\nabla_{ch}\mathbf{e} = \left(-A - d_s \int_\sigma^z A\right)\mathbf{e} + A\mathbf{e} = \left(-d_s \int_\sigma^z A\right)\mathbf{e}$. By definition of the integral in (33), $\left(-d_s \int_{\sigma(s)}^z A\right)\Big|_{s=0} = \left(-\int_\sigma^z \nabla_{\mathrm{GM}}\nu_{ch}\right)\Big|_{s=0}$, where $\nabla_{\mathrm{GM}}\nu_{ch}$ is the Gauss-Manin invariant for the Chern connection. We conclude that $\nabla_{ch}\mathbf{e}/\mathbf{e} = -\int_\sigma^z \nabla_{\mathrm{GM}}\nu_{ch}$ at $s = 0$. On $\widetilde{X}$, the equivariant function $\tilde{\ell}$ on $X \times S^\circ$ associated with a section ℓ satisfies: $\ell = \tilde{\ell} \cdot \mathbf{e}$, and so

$$\frac{\nabla_{ch}\ell}{\ell} = \frac{d\tilde{\ell}}{\tilde{\ell}} + \frac{\nabla_{ch}\mathbf{e}}{\mathbf{e}} = \frac{d\tilde{\ell}}{\tilde{\ell}} - \int_\sigma^z \nabla_{\mathrm{GM}}\nu_{ch} \tag{35}$$

at $s = 0$. Since the choice of base point $0 \in S$ was arbitrary, this completes the proof. □

Theorem 4.6 (Canonical extension)**.** *The canonical extension ∇ is compatible with the holomorphic structure on $\mathcal{L}$, and it satisfies (WR) universally. Moreover, it is rigidified along the section σ.*

Proof. Again we work locally on contractible open subsets S° of S. For the first statement, it suffices to show that for any meromorphic section ℓ of $\mathcal{L}$, $(\nabla\ell/\ell)^{0,1} = 0$. The restriction of this form to the fibers of $\mathcal{X}^\circ$ vanishes; hence, with respect to local holomorphic coordinates $\{s_i\}$ on S° we may write

$$\left(\frac{\nabla\ell}{\ell}\right)^{0,1} = \sum_i \varphi_i d\bar{s}_i \tag{36}$$

for functions φ_i on $\mathcal{X}^\circ$. We wish to prove that the φ_i vanish identically. Write $\nabla = \nabla_0 + \theta$, where ∇_0 is the Chern connection as in the proof of Lemma 4.5. By the construction (34) of the canonical extensions,

$$\theta^{0,1} = \frac{\bar{\partial}\tilde{\ell}}{\tilde{\ell}} - \frac{\bar{\partial}\tilde{\ell}_0}{\tilde{\ell}_0} + \int_{\sigma(s)}^z (\nabla_{\mathrm{GM}}\nu_0)^{0,1} - \int_{\sigma(s)}^z (\nabla_{\mathrm{GM}}\nu)^{0,1} \tag{37}$$

Now from the definition of the Gauss-Manin integral (33), for fixed s the expression $\int_{\sigma(s)}^{z}(\nabla_{\mathrm{GM}}\nu)^{0,1}$ is a harmonic function of z. Similarly for ν_0. Taking $\partial\bar{\partial}$ of (37), it then follows that the $dz\wedge d\bar{z}$ term of $\partial\bar{\partial}\theta^{0,1}$ vanishes. But by Lemma 4.5, $\theta^{0,1}=(\nabla\ell/\ell)^{0,1}-(\nabla_0\ell/\ell)^{0,1}=(\nabla\ell/\ell)^{0,1}$, and so in (36), $\partial_z\partial_{\bar{z}}\varphi_i=0$ for all i. Hence, the φ_i are harmonic, and therefore constant along the fibers of $\mathcal{X}^\circ$. But they also vanish along σ, and so vanish identically, and the first statement of the theorem follows.

It remains to prove that ∇ satisfies (WR) universally. By the compatibility of the construction of ∇ with base change (Lemma 4.4), it is enough to work over S°. Also, in the proof we are allowed to do base changes of S° induced by étale base changes of S, since equalities of differential forms are local for this topology. For the proof we follow the argument for classical Weil reciprocity for Riemann surfaces. Recall that for a holomorphic differential ω and nonzero meromorphic function f on a Riemann surface X with homology basis $\{\alpha_i,\beta_i\}$, we have (*cf.* [22, Reciprocity Law I, p. 230])

$$2\pi i\sum_{p\in\mathrm{div}(f)}\mathrm{ord}_p(f)\int_\sigma^p\omega=\sum_{i=1}^{g}\left(\int_{\alpha_i}\omega\int_{\beta_i}\frac{df}{f}-\int_{\alpha_i}\frac{df}{f}\int_{\beta_i}\omega\right)\tag{38}$$

where the left hand side is independent of the base point σ because $\deg\mathrm{div}(f)=0$. The divisor of f is understood to be restricted to the fundamental domain delimited by the curves representing the homology basis. We note two generalizations of this type of formula:

(i) in (38), we may use an anti-holomorphic form $\overline{\omega}$ instead of ω. Indeed, the periods of df/f are pure imaginary, and the assertion follows by conjugating both sides;

(ii) in families, if $\mathrm{div}\, f/S$ is finite étale over S, then after étale base change we can assume the irreducible components are given by sections. Applying this, the previous comment, and the Hodge decomposition to the cohomological part of $\nabla_{\mathrm{GM}}\nu$, we have for each s,

$$\begin{aligned}2\pi i\sum_{p(s)\in\mathrm{div}(f(\cdot,s))}\mathrm{ord}_{p(s)}(f)\int_{\sigma(s)}^{p(s)}\nabla_{\mathrm{GM}}\nu=\\ \sum_{i=1}^{g}\left(\int_{\alpha_i}\nabla_{\mathrm{GM}}\nu\int_{\beta_i}\frac{df}{f}-\int_{\alpha_i}\frac{df}{f}\int_{\beta_i}\nabla_{\mathrm{GM}}\nu\right).\end{aligned}\tag{39}$$

Here $\{\alpha_i,\beta_i\}$ is a parallel symplectic basis of $(R^1\pi_*\mathbb{Z})^\vee$ on S° as fixed in the beginning of Section 4.1.

There is a second version of Weil reciprocity (*cf.* [22, p. 243]) for pairs f, g of meromorphic functions

$$2\pi i \sum_{q\in \mathrm{div}(g)} \mathrm{ord}_q(g) \log f(q) - 2\pi i \sum_{p\in \mathrm{div}(f)} \mathrm{ord}_p(f) \log g(p) = \sum_{i=1}^{g} \left(\int_{\alpha_i} \frac{df}{f} \int_{\beta_i} \frac{dg}{g} - \int_{\alpha_i} \frac{dg}{g} \int_{\beta_i} \frac{df}{f} \right) \tag{40}$$

which applies also to families (after possible étale base change, to assume the components of the divisors are given by sections). Again, the divisors are taken in the fundamental domain delimited by the homology basis.

We now apply (40) in the case where $\mathrm{div}\, f/S$ is finite étale and $g = \tilde{\ell}$ (regarded as an equivariant meromorphic function fiberwise). We observe that the periods of df/f are constant functions on the base S° (they belong to $2\pi i\mathbb{Z}$). Taking derivatives and appealing to (32) and (39), results in the string of equalities

$$\begin{aligned} 2\pi i\, \mathrm{tr}_{\mathrm{div}\,\ell/S^\circ}\left(\frac{df}{f}\right) &- 2\pi i\, \mathrm{tr}_{\mathrm{div}\, f/S^\circ}\left(\frac{d\tilde{\ell}}{\tilde{\ell}}\right) \\ &= d\left[\sum_{i=1}^{g} \left(\int_{\alpha_i} \frac{df}{f} \int_{\beta_i} \frac{d\tilde{\ell}}{\tilde{\ell}} - \int_{\alpha_i} \frac{d\tilde{\ell}}{\tilde{\ell}} \int_{\beta_i} \frac{df}{f} \right)\right] \\ &= \sum_{i=1}^{g} \left(\int_{\alpha_i} \frac{df}{f} \left\{ d\int_{\beta_i} \tilde{\nu}_s \right\} - \left\{ d\int_{\alpha_i} \tilde{\nu}_s \right\} \int_{\beta_i} \frac{df}{f} \right) \\ &= \sum_{i=1}^{g} \left(\int_{\alpha_i} \frac{df}{f} \int_{\beta_i} \nabla_{\mathrm{GM}}\nu - \int_{\alpha_i} \nabla_{\mathrm{GM}}\nu \int_{\beta_i} \frac{df}{f} \right) \\ &= 2\pi i\, \mathrm{tr}_{\mathrm{div}\, f/S^\circ}\left(\int_{\sigma}^{z} \nabla_{\mathrm{GM}}\nu \right). \end{aligned}$$

Therefore by (34), $\mathrm{tr}_{\mathrm{div}\,\ell/S^\circ}(df/f) = \mathrm{tr}_{\mathrm{div}\, f/S^\circ}(\nabla\ell/\ell)$. In other words, ∇ satisfies (WR). The rigidification property is immediate from the construction (34). This completes the proof of Theorem 4.6. □

4.2. **The canonical extension: uniqueness.** In this section we prove the uniqueness of the extension obtained in the previous section. In fact, we will prove a little more. Let $\theta \in \mathcal{A}^i_{\mathcal{X}}$ satisfying the following properties:

(V1) rigidification: $\sigma^*(\theta) = 0$;
(V2) the pull-back of θ to any fiber $\mathcal{X}_s$, $s \in S$, vanishes;
(V3) vanishing along rational divisors, universally: given a *smooth* morphism of quasi-projective complex varieties $p : T \to S$, and a meromorphic function f on the base change $\mathcal{X}_T$ whose divisor is finite étale over T, we have $\mathrm{tr}_{\mathrm{div}\, f/T}(p^*\theta) = 0$. Here we write $p^*\theta$ for the pull-back of θ to $\mathcal{X}_T$ by the induced morphism $\mathcal{X}_T \to \mathcal{X}$. We call this the *Weil vanishing property*, and it appeared already in the proof of Theorem 3.12.

Proposition 4.7 (Vanishing Property). *Let θ be a smooth complex differential 1-form on $\mathcal{X}$, satisfying properties (V1)–(V3) above. Then θ vanishes identically on $\mathcal{X}$.*

Proof. The vanishing of a differential form is a local property, so we may assume that Ω^1_S is a free sheaf on S. Let $\theta_1, \ldots, \theta_d$ be a holomorphic frame for Ω^1_S. Then, $\theta_1, \ldots, \theta_n, \overline{\theta}_1, \ldots, \overline{\theta}_n$ is a frame for $\mathcal{A}^1_S$. Because θ vanishes on fibers by (V2), on $\mathcal{X}$ we can write $\theta = \sum_i f_i \pi^* \theta_i + \sum_i g_i \pi^* \overline{\theta}_i$, for some smooth functions f_i, g_i on $\mathcal{X}$. Observe that f_i, g_i vanish along the section σ (V1) (by the independence of $\theta_1, \ldots, \theta_n, \overline{\theta}_1, \ldots, \overline{\theta}_n$). They also satisfy the Weil vanishing property (V3). For this, we need to observe that for a smooth morphism $p : T \to S$, the differential forms $p^*\theta_i$, $p^*\overline{\theta}_i$ are still stalk-wise independent, so are their pull-backs to $\mathcal{X}_T$ (because $\mathcal{X}_T \to T$ is smooth).[2] We want to show these functions identically vanish. We are thus required to prove that a smooth complex function $\varphi : \mathcal{X} \to \mathbb{C}$ satisfying (V1) and (V3) automatically satisfies (V2), and therefore vanishes.

Let us observe that Weil vanishing for functions implies something more. Let D be a divisor in $\mathcal{X}_T$, finite and flat over T. Then the trace: $\mathrm{tr}_{D/T}(\varphi)$, can still be defined as a continuous function on T, by averaging on fibers and taking multiplicities into account (for this one does not even need $T \to S$ to be smooth). Hence, if $\mathrm{tr}_{D/T}(\varphi)$ vanishes over a Zariski dense open subset of T, then it vanishes everywhere by continuity. This is the case for $D = \mathrm{div}\, f$, where f is a rational function on $\mathcal{X}_T$ with finite flat divisor over T. Indeed, there is a dense (Zariski) open subset $U \subseteq T$ such that D is finite étale over U. This means that the Weil vanishing property holds for rational divisors whose components are only finite and flat.

Recall that the relative jacobian $J := J(\mathcal{X}/S) \to S$ is a fibration of abelian varieties over S, representing the functor $T \mapsto J(T)$ of line bundles on $\mathcal{X}_T$ of relative degree 0, modulo line bundles coming from the base. Here, we will exploit the fact that the total space J is smooth (because S is smooth), and therefore can be covered by Zariski open subsets U, which are smooth and quasi-projective over S! The natural inclusion of a Zariski open subset $U \hookrightarrow J$ corresponds to the universal rigidified (along σ) Poincaré bundle restricted to $\mathcal{X}_U$, and for small enough U, one can suppose this line bundle is associated to a divisor in $\mathcal{X}_U$, finite flat over U. We will call this "a universal" finite flat divisor over U. It is well defined only up to rational equivalence (through rational divisors which are finite flat over the base).

We proceed to extend $\varphi : \mathcal{X} \to \mathbb{C}$ to a continuous function $\widetilde{\varphi} : J \to \mathbb{C}$, whose restriction on fibers is a continuous morphism of (topological) groups (for the analytic topology). Let U be a Zariski open subset of J, such that $\mathcal{X}_U$ affords a "universal" finite flat divisor of degree 0 over U. Denote this divisor D_U. Then, $\mathrm{tr}_{D_U/U}(\varphi)$ is a continuous function on U. Moreover, it only depends on the

[2]Note, however, that this property is lost in general if $p : T \to S$ is not smooth. This explains the restriction to smooth base change in (V3).

rational equivalence class of D_U, by the Weil vanishing property (extended to finite flat rational divisors). Because of this, given U and V intersecting open subsets in $J(\mathcal{X}/S)$, we also have $\mathrm{tr}_{D_U/U}(\varphi)\mid_{U\cap V}=\mathrm{tr}_{D_V/V}(\varphi)\mid_{U\cap V}$. Therefore these functions glue into a continuous function $\widetilde{\varphi}: J\to\mathbb{C}$. The linearity of the trace function with respect to sums of divisors, guarantees that $\widetilde{\varphi}$ is compatible with the group scheme structure. Namely, given the addition: $\mu: J\times_S J\to J$, and the two projections $p_i: J\times_S J\to J$, the following relation holds: $\mu^*\widetilde{\varphi}=p_1^*\widetilde{\varphi}+p_2^*\widetilde{\varphi}$. This in particular implies that $\widetilde{\varphi}$ is a topological group morphism on fibers and immediately leads to the vanishing of $\widetilde{\varphi}$ on fibers; hence, everywhere. Indeed, a given fiber J_s ($s\in S$) can be uniformized as $\mathbb{C}^g/\Lambda$, for some lattice Λ. The corresponding arrow $\mathbb{C}^g\to\mathbb{C}$ induced from $\widetilde{\varphi}$ is a continuous morphism of topological abelian groups, and it is therefore a linear map of real vector spaces! Because the map factors through J_s, which is compact, its image is compact, and hence is reduced to $\{0\}$.

Finally, let $\iota:\mathcal{X}\hookrightarrow J$ be the closed immersion given by the section σ. Because of the rigidification of φ, we have $\varphi=\iota^*\widetilde{\varphi}=0$. This concludes the proof of the proposition. □

Corollary 4.8 (Uniqueness)**.** *Suppose that we are given ∇_1,∇_2 smooth connections on $L\to\mathcal{X}$ (hence non-necessarily compatible with the holomorphic structure) satisfying the following properties:*

(E1) *they are both rigidified along the section σ;*
(E2) *they coincide on fibers $\mathcal{X}_s$, $s\in S$;*
(E3) *they satisfy the Weil reciprocity for connections, universally.*

Then $\nabla_1=\nabla_2$. Therefore, the canonical extension is unique.

Proof. Indeed, we can write $\nabla_1=\nabla_2+\theta$, where θ is a smooth 1-form. Then properties (E1)–(E3) ensure that θ satisfies (V1)–(V3). By the vanishing lemma, $\theta=0$, so $\nabla_1=\nabla_2$ as required. The consequences for the canonical extension follow, since they satisfy (E1)–(E3). □

A second application of the vanishing lemma is an alternative proof of the compatibility of a connection ∇ on $\mathcal{L}$ with the holomorphic structure (see Theorem 4.6).

Corollary 4.9 (Compatibility)**.** *Let $\mathcal{L}\to\mathcal{X}$ be a holomorphic line bundle with smooth connection ∇ (non-necessarily compatible with the holomorphic structure on $\mathcal{L}$) which:*

(H1) *is rigidified along the section σ;*
(H2) *is holomorphic on fibers;*
(H3) *and satisfies the Weil reciprocity for connections, universally.*

Then ∇ is compatible with the holomorphic structure on $\mathcal{L}$. In particular, the canonical extension is compatible with the holomorphic structure of $\mathcal{L}$.

Proof. Because ∇ is holomorphic on fibers, $\mathcal{L}$ is of relative degree 0 and can be endowed with a Chern connection ∇_{ch}. We can suppose ∇_{ch} is rigidified along the section σ (H1) (because $\mathcal{L}$ is rigidified). We already know that

∇_{ch} satisfies (H2)–(H3) (see Example 2.12). Also, ∇_{ch} is compatible with the holomorphic structure $\mathcal{L}$, by definition of Chern connections. Hence it is enough to compare ∇ and ∇_{ch}. Let us write $\nabla = \nabla_{ch} + \theta$. We decompose θ into types $(1,0)$ and $(0,1)$: $\theta = \theta' + \theta''$, and we wish to see that $\theta'' = 0$. But now, observe the following facts:

- $\sigma^*\theta = 0$ and pull-back by σ respects types, so that $\sigma^*\theta' = -\sigma^*\theta''$ has to vanish;
- θ'' vanishes along the fibers, because ∇ and ∇_{ch} are holomorphic along the fibers;
- θ satisfies the Weil vanishing universally. Because the trace along divisors $\mathrm{tr}_{D/T}$ respects types of differential forms, we deduce that it vanishes for θ''.

Hence, θ'' satisfies the properties (V1)–(V3) above, and it therefore vanishes. It follows that $\nabla = \nabla_{ch} + \theta'$ is compatible with the holomorphic bundle $\mathcal{L}$. □

4.3. **Variant in the absence of rigidification.** In case the morphism $\pi : \mathcal{X} \to S$ does not come with a rigidification, we can still pose the problem of extending connections and impose (WR) universally. We briefly discuss this situation. Locally for the étale topology, the morphism π admits sections. Étale morphisms are local isomorphisms in the analytic topology. Therefore, given a relative connection $\nabla_{\mathcal{X}/S}$, there is an analytic open covering U_i of S, and connections ∇_i on $\mathcal{L}_{\mathcal{X}_{U_i}}$ extending $\nabla_{\mathcal{X}/S}$ and satisfying (WR) universally. On an overlap $U_{ij} := U_i \cap U_j$, the connections ∇_i and ∇_j differ by a smooth $(1,0)$-form θ_{ij} on $\mathcal{X}_{U_{ij}}$. The differential form θ_{ij} satisfies the vanishing properties (V2)–(V3) of Section 4.2. By the vanishing lemma (Proposition 4.7), θ_{ij} comes from a differential form on U_{ij}: $\theta_{ij} \in \Gamma(U_{ij}, \mathcal{A}_S^{1,0})$. This family of differential forms obviously verifies the 1-cocycle condition, and hence gives a cohomology class in $H^1(S, \mathcal{A}_S^{1,0})$. But $\mathcal{A}_S^{1,0}$ is a fine sheaf, because it is a $C^\infty(S)$-module. Therefore, this cohomology group vanishes and the cocycle $\{\theta_{ij}\}$ is trivial. This means that, after possibly modifying the connections ∇_i by suitable $(1,0)$ differential forms coming from the base, we can glue them together into a connection ∇ on $\mathcal{L}$, extending $\nabla_{\mathcal{X}/S}$. Because any differential form coming from the base S has vanishing trace along divisors of rational functions (more generally, along relative degree 0 divisors), this connection ∇ still satisfies (WR) universally. Two such connections differ by a differential form in $\Gamma(S, \mathcal{A}_S^{1,0})$. Again, differential forms coming from S have zero trace along degree zero divisors, and so this implies the induced trace connections on Deligne pairings $\langle \mathcal{L}, \mathcal{M} \rangle$ don't depend on the particular extension, as long as $\mathcal{M}$ has relative degree 0. We summarize the discussion in a statement.

Proposition 4.10. *In the absence of a section of $\pi : \mathcal{X} \to S$, the space of extensions of $\nabla_{\mathcal{X}/S}$ satisfying (WR) universally is a torsor under $\Gamma(S, \mathcal{A}_S^{1,0})$. To $\nabla_{\mathcal{X}/S}$ there is an intrinsically attached trace connection on relative degree zero line bundles $\mathcal{M}$ (still denoted $\nabla_{\langle \mathcal{L}, \mathcal{M} \rangle^{tr}}$).*

Remark 4.11. The proposition refines Theorem 3.11, when we are interested in connections on $\mathcal{L}$ satisfying (WR) universally and extending a given flat relative connection $\nabla_{\mathcal{X}/S}$.

Completion of the Proof of Theorem 1.2 (i). Let now $\nabla_{\mathcal{X}/S}$ be a flat relative connection, and let ∇ be any extension satisfying (WR) universally. Attached to $(\mathcal{L}, \nabla)$ there is a trace connection for the Deligne pairings against line bundles of relative degree 0. This trace connection does not depend on the choice of extension $\nabla_{\mathcal{X}/S}$, as we saw above. By a similar argument, if $(\mathcal{L}, \nabla^{L}_{\mathcal{X}/S})$ and $(\mathcal{M}, \nabla^{M}_{\mathcal{X}/S})$ are line bundles with relatively flat connections on $\mathcal{X} \to S$, Propositions 4.10, 3.15 and 3.16 together show that there is an intrinsically attached intersection connection $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ on $\langle\mathcal{L}, \mathcal{M}\rangle$. □

4.4. **Relation between trace and intersection connections.** We now fill in the proof of Theorem 1.2 (iii), which asserts that if $\mathcal{M} \to \mathcal{X}$ has relative degree 0, then the trace connection on $\langle\mathcal{L}, \mathcal{M}\rangle$ is a special case of an intersection connection. We state the precise result in the following

Proposition 4.12. *Let $\mathcal{L} \to \mathcal{X}$ be equipped with a flat relative connection $\nabla^{L}_{\mathcal{X}/S}$. Let $\mathcal{M} \to \mathcal{X}$ be a hermitian, holomorphic line bundle with Chern connection ∇^{M}_{ch} whose restriction $\nabla^{M}_{\mathcal{X}/S}$ to the fibers of $\pi : \mathcal{X} \to S$ is flat. Let $\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}$ be the trace connection associated to $\nabla^{L}_{\mathcal{X}/S}$, and $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ the intersection connection associated to $\nabla^{L}_{\mathcal{X}/S}$ and $\nabla^{M}_{\mathcal{X}/S}$. Then $\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle} = \nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$.*

Proof. An equality of connections is local for the étale topology, and our constructions are compatible with base change. Therefore, we can assume there is a section σ and that $\mathcal{L}$ is rigidified along σ. Let ∇_L be the canonical extension of $\nabla^{L}_{\mathcal{X}/S}$. By the definition eq. (25), it suffices to show that for all rational sections m of $\mathcal{M}$,

$$\pi_*\left(\frac{\nabla^{M}_{ch}(m)}{m} \wedge F_{\nabla_L}\right) = 0 \tag{41}$$

Let $\|\cdot\|$ denote the metric on $\mathcal{M}$, and write $\nabla_L = \nabla^{L}_{ch} + \theta$, where ∇^{L}_{ch} is the Chern connection, and θ is of type $(1,0)$. Then using the fact that $F_{\nabla^{L}_{ch}}$ is ∂-closed, we have

$$\begin{aligned}\pi_*\left(\frac{\nabla^{M}_{ch}(m)}{m} \wedge F_{\nabla_L}\right) &= \pi_*\left(\partial \log \|m\|^2 \wedge F_{\nabla_L}\right)\\ &= \partial\pi_*\left(\log\|m\|^2 \cdot F_{\nabla^{L}_{ch}}\right) + \pi_*\left(\partial\log\|m\|^2 \wedge d\theta\right)\end{aligned}$$

The first term vanishes, since ∇^{L}_{ch} is flat on the fibers. For the second term, as in the proof of Theorem 1.2 we find

$$\begin{aligned}\pi_*\left(\partial\log\|m\|^2 \wedge d\theta\right) &= \pi_*\left(\partial\log\|m\|^2 \wedge \bar\partial\theta\right)\\ &= -\bar\partial\pi_*\left(\partial\log\|m\|^2 \wedge \theta\right) + \pi_*\left(\bar\partial\partial\log\|m\|^2 \wedge \theta\right) \quad (42)\\ &= \pi_*(F_{\nabla^{M}_{ch}} \wedge \theta) + 2\pi i\, \mathrm{tr}_{\mathrm{div}\, m/S}(\theta) \quad (43)\end{aligned}$$

where we have used that the first term on the right hand side of (42) vanishes because of type, and we apply the Poincaré-Lelong formula to the second term on the right hand side of (42) to derive (43). Locally on contractile open subsets of S, we can apply (38) to obtain

$$2\pi i \sum_{p\in \operatorname{div}(m)} \operatorname{ord}_p(m) \int_\sigma^p \theta = \sum_{i=1}^g \left(\int_{\alpha_i} \theta \int_{\beta_i} \frac{d\tilde{m}}{\tilde{m}} - \int_{\alpha_i} \frac{d\tilde{m}}{\tilde{m}} \int_{\beta_i} \theta \right) \tag{44}$$

Let $\nabla_{\mathrm{GM}} \nu_M$ denote the Gauss-Manin invariant for $\nabla^M_{\mathcal{X}/S}$. We now differentiate the equation above, and obtain:

$$\begin{aligned} 2\pi i \operatorname{tr}_{\operatorname{div} m}(\theta) + 2\pi i \sum_{p\in \operatorname{div}(m)} \operatorname{ord}_p(m) \int_\sigma^p \nabla_{\mathrm{GM}}\theta = \\ \sum_{i=1}^g \left(\int_{\alpha_i} \nabla_{\mathrm{GM}}\theta \int_{\beta_i} \frac{d\tilde{m}}{\tilde{m}} - \int_{\alpha_i} \frac{d\tilde{m}}{\tilde{m}} \int_{\beta_i} \nabla_{\mathrm{GM}}\theta \right) \\ + \sum_{i=1}^g \left(\int_{\alpha_i} \theta \int_{\beta_i} \nabla_{\mathrm{GM}}\nu_M - \int_{\alpha_i} \nabla_{\mathrm{GM}}\nu_M \int_{\beta_i} \theta \right). \end{aligned} \tag{45}$$

A comment is in order to clarify the meaning of $\nabla_{\mathrm{GM}}\theta$ and its path integrations. By construction, the differential form θ is closed on fibers, and even holomorphic. Hence, it defines a relative cohomology class to which we can apply ∇_{GM}. This we write $\nabla_{\mathrm{GM}}\theta$. The meaning of integration along non-closed paths involves representing $\nabla_{\mathrm{GM}}\theta$ in terms of a family of harmonic forms, exactly as in Section 4.1. After this clarification, we also note that equation (44) holds for $\nabla_{\mathrm{GM}}\theta$ as well. We subtract this variant from (45), and deduce

$$\begin{aligned} 2\pi i \operatorname{tr}_{\operatorname{div} m/S}(\theta) &= \sum_{i=1}^g \left(\int_{\alpha_i} \theta \int_{\beta_i} \nabla_{\mathrm{GM}}\nu_M - \int_{\alpha_i} \nabla_{\mathrm{GM}}\nu_M \int_{\beta_i} \theta \right) \\ &= \pi_* \left(\theta \wedge \nabla_{\mathrm{GM}}\nu_M\right) = -\pi_* \left(\theta \wedge F_{\nabla^M_{ch}}\right) , \end{aligned}$$

where to obtain the last equality we use the fact that ∇^M_{ch} corresponds to the canonical extension of $\nabla^M_{\mathcal{X}/S}$ (Lemma 4.5). Hence, the right hand side of (43) vanishes, and so therefore does (41). This completes the proof. □

4.5. **Curvatures.** In this section we compute the curvature of intersection and trace connections on $\langle \mathcal{L}, \mathcal{M}\rangle$. In particular we establish Theorem 1.2 (ii). Let $\mathcal{L}, \mathcal{M} \to \mathcal{X}$ be rigidified line bundles with flat relative connections $\nabla^L_{\mathcal{X}/S}$ and $\nabla^M_{\mathcal{X}/S}$, respectively (the rigidification is not essential here). Denote the Gauss-Manin invariants by $\nabla_{\mathrm{GM}}\nu_L$ and $\nabla_{\mathrm{GM}}\nu_M$, and recall from Section 2.2 that $\operatorname{Re}\nu_L$, $\operatorname{Re}\nu_M$ are well-defined. We will also need the following. Let

$$\mathrm{KS}(\mathcal{X}/S) = \nabla_{GM}\Pi' = -\nabla_{\mathrm{GM}}\Pi'' \in \operatorname{End}\left(H^1_{dR}(\mathcal{X}/S)\right) \otimes \mathcal{A}^{1,0}_S \tag{46}$$

denote the derivative of the period map of the fibration $\mathcal{X} \to S$, where Π', Π'' are as in (8) and (9). Finally, define the operation

$$(H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^i_S) \times (H^1_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^j_S) \xrightarrow{\cup} H^2_{dR}(\mathcal{X}/S) \otimes \mathcal{A}^{i+j}_S \tag{47}$$

is given by the cup product on relative cohomology classes and the wedge product on forms, whereas π_* denotes the fiber integration: $H^2_{dR}(\mathcal{X}/S) \to \mathcal{C}^\infty(S)$. One easily verifies that

$$\overline{\mathrm{KS}(\alpha)} = -\mathrm{KS}(\overline{\alpha}) \tag{48}$$

$$\pi_*\left(\alpha \cup \mathrm{KS}(\beta)\right) = -\pi_*\left(\mathrm{KS}(\alpha) \cup \beta\right) \tag{49}$$

With this understood, we have the following

Proposition 4.13. *The curvature of the intersection connection $\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}$ on $\langle\mathcal{L},\mathcal{M}\rangle$ is given by: $2\pi i \cdot F_{\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}} = \pi_*\left(\nabla_{\mathrm{GM}}\nu_L \cup \nabla_{\mathrm{GM}}\nu_M\right)$ (see (3)).*

Proof. By (34) and (33), the curvature of the canonical extension is given by

$$-d\int_{\tilde{\sigma}(s)}^{\tilde{z}} \nabla_{\mathrm{GM}}\nu = -\sum_{i=1}^{2g} \eta_i \wedge \theta_i + \cdots$$

where the $\cdots$ indicates forms that are annihilated by vertical tangent vectors. The first term on the right hand side represents $\nabla_{\mathrm{GM}}\nu$. From Proposition 3.14, the only term that survives the fiber integration is $\nabla_{\mathrm{GM}}\nu_L \cup \nabla_{\mathrm{GM}}\nu_M$. ◻

The following is then an immediate consequence of the curvature formula.

Corollary 4.14. *If the flat relative connections on $\mathcal{L}$ and $\mathcal{M}$ are of type $(1,0)$ (see Definition 2.6), then the intersection connection on $\langle\mathcal{L},\mathcal{M}\rangle$ is holomorphic, i.e. its curvature is of type $(2,0)$.*

Next, we turn to trace connections, where the calculation is a bit more involved.

Proposition 4.15. *Assume $\mathcal{L}$ is given a rigidification. Then the trace connection $\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}$ on $\langle\mathcal{L},\mathcal{M}\rangle$ has curvature:*

$$\begin{aligned} F_{\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}} = \frac{1}{2\pi i}\pi_*\Big\{&(\nabla_{\mathrm{GM}}\nu_L)' \cup (\nabla_{\mathrm{GM}}\nu_M)'' - (\nabla_{\mathrm{GM}}\nu_L)'' \cup \overline{(\nabla_{\mathrm{GM}}\nu_M)''} \\ &- 2\,\mathrm{Re}\,\nu_M \cup (\mathrm{KS}(\mathcal{X}/S) \wedge \nabla_{\mathrm{GM}}\nu_L)\Big\}. \end{aligned} \tag{50}$$

Remark 4.16. As shown in Section 4.4, the trace connection is a special case of an intersection connection. However, the above gives a more general formula where ν_M is not necessarily associated to a unitary connection. The trace connection on $\langle\mathcal{L},\mathcal{M}\rangle$ is independent of a choice of relative connection on $\mathcal{M}$. Using (48) and (49), one verifies that (50) is indeed independent of the choice of $\nabla^M_{\mathcal{X}/S}$. Moreover, by specializing to the Chern connection on $\mathcal{M}$, (50) reduces to (3).

We also point out the following

Corollary 4.17. *If $(\nabla_{\mathrm{GM}}\nu)''$ vanishes identically then the trace connection on $\langle\mathcal{L},\mathcal{L}\rangle$ is flat.*

Proof. Differentiate the equation $0 = (\nabla_{\mathrm{GM}}\nu)'' = \Pi''\nabla_{\mathrm{GM}}\nu$, to find

$$0 = \nabla_{\mathrm{GM}}\Pi'' \wedge \nabla_{\mathrm{GM}}\nu + \Pi''\nabla^2_{\mathrm{GM}}\nu = -\mathrm{KS}(\mathcal{X}/S) \wedge \nabla_{\mathrm{GM}}\nu + \Pi''\nabla^2_{\mathrm{GM}}\nu\,.$$

Since $\nabla^2_{\mathrm{GM}} = 0$, the result follows. □

Proof of Proposition 4.15. First, since the calculation is local in S for the analytic topology, we can work over a contractible open subsets S°. Let $\tilde{\nu}_L$ be a lift of ν_L (the classifying morphism of $(\mathcal{L},\nabla^L_{\mathcal{X}/S})$), and $\chi_s : \pi_1(\mathcal{X}_s,\sigma(s)) \to \mathbb{C}^\times$ denote the associated character at $s \in S^\circ$: $\chi_s(\gamma) = \exp(-\int_\gamma \tilde{\nu}_s)$, namely the holonomy character. Similarly for $\mathcal{M}$. Choose a homology basis as in Section 4.1. Let m be a meromorphic section of $\mathcal{M}$, whose divisor is finite and étale over (an open subset of) S. After étale base change, we may assume that $\operatorname{div} m$ is given by sections. Using (39), we then have

$$2\pi i \sum_j \mathrm{ord}_{p_j} m \int_{\sigma(s)}^{p_j} (\nabla_{\mathrm{GM}}\nu_L)' = \sum_{i=1}^{g} \left\{ \int_{\alpha_i} (\nabla_{\mathrm{GM}}\nu_L)' \int_{\beta_i} \frac{dm}{m} - \int_{\alpha_i} \frac{dm}{m} \int_{\beta_i} (\nabla_{\mathrm{GM}}\nu_L)' \right\}$$

$$2\pi i \sum_j \mathrm{ord}_{p_j} m \int_{\sigma(s)}^{p_j} (\nabla_{\mathrm{GM}}\nu_L)'' = -\sum_{i=1}^{g} \left\{ \int_{\alpha_i} (\nabla_{\mathrm{GM}}\nu_L)'' \int_{\beta_i} \frac{\overline{dm}}{m} - \int_{\alpha_i} \frac{\overline{dm}}{m} \int_{\beta_i} (\nabla_{\mathrm{GM}}\nu_L)'' \right\}.$$

Hence,

$$\begin{aligned} -2\pi i \sum_j \mathrm{ord}_{p_j} m \int_{\sigma(s)}^{p_j} \nabla_{\mathrm{GM}}\nu_L = \sum_{i=1}^{g} \Big\{ &\mathrm{Re}\log \chi_M(\beta_i) \int_{\alpha_i} [(\nabla_{\mathrm{GM}}\nu_L)' - (\nabla_{\mathrm{GM}}\nu_L)''] \\ &- \mathrm{Re}\log \chi_M(\alpha_i) \int_{\beta_i} [(\nabla_{\mathrm{GM}}\nu_L)' - (\nabla_{\mathrm{GM}}\nu_L)''] \Big\} \\ &+ \sum_{i=1}^{g} \left\{ i\,\mathrm{Im}\log \chi_M(\beta_i) \int_{\alpha_i} \nabla_{\mathrm{GM}}\nu_L - i\,\mathrm{Im}\log \chi_M(\alpha_i) \int_{\beta_i} \nabla_{\mathrm{GM}}\nu_L \right\} \end{aligned}$$

(the choice of log is immaterial). Using the flatness of the Gauss-Manin connection, $d\int_\gamma \nabla_{\mathrm{GM}}\nu_L = \int_\gamma \nabla^2_{\mathrm{GM}}\nu_L = 0$, and eqs. (13) and (34), we find

$$\begin{aligned} 2\pi i F_{\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}} &= 2\pi i d\, \mathrm{tr}_{\operatorname{div} m/S^\circ} \frac{d\langle\ell,m\rangle}{\langle\ell,m\rangle} = -2\pi i \sum_j \mathrm{ord}_{p_j} m \int_{\sigma(s)}^{p_j} \nabla_{\mathrm{GM}}\nu_L \\ &= \sum_{i=1}^{g} \Big\{ d\,\mathrm{Re}\log \chi_M(\beta_i) \wedge \int_{\alpha_i} [(\nabla_{\mathrm{GM}}\nu_L)' - (\nabla_{\mathrm{GM}}\nu_L)''] \qquad (51) \\ &\qquad - d\,\mathrm{Re}\log \chi_M(\alpha_i) \wedge \int_{\beta_i} [(\nabla_{\mathrm{GM}}\nu_L)' - (\nabla_{\mathrm{GM}}\nu_L)''] \Big\} \\ &\quad + \sum_{i=1}^{g} \left\{ id\,\mathrm{Im}\log \chi_M(\beta_i) \wedge \int_{\alpha_i} \nabla_{\mathrm{GM}}\nu_L - id\,\mathrm{Im}\log \chi_M(\alpha_i) \wedge \int_{\beta_i} \nabla_{\mathrm{GM}}\nu_L \right\} \end{aligned}$$

$$+\sum_{i=1}^{g}\Big\{\log|\chi_M(\beta_i)|\int_{\alpha_i}\nabla_{\mathrm{GM}}\left[(\nabla_{\mathrm{GM}}\nu_L)'-(\nabla_{\mathrm{GM}}\nu_L)''\right]$$
$$-\log|\chi_M(\alpha_i)|\int_{\beta_i}\nabla_{\mathrm{GM}}\left[(\nabla_{\mathrm{GM}}\nu_L)'-(\nabla_{\mathrm{GM}}\nu_L)''\right]\Big\}.$$

Now

$$d\,\mathrm{Re}\log\chi_M(\gamma)=-\frac{1}{2}\int_\gamma \nabla_{\mathrm{GM}}\nu_M+\overline{\nabla_{\mathrm{GM}}\nu_M}$$
$$i d\,\mathrm{Im}\log\chi_M(\gamma)=-\frac{1}{2}\int_\gamma \nabla_{\mathrm{GM}}\nu_M-\overline{\nabla_{\mathrm{GM}}\nu_M}.$$

Substituting this into (51), and using (46), we have

$$\begin{aligned}
-2\pi i\cdot F_{\nabla^{tr}_{\langle\mathcal{L},\mathcal{M}\rangle}} &= \frac{1}{2}\sum_{i=1}^{g}\Big\{\int_{\beta_i}\left[\nabla_{\mathrm{GM}}\nu_M+\overline{\nabla_{\mathrm{GM}}\nu_M}\right]\wedge\int_{\alpha_i}\left[(\nabla_{\mathrm{GM}}\nu_L)'-(\nabla_{\mathrm{GM}}\nu_L)''\right]\\
&\quad -\int_{\alpha_i}\left[\nabla_{\mathrm{GM}}\nu_M+\overline{\nabla_{\mathrm{GM}}\nu_M}\right]\wedge\int_{\beta_i}\left[(\nabla_{\mathrm{GM}}\nu_L)'-(\nabla_{\mathrm{GM}}\nu_L)''\right]\Big\}\\
&\quad +\frac{1}{2}\sum_{i=1}^{g}\Big\{\int_{\beta_i}\left[\nabla_{\mathrm{GM}}\nu_M-\overline{\nabla_{\mathrm{GM}}\nu_M}\right]\wedge\int_{\alpha_i}\nabla_{\mathrm{GM}}\nu_L\\
&\quad -\int_{\alpha_i}\left[\nabla_{\mathrm{GM}}\nu_M-\overline{\nabla_{\mathrm{GM}}\nu_M}\right]\wedge\int_{\beta_i}\nabla_{\mathrm{GM}}\nu_L\Big\}\\
&\quad -2\sum_{i=1}^{g}\left\{\log|\chi_M(\beta_i)|\int_{\alpha_i}\mathrm{KS}(\mathcal{X}/S)\wedge\nabla_{\mathrm{GM}}\nu_L-\log|\chi_M(\alpha_i)|\int_{\beta_i}\mathrm{KS}(\mathcal{X}/S)\wedge\nabla_{\mathrm{GM}}\nu_L\right\}\\
&= \sum_{i=1}^{g}\left\{\int_{\beta_i}\nabla_{\mathrm{GM}}\nu_M\wedge\int_{\alpha_i}(\nabla_{\mathrm{GM}}\nu_L)'-\int_{\beta_i}\overline{\nabla_{\mathrm{GM}}\nu_M}\wedge\int_{\alpha_i}(\nabla_{\mathrm{GM}}\nu_L)''\right\}\\
&\quad -\sum_{i=1}^{g}\left\{\int_{\alpha_i}\nabla_{\mathrm{GM}}\nu_M\wedge\int_{\beta_i}(\nabla_{\mathrm{GM}}\nu_L)'-\int_{\alpha_i}\overline{\nabla_{\mathrm{GM}}\nu_M}\wedge\int_{\beta_i}(\nabla_{\mathrm{GM}}\nu_L)''\right\}\\
&\quad -2\sum_{i=1}^{g}\left\{\log|\chi_M(\beta_i)|\int_{\alpha_i}\mathrm{KS}(\mathcal{X}/S)\wedge\nabla_{\mathrm{GM}}\nu_L-\log|\chi_M(\alpha_i)|\int_{\beta_i}\mathrm{KS}(\mathcal{X}/S)\wedge\nabla_{\mathrm{GM}}\nu_L\right\}.
\end{aligned}$$

By the Riemann bilinear relations the right hand side is

$$\begin{aligned}
&= \sum_{i=1}^{g}\left\{\int_{\beta_i}(\nabla_{\mathrm{GM}}\nu_M)''\wedge\int_{\alpha_i}(\nabla_{\mathrm{GM}}\nu_L)'-\int_{\beta_i}\overline{(\nabla_{\mathrm{GM}}\nu_M)''}\wedge\int_{\alpha_i}(\nabla_{\mathrm{GM}}\nu_L)''\right\}\\
&\quad -\sum_{i=1}^{g}\left\{\int_{\alpha_i}(\nabla_{\mathrm{GM}}\nu_M)''\wedge\int_{\beta_i}(\nabla_{\mathrm{GM}}\nu_L)'-\int_{\alpha_i}\overline{(\nabla_{\mathrm{GM}}\nu_M)''}\wedge\int_{\beta_i}(\nabla_{\mathrm{GM}}\nu_L)''\right\}\\
&\quad -2\sum_{i=1}^{g}\left\{\log|\chi_M(\beta_i)|\int_{\alpha_i}\mathrm{KS}(\mathcal{X}/S)\wedge\nabla_{\mathrm{GM}}\nu_L-\log|\chi_M(\alpha_i)|\int_{\beta_i}\mathrm{KS}(\mathcal{X}/S)\wedge\nabla_{\mathrm{GM}}\nu_L\right\}.
\end{aligned}$$

Collecting terms and applying the bilinear relations again, the formula now follows. □

5. Examples and Applications

5.1. **Reciprocity for trivial fibrations.** Throughout this section, we consider trivial families $\mathcal{X} = X \times S$, where X is a fixed compact Riemann surface X of genus $g \geq 1$ with a prescribed point $\sigma \in X$. Using the Hodge splitting we shall give explicit formulas illustrating the main construction of this paper in this simple case. In particular, we shall give a direct proof of Weil reciprocity for the connection defined in Section 4.1.

The *de Rham moduli space* $M_{dR}(X)$ of rank 1 flat connections on X is isomorphic to $H^1(X, \mathbb{C})/H^1(X, 2\pi i\mathbb{Z})$ (see (4)). We shall always assume a rigidification, or trivialization of our bundles at σ. If we take as base $S = M_{dR}(X)$, then there is a universal line bundle $\mathcal{L} \to \mathcal{X}$ equipped a universal relative connection. Choose a symplectic homology basis $\{\alpha_j, \beta_j\}_{j=1}^g$, and normalized abelian differentials ω_j with period matrix Ω. Then $J(X) = \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\Omega$. Given $[\nabla] \in M_{dR}(X)$, we have its associated holonomy character $\chi_\nu : \pi_1(X, \sigma) \to \mathbb{C}^\times : \gamma \mapsto \exp\left(-\int_\gamma \nu\right)$. We regard χ_ν as an element of the *Betti moduli space*,

$$M_B(\pi_1(X, \sigma)) := \mathrm{Hom}(\pi_1(X, \sigma), \mathsf{GL}(1, \mathbb{C})) \simeq (\mathbb{C}^\times)^{2g}, \tag{52}$$

with its structure as an algebraic variety. The Riemann-Hilbert correspondence above gives a complex analytic (though not algebraic) isomorphism $M_{dR}(X) \simeq M_B(\pi_1(X, \sigma))$. As before, we have chosen a lift of ν from $H^1(X, \mathbb{C})/H^1(X, 2\pi i\mathbb{Z})$ to $H^1(X, \mathbb{C})$. In fact, we choose a harmonic representative of this class in $\mathcal{A}^1_X$, and continue to denote this by ν. Since we have chosen a basis $\{\omega_i\}$ for $H^{1,0}(X)$, we have local holomorphic coordinates (t_i, s_i) for $M_{dR}(X)$; namely, we write

$$\nu = \sum_{i=1}^{g} t_i\omega_i + s_i\overline{\omega_i} \tag{53}$$

for the flat connection $\nabla = d + \nu$. It follows that

$$\nabla_{\mathrm{GM}}\nu = \sum_{i=1}^{g} \omega_i \otimes dt_i + \bar{\omega}_i \otimes ds_i. \tag{54}$$

Let $\widetilde{X}$ be the universal cover of X. According to the discussion in Section 4.1, we view sections ℓ of $\mathcal{L}$ as functions $\tilde{\ell}$ on $\widetilde{X} \times M_{dR}(X)$ that satisfy

$$\tilde{\ell}(\gamma z, \nu) = \exp\left(\int_\gamma \nu\right) \tilde{\ell}(z, \nu) \tag{55}$$

(note that the bundle is invariant with respect to the integral lattice $H^1(X, 2\pi i\mathbb{Z})$). The universal connection $\nabla : \Omega^0(\mathcal{X}, \mathcal{L}) \to \Omega^1(\mathcal{X}, \mathcal{L})$ is defined as follows (see

(34)): given $\tilde{\ell}$ satisfying (55), let

$$\nabla\ell(z,\nu) = d\tilde{\ell}(z,\nu) - \int_\sigma^z \nabla_{\mathrm{GM}}\nu\cdot\tilde{\ell}(z,\nu). \tag{56}$$

One can check directly that $\nabla\ell(z,\nu)$ indeed satisfies the correct equivariance, and that the connection is independent of the choice of fundamental domain. A change of homology basis has the same effect as pulling $\nabla_{\mathrm{GM}}\nu$ back by the corresponding action on $M_{dR}(X)$; and therefore ∇ is independent of this choice.

Remark 5.1. This is the connection defined in (34). Notice that this is *not* a holomorphic connection (see Remark 2.4): $(\nabla_{\mathrm{GM}}\nu)'' = \sum_{i=1}^g \bar{\omega}_i(z)\otimes ds_i$.

The flat connection corresponding to ν is $\nabla = d+\nu$, and $\bar\partial_\nabla = \bar\partial + \nu''$ is the corresponding $\bar\partial$-operator for the holomorphic line bundle $\mathcal{L}$ defined by ∇. The map $\pi : M_{dR}(X)\to J(X)$ which takes a holomorphic connection to its underlying holomorphic line bundle realizes $M_{dR}(X)$ as an affine bundle over $J(X)$. We wish to write this map explicitly. First, we identify the Jacobian variety $J(X)$ with the space of flat $\mathsf{U}(1)$-connections, or equivalently, as the space of $\mathsf{U}(1)$-representations of $\pi_1(X,\sigma)$. The Chern connection on $\mathcal{L}_\nu$ is $d_A = d + \nu'' - \overline{\nu''}$, and this defines a *unitary* character $\chi_u : \pi_1(X)\to \mathsf{U}(1)$. Let

$$2\pi i a_j = -\log\chi_u(\alpha_j) = \int_{\alpha_j} \nu'' - \overline{\nu''}$$
$$2\pi i b_j = -\log\chi_u(\beta_j) = \int_{\beta_j} \nu'' - \overline{\nu''}.$$

Then the point $[u]\in J(X)$ corresponding to $[\nabla]\in M_{dR}(X)$ is given by

$$u = b - a^t\Omega. \tag{57}$$

In terms of these coordinates, one calculates:

$$2\pi i a_j = s_j - \bar{s}_j \tag{58}$$
$$2\pi i b_j = \sum_{k=1}^g s_k\overline{\Omega}_{kj} - \bar{s}_k\Omega_{kj} \tag{59}$$
$$u_j = -\frac{1}{\pi}s_k \operatorname{Im}\Omega_{kj} \tag{60}$$
$$s_j = -\pi u_k(\operatorname{Im}\Omega)^{-1}_{kj} \tag{61}$$

Remark 5.2. Notice that there is a smooth section $\jmath : J(X)\to M_{dR}(X)$ defined by $[u]\mapsto[\nu] : \nu = \sum_{i=1}^g(-\bar{s}_i)\omega_i + s_i\bar{\omega}_i$, where s_j is given by (61). The image $U_{dR}(X)\subset M_{dR}(X)$, which consists of the unitary connections, is a *totally real submanifold*.

Since the notion above will be used later on, we recall the definition.

Definition 5.3. Let M be a smooth manifold, $\dim_{\mathbb{R}} M = 2m$, with an integrable almost complex structure J. A smooth submanifold $U\subset M$ is called *totally real* if at each point $p \in U$, $T_pU \cap JT_pU = \{0\}$. In particular, if $\dim_{\mathbb{R}} U = m$ and U is totally real, then $T_pU \oplus JT_pU = T_pM$.

Next, we express meromorphic sections of $\mathcal{L}_\nu \to X$ in terms of meromorphic functions on $\widetilde{X}$ satisfying the equivariance (55). Let $E(z, w)$ be the Schottky prime form associated with $\{\alpha_i, \beta_i\}$ (*cf.* [16, eq. 19]). For a meromorphic section ℓ of $\mathcal{L}_\nu$ with divisor $\sum_{i=1}^N p_i - q_i$ we have

$$\sum_{i=1}^{N} \int_{q_i}^{p_i} \vec{\omega} = u + m + n^t\Omega \quad , \quad m, n \in \mathbb{Z}^g \tag{62}$$

Lemma 5.4. *Define*

$$\tilde{\ell}(z) = \frac{\prod_{i=1}^N E(z, p_i)}{\prod_{i=1}^N E(z, q_i)} \exp\left\{2\pi i(a-n)^t \int_\sigma^z \vec{\omega} + \int_\sigma^z \nu' + \overline{\nu''}\right\}$$

Then $\tilde{\ell}$ is a meromorphic function on $\widetilde{X}$ with multipliers χ_ν^{-1} and divisor projecting to $\mathrm{div}(\ell)$.

A particular case of the above formula is a meromorphic function $f(z)$ with divisor $\sum_{i=1}^M x_i - y_i$ and $\sum_{i=1}^M \int_{y_i}^{x_i} \vec{\omega} = \tilde{m} + \tilde{n}^t\Omega$, $\tilde{m}, \tilde{n} \in \mathbb{Z}^g$. Then f can be expressed

$$f(z) = \frac{\prod_{i=1}^M E(z, x_i)}{\prod_{i=1}^M E(z, y_i)} \exp\left\{-2\pi i \tilde{n}^t \int_\sigma^z \vec{\omega}\right\} \tag{63}$$

With this understood, we are ready to give a direct proof of (WR) in this setting:

Proposition 5.5. *Let ℓ be a meromorphic section of the universal bundle $\mathcal{L} \to \mathcal{X}$ and f a meromorphic function on $\mathcal{X}$. Then*

$$\mathrm{tr}_{\mathrm{div}\, f/M_{dR}}\left(\frac{\nabla \ell}{\ell}\right) = \mathrm{tr}_{\mathrm{div}\, \ell/M_{dR}}\left(\frac{df}{f}\right)$$

Proof. Fix $\nu = \sum_{i=1}^g t_i\omega_i + s_i\bar{\omega}_i$. Then $\nu' + \overline{\nu''} = \sum_{i=1}^g (t_i + \bar{s}_i)\omega_i$. Using (58) we see that, up to a nonzero multiplicative constant,

$$\tilde{\ell}(z) = \frac{\prod_{i=1}^N E(z, p_i)}{\prod_{i=1}^N E(z, q_i)} \exp\left\{(t+s)^t \int_\sigma^z \vec{\omega} - 2\pi i n^t \int_\sigma^z \vec{\omega}\right\} \tag{64}$$

(note that m, n and $\tilde{m}, \tilde{n}$ are locally constant in the t_i and s_i). Hence

$$\begin{aligned}\tilde{\ell}(\mathrm{div}\, f) &= \frac{\prod_{i=1}^N \prod_{j=1}^M E(x_j, p_i)E(y_j, q_i)}{\prod_{i=1}^N E(x_j, q_i)E(y_j, p_i)} \exp\left\{(t+s)^t \sum_{i=1}^M \int_{y_i}^{x_i} \vec{\omega} - 2\pi i n^t \sum_{i=1}^M \int_{y_i}^{x_i} \vec{\omega}\right\} \\ &= \frac{\prod_{i=1}^N \prod_{j=1}^M E(x_j, p_i)E(y_j, q_i)}{\prod_{i=1}^N \prod_{j=1}^M E(x_j, q_i)E(y_j, p_i)} \exp\left\{(t+s)^t(\tilde{m} + \tilde{n}^t\Omega) - 2\pi i n^t(\tilde{n}^t\Omega)\right\}\end{aligned}$$

since $n^t\tilde{m} \in \mathbb{Z}$. Similarly, using (60),

$$f(\mathrm{div}\, \ell) = \frac{\prod_{i=1}^N \prod_{j=1}^M E(x_j, p_i)E(y_j, q_i)}{\prod_{i=1}^N \prod_{j=1}^M E(x_j, q_i)E(y_j, p_i)} \exp\left\{2i\tilde{n}^t(s^t \operatorname{Im}\Omega) - 2\pi i \tilde{n}^t(n^t\Omega)\right\}.$$

Now $(t+s)^t(\tilde{m}+\tilde{n}^t\Omega)-2i\tilde{n}^t(s^t\operatorname{Im}\Omega)=\sum_{i=1}^g t_jv_j+s_j\bar{v}_j$, where $v=\tilde{m}+\tilde{n}^t\Omega$, so that $\tilde{\ell}(\operatorname{div} f)=f(\operatorname{div}\ell)\exp\left(\sum_{i=1}^g t_jv_j+s_j\bar{v}_j\right)$. Differentiating with respect to (z,ν),

$$d\tilde{\ell}(\operatorname{div} f)=\frac{d(f(\operatorname{div}\ell))}{f(\operatorname{div}\ell)}\ell(\operatorname{div} f)+\left(\sum_{i=1}^g dt_jv_j+ds_jv_j\right)\ell(\operatorname{div} f).$$

On the other hand,

$$(\int_{z_0}^z\nabla_{\mathrm{GM}}\nu)(\operatorname{div} f)=\sum_{i=1}^M\int_{y_i}^{x_i}dt_i\omega_i+ds_i\bar{\omega}_i=\sum_{i=1}^g dt_jv_j+ds_jv_j.$$

The result now follows from the definition of ∇. □

5.2. **Holomorphic extension of analytic torsion.** As mentioned in the Introduction, one motivation for this paper was to derive an interpretation of the holomorphic extension of analytic torsion in terms of Deligne pairings. In this section, we review the construction of torsion and give explicit formulas, generalizing those in [17] and [23]. In the next section, we explain the relationship with our construction.

First, we review the definition of analytic torsion for the non-self-adjoint operators we consider. Fix an arbitrary hermitian, holomorphic line bundle $\mathcal{M}\to X$ with Chern connection $\nabla=\partial_\nabla+\bar{\partial}_\nabla$. Given a flat connection on $\mathcal{L}\to X$ with holonomy χ, we regard smooth sections ℓ of $\mathcal{L}\otimes\mathcal{M}$ as χ^{-1}-equivariant sections $\tilde{\ell}$ of the pull-back of $\mathcal{M}$ to $\widetilde{X}$ satisfying (55). Pick a lift $\nu\in H^1(X,\mathbb{C})$ of the character, hence $\chi(\gamma)=\exp(-\int_\gamma\nu)$, and set $G_\nu(z)=\exp\left(-\int_\sigma^z\nu\right)$ (*cf.* [26]). Then for any $\tilde{\ell}$, notice that $G_\nu(z)\tilde{\ell}(z)$ is a well-defined smooth section of $\mathcal{M}\to X$. Define the operators

$$D'':\Omega^{p,0}(X,\mathcal{M})\longrightarrow\Omega^{p,1}(X,\mathcal{M}):\alpha\mapsto G_\nu^{-1}\bar{\partial}_\nabla\left(G_\nu\alpha\right)$$
$$D':\Omega^{0,q}(X,\mathcal{M})\longrightarrow\Omega^{1,q}(X,\mathcal{M}):\beta\mapsto G_\nu^{-1}\partial_\nabla\left(G_\nu\beta\right).$$

Fix a conformal metric on X, and let $*$ denote the Hodge operator. We define the *laplacian* associated to ν and $\mathcal{M}$ by $\square_{\chi_\nu\otimes\mathcal{M}}(s)=-2i*D'D''(s)$, for smooth sections s of $\mathcal{M}$. This is an elliptic operator that is independent of the choice of base point σ. In case ν is unitary, G_ν has absolute value $=1$, and via (55) gives a unitary equivalence between $\square_{\chi_\nu\otimes\mathcal{M}}$ and the ordinary $\bar{\partial}$-laplacian for $\mathcal{L}\otimes\mathcal{M}$. In particular, the spectra of these two operators is the same in this case. For ν not unitary, $\square_{\chi_\nu\otimes\mathcal{M}}$ is not a symmetric operator. Since the symbol of $\square_{\chi_\nu\otimes\mathcal{M}}$ is the same as that of the scalar laplacian, however, the zeta regularization procedure applies to give a well-defined determinant $\det\square_{\chi_\nu\otimes\mathcal{M}}$. For a nice explanation of this, we refer to [1, Section 2.5]; we sketch the ideas here for convenience. First, the following holds (*cf.* [2] and [12, Lemma 4.1]).

Lemma 5.6. *For ν in a compact set there are at most finitely many eigenvalues λ of $\square_{\chi_\nu\otimes\mathcal{M}}$ with $\operatorname{Re}\lambda\le 0$. Moreover, there is $B>0$ such that $-B\le\operatorname{Im}\lambda\le B$.*

Assume $\square_{\chi_\nu\otimes\mathcal{M}}$ has no zero eigenvalues. Then by Lemma 5.6, we wish to show that $\det\square_{\chi_\nu\otimes\mathcal{M}}$ is independent of a choice of *Agmon* angle θ, by which we mean a ray from the origin into the half plane $\operatorname{Re} z \leq 0$ which misses the eigenvalues. Indeed, if $\{\lambda_i\}_{i=1}^N$ are the eigenvalues of $\square_{\chi_\nu\otimes\mathcal{M}}$ with negative real part, $\{\mu_i\}$ the eigenvalues with positive real part. Then by Lidskiĭ's theorem [29] which guarantees that the trace is a sum over eigenvalues,

$$\zeta_{(\square_{\chi_\nu\otimes\mathcal{M}},\theta)}(s) := \operatorname{tr}_\theta\left(\square_{\chi_\nu\otimes\mathcal{M}}\right)^{-s} = \lambda_1^{-s} + \cdots + \lambda_N^{-s} + \sum_{i=1}^{\infty}\mu_i^{-s}\,,$$

for $\operatorname{Re} s > 1$. Here, the eigenvalues are counted with their algebraic multiplicities, that is, the dimension of the generalized eigenspace. The cut θ gives a branch of the logarithm which is used to define the powers λ_i^{-s} and the usual logarithm (real on the positive real axis) is used to define the rest. By Lemma 5.6 and the result of Seeley [32], $\zeta_{(\square_{\chi_\nu\otimes\mathcal{M}},\theta)}(s)$ has a meromorphic continuation to the plane that is regular at $s = 0$. Any other choice $\tilde\theta$ of Agmon angle gives a zeta function of the form

$$\zeta_{(\square_{\chi_\nu\otimes\mathcal{M}},\tilde\theta)}(s) = \lambda_1^{-s}e^{-2\pi i k_1 s} + \cdots + \lambda_N^{-s}e^{-2\pi i k_N s} + \sum_{i=1}^{\infty}\mu_i^{-s}$$

for integers k_i. But then $-\zeta'_{(\square_{\chi_\nu\otimes\mathcal{M}},\tilde\theta)}(0) = -\zeta'_{(\square_{\chi_\nu\otimes\mathcal{M}},\theta)}(0) + 2\pi i\sum_{i=1}^N k_i$, and so $\det\square_{\chi_\nu\otimes\mathcal{M}} := \exp(-\zeta'_{(\square_{\chi_\nu\otimes\mathcal{M}},\theta)}(0))$ is independent of the choice of θ. We also note that a different choice of lift $\tilde\nu$ gives $G_{\tilde\nu} = G_\nu\cdot F$, where pointwise $|F| = 1$. Hence, F gives a unitary automorphism of $L^2(X)$ such that $\square_{\chi_{\tilde\nu}\otimes\mathcal{M}} = F\circ\square_{\chi_\nu\otimes\mathcal{M}}\circ F^{-1}$; hence, the eigenvalues of $\square_{\chi_{\tilde\nu}\otimes\mathcal{M}}$ are the same as those of $\square_{\chi_\nu\otimes\mathcal{M}}$, and so the determinants agree. Finally, since $\square_{\chi_\nu\otimes\mathcal{M}}$ depends holomorphically on ν, so does $\det\square_{\chi_\nu\otimes\mathcal{M}}$ (see [28]), and since it agrees with the usual determinant when ν is unitary, $\det\square_{\chi_\nu\otimes\mathcal{M}}$ is a holomorphic extension of the usual analytic torsion.

As in Section 5.1, let X be a compact genus $g \geq 1$ Riemann surface with a conformal metric and a choice of symplectic homology basis $\{\alpha_j,\beta_j\}_{j=1}^g$. This gives a period matrix Ω, theta function $\vartheta(Z,\Omega)$, and a Riemann divisor κ_0 of degree $g-1$, $2\kappa_0 = \omega_X$ (cf. [17, Theorem 1.1]). Let $\chi_u : \pi_1(X)\to \mathsf{U}(1)$ be a unitary character whose holomorphic line bundle corresponds to the point $u\in J(X)$ as in (57). The choice of conformal metric gives κ_0 a hermitian structure. Then the torsion of the $\bar\partial$-laplacian on $\chi_u\otimes\kappa_0$ is given by

$$T(\chi_u\otimes\kappa_0) = \det\square_{\chi_u\otimes\kappa_0} = C(X)\|\vartheta\|^2(u,\Omega) \tag{65}$$

where $C(X)$ is a constant depending on the Riemann surface X and the conformal metric. Recall the definition of the norm:

$$\|\vartheta\|^2(u,\Omega) = \exp\left(-2\pi\operatorname{Im} u^T(\operatorname{Im}\Omega)^{-1}\operatorname{Im} u\right)|\vartheta|^2(u,\Omega).$$

In terms of periods, this is

$$\|\vartheta\|^2(u,\Omega) = \exp\left(-2\pi a^T(\operatorname{Im}\Omega)a\right)|\vartheta|^2(b - a^T\Omega,\Omega). \tag{66}$$

Now suppose $\chi : \pi_1(X,\sigma) \to \mathbb{C}^\times$ is a complex character with periods $\chi(\alpha_j) = \exp(-2\pi i a_j)$, $\chi(\beta_j) = \exp(-2\pi i b_j)$, $a_j, b_j \in \mathbb{C}/\mathbb{Z}$. Then we have the following definition:

$$T(\chi\otimes\kappa_0) := C(X)\exp\left(-2\pi a^T(\operatorname{Im}\Omega)a\right)\vartheta(b-a^T\Omega,\Omega)\vartheta(b-a^T\overline{\Omega},-\overline{\Omega}). \quad (67)$$

By the transformation properties of the theta function, one verifies that the expression in (67) indeed depends on the values of a_j, b_j modulo $\mathbb{Z}$. The subspace $U_B(\pi_1(X,\sigma)) \subset M_B(\pi_1(X,\sigma))$ of unitary characters $(S^1)^{2g} \subset (\mathbb{C}^\times)^{2g}$ is totally real. The following is clear:

Proposition 5.7 (*cf.* [24]). *The function $\nu \mapsto T(\chi_\nu\otimes\kappa_0)$ is a holomorphic extension of the torsion on unitary characters. In particular, $T(\chi_\nu\otimes\kappa_0) = \det\Box_{\chi_\nu\otimes\kappa_0}$.*

Next we consider the holomorphic extension of the torsion $T(\chi)$. For this we need to choose a basis of Prym differentials $\eta_i(z,\chi^{-1})$ on X and $\eta_i(\bar z,\chi)$ on $\overline{X}$, $i = 1,\dots,g-1$ (χ nontrivial). We choose these to vary holomorphically in χ, and for convenience we require $\eta_i(\bar z,\chi) = \overline{\eta_i(z,\chi^{-1})}$ for χ unitary. For χ unitary we have a natural inner product

$$\langle\eta_i(\chi^{-1}),\eta_j(\chi^{-1})\rangle = \int_X \eta_i(z,\chi^{-1})\wedge\overline{\eta_j(z,\chi^{-1})}. \quad (68)$$

For general characters χ, define the pairing on Prym differentials on X and $\overline{X}$ by

$$(\eta_i(\chi^{-1}),\eta_j(\chi)) = \int_X \eta_i(z,\chi^{-1})\wedge\eta_j(\bar z,\chi). \quad (69)$$

Choose generic points $p_1,\dots,p_g$, and set $u_0 = \kappa_0 - \sum_{i=1}^{g-1} p_i$. Then for χ_u unitary, the torsion is given by

$$\begin{aligned} T(\chi_u) = 4\pi^2 C(X)\left|\det\omega_i(p_j)\right|^2 \exp\left(4\pi\operatorname{Im}u_0\cdot a - 2\pi a^T(\operatorname{Im}\Omega)a\right) \\ \times\frac{\det\langle\eta_i(\chi^{-1}),\eta_j(\chi^{-1})\rangle}{\left|\det\eta_i(p_j,\chi^{-1})\right|^2}\frac{|\vartheta(u+u_0,\Omega)|^2}{\left|\sum_{i=1}^g\partial_{Z_i}\vartheta(u_0,\Omega)\omega_i(p_g)\right|^2} \end{aligned} \quad (70)$$

where it is understood that in the expression, $\det\eta_i(p_j)$, $1 \le j \le g-1$. As before this leads to the definition of holomorphic torsion. For χ an arbitrary character, define

$$\begin{aligned} T(\chi) = 4\pi^2 C(X)\left|\det\omega_i(p_j)\right|^2 \exp\left(4\pi\operatorname{Im}u_0\cdot\alpha - 2\pi\alpha^T(\operatorname{Im}\Omega)\alpha\right) \\ \times\frac{\det(\eta_i(\chi^{-1}),\eta_j(\chi))}{\det\eta_i(p_j,\chi^{-1})\det\eta_i(\bar p_j,\chi)}\frac{\vartheta(\beta-\alpha^T\Omega+u_0,\Omega)\vartheta(\beta-\alpha^T\overline{\Omega}-\overline{u}_0,-\overline{\Omega})}{\left|\sum_{i=1}^g\partial_{Z_i}\vartheta(u_0,\Omega)\omega_i(p_g)\right|^2}. \end{aligned} \quad (71)$$

Proposition 5.8 (*cf.* [17]). *The function $\chi \mapsto T(\chi)$ is a holomorphic extension to $M_{dR}(X)$ of the torsion on unitary characters.*

5.3. **Holomorphic torsion and the Deligne isomorphism.** We next explain how the holomorphic extension of analytic torsion is related to the Deligne isomorphism (1) and the intersection connection. To begin, from (54) and (56) we have that the curvature of the universal connection is

$$F_\nabla = -\sum_{i=1}^{g} \omega_i \wedge dt_i + \bar{\omega}_i \wedge ds_i \in \mathcal{A}^2_{\mathfrak{X}}$$

(recall the coordinates (53)). Computing directly from this, or alternatively using Proposition 4.13, it follows that the curvature of the intersection connection on $\langle \mathcal{L}, \mathcal{L} \rangle$ is given by

$$F_{\nabla^{int}_{\langle \mathcal{L}, \mathcal{L}\rangle}} = -\frac{2}{\pi} \sum_{i,j=1}^{g} \operatorname{Im} \Omega_{ij}(dt_i \wedge ds_j). \tag{72}$$

Note that the intersection connection is holomorphic, coming from the fact that $\nabla_{\mathrm{GM}}\nu$ is of type $(1,0)$, as in Corollary 4.14.

Let us suppose, to simplify the following discussion, that the genus $g \geq 2$. Choose a uniformization $X = \Gamma\backslash\mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half plane and $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a cocompact lattice $\simeq \pi_1(X,\sigma)$. Then let $\overline{X} = \Gamma\backslash\mathbb{L}$, where $\mathbb{L} \subset \mathbb{C}$ is the lower half plane. If $\overline{\mathfrak{X}} = \overline{X} \times M_{dR}(\overline{X})$, and $\overline{\pi} : \overline{\mathfrak{X}} \to M_{dR}(\overline{X})$ the projection, we define the universal bundle $\overline{\mathcal{L}} \to \overline{\mathfrak{X}}$, where the fiber over $\overline{X} \times \{\nu\}$ is the line bundle associated to the character χ_ν^{-1}. Then $\langle \overline{\mathcal{L}}, \overline{\mathcal{L}} \rangle$ is also a holomorphic line bundle on $M_{dR}(\overline{X})$. By the Riemann-Hilbert correspondence, there are complex analytic isomorphisms: $M_{dR}(X) \xrightarrow{\sim} M_B(\Gamma) \xleftarrow{\sim} M_{dR}(\overline{X})$, where $M_B(\Gamma)$ is defined in (52). We therefore regard $\langle \mathcal{L}, \mathcal{L} \rangle$ and $\langle \overline{\mathcal{L}}, \overline{\mathcal{L}} \rangle$ as holomorphic bundles on $M_B(\Gamma)$. On $\overline{X}$, the imaginary part of the period matrix $\operatorname{Im}\Omega$ is unchanged, but the coordinates $(t_i, s_j) \mapsto (-s_j, -t_i)$. Hence, by (72),

$$F_{\nabla^{int}_{\langle \overline{\mathcal{L}}, \overline{\mathcal{L}}\rangle}} = \frac{2}{\pi} \sum_{i,j=1}^{g} \operatorname{Im} \Omega_{ij}(dt_i \wedge ds_j). \tag{73}$$

In particular, the intersection connection on $\langle \mathcal{L}, \mathcal{L} \rangle \otimes \langle \overline{\mathcal{L}}, \overline{\mathcal{L}} \rangle$ is flat! Next, we have

Lemma 5.9. *For any choice of theta characteristic κ (i.e. $2\kappa = \omega_X$), there is the following functorial isomorphism*

$$\left[\det R\pi_*(\mathcal{L} \otimes \kappa) \otimes \det R\pi_*(\kappa)^{-1}\right]^{\otimes 12} \xrightarrow{\sim} \langle \mathcal{L}, \mathcal{L} \rangle^{\otimes 6}. \tag{74}$$

Proof. From compatibility of the Deligne pairing with tensor products,

$$\begin{aligned}\langle \mathcal{L} \otimes \kappa, \mathcal{L} \otimes \kappa \otimes \omega_{\mathfrak{X}/S}^{-1} \rangle &\simeq \langle \mathcal{L} \otimes \kappa, \mathcal{L} \otimes \kappa^{-1} \rangle \simeq \langle \mathcal{L}, \mathcal{L} \otimes \kappa^{-1} \rangle \otimes \langle \kappa, \mathcal{L} \otimes \kappa^{-1} \rangle \\ &\simeq \langle \mathcal{L}, \mathcal{L} \rangle \otimes \langle \mathcal{L}, \kappa^{-1} \rangle \otimes \langle \kappa, \mathcal{L} \rangle \otimes \langle \kappa, \kappa^{-1} \rangle \simeq \langle \mathcal{L}, \mathcal{L} \rangle \otimes \langle \kappa, \kappa \rangle^{-1}.\end{aligned}$$

Similarly, $\langle \omega_{\mathfrak{X}/S}, \omega_{\mathfrak{X}/S} \rangle \simeq \langle \kappa, \kappa \rangle^{\otimes 4}$. By (1),

$$\det R\pi_*(\mathcal{L} \otimes \kappa)^{\otimes 12} \simeq \langle \omega_{\mathfrak{X}/S}, \omega_{\mathfrak{X}/S} \rangle \otimes \langle \mathcal{L} \otimes \kappa, \mathcal{L} \otimes \kappa \otimes \omega_{\mathfrak{X}/S}^{-1} \rangle^{\otimes 6}$$

$$\simeq \langle \kappa,\kappa\rangle^4 \otimes \langle \mathcal{L},\mathcal{L}\rangle^{\otimes 6} \otimes \langle \kappa,\kappa\rangle^{-6} \simeq \langle \mathcal{L},\mathcal{L}\rangle^{\otimes 6} \otimes \langle \kappa,\kappa\rangle^{-2}.$$

On the other hand, $\det R\pi_*(\kappa)^{\otimes 12} \simeq \langle \kappa,\kappa\rangle^{\otimes 4} \otimes \langle \kappa,\kappa^{-1}\rangle^{\otimes 6} \simeq \langle \kappa,\kappa\rangle^{-2}$. The result follows. □

Remark 5.10. There is a refinement of Deligne's isomorphism [13, Théorème 11.4]) to virtual bundles of virtual rank 0, such as $\mathcal{L}\otimes\kappa - \kappa$. In this case, the lemma can be refined to a more natural looking isomorphism, canonical up to sign: $\left[\det R\pi_*(\mathcal{L}\otimes\kappa)\otimes \det R\pi_*(\kappa)^{-1}\right]^{\otimes 2} \xrightarrow{\sim} \langle \mathcal{L},\mathcal{L}\rangle$. Consequently, the isomorphism of the lemma is canonical and there is no sign ambiguity (since we take the 6th power of the latter).

We may now give a geometric interpretation of the holomorphic extension of torsion. To simplify the notation, let $\lambda(X,\kappa) = \det R\pi_*(\mathcal{L}\otimes\kappa)\otimes\det R\pi_*(\kappa)^{-1}$. Considering both X and $\overline{X}$, by (74) we have a canonical isomorphism

$$\phi : \left[\lambda(X,\kappa)\otimes\lambda(\overline{X},\bar\kappa)\right]^{\otimes 12} \xrightarrow{\sim} \left[\langle \mathcal{L},\mathcal{L}\rangle \otimes \langle \overline{\mathcal{L}},\overline{\mathcal{L}}\rangle\right]^{\otimes 6}. \tag{75}$$

By (72) and (73), the intersection connections give a flat connection on right hand side of (75). On the other hand, $\lambda(X,\kappa)\otimes\lambda(\overline{X},\bar\kappa)$ has a canonical flat connection given by the form $-\partial \log T(\chi\otimes\kappa)$, in the canonical (up to a constant) frame determined by the relation with theta functions (see 67). With this understood, we have the following

Theorem 5.11. *The Deligne isomorphism ϕ in (75) is flat with respect to the connections defined above.*

Proof. We first show, by explicit calculation, that ϕ is flat when restricted to the unitary connections $U_{dR}(X)\subset M_{dR}(X)$. Recall from Lemma 4.5 that the connection on $\langle\mathcal{L},\mathcal{L}\rangle$ coincides with the Chern connection along $U_{dR}(X)$. The difference between the connection we have defined on $\lambda(X,\kappa)\otimes\lambda(\overline{X},\bar\kappa)$ and the Quillen connection is given by the $(1,0)$ form

$$\partial \log\left[\frac{T_X(\chi_u\otimes\kappa)T_{\overline{X}}(\chi_u^{-1}\otimes\bar\kappa)}{T(\chi\otimes\kappa)}\right]. \tag{76}$$

Since the Deligne isomorphism (for the Quillen metric) is an isometry, it suffices to show that the expression in (76) vanishes when χ is unitary. Let $\nabla = d + B$ be a flat connection, where $B = \sum_{i=1}^g t_i\omega_i + s_i\bar\omega_i$. Let $\chi_B\in M_{dR}(X)$ be the associated character. Notice that

$$2\pi i a_j = \int_{A_j} B = t_j + s_j\ ,\ 2\pi i b_j = \int_{B_j} B = \sum_{k=1}^g t_k\Omega_{kj} + s_k\overline{\Omega}_{kj}\ .$$

From this expression and the interchange $\Omega\mapsto -\overline{\Omega}$, we see that the character χ^{-1} on $\overline{X}$ corresponds to the change of coordinates $(t_j,s_j)\mapsto(-s_j,-t_j)$.

Next, consider the map $M_{dR}(X)\to J(X)$. This takes $[\nabla]$ to the isomorphism class $[\nabla'']$ of the underlying holomorphic line bundle. In terms of flat connections, $[d+B] \mapsto [d+B''-\overline{B''}]$. Rewriting (66) and (67) in the coordinates above, we have

$$T_X(\chi_u\otimes\kappa) = C(X)\exp\left((1/2\pi)(s-\bar{s})^T(\operatorname{Im}\Omega)(s-\bar{s})\right)|\vartheta|^2((1/\pi)(\operatorname{Im}\Omega)s,\Omega)$$

$$T_{\overline{X}}(\chi_u^{-1}\otimes\overline{\kappa}) = C(X)\exp\left((1/2\pi)(t-\bar{t})^T(\operatorname{Im}\Omega)(t-\bar{t})\right)|\vartheta|^2((1/\pi)(\operatorname{Im}\Omega)t,-\overline{\Omega})$$

$$T(\chi) = C(X)\exp\left((1/2\pi)(t+s)^T(\operatorname{Im}\Omega)(t+s)\right)\vartheta((1/\pi)(\operatorname{Im}\Omega)s,\Omega)\vartheta((1/\pi)(\operatorname{Im}\Omega)t,-\overline{\Omega}).$$

We now calculate:

$$\partial_\chi \log T(\chi_u\otimes\kappa) = \frac{1}{\pi}\sum_{i,j=1}^{g}(\operatorname{Im}\Omega)_{ij}(s_i-\bar{s}_i)ds_j + \frac{1}{\pi}\sum_{i,j=1}^{g}\partial_{Z_i}\vartheta((1/\pi)(\operatorname{Im}\Omega)s,\Omega)(\operatorname{Im}\Omega)_{ij}ds_j$$

$$\partial_\chi \log T_{\overline{X}}(\chi_u^{-1}\otimes\overline{\kappa}) = \frac{1}{\pi}\sum_{i,j=1}^{g}(\operatorname{Im}\Omega)_{ij}(t_i-\bar{t}_i)dt_j + \frac{1}{\pi}\sum_{i,j=1}^{g}\partial_{Z_i}\vartheta((1/\pi)(\operatorname{Im}\Omega)t,-\overline{\Omega})(\operatorname{Im}\Omega)_{ij}dt_j$$

$$\begin{aligned}\partial_\chi \log T(\chi\otimes\kappa) = \frac{1}{\pi}\sum_{i,j=1}^{g}(\operatorname{Im}\Omega)_{ij}(t_i+s_i)(dt_j+ds_j) &+ \frac{1}{\pi}\sum_{i,j=1}^{g}\partial_{Z_i}\vartheta((1/\pi)(\operatorname{Im}\Omega)s,\Omega)(\operatorname{Im}\Omega)_{ij}ds_j\\ &+ \frac{1}{\pi}\sum_{i,j=1}^{g}\partial_{Z_i}\vartheta((1/\pi)(\operatorname{Im}\Omega)t,-\overline{\Omega})(\operatorname{Im}\Omega)_{ij}dt_j.\end{aligned}$$

Hence, restricted to the unitary connections $U_{dR}(X)\subset M_{dR}(X)$ defined by $t_i=-\bar{s}_i$ (see Remark 5.2),

$$\partial_\chi \log T(\chi\otimes\kappa) = \partial_\chi \log T(\chi_u\otimes\kappa) + \partial_\chi \log T_{\overline{X}}(\chi_u^{-1}\otimes\overline{\kappa})$$

It follows that $\nabla\phi$ restricted to $U_{dR}(X)$ vanishes. Now there is a $(1,0)$ form Ω such that for any local section σ of $\left[\lambda(X,\kappa)\otimes\lambda(\overline{X},\bar{\kappa})\right]^{\otimes 12}$, $(\nabla\phi)(\sigma)=\Omega\cdot\phi(\sigma)$. Moreover, since the connections defined on the left and right hand side of (75) are flat, Ω is closed. We may therefore locally write $\Omega=\partial f$ for a holomorphic function f, where by the result above, $f|_{U_{dR}(X)}$ is constant. Next, we appeal to the following standard result.

Lemma 5.12. *Let $U\subset M$ be a totally real submanifold of a connected complex manifold M with* $\dim_{\mathbb{R}} U = \dim_{\mathbb{C}} M$ *(see Definition 5.3). Then a holomorphic function on M that is constant on U is constant.*

Since $U_{dR}(X)$ is totally real (Remark 5.2), we conclude from the lemma that f is constant, and so $\nabla\phi \equiv 0$. □

Remark 5.13. An analogous result to Theorem 5.11 holds for the holomorphic torsion $T(\chi)$ and the determinant bundle $\det R\pi_*(\mathcal{L})$. The idea of the proof is the same, where the calculation making use of (71) is somewhat more lengthy.

5.4. **The hyperholomorphic line bundle on twistor space.** In this section we show how the intersection connection leads quite naturally to the construction of a meromorphic connection on the hyperholomorphic line bundle over the twistor space of $M_{dR}(X)$. This result is inspired by Hitchin's exposition in [23, 24], to which we refer for more context and detail.

We begin with a quick review of the basic set-up. Recall that $M_{dR}(X)$ has a hyperkähler structure (for much more on this, see [21]). In terms of the coordinates introduced above, the symplectic structures are:

$$\Phi_1 = \frac{i}{2\pi}\sum_{i,j=1}^{g} \operatorname{Im}\Omega_{ij}(dt_i\wedge d\bar{t}_j + ds_i\wedge d\bar{s}_j)$$
$$\Phi_2 + i\Phi_3 = \frac{1}{\pi}\sum_{i,j=1}^{g} \operatorname{Im}\Omega_{ij} ds_i\wedge dt_j.$$

Let $Z = M_{dR}(X)\times\mathbb{P}^1$ denote the twistor space of $M_{dR}(X)$, and $\lambda : Z \to \mathbb{P}^1$ the projection. Then Z has the structure of a complex manifold with respect to which λ is holomorphic, but the tautological complex structure is not a product. The fiber $\lambda^{-1}(1)$ is biholomorphic to $M_{dR}(X)$, whereas the fiber $\lambda^{-1}(0)$ is biholomorphic to $T^*J(X)$, the space of rank 1 Higgs bundles on X. Similarly, $\lambda^{-1}(\infty)\simeq T^*J(\overline{X})$. Each fiber has a holomorphic symplectic form given by

$$\Phi = \Phi_2 + i\Phi_3 + 2i\lambda\Phi_1 + \lambda^2(\Phi_2 - i\Phi_3) \tag{77}$$

(see [25, Theorem 3.3]).

Next, recall the following (see [33]).

Definition 5.14 (Deligne)**.** Let S be smooth algebraic, and set $\mathcal{X} = X\times S$. Suppose we are given a function $\lambda : S\to\mathbb{A}^1$. Then a *$\lambda$-connection* on a line bundle $\mathcal{L}\to\mathcal{X}$ is a $\mathbb{C}$-linear map $\nabla_\lambda : \mathcal{L}\to\mathcal{L}\otimes\Omega^1_{\mathcal{X}/S}$ of $\mathcal{O}_{\mathcal{X}}$-modules satisfying $\nabla_\lambda(f\ell) = \lambda df\otimes\ell + f\cdot\nabla_\lambda\ell$, for $f\in\mathcal{O}_{\mathcal{X}}$ and $\ell\in\mathcal{L}$.

By a result of Simpson, the functor which associates to $\lambda : S\to\mathbb{A}^1$ the set of rank one λ-connections on $\mathcal{X}$ is representable by a scheme $M_{Hod}(X)$ with a morphism $\lambda : M_{Hod}(X)\to\mathbb{A}^1$. By considering $M_{Hod}(\overline{X})$ and a gluing procedure with respect to the anti-holomorphic involution $\lambda\mapsto -\bar{\lambda}^{-1}$, one constructs the *Deligne moduli space of λ-connections* $\lambda : M_{Del}(X)\to\mathbb{P}^1$. Moreover, there is a biholomorphism $M_{Del}(X)\simeq Z$. This is achieved by finding holomorphic sections $\mathbb{A}^1\to M_{Hod}(X)$ of λ, compatible with the anti-holomorphic involution. For example, in the case of the flat connection $\nabla = d + \nu$, ν harmonic, the family of λ-connections is given by:

$$\begin{aligned}\nabla^{0,1}_\lambda &= \bar\partial + \frac{1}{2}((\lambda+1)\nu'' + (\lambda-1)\overline{\nu'})\\ \nabla^{1,0}_\lambda &= \lambda\partial + \frac{1}{2}((1+\lambda)\nu' + (1-\lambda)\overline{\nu''}).\end{aligned} \tag{78}$$

Let $\mathcal{X} = X \times M_{Del}(X)$, $\pi : \mathcal{X} \to M_{Del}(X)$ the projection. We furthermore assume a rigidification. Then the universal bundle $\mathcal{L} \to \mathcal{X}$ admits a universal λ-connection. Let κ be a theta characteristic as in the previous section, and use the same notation for the pull-back to $\mathcal{X} \to X$. We define the *hyperholomorphic line bundle* on $M_{Del}(X)$ by

$$\mathcal{L}_Z := \det R\pi_*(\mathcal{L}\otimes\kappa)\otimes \det R\pi_*(\kappa)^{-1}. \tag{79}$$

Consider the divisor $D = D_0 \cup D_\infty = \lambda^{-1}(0)\cup\lambda^{-1}(\infty)$. We shall use the construction of this paper to obtain an explicit realization of the following property of the hyperholomorphic line bundle (Theorem 1.3 of the introduction).

Theorem 5.15 (Hitchin, *cf.* [24, Theorem 3])**.** *The line bundle $\mathcal{L}_Z$ admits a meromorphic connection with logarithmic singularities along the divisor D. The curvature of this connection restricted to the fibers of $Z - D \to \mathbb{C}^\times$ is $\lambda^{-1}\Phi$, where Φ is the HKLR form (77). The residue of the connection at $\lambda = 0$ (resp. $\lambda = \infty$) is the Liouville or tautological 1-form on $T^*J(X)$ (resp. $T^*J(\overline{X})$).*

Proof. There is a holomorphic map $Z - D \to M_{dR}(X)$ obtained by sending a holomorphic bundle with λ-connection ∇_λ to the same holomorphic bundle with holomorphic connection $\lambda^{-1}\nabla_\lambda$. By Remark 5.10, $\mathcal{L}_Z^{\otimes 12}$ is naturally isomorphic to the pull-back of $\langle\mathcal{L},\mathcal{L}\rangle^{\otimes 6}$, and therefore the pull-back of the intersection connection gives a holomorphic connection on $\mathcal{L}_Z$ over $Z - D$. The statement about the curvature follows from the fact that the HKLR form is the pull-back of the holomorphic symplectic form on $M_{dR}(X)$. We shall verify this directly using the coordinates above. Let (τ_i, σ_i) be holomorphic coordinates on $M_{dR}(X)$. It will be convenient to locally parametrize $Z - D_\infty$ by (t_i, s_i, λ), where (t_i, s_i) are the holomorphic coordinates on $T^*J(X)$, and the λ-connection is given by, $\nabla^{0,1}_\lambda = \bar\partial + a'' + \lambda\psi''$, $\nabla^{1,0}_\lambda = \lambda(\partial + a') + \psi'$. Here, $a'' = \sum_{i=1}^g s_i\bar\omega_i$, $a' = -\overline{a''}$, $\psi' = \sum_{i=1}^g t_i\omega_i$, $\psi'' = \overline{\psi'}$. In these coordinates, the map $Z - D \to M_{dR}(X)$ is given by $\tau_i = -\bar s_i + \lambda^{-1}t_i$, $\sigma_i = s_i + \lambda\bar t_i$. Then

$$d\tau_i = -d\bar s_i + \lambda^{-1}dt_i - \lambda^{-2}t_i d\lambda\ ,\ d\sigma_i = ds_i + \lambda d\bar t_i + \bar t_i d\lambda\ ,$$

from which

$$\begin{aligned}d\tau_i\wedge d\sigma_j = {}& ds_i\wedge d\bar s_j + dt_i\wedge d\bar t_j + \lambda^{-1}dt_i\wedge ds_j - \lambda d\bar s_i\wedge d\bar t_j\\ &+ \bar t_j(-d\bar s_i + \lambda^{-1}dt_i)\wedge d\lambda + \lambda^{-2}t_i(ds_j + \lambda d\bar t_j)\wedge d\lambda.\end{aligned}$$

Using (72), it follows that restricted to the fibers,

$$F_{\mathcal{L}_Z}\Big|_{\text{fiber}} = 2i\Phi_1 + \lambda^{-1}(\Phi_2 + i\Phi_3) + \lambda(\Phi_2 - i\Phi_3).$$

For the residue at $\lambda = 0$, note that from (64),

$$\frac{d\tilde{\ell}}{\tilde{\ell}} = (d\tau)^t \int_\sigma^t \vec{\omega} + \tau^t d\int_\sigma^z \vec{\omega} + \text{ regular terms}$$

while $-\int_\sigma^z \nabla_{\mathrm{GM}}\nu = -(d\tau)^t \int_\sigma^z \vec{\omega} - (d\sigma)^t \int_\sigma^z \overline{\vec{\omega}}$. It follows that

$$\begin{aligned}\mathrm{tr}_{\operatorname{div} m/S}\left(\frac{\nabla\ell}{\ell}\right) &= \tau^t d\left(\mathrm{tr}_{\operatorname{div} m/S}\int_\sigma^z \vec{\omega}\right) + \text{ regular terms} \\ &= -\frac{\lambda^{-1}}{\pi}\sum_{i,j=1}^{g}(\operatorname{Im}\Omega)_{ij}t_i ds_j + \text{ regular terms}\end{aligned}$$

The residue of the connection at ∞ is calculated similarly. This concludes the proof. ∎

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