J. reine angew. Math., Galley Proof, 45 pages
DOI 10.1515/crelle–

Journal für die reine und angewandte Mathematik

A Hitchin connection on nonabelian theta functions for parabolic G-bundles

By *Indranil Biswas* at Greater Noida, *Swarnava Mukhopadhyay* at Mumbai and *Richard Wentworth* at College Park

Abstract. For a simple, simply connected complex affine algebraic group G, we prove the existence of a flat projective connection on the bundle of nonabelian theta functions on the moduli spaces of semistable parabolic G-bundles for families of smooth projective curves with marked points.

1. Introduction

In this paper, we prove the existence of a flat projective connection on spaces of generalized theta functions on the moduli spaces of parabolic H-bundles for a family of smooth projective curves with marked points, where H is a connected, complex, simple, affine algebraic group. Before stating the precise results, and since it is part of the larger and well-studied program of geometric quantization, we first provide a brief historical context to this subject.

Quantization as envisioned by Dirac, et al., can be thought of as a deformation of a classical mechanical system depending on a parameter $\hbar$ that recovers the original classical system in the limit. Kostant-Kirillov-Souriau developed and generalized this notion of "quantizing a function", and Auslander-Kostant [6] used it to construct unitary representations of a connected Lie group (see also Kirillov [45]).

Geometric Quantization. The starting point of the theory is a symplectic manifold (M, ω) where the symplectic form ω is the curvature of a Hermitian line bundle $\mathcal{L}$ with connection ∇. The quantum Hilbert space $\mathscr{H}$ is then the L^2-completion of the space of global sections $\Gamma(M, \mathcal{L})$ of this line bundle. The Lie algebra of functions on M, under the Poisson bracket given by the form ω, acts naturally on $\mathscr{H}$. This process of assigning a function to this Lie algebra satisfying certain commutativity constraints depending on $\hbar$ is known as quantization in the present literature. However, it is not possible to achieve these commutativity constraints in practice. To remedy this, Kostant [46] and Souriau [68] further consider a

I.B. and S.M. are supported in part by a J.C. Bose fellowship and by DAE, India under project no. 1303/3/2019/R&D/IIDAE/13820. S.M. received additional funding from the Science and Engineering Research Board, India (SRG/2019/000513). R.W. is supported in part by National Science Foundation grant DMS-1906403.

compatible almost complex structure I on M such that (M, ω, I) is a Kähler manifold. This induces a holomorphic structure on the line bundle $\mathcal{L}$ and leads to the notion of *geometric quantization*, where the Hilbert space $\mathscr{H}_I$ is reduced to the space of holomorphic L^2-sections of $\mathcal{L}$. Because the quantization process should arrive at a unique answer, it is natural to investigate the dependence of the geometric quantization on the choice of almost complex structure I on M.

In [43], Hitchin analyzes this question in a very important setting (see also [7], [33], [35], [2]). Here, $M = \mathrm{Hom}^{irr}(\pi_1(\Sigma), K)/K$, is the moduli space of a class of representations of the fundamental group $\pi_1(\Sigma)$ to K, where Σ is a closed oriented surface and $K \subset G$ is a maximal compact subgroup of the earlier mentioned simple, simply connected group G. The group K acts by conjugation on a representation $\rho : \pi_1(\Sigma) \to K$, and $\rho \in \mathrm{Hom}^{irr}(\pi_1(\Sigma), K)$ if the stabilizer of ρ under this action is exactly the center of K. This space has a symplectic form defined by Atiyah-Bott [5], Narasimhan [54], and Goldman. A choice of a complex structure I on Σ endows M with a Kähler structure, and via the Narasimhan-Seshadri-Ramanathan theorem this complex manifold, which we call M_I, can be identified with the space of regularly stable holomorphic principal G-bundles on $C := (\Sigma, I)$ (see [61, Prop. 7.7 and Thm. 7.1]). The role of $\mathcal{L}$ is played by a determinant of cohomology line bundle defined via some linear representation of G, and $\mathscr{H}_I := H^0(M_I, \mathcal{L}^{\otimes k})$ is the space of nonabelian theta functions of level k. The connection ∇ is the Chern connection of the Quillen metric. Hitchin found a flat projective connection on the bundle of nonabelian theta functions over a family of curves of fixed genus. His construction may be interpreted as a natural identification between the spaces $\mathbb{P}(H^0(M_I, \mathcal{L}^{\otimes k}) \cong \mathbb{P}(H^0(M_{I'}, \mathcal{L}^{\otimes k})$ via parallel transport along a path connecting I and I' in the Teichmüller space.

TUY/WZW connection. As mentioned above, the vector spaces $\mathscr{H}_I$ that appear in Hitchin's geometric quantization have a counterpart in the WZNW-model of a $2d$ rational conformal field theory constructed by Tsuchiya-Ueno-Yamada [74], which appears in the quantization of a $3d$-Chern-Simons theory to a $3d$-TQFT as considered by Witten [77]. Let $\mathfrak{g}$ denote the Lie algebra of G. Given a positive integer k and an n-tuple $\boldsymbol{\lambda}$ of dominant weights for $\mathfrak{g}$ satisfying a certain integrability condition depending on k, the paper [74] constructs a vector bundle $\mathbb{V}^{\dagger}_{\boldsymbol{\lambda}}(\mathfrak{g}, k)$ on the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g. Over the interior $\mathcal{M}_{g,n}$ parametrizing smooth curves, $\mathbb{V}^{\dagger}_{\boldsymbol{\lambda}}(\mathfrak{g}, k)$ admits a flat projective connection. These vector bundles of conformal blocks satisfy the axioms of a $2d$-rational conformal field theory. Moreover, due to work of Beauville-Laszlo [12] and Kumar-Narasimhan-Ramanathan [48], in the case of a single puncture with trivial weight, we get a canonical (up to a scalar) identification of $\mathscr{H}_I$ with the fiber of $\mathbb{V}^{\dagger}_{\boldsymbol{\lambda}}(\mathfrak{g}, k)$ at the point $C = (\Sigma, I)$ in $\mathcal{M}_{g,n}$. It is natural to ask whether the connections of Hitchin [43] and Tsuchiya-Ueno-Yamada [74] coincide. That this is indeed the case was proven by Laszlo [49].

A generalization of the identification of $\mathscr{H}_I$ with conformal blocks also holds for smooth C with an n-tuple of marked points $\boldsymbol{p}$. Consider the moduli space $M_G^{par,rs} = M_G^{par,rs}(C, \boldsymbol{p}, \boldsymbol{\lambda})$ of regularly stable parabolic G bundles on a compact Riemann surface C with n-marked points $\boldsymbol{p}$ and parabolic structures $\boldsymbol{\lambda}$ at $\boldsymbol{p}$. Let $\mathcal{L}_{\boldsymbol{\lambda},k}$ be a parabolic "determinant of cohomology" line bundle on $M_G^{par,rs}$. Then there is a canonical (up to scalars) isomorphism between the finite dimensional vector space of holomorphic sections $H^0(M_G^{par,rs}, \mathcal{L}_{\boldsymbol{\lambda},k})$ and the fiber of the space of conformal blocks $\mathbb{V}^{\dagger}_{\boldsymbol{\lambda}}(\mathfrak{g}, k)\big|_{(C,\boldsymbol{p})}$ (see [58] and [50]). This identification between conformal blocks and nonabelian theta functions is a mathematical analog of the Chern-Simons/WZNW correspondence of Witten [77]. Since the vector bundle of conformal blocks is endowed with a flat projective connection, it is very natural to ask the following question:

Question. *Is there a natural flat projective connection on the family of spaces $H^0(M_G^{par,rs}, \mathcal{L}_{\boldsymbol{\lambda},k})$ as the pointed Riemann surface structure of C moves in a holomorphic family?*

For parabolic vector bundles, a construction of the projectively flat connection was given by Scheinost-Schottenloher in [64] for those *special cases* of weights $\boldsymbol{\lambda}$ such that the canonical class of the corresponding parabolic moduli space, which depends only on the rank, number of points, and the flag types of $\boldsymbol{\lambda}$, admits a square root. This condition often appears in the context of geometric quantization under the term *metaplectic correction* (see also [4]) and it produces a projective connection on the push-forward of the line bundle obtained by modifying $\mathcal{L}_{\boldsymbol{\lambda},k}$ by the square root. The proof in the above reference makes use of a correspondence between parabolic bundles on a curve with rational weights, and holomorphic bundles on an associated elliptically fibered complex surface. However, for moduli spaces of parabolic bundles, the condition on the existence of a square root of the canonical bundle is *not always satisfied*.

In [26], Bjerre proved the existence of a (unique) flat projective connection for the moduli space of parabolic vector bundles via a gauge theoretic description of the moduli space. An important step in the proof was to remove the condition on the existence of a square root by passing to a different moduli space with altered weights.[1)] [2)] The results of Bjerre, Scheinost-Schottenloher, and Zakaria stated above work only for curves of genus $g \geq 2$ and exclude the important case of genus zero curves with marked points. The connection on conformal blocks for genus zero curves is known as the Knizhnik-Zamolodchikov connection, and it has been extensively studied from different perspectives.

The motivation of the present paper is to give an affirmative answer to the above question for general G and curves of all genus using algebro-geometric methods applied directly to the moduli spaces in question. To state the result precisely, first note that the curve C and parabolic weights $\boldsymbol{\lambda}$ determine an orbifold curve $\mathscr{C}$ (cf. Appendix C and Lemma C.1). Our main result is the following:

Main Theorem. *Let $\mathcal{C} \to S$ be a versal family of n-pointed smooth projective curves, and let G be a simple, simply connected complex algebraic group. Assume that the genus $g(\mathscr{C})$ of the orbifold curve determined by the weights $\boldsymbol{\lambda}$ satisfies $g(\mathscr{C}) \geq 2$, and if $G = \mathsf{SL}_2$ or Sp_4, $g(\mathscr{C}) \geq 3$. Let $\pi : M_G^{par,rs} \to S$ be the relative moduli space of regularly stable parabolic G bundles on $\mathcal{C}$ for some fixed parabolic weights $\boldsymbol{\lambda}$. Let $\mathcal{L}_\phi$ be the determinant of cohomology line bundle on $M_G^{par,rs}$ determined by a choice of representation $\phi : G \to \mathsf{SL}_r$. Then for any $a \in \mathbb{Q}$, for which $\mathcal{L}_\phi^{\otimes a}$ defines a line bundle on $M_G^{par,rs}$, the coherent sheaf $\pi_*(\mathcal{L}_\phi^{\otimes a})$ has a natural flat projective connection.*

Observe that we can allow the genus of C to be zero or one in the above theorem, provided some inequalities are satisfied (cf. Example B.2 below). It is reasonable to expect that the TUY connection for conformal blocks and the parabolic Hitchin connection constructed in the Main Theorem coincide under the identification mentioned above. We postpone this question for a future work.

[1)] After the present paper was posted to the arXiv we received a preliminary version of the work of Andersen-Bjerre attributed here [3].

[2)] Subsequent to the submission of this paper, in May 2023 a draft of the thesis of Zakaria Ouaras [56] appeared in which the author proves the existence of a unique flat projective connection in the case of moduli spaces of parabolic vector bundles with arbitrary fixed determinant and genus $g \geq 2$.

Key difference in the parabolic case. Before proceeding further, we describe the key difference in the parabolic set-up. The moduli space of principal G-bundles satisfies a "monotone" condition: the first Chern class of the moduli space is a multiple of the Chern class of the prequantum line bundle. This property is an important technical point in Hitchin's construction of the connection (cf. [43, eqs. (2.8) and (3.9)]), and it leads to a solution to the van Geemen-de Jong condition in Theorem 2.2 (i) below.

The main new feature in the case of *parabolic bundles* is the higher rank of the Picard group of the moduli space, and because of this *monotonicity no longer holds*.

Main Ideas. The key ideas and methods used this paper to address the lack of monotonicity mentioned above are the following:

- The fiducial symbol coming from the usual construction of Hitchin connection can be naturally modified to a new condition that now satisfies the van Geemen-de Jong condition (see (50)).
- This modification is facilitated by another crucial ingredient, which is a categorical equivalence of "π-bundles" on a ramified cover $\widehat{C} \to C$ with parabolic bundles on C ([66], [19], [8] and [67]).
- We prove and use an equivariant analog of a result of Beilinson-Schechtman [16] connecting classes of *Atiyah algebras* obtained as equivariant push-forwards of a differential graded Lie algebra with those associated to the determinant of cohomology of the universal bundle.
- Finally we use the fact that the line bundles on moduli space of parabolic bundles adapted to the parabolic weights correspond exactly to the restriction of the determinants of cohomology to the locus of orbifold bundles (cf. [24], [28]).

We now discuss some applications of the main theorem mentioned above. Let H be a simple algebraic group with nontrivial fundamental group, and let $\widetilde{H}$ be its simply connected cover. Let $\pi : M_H^{par,rs,0} \to S$ be the neutral component of the relative moduli space of regularly stable parabolic H bundles on $\mathcal{C} \to S$ for some fixed parabolic weights $\boldsymbol{\lambda}$, which we assume lift to weights for $\widetilde{H}$. As before, let $\mathcal{L}_{\boldsymbol{\lambda},k}$ be the parabolic determinant of cohomology. It is natural to ask whether the coherent sheaf $\pi_*\mathcal{L}_{\boldsymbol{\lambda},k}$ carries a projectively flat connection. A direct corollary of the main theorem is the following:

Corollary 1.1. *For any simple group H, the coherent sheaf $\pi_*\mathcal{L}_{\boldsymbol{\lambda},k}$ is locally free and carries a flat projective connection whose symbol is the same that for that for the simply connected cover $\widetilde{H}$.*

Observe that moduli spaces of parabolic bundles are not necessarily Fano, and hence we cannot use a Grauert-Riemenschneider type vanishing theorem as in the nonparabolic case to conclude local freeness via vanishing of higher cohomologies. Furthermore, since H is not simply connected, we cannot reconstruct these space via affine Lie algebraic methods.

We now briefly recall the earlier constructions of the Hitchin/WZW/TUY connections in the nonparabolic setting. Hitchin's construction of a projective connection in the closed (nonparabolic) case draws parallels with Welters' work on theta functions for abelian varieties [76]. The starting point is the description of first order deformations of the triple $(M_I, \mathcal{L}^{\otimes k}, s)$, where $s \in H^0(M_I, \mathcal{L}^{\otimes k})$, in terms of the first hypercohomology group of the complex $\mathrm{At}(\mathcal{L}^{\otimes k}) \to \mathcal{L}^{\otimes k}$ constructed using s. Here, $\mathrm{At}(\mathcal{L})$ denotes the Atiyah algebra of $\mathcal{L}$. Though Hitchin's methods were differential geometric in nature, in [35] van Geemen and de Jong reinterpreted the construction in an algebraic manner closer to that of [76]. Using this framework, along with the fundamental results of Beilinson-Schechtman [16] and Bloch-Esnault [27], Baier-Bolognesi-Martens-Pauly [59] reproduced Hitchin's connection for $G = \mathsf{SL}_r$. Moreover, their proof works over fields of positive characteristic, with a few extra assumptions.

The Hitchin connection for $G = \mathsf{GL}_r$ bundles had previously been found by Belkale [15]. Other algebro-geometric constructions of the Hitchin connection are given in [33], [63], [60], and by Ginzburg in [36]. Ref. [70] uses the results of [16] to extend Hitchin's connection for logarithmic connections and the moduli space of semistable torsion-free sheaves on nodal curves. The approach in the present paper is strongly motivated by [59] and [36].

Further generalizations. In fact, it is possible to work in the general setting of Γ-$\mathrm{Aut}(G)$-bundles. A moduli space of such pairs with a fixed local type has been constructed by Balaji-Seshadri [9] (in the case of Γ-G-bundles in characteristic zero) and by Heinloth [40] (in the more general settings of Bruhat-Tits torsors in the sense of Pappas-Rapoport [57], and over fields of arbitrary characteristic). We note that it has been not verified whether the stability conditions of [9] and [40] coincide. Nevertheless, the results in Section 3 generalize verbatim to moduli spaces of Γ-$\mathrm{Aut}(G)$-bundles. However, in order to produce a Hitchin connection (as described in Section 5), the following additional information would be required:

- the base of the Hitchin map for the moduli of parahoric Higgs bundles for $(\Gamma, \mathrm{Aut}(G))$ is affine, and the fibers of the Hitchin map are connected;
- the complement of the cotangent bundle of the moduli space of Γ-$\mathrm{Aut}(G)$-bundles in the parahoric Higgs bundles moduli space has codimension at least 2.

There are some results in the direction of the first point by B. Wang [75], who extends the result of Donagi-Pantev [29] to the set-up of parahoric Γ-G-Higgs bundles. In full generality, however, the two items above are not presently available in the literature, and we therefore restrict ourselves here to the setting of parabolic bundles.

Outlook. The paper [59] cited above argues that it is of independent interest to consider the Hitchin connection over field of positive characteristics from the view point of the Grothendieck-Katz p-curvature conjecture and the modular representations of the mapping class group. The constructions in this paper follow those of [59] and are likely to work (after suitable modifications of the techniques used here) over fields of characteristic $p > 0$, unless $p \in \{2, 3, h^\vee(\mathfrak{g}), k, k + h^\vee\}$. But even given these constraints on p it is not clear whether $\pi_* \mathcal{L}_{\boldsymbol{\lambda},k}$ is locally free. For this, it would be enough to show that $H^1(M_G^{par,ss}, \mathcal{L}_{\boldsymbol{\lambda},k})$ vanishes. However, in the parabolic case the moduli spaces $M_G^{par,ss}$ are not Fano in general, even in characteristic zero. Moreover, there is no suitable Grauert-Riemenschneider vanishing theorem.

A uniform approach to this vanishing result would follow if one can show that $M_G^{par,ss}$ are Frobenius-split. There is some work in this direction for $G = \mathsf{SL}_2$ by Mehta-Ramadas [52] and by Sun-Zhou [71], who show that semistable parabolic bundles of rank r and fixed determinant are globally F-regular type. A general result on Frobenius splitting for moduli of parabolic bundles is presently missing in the literature.

Organization. This paper is organized as follows: In Section 1, we review the construction of the projectively flat connection in the general set-up following Hitchin [43] and van Geemen-de Jong [35]. In Section 3, we review the generalizations of Hitchin's symbol and Kodaira-Spencer maps in the parabolic bundle context. The important result here is Theorem 3.3, which relates the fiducial Hitchin symbol to the relative extension classes of the Atiyah algebras of the G-bundle and the determinant of cohomology.

Finally, in Section 5 we prove that the modified Hitchin symbol satisfies the constraint equations of van Geemen-de Jong. This leads to the proof of the Main Theorem. The last three sections contains some definitions and technical results on parabolic bundles, invariant push-forwards, and vanishing theorems, that are used at various points in the paper. In particular, the determinant of cohomology line bundles $\mathcal{L}_\phi$ associated to a linear representation ϕ of G are defined there. *Parabolic* determinant of cohomology line bundles are defined in A.12 and A.16. We also explain the admissible values of k, how to realize the parabolic determinant of cohomology bundles via the moduli space of Γ-G-bundles, and the invariant push-forward functor construction.

For the rest of the paper we emphasize that the ground field of varieties and schemes is always $\mathbb{C}$, and we shall freely go back and forth between Zariski and analytic topologies.

2. Flat projection connection following Hitchin–van Geemen–de Jong

Let $\pi : M \to S$ be a smooth surjective proper map of smooth varieties with connected fibers and $\mathcal{L} \to M$ a line bundle. In this section we briefly recall a general approach for constructing connections on the coherent sheaf $\pi_*\mathcal{L}$. This is due to Hitchin [43] in the Kähler setting (generalizing Welters [76]) and to van Geemen–de Jong [35] in the algebro-geometric setting.

2.1. Heat operators. From [35, Sec. 2.3] we recall the notion of a heat operator and associated connections. For $i \geq 1$, let $\mathcal{D}^{\leq i}(\mathcal{L})$ (resp. $\mathcal{D}^{\leq i}_{M/S}(\mathcal{L})$) denote the sheaf of differential operators (resp. relative differential operators) of order at most i on the line bundle $\mathcal{L}$.

Consider the subsheaf $\mathcal{W}_{M/S}(\mathcal{L}) = \mathcal{D}^{\leq 1}(\mathcal{L}) + \mathcal{D}^{\leq 2}_{M/S}(\mathcal{L})$ of the sheaf of second order differential operators on $\mathcal{L}$. It fits into the following short exact sequence:

$$0 \to \mathcal{D}^{\leq 1}_{M/S}(\mathcal{L}) \to \mathcal{W}_{M/S}(\mathcal{L}) \to \pi^*\mathcal{T}_S \oplus \mathrm{Sym}^2\,\mathcal{T}_{M/S} \to 0\,. \tag{1}$$

Note that $\mathcal{O}_S \subset \mathcal{D}^{\leq 1}_{M/S}(\mathcal{L}) \subset \mathcal{W}_{M/S}(\mathcal{L})$.

Definition 2.1. *A heat operator D on $\mathcal{L}$ is a map $D : \pi^*\mathcal{T}_S \to \mathcal{W}_{M/S}(\mathcal{L})$ whose composition with the natural projection map $\mathcal{W}_{M/S}(\mathcal{L}) \to \pi^*\mathcal{T}_S$, given by* (1)*, is the identity map of $\pi^*\mathcal{T}_S$. A projective heat operator $\overline{D}$ on $\mathcal{L}$ is an $\mathcal{O}_S$-linear map $\overline{D} : \mathcal{T}_S \to (\pi_*\mathcal{W}_{M/S}(\mathcal{L}))/\mathcal{O}_S$ such that any local lifting gives a heat operator.*

Given a heat operator D, we can construct a connection $\nabla(D) : \pi_*\mathcal{L} \to \pi_*\mathcal{L} \otimes \Omega^1_S$ on the coherent sheaf $\pi_*\mathcal{L}$ as follows: Let $\theta \in \mathcal{T}_S(U)$, where $U \subset S$ an open subset. Then by definition, $D(\pi^{-1}\theta)$ is a second order differential operator on $\mathcal{L}(\pi^{-1}(U))$. Let s be a section of $\pi_*\mathcal{L}(U)$ and $f \in \mathcal{O}_S(U)$. Then $D(\pi^{-1}\theta)((f \circ \pi)s) = f \cdot D(\pi^{-1}\theta)(s) + \theta(f) \cdot s$, in other words, $D(\pi^{-1}\theta)$ satisfies the Leibniz rule. Indeed, this follows from the requirement in

Definition 2.1 that the heat operator is the standard first order operator on the base. Hence, we get a connection $\nabla(D)$.

2.2. Existence of a heat operator. The Kodaira-Spencer map is given by:

$$KS_{M/S} : \mathcal{T}_S \longrightarrow R^1\pi_*\mathcal{T}_{M/S}.$$

On the other hand, we have the coboundary map

$$\mu_{\mathcal{L}} : \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow R^1\pi_*\mathcal{T}_{M/S},$$

occurring in the long exact sequence obtained from the push forward π_* of the fundamental short exact sequence of differential operators

$$0 \longrightarrow \mathcal{T}_{M/S} \cong \mathcal{D}^{\leq 1}_{M/S}(\mathcal{L})/\mathcal{O}_M \longrightarrow \mathcal{D}^{\leq 2}_{M/S}(\mathcal{L})/\mathcal{O}_M \xrightarrow{s_2} \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow 0\ ,$$

where s_2 is the symbol map. Given $\rho : \mathcal{T}_S \to \pi_*(\operatorname{Sym}^2 \mathcal{T}_{M/S})$, van Geemen and de Jong [35] analyze necessary conditions so that this map ρ arises as a symbol of a projective heat operator. More precisely, one seeks a map $\overline{D} : \mathcal{T}_S \to \big(\pi_*\big(\mathcal{D}^{\leq 1}(\mathcal{L}) + \mathcal{D}^{\leq 2}_{M/S}(\mathcal{L})\big)\big)/\mathcal{O}_S$, such the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T}_S \arrow[r, "\overline{D}"] \arrow[drr, bend right=10, "\rho"'] & \big(\pi_*\big(\mathcal{D}^{\leq 1}(\mathcal{L}) + \mathcal{D}^{\leq 2}_{M/S}(\mathcal{L})\big)\big)/\mathcal{O}_S \arrow[r, hook] & \big(\pi_*\mathcal{D}^{\leq 2}(\mathcal{L})\big)/\mathcal{O}_S \arrow[d, "s_2"] \\
 & & \pi_*(\operatorname{Sym}^2 \mathcal{T}_{M/S})\ .
\end{tikzcd}$$

The following theorem is one of the main results in [35] (see [35, Sec. 2.3.7]). It gives an algebro-geometric perspective on Hitchin's construction of the flat projective connections for a family of Kähler polarizations in [43, Thm. 1.20].

Theorem 2.2 (EXISTENCE CRITERIA). *Given a symbol map $\rho : \mathcal{T}_S \to \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S}$, with M, $\mathcal{L}$ and S as above, there exists a unique projective heat operator $\overline{D}$ who symbol is ρ if the following three conditions are satisfied:*

(i) (Hitchin, van Geemen-de Jong equation)*: $KS_{M/S} + \mu_{\mathcal{L}} \circ \rho = 0$ in $\mathcal{T}_S$;*

(ii) (Welters condition) *the cup product: $\cup [\mathcal{L}] : \pi_*\mathcal{T}_{M/S} \to R^1\pi_*\mathcal{O}_M$ is an isomorphism;*

(iii) $\pi_*\mathcal{O}_M = \mathcal{O}_S$.

In particular, if the coherent sheaf $\pi_*\mathcal{L}$ is locally free, then $\mathbb{P}(\pi_*\mathcal{L})$ is equipped with a connection.

In [59], the authors translate Hitchin's proof of flatness of projective connections into the abstract formalism of [35]. In the set-up of Theorem 2.2, they prove the following (see [59, Thm. 4.8.2]):

Theorem 2.3 (FLATNESS CRITERIA). *If the following three conditions are satisfied, then the projective connection that is a consequence of Theorem 2.2 is flat.*

(i) *For any local sections θ_1 and θ_2 of $\mathcal{T}_S$, the symmetric vector fields $\rho(\theta_i)$ considered as functions on $\mathcal{T}^\vee_{M/S}$ Poisson commute (for the standard symplectic structure).*

(ii) *The map $\mu_{\mathcal{L}}$ is injective.*

(iii) $\pi_*\mathcal{T}_{M/S} = 0$.

3. Towards a parabolic Hitchin Symbol

In this section we discuss the parabolic analog of the Hitchin symbol. This will turn out to be the symbol of a natural second order differential operator. The original case of (non-parabolic) vector bundles is due to Hitchin. We follow and generalize the discussion in [59]. We begin by recalling the notion of a parabolic Atiyah algebra.

3.1. Parabolic bundles and their Atiyah algebras. Let $q : \mathcal{C} \to S$ be a family of smooth projective curves with n marked points given by disjoint sections $p_1, \cdots, p_n : S \to \mathcal{C}$ of q, and let $D = p_1 + \cdots + p_n$ be the corresponding relative divisor in $\mathcal{C}$. Let $\widehat{\pi} : \widehat{\mathcal{C}} \to S$ be a family of Γ-Galois covers of the fibers of $\mathcal{C}$, ramified along $\widehat{D}$. In particular, this comes with a natural projection map $p : \widehat{\mathcal{C}} \to \mathcal{C}$ such that $p(\widehat{D}) = D$. In order to analyze parabolic Atiyah algebras for families of parabolic bundles on $\mathcal{C}$, we shall use the notion of Γ-linearized bundles on the Galois cover $\widehat{\mathcal{C}}$. The reader is referred to Appendix B for more details.

Let $\widehat{\mathcal{P}}$ be a family of Γ-G-bundles on $\widehat{\mathcal{C}}$, and let $\mathcal{P}$ be the family of parabolic G-bundles obtained by applying the invariant push-forward functor. The *relative parabolic Atiyah algebra* is given by:

$$
{}^{par}\mathrm{At}_{\mathcal{C}/S}(\mathcal{P}) := p_*^{\Gamma}(\mathrm{At}_{\widehat{\mathcal{C}}/S}(\widehat{\mathcal{P}})) ,
$$

and the *strongly parabolic Atiyah algebra* is given by:

$$
{}^{spar}\mathrm{At}_{\mathcal{C}/S}(\mathcal{P}) := p_*^{\Gamma}(\mathrm{At}_{\widehat{\mathcal{C}}/S}(\widehat{\mathcal{P}})(-\widehat{D})) .
$$

Similarly, we define the sheaf of *parabolic endomorphisms* $\mathrm{Par}(\mathcal{P})$ by $p_*^{\Gamma}(\mathrm{ad}(\widehat{\mathcal{P}}))$, and the *strongly parabolic endomorphisms* $\mathrm{SPar}(\mathcal{P})$ by $p_*^{\Gamma}(\mathrm{ad}(\widehat{\mathcal{P}})(-\widehat{D}))$.

Just as in the case of parabolic vector bundles, these sheaves fit into the following fundamental exact sequences

$$
\begin{aligned}
0 \longrightarrow \mathrm{Par}(\mathcal{P}) \longrightarrow {}^{par}\mathrm{At}_{\mathcal{C}/S}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow 0 \, ; \\
0 \longrightarrow \mathrm{SPar}(\mathcal{P}) \longrightarrow {}^{spar}\mathrm{At}_{\mathcal{C}/S}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow 0 \, .
\end{aligned}
\tag{2}
$$

Also, as in the case of parabolic vector bundles we get the following quasi-Lie algebra:

$$
0 \longrightarrow \Omega_{\mathcal{C}/S} \longrightarrow ({}^{par}\mathrm{At}_{\mathcal{C}/S}(\mathcal{P})(D))^{\vee} \longrightarrow (\mathrm{SPar}(\mathcal{P})(D))^{\vee} \longrightarrow 0 \, . \tag{3}
$$

The Cartan-Killing form $\kappa_{\mathfrak{g}}$ on $\mathfrak{g} = \mathrm{Lie}(G)$ gives an identification

$$
\nu_{\mathfrak{g}}^{-1} : (\mathrm{SPar}(\mathcal{P})(D))^{\vee} \xrightarrow{\sim} \mathrm{Par}(\mathcal{P}) \, . \tag{4}
$$

A more explicit description of these bundles in Lie theoretic terms goes as follows: Let $\mathfrak{n}_i$ be the nilradical of the Lie algebra of the parabolic subgroup P_i. Consider the adjoint bundle $\mathrm{ad}(\mathcal{P})$ of the parabolic bundle $\mathcal{P}$. The sheaf of strongly parabolic (respectively, parabolic) endomorphisms is the subsheaf $\mathrm{ad}(\mathcal{P})$ such that the residue at p_i lies in the Lie algebra $\mathfrak{n}_i$ (respectively, in Lie algebra of P_i) for each $1 \leq i \leq n$.

3.2. Some canonical maps. Now assume the family $\mathcal{C} \to S$ to be versal with respect to the divisor D. Universal bundles on relative moduli spaces of bundles exist locally in the étale topology, and moreover both the associated Atiyah algebra and the adjoint bundle glue together to extend globally. For convenience of exposition we can therefore assume the existence of a universal bundle $\mathcal{P}$ on the family of curves $\mathfrak{X}_G^{par}/M_G^{par,rs}$ with parabolic structure supported on a relative divisor D base changed to $M_G^{par,rs}$. We have the following useful diagram:

$$\begin{tikzcd} \mathfrak{X}_G^{par} := \mathcal{C} \times_S M_G^{par,rs} \arrow[r, "\pi_n"] \arrow[d, "\pi_w"] \arrow[dr, "\pi_c"] & M_G^{par,rs} \arrow[l, bend right, "p_i"'] \arrow[d, "\pi_e"] \\ \mathcal{C} \arrow[r, "\pi_s"] & S. \arrow[l, bend left, "p_i"] \end{tikzcd} \tag{5}$$

The above map $\pi_c : \mathfrak{X}_G^{par} \to S$ is defined by $\pi_c := \pi_s \circ \pi_w = \pi_e \circ \pi_n$. Recall the duality in (4). There is a canonical inclusion map

$$\mathrm{SPar}(\mathcal{P}) \hookrightarrow \mathrm{Par}(\mathcal{P}) \tag{6}$$

whose quotient is supported on D. Composing the evaluation map

$$\pi_n^* \pi_{n*} \big(\mathrm{SPar}(\mathcal{P}) \otimes \pi_w^* \Omega_{\mathcal{C}/S}(D)\big) \longrightarrow \mathrm{SPar}(\mathcal{P}) \otimes \pi_w^* \Omega_{\mathcal{C}/S}(D)$$

followed by (6) (tensored with $\pi_w^* \Omega_{\mathcal{C}/S}$), we obtain the following:

$$\pi_n^* \pi_{n*} \big(\mathrm{SPar}(\mathcal{P}) \otimes \pi_w^* \Omega_{\mathcal{C}/S}(D)\big) \longrightarrow \mathrm{Par}(\mathcal{P}) \otimes \pi_w^* \Omega_{\mathcal{C}/S}(D)\,.$$

Taking duals and applying Serre duality, and then using the identification via $\nu_{\mathfrak{g}}^{-1}$ in eq. (4), we get that

$$\begin{aligned} (\mathrm{Par}(\mathcal{P}))^\vee \otimes \pi_w^* \mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow \pi_n^* &\Big(\pi_{n*}\big(\mathrm{SPar}(\mathcal{P}) \otimes \pi_w^* \Omega_{\mathcal{C}/S}(D)\big)^\vee\Big) \\ &\cong \pi_n^* R^1 \pi_{n*} \Big(\big(\mathrm{SPar}(\mathcal{P})(D)\big)^\vee\Big) \cong \pi_n^* \big(R^1 \pi_{n*} \,\mathrm{Par}(\mathcal{P})\big)\,. \end{aligned}$$

This, in turn, gives a map $\pi_w^* \mathcal{T}_{\mathcal{C}/S}(-D) \to \mathrm{Par}(\mathcal{P}) \otimes \pi_n^* \big((R^1 \pi_{n*} \,\mathrm{Par}(\mathcal{P}))\big)$. Applying $R^1 \pi_{n*}$ and the push-pull formula, we obtain a morphism

$$R^1 \pi_s^* \mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow R^1 \pi_{n*} \left(\mathrm{Par}(\mathcal{P})\right) \otimes \big(R^1 \pi_{n*} \left(\mathrm{Par}(\mathcal{P})\right)\big)\,.$$

Further applying π_{e*} and identifying $\pi_{e*} \mathcal{T}_{M^{par}/S}$, we get a map

$$\rho_{sym} : R^1 \pi_{s*} \mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow \pi_{e*} \left(\mathcal{T}^{\otimes 2}_{M_G^{par,rs}/S}\right). \tag{7}$$

We briefly recall the notion of a strongly parabolic Higgs bundle on the family $\mathcal{C} \to S$. Let $\mathcal{P}$ be a parabolic G bundle on a curve C with weights $\boldsymbol{\alpha}$, and consider the sheaf of strongly parabolic endomorphisms $\mathrm{SPar}(\mathcal{P})$. A strongly parabolic Higgs pair $(\mathcal{P}, \theta)$ consists of a parabolic bundle $\mathcal{P}$ and a section θ of $\mathrm{SPar}(\mathcal{P}) \otimes \Omega_{\mathcal{C}/S}(D)$. We refer the reader to [21, Sec. 3–4] for the notion of semistability and the construction of the moduli space $\mathcal{H}^{par,ss}_{\boldsymbol{\alpha},G}$ (or simply denoted by $\mathcal{H}^{par,ss}_G$) (see also [11, Sec. 5], [34, Sec. 5]).

The Hitchin map assigns to a parabolic Higgs pair $(\mathcal{P},\theta)$ the evaluation on θ of a basis of invariant polynomials on $\mathfrak{g}$. Since G is simple, the lowest degree is quadratic; it produces a map:

$$\mathrm{Hit} : \mathcal{H}_G^{par,ss} \longrightarrow \pi_{s*}\Omega^{\otimes 2}_{\mathcal{C}/S}(D) \ ,$$

where $\Omega^{\otimes 2}_{\mathcal{C}/S}(D)$ is the space of holomorphic relative quadratic differentials with simple poles along the divisor D. Now consider the multiplication map

$$R^1\pi_{n*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D) \otimes \pi_{n*}\big(\mathrm{SPar}(\mathcal{P}) \otimes \Omega_{\mathfrak{X}_G^{par}/M^{par,rs}}(D)\big) \longrightarrow R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \ .$$

This gives the following map:

$$R^1\pi_{n*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D) \longrightarrow \big(\pi_{n*}\big(\mathrm{SPar}(\mathcal{P}) \otimes \Omega_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(D)\big)\big)^\vee \otimes R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \ ,$$

which, by relative Serre duality (4), and after applying π_{e*} (see (5)) together with symmetrization, gives a map

$$\rho_{Hit} : R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow \pi_{e*}\,\mathrm{Sym}^2\,\mathcal{T}_{M_G^{par,rs}/S} \ . \tag{8}$$

Observe that the cotangent bundle $\mathcal{T}^\vee_{M_G^{par,rs}/S}$ embeds into $\mathcal{H}_G^{par,ss}$. We rewrite the Hitchin map via the following commutative diagram as in the nonparabolic case:

$$\begin{tikzcd} \mathcal{T}^\vee_{M_G^{par,rs}/S} \arrow[r, "\Delta"] \arrow[dr, "\mathrm{Hit}"'] & \mathcal{T}^\vee_{M_G^{par,rs}/S} \otimes \mathcal{T}^\vee_{M_G^{par,rs}/S} \arrow[d, "\mathrm{Tr}"] \\ & \pi_{n*}\Omega^{\otimes 2}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(D) \ . \end{tikzcd} \tag{9}$$

Here, Δ is the diagonal map, and the operator Tr is the pairing given by symmetric form on $\mathrm{SPar}(\mathcal{P})$ defined by the Killing form $\kappa_{\mathfrak{g}}$; recall that $\mathcal{T}^\vee_{M_G^{par,rs}/S}$ is given by sections of $\mathrm{SPar}(\mathcal{P}) \otimes \Omega_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(D)$. Composing with π_{e*} and applying relative Serre duality we get that the dual of the vertical map Tr in (9) is ρ_{Hit} in (8). The two maps ρ_{Hit} and ρ_{sym} (constructed in (7)) are hence identified.

Proposition 3.1. *The map ρ_{Hit} in* (8) *coincides with ρ_{sym} given in* (7).

Proposition 3.1 was proven in the (nonparabolic) vector bundle case in [59, Lemma 4.3.2].

3.3. Deformation of $M_G^{par,rs}$ via pointed curves. Recall that we have an isomorphism between the moduli space of parabolic bundles with fixed parabolic weights $\boldsymbol{\lambda}$ on a curve C and the moduli space of Γ-G-bundles on a Galois cover $\widehat{C} \to C$ of type $\boldsymbol{\tau}$. Here, the cover $\widehat{C}$ and type are related to the parabolic weights. We refer the reader to Appendix B for more details. We will need the following lemma, the proof of which is straightforward.

Lemma 3.2. *There is a natural isomorphism $\pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D) \xrightarrow{\sim} \mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)$, where π_w is the map in* (5). *Furthermore:*

(i) $R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big) \cong R^1\pi_{s*}\big(\mathcal{T}_{\mathcal{C}/S}(-D)\big)$;

(ii) $R^1\pi_{n*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big) \cong \pi_e^* R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big)$,

where the maps are as in (5).

Consider the relative parabolic Atiyah algebra:

$$ {}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) := p_*^{\Gamma}\big(\mathrm{At}_{\widehat{C}\times_S M_G^{par,rs}/M_G^{par,rs}}(\widehat{\mathcal{P}})\big)\ , $$

and the fundamental exact sequence (cf. (2)) known as the relative Atiyah sequence:

$$ 0 \longrightarrow \mathrm{Par}(\mathcal{P}) \longrightarrow {}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D) \longrightarrow 0\ . \tag{10} $$

Now since $\pi_{n*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D) = 0$ and $R^2\pi_{n*}\,\mathrm{Par}(\mathcal{P}) = 0$, applying $R^1\pi_{n*}$ to the above we get the short exact sequence

$$ 0 \to R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \longrightarrow R^1\pi_{n*}\big({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})\big) \longrightarrow R^1\pi_{n*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D) \to 0\ . \tag{11} $$

The relative extension class of the exact sequence in (11) is an element

$$ \begin{aligned} \alpha(\mathcal{P},\boldsymbol{\lambda}) &\in R^1\pi_{e*}((R^1\pi_{n*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D))^\vee \otimes R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P})) \\ &\cong R^1\pi_{e*}(\pi_e^*(R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D))^\vee \otimes R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P})) \\ &\cong R^1\pi_{e*}(\pi_e^*(R^1\pi_{c*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D))^\vee \otimes R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}))\ . \end{aligned} \tag{12} $$

The last two isomorphisms are constructed using Lemma 3.2. The exact sequence of tangent sheaves induced by the map $\pi_e : M_G^{par,rs} \to S$ is:

$$ 0 \longrightarrow \mathcal{T}_{M_G^{par,rs}/S} \longrightarrow \mathcal{T}_{M_G^{par,rs}} \longrightarrow \pi_e^*\mathcal{T}_S \longrightarrow 0\ . \tag{13} $$

Since by assumption the family of pointed curves is versal, the Kodaira-Spencer map gives an isomorphism $KS_{\mathcal{C}/S} : \mathcal{T}_S \cong R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D)$, which, pulling back via π_e and using Lemma 3.2, gives

$$ \pi_e^*\mathcal{T}_S \cong \pi_e^* R^1\pi_{s*}\big(\mathcal{T}_{\mathcal{C}/S}(-D)\big) \cong R^1\pi_{n*}(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D))\ . \tag{14} $$

The identification in (14) and the equivariant version of [70, eq. (3.10)] together produce the following commutative diagram, which relates (11) and (13):

$$ \begin{tikzcd} R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \arrow[r, hook] & R^1\pi_{n*}{}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \arrow[r, two heads] & R^1\pi_{n*}\Big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\Big) \\ \mathcal{T}_{M_G^{par,rs}/S} \arrow[u, "\cong"] \arrow[r, hook] & \mathcal{T}_{M_G^{par,rs}} \arrow[u] \arrow[r, two heads] & \pi_e^*\mathcal{T}_S \arrow[u, "\cong"'] \end{tikzcd} $$

The Kodaira-Spencer class for the family $\pi_e : M_G^{par,rs} \to S$ gives a map

$$ KS_{M_G^{par,rs}/S} : \mathcal{T}_S \longrightarrow R^1\pi_{e*}\mathcal{T}_{M_G^{par,rs}/S} \cong R^1\pi_{e*}\left(R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P})\right)\ . $$

The cup product by $\alpha := \alpha(\mathcal{P}, \boldsymbol{\lambda})$ produces maps

$$
\begin{array}{c}
R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big) \\
\Big\downarrow {\scriptstyle \cup\,\alpha} \\
R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big) \otimes R^1\pi_{e*}(\pi_e^*(R^1\pi_{c*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D))^\vee \otimes R^1\pi_{n*}\operatorname{Par}(\mathcal{P})) \\
\Big\downarrow {\scriptstyle \cong} \\
\Big(R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big) \otimes \big((R^1\pi_{c*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big)\Big)^\vee \otimes R^1\pi_{e*}(R^1\pi_{n*}\operatorname{Par}(\mathcal{P})) \\
\Big\downarrow \\
R^2\pi_{c*}\operatorname{Par}(\mathcal{P}) \cong R^1\pi_{e*}\big(R^1\pi_{n*}\operatorname{Par}(\mathcal{P})\big) \ .
\end{array}
$$

The isomorphism in the last step uses the identification $R^1\pi_{n*}\operatorname{Par}(\mathcal{P}) \cong \mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}$, along with the facts that $M_G^{par,rs}$ has no global tangent vector fields relative to S (cf. Lemma 5.6) and $\pi_{n*}\operatorname{Par}(\mathcal{P})$ is zero. This forces the Grothendieck spectral sequence to collapse.

We may summarize the discussion and identifications above with the following commutative diagram:

(15)

$$
\begin{tikzcd}
\mathcal{T}_S \arrow[r, "KS_{\mathcal{C}/S}", "\cong"'] \arrow[dr, "KS_{M_G^{par,rs}/S}"'] & R^1\pi_{s*}(\mathcal{T}_{\mathcal{C}/S}(-D)) \arrow[r, "\cong"] \arrow[d, "\Phi"] & R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big) \arrow[dl, "\cup\,\alpha"] \\
& R^1\pi_{e*}\mathcal{T}_{M_G^{par,rs}/S} \cong R^1\pi_{e*}\big(R^1\pi_{n*}\operatorname{Par}(\mathcal{P})\big) \ . &
\end{tikzcd}
$$

Here Φ is the map induced by the cup product with the class $\alpha(\mathcal{P}, \boldsymbol{\lambda})$ (see eq. (12)) preceded by the isomorphism of $R^1\pi_{s*}\big(\mathcal{T}_{\mathcal{C}/S}(-D)\big)$ with $R^1\pi_{c*}\big(\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(-D)\big)$ given in Lemma 3.2.

3.4. A fundamental commutative diagram. Consider $R^1\pi_{n*}$ of the sequence (2) applied to ${}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})$, where π_n is the map in (5):

(16)

$$
0 \longrightarrow R^1\pi_{n*}(\pi_w^*\Omega_{\mathcal{C}/S}) \longrightarrow R^1\pi_{n*}(({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee) \longrightarrow R^1\pi_{n*}((\mathrm{SPar}(\mathcal{P})(D))^\vee) \longrightarrow 0 \ .
$$

Let $\beta := \beta(\mathcal{P}, \boldsymbol{\lambda})$ be the relative extension class with respect to π_e (see (5)) of the extension (16). Then we have a diagram:

(17)

$$
\begin{tikzcd}
R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D) \arrow[r, "-\Phi"] \arrow[dr, "\rho_{sym}"'] & R^1\pi_{e*}\mathcal{T}_{M_G^{par,rs}/S} \\
& \pi_{e*}\big(\mathrm{Sym}^2\,\mathcal{T}_{M_G^{par,rs}/S}\big) \ . \arrow[u, "\cup\,\beta"']
\end{tikzcd}
$$

We have the following key result which relates all three maps. In the (nonparabolic) vector bundle case, this was proven in [59, Prop. 4.7.1].

Theorem 3.3. *The diagram* (17) *commutes. In other words,*

$$\Phi + \cup \beta(\mathcal{P}, \boldsymbol{\lambda}) \circ \rho_{sym} = 0$$

as a morphism $R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S} \to R^1\pi_{e*}\mathcal{T}_{M_G^{par,rs}/S}$.

Proof. Pull back the short exact sequence in (16) to $\mathfrak{X}_G^{par}$ via the map π_n in (5). Tensoring the resulting sequence with $\mathrm{Par}(\mathcal{P})$ we obtain the following exact sequence

$$\begin{array}{ccc} \mathrm{Par}(\mathcal{P}) \otimes \pi_n^* \left(R^1\pi_{n*}\pi_w^*\Omega_{\mathcal{C}/S}\right) & \hookrightarrow & \mathrm{Par}(\mathcal{P}) \otimes \pi_n^*\left(R^1\pi_{n*}\left(({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee\right)\right) \\ & & \downarrow \\ & & \mathrm{Par}(\mathcal{P}) \otimes \pi_n^*\left(R^1\pi_{n*}\left((\mathrm{SPar}(\mathcal{P})(D))^\vee\right)\right) . \end{array}$$

Using $\kappa_{\mathfrak{g}}$, we can rewrite this as

$$\begin{array}{ccc} \mathrm{Par}(\mathcal{P}) \otimes \pi_n^* \left(R^1\pi_{n*}\pi_w^*\Omega_{\mathcal{C}/S}\right) & \hookrightarrow & \mathrm{Par}(\mathcal{P}) \otimes \pi_n^*\left(R^1\pi_{n*}\left(({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee\right)\right) \\ & & \downarrow \\ & & \mathrm{Par}(\mathcal{P}) \otimes \pi_n^* \left(R^1\pi_{n*}\left(\mathrm{Par}(\mathcal{P})\right)\right) . \end{array}$$

The assumptions ensure that $R^1\pi_{n*}\pi_w^*\Omega_{\mathcal{C}/S} = \mathcal{O}_{M_G^{par,rs}}$. Dualize (10) to get

$$0 \longrightarrow \pi_w^*\Omega_{\mathcal{C}/S}(D) \longrightarrow \left({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})\right)^\vee \longrightarrow \mathrm{Par}(\mathcal{P})^\vee \longrightarrow 0 .$$

Tensoring by $\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D)$ and taking the duals (outside bracket) we get the short exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Par}(\mathcal{P}) \longrightarrow \mathrm{Par}(\mathcal{P}) \otimes \left(({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\right)^\vee \\ \longrightarrow \mathrm{Par}(\mathcal{P}) \otimes \left(\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\right)^\vee \longrightarrow 0 . \end{array}$$

Now observe that the dual of the evaluation gives maps

$$\begin{aligned} \left({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\right)^\vee &\longrightarrow \left(\pi_n^*\pi_{n*}\left({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\right)\right)^\vee \\ &= \pi_n^*\left(\pi_{n*}\left({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\right)\right)^\vee \\ &\cong \pi_n^*\left(R^1\pi_{n*}\left(({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee\right)\right) \\ &\longrightarrow \pi_n^*\left(R^1\pi_{n*}\left(({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee\right)\right) . \end{aligned}$$

In the above equation we have used the isomorphism

$$R^1\pi_{n*}(({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee) \cong (\pi_{n*}({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D) \otimes \pi_w^*\Omega_{\mathcal{C}/S}))^\vee$$

coming from relative Serre duality and the dual of the natural inclusion map

$$ {}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \hookrightarrow {}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \, . $$

We now reverse engineer the construction of the Hitchin morphism ρ_{sym}:

$$ \begin{aligned} \big(\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\big)^\vee &\longrightarrow \pi_n^*\pi_{n*}\left(\big(\mathrm{Par}(\mathcal{P})(D) \otimes \pi_w^*\Omega_{\mathcal{C}/S}\big)\right)^\vee \\ &\longrightarrow \pi_n^*\pi_{n*}\left(\big(\mathrm{SPar}(\mathcal{P})(D) \otimes \pi_w^*\Omega_{\mathcal{C}/S}\big)\right)^\vee \\ &\cong \pi_n^*\left(R^1\pi_{n*}\left((\mathrm{SPar}(\mathcal{P})(D))^\vee\right)\right) \qquad \text{(by relative Serre duality)} \\ &\cong \pi_n^*\left(R^1\pi_{n*}\left(\mathrm{Par}(\mathcal{P})\right)\right) \qquad \text{(by trace pairing).} \end{aligned} $$

Consider the natural inclusion map $\pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D) \hookrightarrow \mathrm{Par}(\mathcal{P}) \otimes \mathrm{Par}(\mathcal{P})^\vee \otimes \pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D)$, and pull back the short exact sequence

$$ \mathrm{Par}(\mathcal{P}) \hookrightarrow \mathrm{Par}(\mathcal{P}) \otimes ({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}))^\vee \otimes \pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D) \twoheadrightarrow (\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D))^\vee. $$

Finally, by [59, Lemma 4.5.1], we obtain an isomorphism of the extensions:

(18)

$$ \begin{tikzcd} \mathrm{Par}(\mathcal{P}) \arrow[r, hook] \arrow[d, equal] & {}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \arrow[r, two heads, "(-1)"] \arrow[d, hook] & \pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D) \arrow[d, hook] \\ \mathrm{Par}(\mathcal{P}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Par}(\mathcal{P}) \otimes ({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}))^\vee \otimes \pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D) \arrow[r, two heads] \arrow[d, equal] & \mathrm{Par}(\mathcal{P}) \otimes (\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D))^\vee \arrow[d, equal] \\ \mathrm{Par}(\mathcal{P}) \arrow[r, hook] & \mathrm{Par}(\mathcal{P}) \otimes ({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D))^\vee \arrow[r, two heads] & \mathrm{Par}(\mathcal{P}) \otimes (\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D))^\vee . \end{tikzcd} $$

Here, the minus sign (-1) indicates the negative of the projection map. Following the case of vector bundles in [59], after composing we arrive at a commutative diagram

(19)

$$ \begin{tikzcd} \mathrm{Par}(\mathcal{P}) \arrow[r, hook] \arrow[d, equal] & {}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \arrow[r, two heads, "(-1)"] \arrow[d] & \pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D) \arrow[dd] \\ \mathrm{Par}(\mathcal{P}) \arrow[r, hook] \arrow[dd, equal] & \mathrm{Par}(\mathcal{P}) \otimes \left(\pi_n^*\pi_{n*}({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D))\right)^\vee \arrow[dd] \arrow[dr, two heads, bend right] & \\ & & \mathrm{Par}(\mathcal{P}) \otimes \left(\pi_n^*\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \otimes \pi_w^*\Omega_{\mathcal{C}/S}(D)\right)^\vee \arrow[dd] \\ \mathrm{Par}(\mathcal{P}) \arrow[r, hook] & \mathrm{Par}(\mathcal{P}) \otimes \pi_n^*\left(R^1\pi_{n*}\left(({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee\right)\right) \arrow[dr, two heads, bend right] & \\ & & \mathrm{Par}(\mathcal{P}) \otimes \pi_n^*\left(R^1\pi_{n*}(\mathrm{Par}(\mathcal{P}))\right). \end{tikzcd} $$

Now we take $R^1\pi_{n*}$ of the exact sequences in the first and third rows in (19) to obtain

(20)

$$ \begin{tikzcd} R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \arrow[r, hook] \arrow[d, equal] & R^1\pi_{n*}({}^{par}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})) \arrow[r, two heads, "(-1)"] \arrow[d] & R^1\pi_{n*}(\pi_w^*\mathcal{T}_{\mathcal{C}/S}(-D)) \arrow[d] \\ R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \arrow[r, hook] & R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \otimes R^1\pi_{n*}({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee \arrow[r, two heads] & R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \otimes R^1\pi_{n*}(\mathrm{Par}(\mathcal{P})). \end{tikzcd} $$

The connecting homomorphism for $(\pi_e)_*$ gives

$$\begin{array}{ccc} R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D) & \xrightarrow{\ -\Phi\ } & R^1\pi_{e*}\mathcal{T}_{M_G^{par,rs}/S} \\ \big\downarrow{\scriptstyle \rho_{sym}} & & \big\| \\ \pi_{e*}\big(\mathcal{T}_{M_G^{par,rs}/S}\otimes\mathcal{T}_{M_G^{par,rs}/S}\big) & \longrightarrow & R^1\pi_{e*}\mathcal{T}_{M_G^{par,rs}/S}\,. \end{array}$$

The negative sign $-\Phi$ appears above due to the factor (-1) in (18); recall that Φ (see eq. (15)) is the connecting homomorphism for the direct image by π_e of the exact sequence in (11). The proof of the theorem will be complete if we can show that the underlying map is $\cup\,\beta(\mathcal{P},\boldsymbol{\lambda})$. But this follows from the fact that the bottom row of (20) is just the exact sequence

$$0 \longrightarrow \mathcal{O}_{M_G^{par}} \cong R^1\pi_{n*}\Omega_{\mathfrak{X}_G^{par}/M_G^{par,rs}} \longrightarrow R^1\pi_{n*}({}^{spar}\mathrm{At}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}(\mathcal{P})(D))^\vee \longrightarrow R^1\pi_{n*}\,\mathrm{Par}(\mathcal{P}) \longrightarrow 0$$

tensored with $\mathcal{T}_{\mathfrak{X}_G^{par}/M_G^{par,rs}}$, and $\beta(\mathcal{P},\boldsymbol{\lambda})$ is the relative extension class of the above with respect to π_e. $\square$

4. Cupping with the parabolic determinant of cohomology

In this section, we state and prove a key result that compares the cupping map by the class of the parabolic determinant of cohomology to that of the usual determinant of cohomology. This will be crucial for later arguments. Let $\vec{P}=(P_1,\dots,P_n)$ be an n-tuple of standard parabolic subgroups, and consider the stack $\mathcal{P}ar_G(C,\vec{P})$ of quasi parabolic bundles on a curve as recalled in Definition A.2 and let $\mathrm{Det}(\mathcal{V})$ (or simply Det) denote the determinant of cohomology line bundle on a scheme T parametrizing a family $\mathcal{V}$ of vector bundles on a smooth projective curve C. Recall (cf. Proposition A.5) that any line bundle on $\mathcal{P}ar_G(C,\vec{P})$ is of the form $\mathrm{Det}(\mathcal{E}(\mathcal{V}))^{\otimes a}\bigotimes\mathscr{K}$, where $\mathcal{E}(\mathcal{V})$ is a vector bundle associated to a chosen representation $\phi: G\to\mathsf{SL}(V)$, $a\in\mathbb{Q}$ and $\mathscr{K}\in\mathrm{Pic}(G/P_1\times\cdots\times G/P_n)\otimes\mathbb{Q}$. We will refer to the rational number a as the *level* (see Definition A.16).

Theorem 4.1. *Let $\mathbb{L}$ be an element of $\mathrm{Pic}(M_{G,\boldsymbol{\beta}}^{par,rs})\otimes\mathbb{Q}$ of level a. Then as linear maps $\pi_{e*}\,\mathrm{Sym}^2\,\mathcal{T}_{M_{G,\boldsymbol{\beta}}^{par,rs}/S}\to R^1\pi_{e*}\mathcal{T}_{M_{G,\boldsymbol{\beta}}^{par,rs}/S}$, we have: $\cup[\mathbb{L}]=\cup\, a[\mathrm{Det}]$, where* Det *is the determinant of cohomology (nonparabolic) line bundle.*

Theorem 4.1 is proved in several steps. The strategy of the proof is to reduce to the case of parabolic vector bundles with full flags and apply the technique of abelianization by restricting to generic fibers of the Hitchin map.

4.1. Reduction to the SL_r case. Since G is simple (hence semisimple), any short exact sequence of finite dimensional G-modules splits. In particular, for a faithful irreducible G-module V, the G-module $\mathrm{End}(V)$ decomposes as $\mathfrak{g}\oplus W_0$. Fix a complement W_0 of the G-submodule $\mathfrak{g}$. Given an injective homomorphism $G\hookrightarrow\mathsf{SL}_r(\mathbb{C})$, we have an embedding

$M^{par,ss}_{G,\boldsymbol\beta} \hookrightarrow M^{par,ss}_{\mathsf{SL}_r,\boldsymbol\alpha}$ which restricts to a map $f : M^{par,rs}_{G,\boldsymbol\beta} \to M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}$. Using the splitting of the G-module $\mathfrak{sl}_r(\mathbb{C})$, the tangent bundle $f^*M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}$ splits as $f^*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}/S} = \mathcal{T}_{M^{par,rs}_{G,\boldsymbol\beta}/S} \oplus W$. This gives splittings of tensor powers, duals etc. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}/S} & \xrightarrow{\cup\mathbb{L}} & R^1\pi_*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}/S} \\ \big\downarrow & & \big\downarrow \\ \pi_{G*} \operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_{G,\boldsymbol\beta}/S} & \xrightarrow{\cup\mathbb{L}} & R^1\pi_{G*}\mathcal{T}_{M^{par,rs}_{G,\boldsymbol\beta}/S}\ , \end{array} \tag{21}$$

where $\pi : M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha} \to S$ and $\pi_G : M^{par,rs}_{G,\boldsymbol\tau} \to S$ are the projections (this was earlier denoted by π_e, but here we simply write π and π_G); the vertical maps in (21) are given by the above mentioned splittings. Here, $\mathbb{L}$ is an element of the rational Picard group of $M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}$, and $\pi_G = f \circ \pi$. The homomorphism $\pi_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}/S} \to \pi_{G*} \operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_{G,\boldsymbol\beta}/S}$ in (21) is surjective. Thus we have proved the following proposition:

Proposition 4.2. *Consider two elements $\mathbb{L}_1$ and $\mathbb{L}_2$ in $\operatorname{Pic}(M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}) \otimes \mathbb{Q}$. If the maps $\cup[\mathbb{L}_1]$ and $\cup[\mathbb{L}_2]$ agree on $\pi_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol\alpha}/S}$, then they also agree on $\pi_{G*} \operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_{G,\boldsymbol\beta}/S}$.*

4.2. Reduction to the SL_r with full flags. In this step, we will show that in order to prove Theorem 4.1 it is enough to assume that $\boldsymbol\alpha$ corresponds to weights for full flags. This step is only required when $r > 2$.

Changing weights without changing stability. Let $D = \{p_1, \cdots, p_n\} \subset C$ be the parabolic divisor. Consider parabolic vector bundles of rank r. For any $1 \le i \le n$, let

$$\alpha_{i,j} = m_{i,j}/\ell, \quad 1 \le j \le r, \tag{22}$$

be the parabolic weights at p_i, where $m_{i,j}$ and ℓ are nonnegative integers. Note that for any i, the integers $m_{i,j}$, $1 \le j \le r$, need not be distinct and the weights are assigned to full flags. We will reformulate a general notion of parabolic bundles for which the quasiparabolic flags are not necessarily complete in the following way: We will set the quasiparabolic flag at each p_i to be complete flags, but two different terms in the filtration can have same parabolic weight. This reformulation does not alter any of the stability and semistability conditions.

Fix a vector bundle E of rank r on X. Let E_* be a parabolic structure on E of the above type. Let E'_* be another parabolic bundle satisfying the following conditions:

(i) The underlying holomorphic vector bundle for E'_* is E itself,

(ii) the quasiparabolic flag for E'_* coincides with that of E_* at each p_i (recall that the quasi-parabolic flags are complete but two different subspaces of E_{p_i} can have same parabolic weight), and

(iii) for any term $F_{i,j} \subset E_{p_i}$ of the quasiparabolic flag at p_i, if $\alpha_{i,j}$ and $\widetilde{\alpha}_{i,j}$ are the weights of $F_{i,j}$ in E_* and E'_*, respectively, then

$$\left|\alpha_{i,j} - \widetilde{\alpha}_{i,j}\right| \le \frac{1}{3\ell n r^2}\,. \tag{23}$$

Proposition 4.3. *The parabolic vector bundle E'_* is stable if the parabolic vector bundle E_* is stable. Moreover, the parabolic vector bundle E_* is semistable if the parabolic vector bundle E'_* is semistable.*

Proof. Assume that E_* is parabolic stable. Take any subbundle $0 \neq F \subsetneq E$. Let F_* denote the parabolic structure on it induced by E_*. Since E_* is parabolic stable, we have

$$\text{par-deg}(F_*)r \,<\, \text{par-deg}(E_*)r' \,, \tag{24}$$

where $r' = \text{rank}(F)$. From (22) it follows that $\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r$ is an integral multiple of $1/\ell$, and hence (24) implies that

$$\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r \,\geq\, \frac{1}{\ell}. \tag{25}$$

Let F'_* denote the parabolic vector bundle defined by F equipped with the parabolic structure induced by E'_*. From (23) we have

$$\text{par-deg}(F'_*) - \text{par-deg}(F_*) \,\leq\, \frac{nr'}{3\ell nr^2} \quad \text{and} \quad \text{par-deg}(E_*) - \text{par-deg}(E'_*) \,\leq\, \frac{nr}{3\ell nr^2}\,.$$

These imply that

$$(\text{par-deg}(F'_*) - \text{par-deg}(F_*))r \,\leq\, \frac{1}{3\ell} \quad \text{and} \quad (\text{par-deg}(E_*) - \text{par-deg}(E'_*))r' \,\leq\, \frac{1}{3\ell}\,.$$

Adding these

$$(\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r) - (\text{par-deg}(E'_*)r' - \text{par-deg}(F'_*)r) \,\leq\, \frac{2}{3\ell},$$

and hence using (25),

$$\text{par-deg}(E'_*)r' - \text{par-deg}(F'_*)r \,\geq\, \frac{1}{\ell} - \frac{2}{3\ell} \,=\, \frac{1}{3\ell} \,>\, 0.$$

Therefore, E'_* is parabolic stable. Now assume that E'_* is parabolic semistable. So we have

$$\text{par-deg}(F'_*)r \,\leq\, \text{par-deg}(E'_*)r' \,, \tag{26}$$

From (23) we have

$$\text{par-deg}(F_*) - \text{par-deg}(F'_*) \,\leq\, \frac{nr'}{3\ell nr^2} \quad \text{and} \quad \text{par-deg}(E'_*) - \text{par-deg}(E_*) \,\leq\, \frac{nr}{3\ell nr^2}\,.$$

These imply that

$$(\text{par-deg}(F_*) - \text{par-deg}(F'_*))r \,\leq\, \frac{1}{3\ell} \quad \text{and} \quad (\text{par-deg}(E'_*) - \text{par-deg}(E_*))r' \,\leq\, \frac{1}{3\ell}\,.$$

Adding these

$$(\text{par-deg}(E'_*)r' - \text{par-deg}(F'_*)r) - (\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r) \,\leq\, \frac{2}{3\ell},$$

So using (26),

$$\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r \,\geq\, -\frac{2}{3\ell}.$$

But this implies that $\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r \,\geq\, 0$ because $\text{par-deg}(E_*)r' - \text{par-deg}(F_*)r$ is an integral multiple of $1/\ell$. Hence E_* is parabolic semistable. □

Let $\boldsymbol{\alpha}$ be a set of weights defining the parabolic structure. We choose a refinement of $\boldsymbol{\alpha}$, denoted by $\widetilde{\boldsymbol{\alpha}}$, such that for each point p_i, the weight-tuple $\boldsymbol{\alpha}_i$ consists of distinct weights. The weights $\widetilde{\boldsymbol{\alpha}}$ are a choice of weights for full flags such that the corresponding weights for the given partial flags is $\boldsymbol{\alpha}$. By (23), we can always find $\widetilde{\boldsymbol{\alpha}}$ by choosing the missing weights small enough such that the natural forgetful map preserves stability with respect to $\widetilde{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}$. In particular by Proposition 4.3 , we get a natural regular map $F : M^{par,ss}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}} \to M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$ fitting in the following commutative diagram:

$$\begin{tikzcd} M^{par,ss}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}} \arrow[r, "F"] \arrow[dr, "\widetilde{\pi}"'] & M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\alpha}} \arrow[d, "\pi"] \\ & S\,. \end{tikzcd} \tag{27}$$

Let $M_{\widetilde{\boldsymbol{\alpha}}} := F^{-1}(M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}})$. Again, by Proposition 4.3, $M_{\widetilde{\boldsymbol{\alpha}}} \subset M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$. The map F is fibration by product of flag varieties. By Lemma C.1, the codimension of the complement of $M_{\widetilde{\boldsymbol{\alpha}}}$ in $M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$ is at least three. Hence, we have the following isomorphisms (via Hartogs' Theorem):

$$R^1\widetilde{\pi}_*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \cong R^1\widetilde{\pi}_*\mathcal{T}_{M_{\widetilde{\boldsymbol{\alpha}}}/S} \quad , \quad \widetilde{\pi}_*\operatorname{Sym}^2\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \cong \widetilde{\pi}_*\operatorname{Sym}^2\mathcal{T}_{M_{\widetilde{\boldsymbol{\alpha}}}/S}. \tag{28}$$

The differential of F, along with the isomorphisms (28), induces natural maps

$$R^1\widetilde{\pi}_*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \xrightarrow{DF} R^1\widetilde{\pi}_*\big(DF^*(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})\big),$$
$$\widetilde{\pi}_*\operatorname{Sym}^2\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \xrightarrow{\operatorname{Sym}^2 DF} \widetilde{\pi}_*\operatorname{Sym}^2\big(DF^*(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})\big)\ .$$

We have the following lemma:

Lemma 4.4. *The Leray spectral sequence gives natural isomorphisms:*

$$R^1\widetilde{\pi}_*\big(DF^*(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})\big) \;\cong\; R^1\pi_*\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\big),$$
$$\widetilde{\pi}_*\operatorname{Sym}^2\big(DF^*(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})\big) \;\cong\; \pi_*\operatorname{Sym}^2\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\big)\ .$$

Proof. For the map F in (27), space $M_{\widetilde{\boldsymbol{\alpha}}}$ is a fiber bundle over the moduli space $M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$, and moreover, the fibers are products of flag manifolds. Hence, we have

$$F_*\mathcal{O}_{M_{\widetilde{\boldsymbol{\alpha}}}} \;=\; \mathcal{O}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}} \quad \text{and} \quad R^kF_*\mathcal{O}_{M_{\widetilde{\boldsymbol{\alpha}}}} \;=\; 0 \tag{29}$$

for all $k \geq 1$. Given any vector bundle W on $M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$, using (29) and the projection formula we have

$$F_*F^*W \;=\; W \quad \text{and} \quad R^kF_*F^*W \;=\; 0 \tag{30}$$

for all $k \geq 1$. From (30) it follows that

$$R^k\widetilde{\pi}_*F^*W \;=\; R^k\pi_*W\,. \tag{31}$$

Now take $W \;=\; \operatorname{Sym}^2(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})$ in (31). $\square$

As before, let $\mathbb{L}$ be an element of the rational Picard group $\mathrm{Pic}(M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}) \otimes \mathbb{Q}$. Using the isomorphisms in Lemma 4.4 we have the following diagram:

(32)

$$\begin{tikzcd}
 & & R^1\pi_*\left(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\right) \\
R^1\widetilde{\pi}_*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \arrow[r, "\cong"] & R^1\widetilde{\pi}_*\mathcal{T}_{M_{\widetilde{\boldsymbol{\alpha}}}/S} \arrow[r, "DF"] \arrow[ur] & R^1\widetilde{\pi}_*\left(DF^*\left(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\right)\right) \arrow[u, "\cong"'] \\
\widetilde{\pi}_*\,\mathrm{Sym}^2\,\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \arrow[r, "\cong"] \arrow[u, "\cup\mathbb{L}"] & \widetilde{\pi}_*\,\mathrm{Sym}^2\,\mathcal{T}_{M_{\widetilde{\boldsymbol{\alpha}}}/S} \arrow[r, "\mathrm{Sym}^2 DF"] \arrow[u, "\cup\mathbb{L}"] \arrow[dr] & \widetilde{\pi}_*\,\mathrm{Sym}^2\left(DF^*\left(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\right)\right) \arrow[u, "\cup\mathbb{L}"'] \arrow[d, "\cong"'] \\
 & & \pi_*\,\mathrm{Sym}^2\left(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\right). \arrow[uuu, bend right=60, "\cup\mathbb{L}"']
\end{tikzcd}$$

We note that we have used the same notation $\mathbb{L}$ for a line bundle on both $M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$ and also on $M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$. The isomorphisms in Lemma 4.4, composed with the differential maps, give natural maps

$$R^1\widetilde{\pi}_*\mathcal{T}_{M^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \longrightarrow R^1\pi_*\big(\mathcal{T}_{M^{par,s}_{\boldsymbol{\alpha},\mathsf{SL}_r}/S}\big)\ ; \tag{33}$$

$$\widetilde{\pi}_*\,\mathrm{Sym}^2\,\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \longrightarrow \pi_*\,\mathrm{Sym}^2\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\big)\ . \tag{34}$$

With the above notation we have the following proposition:

Proposition 4.5. *The maps in* (33) *and* (34) *are isomorphisms, and the diagram in* (32) *is commutative.*

Proof. Consider the differential $DF\ :\ \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}/S}} \longrightarrow F^*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$, and its second symmetric product

$$\mathrm{Sym}^2(DF)\ :\ \mathrm{Sym}^2\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \longrightarrow \mathrm{Sym}^2(F^*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})\ =\ F^*\mathrm{Sym}^2(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})\,.$$

Let $\beta\ :=\ (DF)^*\ :\ F^*\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S} \to \mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r\widetilde{\boldsymbol{\alpha}}}/S}$ be the dual of the above homomorphism DF. Note that $\mathrm{Sym}^2(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S})$ (respectively, $\mathrm{Sym}^2(F^*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})$ defines fiberwise quadratic functions $\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}$ (respectively, $F^*\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$. Take any $z\ \in\ M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$. For any $w \in \mathrm{Sym}^2(\mathcal{T}_{M^{par,s}_{SL_r,\widetilde{\boldsymbol{\alpha}}}/S})_z$ and $\nu\ \in\ (F^*\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S})_z$, we have: $(\mathrm{Sym}^2(DF))_z(w)(\nu)\ =\ w((DF)^*_z(\nu))$. From this we have the following commutative diagram of homomorphisms (recall (5)):

$$\begin{tikzcd}
\widetilde{\pi}_*\mathrm{Sym}^2(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}) \arrow[r, "\widetilde{\pi}_*\mathrm{Sym}^2(DF)"] \arrow[d, "\cong"] & \widetilde{\pi}_*F^*\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S} \arrow[d, "\cong"] \\
R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D) \arrow[r, "\mathrm{Id}"] & R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D)
\end{tikzcd} \tag{35}$$

in which $\widetilde{\pi}_*\mathrm{Sym}^2(DF)$ is an isomorphism, because all other homomorphisms in (35) are isomorphisms. This proves that the map in (34) is an isomorphism. The proof that the map in (33) is an isomorphism is very similar to the proof of it for (34). Now it is evident that the diagram in (32) is commutative. □

Thus we have proved the following proposition:

Proposition 4.6. *Consider two elements* $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *in* $\mathrm{Pic}(M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}) \otimes \mathbb{Q}$. *If the maps* $\cup[\mathbb{L}_1]$ *and* $\cup[\mathbb{L}_2]$ *agree on* $\widetilde{\pi}_* \,\mathrm{Sym}^2\, \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\alpha}/S}}$, *then they also agree on* $\pi_* \,\mathrm{Sym}^2\, \mathcal{T}_{M^{par,rs}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$.

4.3. Reduction to abelian varieties. This step is essentially the same as in [43, Prop. 5.2] generalized to the parabolic set-up with the additional information about spectral data with one node. For completeness, we include the details by following the exposition in [59].

Hitchin Map. Let $\pi_s : \mathcal{C} \to S$ be a family of n-pointed curves, and let D be the divisor of marked points. Consider the vector bundle $\mathcal{B} := \bigoplus_{i=2}^r \pi_{s*}K^i_{\mathcal{C}/S}((i-1)D)$, and let $\pi_{\mathcal{B}} : \mathcal{B} \to S$ be the natural projection map. Let $\pi_{\mathcal{H}} : \mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r} \to S$ be the relative strongly parabolic Higgs moduli space parametrizing pairs $(\mathcal{P},\theta)$, where $\mathcal{P}$ is a parabolic bundle and θ is a strongly parabolic endomorphism of $\mathcal{P}$ twisted by $K(D)$. We refer the reader to [21] for notions of stability and semistability for strongly parabolic Higgs bundles. Recall the Hitchin morphism $\mathrm{Hit} : \mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r} \to \mathcal{B}$ from Section 3.2. We have the following commutative diagram

$$\begin{tikzcd} \mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r} \arrow[r, "\mathrm{Hit}"] \arrow[dr, "\pi_{\mathcal{H}}"'] & \mathcal{B} \arrow[d, "\pi_{\mathcal{B}}"] \\ & S\,. \end{tikzcd} \tag{36}$$

Let $\mathcal{B}^0$ denote the collection of points in $\mathcal{B}$ such that the corresponding spectral curve (as described in [13, Sec. 3]) is smooth. The complement of $\mathcal{B}^0$ in $\mathcal{B}$ is a divisor, since we are in the case of SL_r-Higgs bundles with full flags. This follows from the fact ([37, Lemma 3.1] and [13, Remark 3.5]) that K^rD^{r-1} is very ample and has sections without multiple zeros in either of the following cases: $g \geq 2$; $g = 1$ and degree of $D \geq \frac{3}{r-1}$; $g = 0$ and degree of $D \geq 2 + \frac{3}{r-1}$. But this is implied by the assumption that the orbifold genus $g(\mathscr{C}) \geq 2$ (see Definition B.1 and also Appendix C). Then via abelianization, it is well-known that the fibers of $\mathrm{Hit}^{-1}(\vec{b})$, $\vec{b} \in \mathcal{B}^0$, are families of abelian varieties $A_{\vec{b}}$ over S.

Consider the divisor $\mathcal{D} := \mathcal{B}\backslash\mathcal{B}^\circ \subsetneq \mathcal{B}$. As in [1, Prop. 4.1], for $x \in D$ let $\mathcal{D}_x$ to be the set of characteristic polynomials whose spectral curves are singular over x, and let $\mathcal{D}_U$ to be the set of characteristic polynomials whose spectral curves are smooth over each $x \in D$, but singular over some $y \notin D$. Then $\mathcal{D} = \overline{\mathcal{D}}_U \cup \bigcup_{x\in D} \mathcal{D}_x$.

Now ([1, p. 28]) $\mathcal{D}_x = \bigoplus_{i=2}^{r-1} H^0(K^iD^{i-1}) \oplus H^0(K^rD^{r-1}(-x))$, and hence is irreducible. By the assumption, K^rD^{r-1} is very ample, which implies that $\dim \mathcal{D}_x < \dim \mathcal{D}$. Similarly, the remaining part of the proof of [1, Prop. 4.1] also goes through under this assumption. We obtain that $\overline{\mathcal{D}}_U$ is the surjective image of an affine bundle over $C\backslash D$ whose fiber at y is given by

$$\oplus_{i=1}^{r-2} H^0(K^iD^{i-1}) \oplus H^0(K^{r-1}D^{r-2}(-y)) \oplus H^0(K^rD^{r-1}(-2y))\,.$$

Hence, $\overline{\mathcal{D}}_U$ is also irreducible.

Thus $\overline{\mathcal{D}}_U$ is the unique irreducible component of highest dimension in $\mathcal{D}$. Now by Bertini's theorem, a generic point of $\mathcal{D}_U$ has an irreducible spectral curve with exactly one node over a point $y \notin D$.

Now we let $\mathcal{B}^{\heartsuit}$ denote the subspace of $\mathcal{B}$ consisting of all points such that the spectral curve is irreducible and has at most one node outside the divisor D. By the previous discussion, we get that the codimension of the complement of $\mathcal{B}^{\heartsuit}$ in $\mathcal{B}$ is at least two. The following lemma determines the fibers of the Hitchin map over points of $\mathcal{B}^{\heartsuit}$.

Proposition 4.7. *The fiber of the Hitchin map $\mathcal{H}^{par,ss}_{\vec{\alpha},\mathsf{SL}_r} \to \mathcal{B}$ over any point $\vec{b} \in \mathcal{B}^{\heartsuit}$ is a quasi-abelian variety.*

Proof. Fix a n-pointed Riemann surface (X, D). Let $\mathcal{B}$ be the base of the strongly parabolic Hitchin map. For any $\vec{b} \in \mathcal{B}$, let $C_{\vec{b}} \subset K_X(D)$ be the corresponding spectral curve; let $p_{\vec{b}} : C_{\vec{b}} \to X$ be the natural projection. By assumptions $\vec{b}$ is such that $C_{\vec{b}}$ is a nodal curve with a single node z which is not contained in $p_{\vec{b}}^{-1}(D)$. Moreover since the curve C_b is integral, we get that the pushforward of a torsion free sheaf to X is locally free.

Consider the compactified Jacobian $\overline{J}^{\delta}(C_{\vec{b}})$ consisting of rank one-torsion free sheaves L such that degree of $p_{\vec{b},*}L$ is zero. Since the node is not a marked point, we get a natural filtration of sheaves with quotients supported on the divisor D.

$$p_{\vec{b},*}(L\otimes\mathcal{O}_{C_{\vec{b}}}(-(r-1)R)) \subset \cdots \subset p_{\vec{b},*}(L\otimes\mathcal{O}_{C_{\vec{b}}}(-(r-i)R)) \subset \cdots \subset p_{\vec{b},*}L, \tag{37}$$

where R is the ramification divisor. As in [13], pushing forward a section ϕ of $p^*_{\vec{b}}(K_X D)$ induces a map $\phi : p_{\vec{b},*}L \to p_{\vec{b},*}L \otimes K_X(D)$. Now since the node and the marked points are disjoint, the section ϕ gives the required Higgs field as in the case of smooth spectral curves [37]. This gives the spectral correspondence in the case of degree zero Higgs bundles. Consider the closed variety of $\overline{J}^{\delta}(C_{\vec{b}})$ defined as follows:

$$\overline{\mathrm{Prym}}(C_{\vec{b}}, C) = \{M \in \overline{J}(C_{\vec{b}}) \mid p_{\vec{b},*}M = \mathcal{O}_X\}.$$

Clearly the variety $\overline{\mathrm{Prym}}(C_{\vec{b}}, C)$ gives the Hitchin fiber at $\vec{b} \in \mathcal{B}^{\heartsuit} \backslash \mathcal{B}^0$ (cf. [38, Thm. 6.1]). To complete the proof we need to show that $\overline{\mathrm{Prym}}(C_{\vec{b}}, C)$ is semi-abelian.

Let $n : Y \to C_{\vec{b}}$ be the normalization and $f = p_{\vec{b}} \circ n$ the projection of Y to X. The points of Y over z are a and b, respectively. Let $P \subset J^{\delta}(Y)$ be the Prym for f. Let $L \to Y \times P$ be a Poincaré line bundle which is just the restriction of a Poincaré bundle on $Y \times J^{\delta}(Y)$. For any point y of Y, the line bundle in P (resp. also on $J^{\delta}(Y)$) obtained by restricting L to $y \times P$ (resp. also on $y \times J^{\delta}(Y)$) will be denoted by L_y. Consider the line bundle $A := L^*_b \otimes L_a$ on P (resp. $J^{\delta}(Y)$); it is independent of the choice of the Poincaré bundle L. Now consider the projective bundle $\mathbb{P}(A \oplus \mathcal{O}_P) \to P$ (also on $J^{\delta}(Y)$) and identify the two sections of it given by A and $\mathcal{O}_A$. The resulting varieties $B_P \subset B_{J^{\delta}}$ are semi-abelian. By [18, Thm. 4], $B_{J^{\delta}}$ is identified with $\overline{J}^{\delta}(C_{\vec{b}})$. Moreover, by the choice of δ, we get $B_P \subseteq \overline{\mathrm{Prym}}(C_{\vec{b}}, C)$. The equality follows from the fact that the dimensions of both B_P and $\overline{\mathrm{Prym}}(C_{\vec{b}}, C)$ are the same. This completes the proof. □

Vector fields tangent to fibers of Hit. We get natural functions on $\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$ obtained by pulling back sections of $\mathcal{B}^*$ to $\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$ via the Hitchin map Hit in (36). Since $\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}} \subset \mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$, and the natural Liouville symplectic form on $\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}}$ extends to $\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$, we get Hamiltonian vector fields on $\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$ tangent to the fibers of the parabolic Hitchin map. As the codimension of the complement of $\mathcal{T}^\vee_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}}$ in $\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$ is at least two, we conclude that any class $\mathbb{L}$ in the rational Picard group of $M^{par,ss}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$ extends to entire $\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$. Now the cup product with the relative Atiyah class of $\mathbb{L}$ gives a natural map

$$\pi_{\mathcal{H}*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \longrightarrow R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}\ . \tag{38}$$

Since the map $\pi_{\mathcal{B}}$ in (36) is affine, it follows that $R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ is isomorphic to the locally free sheaf $\pi_{\mathcal{B}*}\big(R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}\big)$. We also have the inclusion $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}} \hookrightarrow \pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S}$. Now consider the map obtained by restricting (38), which on pushing forward gives

$$\begin{tikzcd} \pi_{\mathcal{B}*}\big(\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}\big) \arrow[r, hook] \arrow[rr, bend right, "f_{\mathbb{L}}"'] & \pi_{\mathcal{H}*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}} \arrow[r] & \pi_{\mathcal{B}*}\big(R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}\big)\ . \end{tikzcd} \tag{39}$$

We have the following proposition:

Proposition 4.8. *The coherent sheaves $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}$ and $R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ are both trivial and isomorphic of same rank, where the fibers are just the vector spaces $H^0(A_{\vec{b}},\,\mathcal{T}_{A_{\vec{b}}})$ and $H^1(A_{\vec{b}},\,\mathcal{O}_{A_{\vec{b}}})$, respectively, for any for $\vec{b}\in\mathcal{B}^0$; the isomorphism is given by cup product by a Kähler class on $A_{\vec{b}}$.*

Proof. Cupping with the first Chern class of the pull back of the ample line bundle $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$ from $M^{par,ss}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$ induces a map between coherent sheaves $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}$ and $R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$. Over $\mathcal{B}^0$, the fibers of the coherent sheaf $R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ have constant dimension which equals $\dim A_{\vec{b}}$. Similarly over $\mathcal{B}^0$, because the fibers of the map π_{Hit} are abelian varieties and the sheaf $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}$ is locally free and trivial. Moreover, there is an isomorphism between $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}$ and $R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ induced by the natural isomorphism between $H^0(A_{\vec{b}},\,\mathcal{T}_{A_{\vec{b}}})$ and $H^1(A_{\vec{b}},\,\mathcal{O}_{A_{\vec{b}}})$ given by a Kähler class.

Now for $\vec{b}\in\mathcal{B}^\heartsuit\backslash\mathcal{B}^0$, by Proposition 4.7, we know that the fibers are quasi-abelian varieties $\overline{A}_{\vec{b}}$ and in particular $\dim H^1(\overline{A}_{\vec{b}},\mathcal{O}_{\overline{A}_{\vec{b}}}) = \dim\overline{A}_{\vec{b}}$. Since the codimension of the complement of $\mathcal{B}^\heartsuit$ in $\mathcal{B}$ is at least two and the Hitchin map is flat [11, Corollary 11], [10, Theorem 1.17], it follows that $R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ is locally free on $\mathcal{B}$. As in the case of Abelian varieties, the cup product by a Kähler form induces an isomorphism of $\mathrm{Ext}^0(\overline{A}_{\vec{b}},\mathcal{O}_{\overline{A}_{\vec{b}}})$ with $H^1(\overline{A}_{\vec{b}},\mathcal{O}_{\overline{A}_{\vec{b}}})$. This shows that the coherent sheaf $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}$ is trivial over $\mathcal{B}^\heartsuit$ with fibers given by functions on $\mathcal{B}$. Moreover cupping with the first Chern class of $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$ induces an isomorphism of $\pi_{\mathrm{Hit}_*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/\mathcal{B}}$ with $R^1\,\mathrm{Hit}_*\,\mathcal{O}_{\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$. Thus the proposition follows from Hartogs' theorem and the fact that codimension of the complement of $\mathcal{B}^\heartsuit$ is at least two. □

The following result is a direct consequence of Proposition 4.8.

Corollary 4.9. *Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two rational line bundles on $M^{par,ss}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}$, and let $\vec{b}$ be a generic point of the Hitchin base. Then $f_{\mathbb{L}_1} = f_{\mathbb{L}_2}$ (see* (39)*) if and only if the two homomorphisms $H^0(A_{\vec{b}},\, \mathcal{T}_{A_{\vec{b}}}) \to H^1(A_{\vec{b}},\, \mathcal{O}_{A_{\vec{b}}})$ induced by cupping with the first Chern class of the restrictions of $\mathbb{L}_1$ and $\mathbb{L}_2$ are the same.*

Now the composition of $f_{\mathbb{L}}$ with the natural Hamiltonian vector fields produces a homomorphism

$$h_{\mathbb{L}} \,:\, \pi_{\mathcal{B}*}\mathcal{O}_{\mathcal{B}} \otimes \mathcal{B}^* \longrightarrow R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}\,. \tag{40}$$

Observe that this map $h_{\mathbb{L}}$ is equivariant with respect to the natural $\mathbb{C}^*$ action on $\pi_{B*}\mathcal{O}_{\mathcal{B}} \otimes \mathcal{B}^*$ and the natural action of $\mathbb{C}^*$ on $R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ is of weight -1. Since $H^0(A_{\vec{b}},\, \mathcal{T}_{A_{\vec{b}}})$ is given by vector fields coming from $\mathcal{B}^*$, we have the following lemma:

Lemma 4.10. *The two homomorphisms $h_{\mathbb{L}_1}$ and $h_{\mathbb{L}_2}$ (see* (40)*) coincide if and only if $f_{\mathbb{L}_1} = f_{\mathbb{L}_2}$ (see* (39)*).*

Finally, we would like to relate the map $\cup[\mathbb{L}] : \pi_* \operatorname{Sym}^2\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}\big) \to R^1\pi_*\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}\big)$ with the map $h_{\mathbb{L}}$ in (40). Observe that $\pi_* \operatorname{Sym}^2\left(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}\right)$ injects into $\pi_{\mathcal{H}_*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ as the degree two part. Since the Hitchin map is proper (Lemma 5.7), and its fibers are connected, functions on the Higgs moduli spaces are all pull-backs of functions on the Hitchin base. As described earlier, these functions give Hamiltonian vector fields and hence we have a map

$$\pi_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \hookrightarrow \pi_{\mathcal{H}_*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}} \longrightarrow \pi_{\mathcal{H}*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}\,. \tag{41}$$

Cupping with any section γ of $R^1\pi_{\mathcal{H}*}\Omega_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$ produces a map

$$\pi_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \hookrightarrow \pi_{\mathcal{H}_*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}} \longrightarrow \pi_{\mathcal{H}*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \xrightarrow{\cup\gamma} R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}\,. \tag{42}$$

Consider the inclusion of $R^1\pi_*\mathcal{T}_{M^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S}$ into $R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}$. On the other hand, we have the following exact sequence

$$0 \longrightarrow \mathcal{T}_{M^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \longrightarrow \mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}/\mathcal{I}^2_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}} \longrightarrow \mathcal{O}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}} \longrightarrow 0\,,$$

where $\mathcal{I}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}}$ is the ideal sheaf of $M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$ in the moduli of parabolic Higgs bundles. Since there are no global tangent vector field on $M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}$, it follows from the long exact sequence of cohomology that $R^1\pi_*\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}\big) \cong R^1\pi_*\big(\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}/\mathcal{I}^2_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}}\big)$. Now the restriction induces another map

$$R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}} \longrightarrow R^1\pi_*\big(\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}}/\mathcal{I}^2_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}}\big) \tag{43}$$

which restricts to the identity map on $R^1\pi_*\big(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}\big)$. Hence, combining eqns. (42) and

(43), we have the following diagram

(44)
$$\begin{tikzcd}
& \pi_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S} \arrow[rr, "\cup\gamma"] \arrow[dl, hook'] & & R^1\pi_*\left(\mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}/S}\right) \arrow[d, bend left] \\
\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}} \arrow[r] & \pi_{\mathcal{H}*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \arrow[r, hook] & \pi_{\mathcal{H}*}\mathcal{T}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}/S} \arrow[r, "\cup\gamma"] & R^1\pi_{\mathcal{H}*}\mathcal{O}_{\mathcal{H}^{par,s}_{\widetilde{\boldsymbol{\alpha}},\mathsf{SL}_r}} \arrow[u]
\end{tikzcd}$$

The same proof as in Hitchin [43, p. 379] (see also [59, Prop. C.2.4]) shows that the above diagram commutes up to a scalar, and, by construction, the horizontal map at the bottom is the map h_γ (cf. (40)). Thus we proved the following.

Proposition 4.11. *Consider two elements* $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *in* $\operatorname{Pic}(M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}) \otimes \mathbb{Q}$ *and let* $A_{\vec{b}}$ *be as in Corollary* 4.9. *If the maps between* $H^0(A_{\vec{b}}, \mathcal{T}_{A_{\vec{b}}}) \to H^1(A_{\vec{b}}, \mathcal{O}_{A_{\vec{b}}})$ *induced by cupping with the first Chern classes of restrictions of* $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *are the same, then they also agree on* $\widetilde{\pi}_* \operatorname{Sym}^2 \mathcal{T}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$.

4.4. Abelianization and determinant of cohomology. It is enough to consider the case of parabolic Higgs bundles of degree zero and rank r with full flag and arbitrary parabolic weights $\widetilde{\alpha}$. Consider a generic point $\vec{b}$ of the Hitchin base for the parabolic Higgs moduli space $\mathcal{H}^{par,ss}_{\widetilde{\boldsymbol{\alpha}}}$ with full flag and weights $\widetilde{\boldsymbol{\alpha}}$, and let $\widetilde{p} : \widetilde{C}_{\vec{b}} \to C$ be the spectral cover of C determined by the chosen point $\vec{b}$ of the Hitchin base. The map $\widetilde{p}$ is of degree r and is fully ramified at the points $\boldsymbol{p} = (p_1, \cdots, p_n)$. Let $\boldsymbol{q} = (q_1, \cdots, q_n)$ be the inverse image $\widetilde{p}^{-1}(\boldsymbol{p})$ of the points $\boldsymbol{p}$. It is known [33, 51] that the generic fiber $A_{\vec{b}}$ of the Hitchin map at $\vec{b}$ is exactly the Jacobian $J(\widetilde{C}_{\vec{b}})$. Let L be a line bundle on $\widetilde{C}$ giving a point of $A_{\vec{b}}$ and consider the push-forward $\widetilde{p}_*L$ on C. Consider the divisor $D = p_1 + \cdots + p_n$. There is a natural inclusion of sheaves

$$\widetilde{p}_*(L \otimes \mathcal{O}_{\widetilde{C}_{\vec{b}}}(-(r-1)R)) \subset \cdots \subset \widetilde{p}_*(L \otimes \mathcal{O}_{\widetilde{C}_{\vec{b}}}(-(r-i)R)) \subset \cdots \subset \widetilde{p}_*L \tag{45}$$

with quotients supported on D giving a quasiparabolic structure on π_*L at the points $\boldsymbol{p}$. Here R is the ramification divisor $(\widetilde{p}^*D)_{\mathrm{red}}$. Hence, this gives a rational map from $A_{\vec{b}}$ to the $M^{par}_{\widetilde{\boldsymbol{\alpha}}}$. The fiber of the pull-back of the the parabolic determinant of cohomology $\operatorname{ParDet}(\boldsymbol{\alpha})$ to the abelian variety at the point $L \in A_{\vec{b}}$ is a rational linear combinations of elements of the form

(i) $H^0(\widetilde{C}_{\vec{b}}, L)^\vee \otimes H^1(\widetilde{C}_{\vec{b}}, L) \otimes \det(\widetilde{p}_*L)_{p_i}^{\frac{\chi(\pi_*L)}{r}}$,

(ii) $\det \operatorname{Gr}^j \mathscr{F}(\widetilde{p}_*L)_{p_i} \otimes \det^{-1}(\widetilde{p}_*L)_{p_i}$ for all $1 \leq i \leq n$.

However, observe that the second expression for each p_i is independent of L and is equal to the line $\mathcal{O}_{\widetilde{C}_{\vec{b}}}(-q_i)_{|q_i} = K_{\widetilde{C}_{\vec{b}}|q_i}$. Indeed, this follows from the facts that

- $\det \operatorname{Gr}^j \mathscr{F}(\widetilde{p}_*L)_{p_i} = L_{q_i} \otimes \mathcal{O}_{\widetilde{C}_{\vec{b}}}(-jq_i)_{|q_i} \otimes \mathcal{O}_{\widetilde{C}_{\vec{b}}}(-(j-1)q_i)^{-1}_{|q_i}$
- $\det(\widetilde{p}_*L)_{p_i} = L_{q_i}$

together with the natural flag structure given by (45).

The calculations above show that the pull-back of $\operatorname{ParDet}(\boldsymbol{\alpha})$ to the abelian variety only depends on the factors of type (1). The map $\cup[\mathbb{L}] : H^0(A_{\vec{b}}, \mathcal{T}_{A_{\vec{b}}}) \to H^1(A_{\vec{b}}, \mathcal{O}_{A_{\vec{b}}})$ thus depends only on the level for all $\mathbb{L} \in \operatorname{Pic}(M^{par}_{\widetilde{\boldsymbol{\alpha}}}) \otimes \mathbb{Q}$. Thus we have proved the following proposition:

Proposition 4.12. *Let $\mathbb{L} \in \operatorname{Pic}(M^{par,s}_{\mathsf{SL}_r,\widetilde{\boldsymbol{\alpha}}}) \otimes \mathbb{Q}$, then the natural map induced by the first Chern class of the restriction of $\mathbb{L}$ between $H^0(A_{\vec{b}}, \mathcal{T}_{A_{\vec{b}}}) \to H^1(A_{\vec{b}}, \mathcal{O}_{A_{\vec{b}}})$ depends only on the level of $\mathbb{L}$.*

4.5. Proof of Theorem 4.1. For the convenience of the reader let us recall the statement of Theorem 4.1 from the beginning of the section.

Theorem 4.13. *Let $\mathbb{L}$ be an element of $\operatorname{Pic}(M^{par,rs}_{G,\boldsymbol{\beta}}) \otimes \mathbb{Q}$ of level a. Then as linear maps $\pi_{e*} \operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_{G,\boldsymbol{\beta}}/S} \to R^1\pi_{e*}\mathcal{T}_{M^{par,rs}_{G,\boldsymbol{\beta}}/S}$, we have: $\cup[\mathbb{L}] = \cup a[\mathrm{Det}]$, where* Det *is the determinant of cohomology (nonparabolic) line bundle.*

Proof. The proof follows from Propositions 4.2, 4.6, 4.11, and 4.12 and fact that any line bundle on $M_{G,\boldsymbol{\beta}}$ is obtained as pulled back of a rational multiple of a line bundle on the moduli space of parabolic bundles for $G = \mathsf{SL}_r$. □

5. The parabolic Hitchin connection

In this section we will use Theorem 2.2 and the results from [23] on Ginzburg dglas and the class of the parabolic determinant of cohomology $\mathcal{L}$ to construct a flat projective connection on the vector bundle $\pi_{e*}\mathcal{L}^k$, where $\pi_e : M^{par,rs}_G \to S$ is the projection.

5.1. Definition of the symbol. We first seek a candidate for the symbol map

$$\rho_{par} : \mathcal{T}_S \to \pi_{e*} \operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_G/S}.$$

As in the nonparabolic case, set $\widetilde{\rho} := \rho_{sym} \circ KS_{\mathcal{C}/S}$. Let $k \geq 1$ be a positive integer, and let $\mathcal{L}_\phi$ be a line bundle on $M^{par,rs}_G$ constructed via its identification with Γ-G-bundles of fixed local type, a representation $\phi : G \to \mathsf{SL}_r$, and the restriction of determinant of cohomology from $\widehat{M}_{\mathsf{SL}_r}$. We first recall the main result [23, Cor. 4.13 and Prop. 4.12] that relates the class $\beta(\mathcal{P}, \boldsymbol{\lambda})$ with the Atiyah class of $[\mathcal{L}_\phi]$ of the line bundle $\mathcal{L}_\phi$.

Theorem 5.1. *Let m_ϕ be the Dynkin index of the map $\phi : G \to \mathsf{SL}_r$. Then*

$$\beta(\mathcal{P}, \boldsymbol{\lambda}) = \frac{1}{m_\phi}[\mathcal{L}_\phi]\,. \tag{46}$$

Now we further expand $\mu_{\mathcal{L}_\phi^{\otimes k}} \circ \frac{1}{m_\phi k}\widetilde{\rho}$ and get the following:

$$\begin{aligned}
\mu_{\mathcal{L}_\phi^{\otimes k}} \circ \frac{1}{m_\phi k}\widetilde{\rho} &= \frac{1}{m_\phi}.\big((\cup(k[\mathcal{L}_\phi] - \frac{1}{2}[\Omega_{M^{par,rs}_G/S}])) \circ \frac{1}{k}\rho_{sym} \circ KS_{\mathcal{C}/S}\big) \\
&= \frac{1}{m_\phi}\big(\cup[\mathcal{L}_\phi] \circ \rho_{sym} \circ KS_{\mathcal{C}/S} - \cup\frac{1}{2k}[\Omega_{M^{par,rs}_G/S}] \circ \rho_{sym} \circ KS_{\mathcal{C}/S}\big) \\
&= \cup\beta(\mathcal{L}_\phi) \circ \rho_{sym} \circ KS_{\mathcal{C}/S} - \cup\frac{1}{2m_\phi k}[\Omega_{M^{par,rs}_G/S}] \circ \rho_{sym} \circ KS_{\mathcal{C}/S} \\
&= -\Phi \circ KS_{\mathcal{C}/S} - \cup\frac{1}{2m_\phi k}[\Omega_{M^{par,rs}_G/S}] \circ \rho_{sym} \circ KS_{\mathcal{C}/S}
\end{aligned}$$

$$= -KS_{M^{par,rs}_{G/S}} - \cup \frac{1}{2m_\phi k}[\Omega_{M^{par,rs}_G/S}] \circ \rho_{sym} \circ KS_{\mathcal{C}/S}\ .$$

In the above, we have used the fundamental equalities

$$\beta(\mathcal{L}_\phi) = \frac{1}{m_\phi}[\mathcal{L}_\phi] \ \text{ and } \ \beta(\mathcal{L}_\phi) \circ \rho_{sym} + \Phi = 0.$$

Thus we get the following equation:

$$KS_{M^{par}_G/S} + \mu_{\mathcal{L}^{\otimes k}_\phi} \circ \frac{1}{m_\phi k}\widetilde{\rho} + \cup \frac{1}{2m_\phi k}[\Omega_{M^{par,rs}_G/S}] \circ \rho_{sym} \circ KS_{\mathcal{C}/S} = 0\ . \tag{47}$$

We now have a key result.

Proposition 5.2. *The map $\mu_{\mathcal{L}^{\otimes k}_\phi} : \pi_{e*}\operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_G/S} \to R^1\pi_{e*}\mathcal{T}_{M^{par,rs}_G/S}$ is an isomorphism.*

Proof. Let $Y^{par,rs}_G := \phi^{-1}(\widehat{M}^{rs}_G) \subset M^{par,rs}_G$, where $\phi : M^{par,ss}_G \to \widehat{M}^{ss}_G$ is the natural forgetful map. By Lemma C.3, the codimension of the complement of $Y^{par,rs}_G$ in $M^{par,rs}_G$ is at least three, so it enough to show that $\mu_{\mathcal{L}_\phi}$ is an isomorphism over $Y^{par,rs}_G$. Now by Theorem 4.1, it follows that it suffices to show that $\cup[\mathcal{L}_\phi]$ is an isomorphism. Observe that in the nonparabolic case, the canonical class is a multiple of the ample generator of the Picard group of M_G. Hence, for the nonparabolic case $\mu_{\mathcal{L}_\phi}$ is a nonzero multiple of $\cup[\mathcal{L}_\phi]$. By construction, the map $\cup\mathcal{L}^{\otimes k}_\phi : \pi_{e*}\operatorname{Sym}^2 \mathcal{T}_{Y^{par,rs}_G/S} \to R^1\pi_{e*}\mathcal{T}_{Y^{par,rs}_G/S}$ is first obtained by restricting the map $\cup\mathcal{L}^{\otimes k}_\phi : \pi_{e*}\operatorname{Sym}^2 \mathcal{T}_{\widehat{M}^{rs}_G/S} \to R^1\pi_{e*}\mathcal{T}_{\widehat{M}^{rs}_G/S}$ to $\mathcal{T}_{Y^{par,rs}_G/S}$ and then taking invariants. Consequently, we will be done if we can show that the following map is an isomorphism: $\cup\mathcal{L}^{\otimes k}_\phi : \pi_{e*}\operatorname{Sym}^2 \mathcal{T}_{\widehat{M}^{rs}_G/S} \to R^1\pi_{e*}\mathcal{T}_{\widehat{M}^{rs}_G/S}$. This is proved in [43] and also in [59] in the algebro-geometric set-up for $G = \mathsf{SL}_r$, where $\mathcal{L}$ is the determinant of cohomology line bundle. For an arbitrary G, we can choose a faithful irreducible representation $\phi : G \to \mathsf{SL}_r$ and get a map $f : \widehat{M}_G \to \widehat{M}_{\mathsf{SL}_r}$ which restricts to a map $f : \widehat{M}^{rs}_G \to \widehat{M}^s_{\mathsf{SL}_r}$. Since any short exact sequence of G-modules splits, this induces a splitting of the tangent bundle of the moduli spaces: $f^*\mathcal{T}_{\widehat{M}^s_{\mathsf{SL}_r}/S} = \mathcal{T}_{\widehat{M}^{rs}_G/S} \oplus W$, along with the diagram

$$\begin{tikzcd}
\pi_*\operatorname{Sym}^2 \mathcal{T}_{\widehat{M}^s_{\mathsf{SL}_r}/S} \arrow[r, "\cup\mathbb{L}"] & R^1\pi_*\mathcal{T}_{\widehat{M}^s_{\mathsf{SL}_r}/S} \arrow[d] \\
\pi_{G*}\operatorname{Sym}^2 \mathcal{T}_{\widehat{M}^{rs}_G/S} \arrow[u, "\operatorname{Sym}^2 Df"] \arrow[r, "\cup\mathbb{L}"] & R^1\pi_{G*}\mathcal{T}_{\widehat{M}^{rs}_G/S},
\end{tikzcd} \tag{48}$$

where $\pi : \widehat{M}^s_{\mathsf{SL}_r} \to S$ and $\pi_G : \widehat{M}^{rs}_G \to S$ are the natural projections. Thus, we are again reduced to the case of $G = \mathsf{SL}_r$. □

Since the map $\mu_{\mathcal{L}^{\otimes k}_\phi}$ is an isomorphism, from (47) we get that

(49)

$$KS_{M^{par,rs}_G/S} + \mu_{\mathcal{L}^{\otimes k}_\phi} \circ \Big(\frac{1}{m_\phi k}\rho_{sym} + \mu^{-1}_{\mathcal{L}^{\otimes k}_\phi} \circ (\cup \frac{1}{2m_\phi k}[\Omega_{M^{par,rs}_G/S}]) \circ \rho_{sym}\Big) \circ KS_{\mathcal{C}/S} = 0.$$

Motivated by (49), we *define* the parabolic Hitchin symbol ρ_{par} to be:

$$\rho_{par} := \Big(\frac{1}{m_\phi k} + \mu^{-1}_{\mathcal{L}_\phi^{\otimes k}} \circ (\cup \frac{1}{2m_\phi k}[\Omega_{M_G^{par,rs}/S}])\Big) \circ \rho_{sym} \circ KS_{\mathcal{C}/S}. \tag{50}$$

Remark 5.3. By Theorem 4.1, we see that $\mu_{\mathcal{L}_\phi}$ is a nonzero multiple of $\cup[\mathcal{L}_\phi]$ and hence $\mu^{-1}_{\mathcal{L}_\phi^{\otimes k}} \circ (\cup \frac{1}{2m_\phi k}[\Omega_{M_G^{par,rs}/S}]$ is a nonzero multiple of identity. This is essentially akin to the nonparabolic situation. In the case of the moduli space of rank r vector bundles with trivial determinant, it turns out that the class of the canonical bundle is $[\Omega_{M^s_{\mathsf{SL}_r}/S}] = -2r[\mathcal{L}]$, where $\mathcal{L}$ is the ample generator of the Picard group. Hence, $\mu^{-1}_{\mathcal{L}^{\otimes k}} = -\frac{2r}{r+k}(\cup [\Omega_{M^s_{\mathsf{SL}_r}/S}])^{-1}$, and ρ_{par} in this case is just $\frac{1}{r+k}\rho_{sym} \circ KS_{\mathcal{C}/S}$ as in [43]. Our results also recover and generalize those of [64].

By construction, we get the following:

Lemma 5.4. *The parabolic Hitchin symbol ρ_{par} defined in* (50) *satisfies the condition in Theorem* 2.2 *(i).*

5.2. Welters' condition. In this subsection, we show that for $M = M_G^{par,rs}$, the condition in Theorem 2.2 (ii) is satisfied. In fact, we will prove a stronger statement in the set-up of parabolic G-bundles.

Lemma 5.5. *Let $M_G^{par,rs}$ be the moduli space of regularly stable parabolic G-bundles on a curve C. Then $H^1(M_G^{par,rs}, \mathcal{O}_{M_G^{par,rs}}) = 0$.*

Proof. It suffices to show that the Picard group of the moduli space $M_G^{par,rs}$ is discrete, since the space $H^1(M_G^{par,rs}, \mathcal{O}_{M_G^{par,rs}})$ can be considered as the Lie algebra of the Picard group of $M_G^{par,rs}$. Hence, it is enough to show that the Picard group of the corresponding moduli stack $\mathcal{P}ar_G^{rs}(C, \vec{P})$ is discrete. By [50], it is known that the Picard group of the moduli stack $\mathcal{P}ar_G(C, \vec{P})$ of quasiparabolic G-bundles is discrete. Thus, we will be done if we can show that the codimension of the complement of the regularly stable locus has codimension at least two, as the inclusion will then induce an isomorphism on the Picard groups (cf. [22, Lemma 7.3]). But this is the content of Lemma C.1 below. □

Lemma 5.6. *With the notation of Lemma* 5.5*, $H^0(M_G^{par,rs}, \mathcal{T}_{M_G^{par,rs}}) = 0$.*

Proof. The proof follows the steps given in [43]. Firstly, $\mathcal{T}^\vee_{M_G^{par,rs}}$ embeds into the moduli space of strongly parabolic G-Higgs bundles $\mathcal{H}_G^{par,ss}$. Now given a global vector field on $M_G^{par,rs}$, pairing it with the cotangent bundles produces a function on $\mathcal{T}^\vee_{M_G^{par,rs}}$, which via Hartogs' theorem extends to a function of degree one (with respect to the standard $\mathbb{C}^*$-action) on the Higgs moduli space $\mathcal{H}_G^{par,ss}$. Since the Hitchin fibration is proper (Lemma 5.7) with connected fibers ([42, Sec. 5], [33, Cor. III.3] and [29, Claim 3.5] for nonparabolic Higgs bundles; [33, Cor. V.5] for strongly parabolic with full flags; [69, Sec. 4.5], [75, Thm. 1.2] for all strongly parabolic cases) it descends to a function on the Hitchin base. This is impossible since the degree of homogeneity is one. Thus we are done. □

The following result is well known ([11, Thm. 13], and also [78] for $G = \mathsf{GL}_r$), but for completeness we include a brief proof of it.

Lemma 5.7. *The Hitchin map* $\mathrm{Hit} : \mathcal{H}_G^{par,ss} \to \mathcal{B}$ *is proper.*

Proof. In [21] strongly parabolic Higgs bundles on a curve C are constructed as Γ-G-Higgs bundles on a Γ-cover $\widehat{C}$ of C. Let $\mathcal{H}_G^{ss}(\widehat{C})$ denote the moduli of semistable Higgs bundles on $\widehat{C}$, with Hitchin base $\widehat{\mathcal{B}}$, and Hitchin map $\mathrm{Hit}_{\widehat{C}}$. Note that in the strongly parabolic setting, we have an inclusion $\imath : \mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_G^{par,ss} & \xrightarrow{F} & \mathcal{H}_G^{ss}(\widehat{C}) \\ {\scriptstyle \mathrm{Hit}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hit}_{\widehat{C}}} \\ \mathcal{B} & \xhookrightarrow{\quad \imath \quad} & \widehat{\mathcal{B}} \end{array}$$

Here, F is the forgetful map sending a Γ-G-bundle on $\widehat{C}$ to the underlying G-bundle. Since F and the Hitchin map $\mathrm{Hit}_{\widehat{C}}$ are proper, and the map $\imath$ is a closed embedding, we conclude that Hit is also proper. □

Finally, we are in a position to prove the main theorem.

Proof of the Main Theorem. For the conditions in Theorem 2.2: (i) is the statement of Lemma 5.4, (ii) follows from Lemma 5.5 and Lemma 5.6, and (iii) is the connectedness of the moduli space. For the conditions in Theorem 2.3: (i) follows as in [43], using integrability results in [51], [11] and [75] (ii) follows from Proposition 5.2, and (iii) is the statement in Lemma 5.6. This completes the proof. □

We now apply the main theorem to extend the result to case of the simple groups which are not necessarily simply connected.

Proof of Corollary 1.1. Take $s \in S$ and let C_s be the corresponding smooth n-pointed curve. Consider the moduli space $M_G^{par,rs,0}(C_s) = \pi^{-1}(s)$ for a connected, simple group H. Let G be the simply connected cover of H and $M_G^{par,rs}(C_s)$ the corresponding moduli space. Consider the map between moduli spaces $M_G^{par,rs}(C_s)$ and $M_H^{par,rs,0}(C_s)$ induced by the quotient map $\widetilde{G} \to G$. This map is étale on the base with Galois group Γ which is a subgroup of the center $Z(G)$ of G. Any element $\gamma \in \Gamma$, acts on $M_G^{par,rs}(C_s)$ by twisting. This action of γ evidently commutes with the Hitchin map. Hence, if we consider the same symbol as in the simply connected case, the same arguments in [14, Cor. 5.2 and Lemma 4.1] tell us that the projective connection constructed for simply connected group commutes with the action on Γ. Thus we see that $\pi_*\mathcal{L}_{\boldsymbol{\lambda},k}$ is a twisted D-module, and so it is locally free. □

A. Parabolic G-bundles

Let G be a simple, simply connected complex algebraic group and $(C, \vec{p})$ an n-pointed smooth projective curve of genus g. Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of the group G. We further let Δ_+ denote the set of simple positive roots α_i, and let θ denote the highest root of $\mathfrak{g}$. Define the fundamental alcove

$$\Phi_0 := \{h \in \mathfrak{h} | \, \alpha_i(h) \geq 0, \text{ and } \theta(h) \leq 1 \, \forall \, \alpha_i\} \, .$$

For $h \in \Phi_0$, we denote by $P(h)$ the standard parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{p}(h)$ will denote the corresponding Lie subalgebra of $\mathfrak{g}$. The following result is standard and can be found in [41, Thm. 7.9]:

Lemma A.1. *Let K be a maximal compact subgroup of G. The exponential map*

$$h \to \exp(2\pi\sqrt{-1}h)$$

induces a natural bijection between Φ_0 and the set of K orbits for the adjoint action of K on itself.

For any one parameter subgroup $\varphi : \mathbb{G}_m \to G$, the *Kempf's parabolic subgroup* is defined as: $P(\varphi) := \{g \in G \mid \lim_{t\to 0} \varphi(t)g\varphi(t)^{-1} \text{ exists in } G\}$. Every $\tau \in \Phi_0$ determines a 1-parameter subgroup of G and hence by above a parabolic subgroup $P(\tau)$. It directly follows that the Lie algebra of $P(\tau)$ is the Kempf's parabolic subalgebra

$$\mathfrak{p}(\tau) := \{X \in \mathfrak{g} \mid \lim_{t\to\infty} \mathrm{Ad}(\exp t\tau) \cdot X \text{ exists in } \mathfrak{g}\} \, .$$

We now recall the definition of the moduli stack of quasi-parabolic bundles. We refer the reader to [47, Ch. 5.1]

Definition A.2. *The quasi parabolic moduli stack $\mathcal{P}ar_G(C, \vec{P})$ is the stack parametrizing pairs $(\mathcal{E}, \vec{\sigma})$, where $\mathcal{E}$ is a principal G-bundle on a smooth curve $C \times T$, with T being any scheme, and σ_i are sections over T of $\mathcal{E}_{|p_i \times T}/P_i$ while $\vec{P} = (P_1, \dots, P_n)$ are an n-tuple of standard parabolic subgroups of G.*

We now recall the definition of a parabolic G-bundle on a smooth pointed curve $(C, \vec{p})$.

Definition A.3. *A parabolic structures on a principal G-bundle $E \to C$ is given by the following data:*

- *A choice of parabolic weights $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \Phi_0^n$, where τ_i is the parabolic weight attached to the point $p_i \in C$.*
- *a section σ_i of the homogeneous space $E_{p_i}/P(\tau_i)$, where $P(\tau_i)$ is the standard parabolic associated to $\tau_i \in \Phi_0$.*

A family of parabolic G-bundles parametrized by a scheme T is defined analogously. of a section σ_i for every $1 \leq i \leq n$. Similarly extend the definitions of parabolic structures when G is connected and reductive.

A.1. Uniformization of quasiparabolic bundles. Most of the results in this section can be easily modified for semi-simple groups, however for simplicity we restrict ourselves to the case when G is simple and simply connected.

For any simple, simply algebraic group G, let L_G be the corresponding loop group and $\mathrm{L}_G^+ \subset \mathrm{L}_G$ the subgroup of positive loops. The affine Grassmannian $\mathcal{Q}_G$ is defined to be $\mathrm{L}_G \,/\, \mathrm{L}_G^+$. Let q be a point on a the curve C.

Consider the functor $\mathscr{S}_{G,C\backslash q}$ from the category of k-algebras $\mathcal{A}lg$ to the category $\mathcal{S}ets$ that assigns to an k-algebra R, isomorphism classes of pairs (E_R, σ_R), where E_R is a principal G bundle over $X \times \operatorname{Spec} R$ and σ is a section of E_R over $(C\backslash q) \times \operatorname{Spec} R$ (cf. [50, Sec. 3.5]). The following statement, which uses a crucial *uniformization* result of Drinfeld-Simpson [31], gives a geometric realization of the affine Grassmannian $\mathcal{Q}_G$.

Proposition A.4. *The affine Grassmannian $\mathcal{Q}_G$ represents the functor $\mathscr{S}_{G,C\backslash q}$. Moreover, there is a universal principal G-bundle $\mathbb{U} \to C \times \mathcal{Q}_G$ and section $\sigma_{\mathcal{Q}_G}$, such for any $[E_R, \sigma_R] \in \mathscr{S}_{G,C\backslash q}$ and any morphism $f : \operatorname{Spec} R \to \mathcal{Q}_G$,*

$$[(\mathrm{id} \times f)^*\mathbb{U}, (id \times f)^*\sigma_{\mathcal{Q}_G}] = [E_R, \sigma_R]\ .$$

Let $\mathrm{L}_{C,\vec{p}}(G)$ be the punctured loop ind-group $\mathrm{Mor}(C\backslash\vec{p}, G)$ that parametrizes morphisms $C\backslash\vec{p} \to G$ from the punctured curve. The following result of Laszlo-Sorger [50] expresses the moduli stack of principal G-bundles as a quotient stack.

Proposition A.5. *The stacks $\mathcal{P}ar_G(C, \vec{P})$ and $\mathrm{L}_{C,q}(G)\backslash\,(\mathcal{Q}_G \times \prod_{i=1}^n G/P_i)$ are isomorphic, where q is a point on $C\backslash\vec{p}$ and $\mathrm{L}_{C,q}(G)$ acts on G/P_i by evaluation at the point p_i. Moreover,* $\mathrm{Pic}(\mathrm{Par}_G(C, \vec{P})) \cong \mathbb{Z} \times \prod_{i=1}^n \mathrm{Pic}(G/P_i)$ *if G is simply connected.*

We now describe another uniformization of the moduli stack $\mathcal{P}ar_G(C, \vec{P})$ that connects directly to the moduli stack of Γ-equivariant bundles of fixed topological type that will be discussed in Appendix B. For an n-tuple of points $\vec{p} = (p_1, \ldots, p_n)$, we choose formal parameters t_i at p_i, i.e., $\widehat{\mathcal{O}}_{C,p_i} = \mathbb{C}((t_i))$. Consider the natural evaluation map at $t_i = 0$, $ev_0 : G[[t_i]] \longrightarrow G$, from the Iwahori subgroup $G[[t_i]]$. For any standard parabolic subgroup $P_i \subset G$, we denote by $\mathcal{P}_j := ev_0^{-1} P_i$ the standard parahoric subgroup of the loop group. Now consider the reduced ind-scheme $\mathrm{L}_{C,\vec{p}}(G)$ as discussed above. Then any element of $\mathrm{L}_{C,\vec{p}}(G)$ acts on $G((t_i))/\mathcal{P}_j$ via Laurent expansion at the point p_i in the local parameter t_i. As in Proposition 2.8 of [48], we have a family of principal G-bundles $\mathbb{U}_{par}$ on $C \times \prod_{i=1}^n G((t_i))/G[[t_i]]$ such that the following three hold:

(i) The bundle $\mathbb{U}_{par}$ is $\mathrm{L}_{C,\vec{p}}(G)$ equivariant.

(ii) There is a section σ_{par} of $\mathbb{U}_{par}$ over $(C\backslash\vec{p}) \times \prod_{i=1}^n G((t_i))/G[[t_i]]$ which extends to a section on a formal disc around the punctures p_i.

(iii) The section σ_{par} satisfies the condition $\gamma \cdot \sigma(q, [g_1], \ldots, [g_n]) = \sigma(q, [g_1], \ldots, [g_n])\gamma(q)$, where $[g_i]$ is the class of an element $g_i \in G((t_i))$, $\gamma \in \mathrm{L}_{C,\vec{p}}(G)$ and $q \in C\backslash\vec{p}$. Moreover, the pair $(\mathbb{U}_n, \sigma)$ is unique up to an unique isomorphism satisfying the above properties.

Now pulling back $\mathbb{U}_{par}$ via the natural $\mathrm{L}_{C,\vec{p}}(G)$-equivariant projection

$$\prod_{i=1}^n G((t_i))/\mathcal{P}_i \to \prod_{i=1}^n G((t_i))/G[[t_i]]\ ,$$

we obtain a natural $\mathrm{L}_{C,\vec{p}}(G)$-equivariant principal G-bundle on $C \times \prod_{i=1}^n G((t_i))/\mathcal{P}_i$. Hence, using this G-bundle $\mathbb{U}_{par}$ and the section σ_{par}, we obtain the following well known result ([47, pp. 181-182], and also [9, Prop. 3.3], [50, Thm. 1.3]):

Proposition A.6. *The stack $\mathcal{P}ar_G(C, \vec{P})$ is isomorphic to $\mathrm{L}_{C,\vec{p}}(G)\backslash \prod_{i=1}^n G((t_i))/\mathcal{P}_i$.*

A.2. Parabolic bundles and associated constructions. Let $P \subset G$ be a standard parabolic subgroup with Levi subgroup L_P containing a maximal torus H. Consider the set S_P of simple roots of the Levi subalgebra L_P of the parabolic P. If $P = P(h)$ for some $h \in \Phi_0$, then $S_P := \{\alpha_i \in \Delta_+ \mid \alpha_i(h) = 0\}$. The group of characters $X(P)$ of the parabolic subgroup P can be identified with the subset of the dual Cartan subalgebra

$$\mathfrak{h}^\vee_{\mathbb{Z},P} := \{\lambda \in \mathfrak{h}^\vee \ \mid \ \lambda(\alpha_i^\vee) \in \mathbb{Z},\ \forall \alpha_i, \text{ and } \lambda(\alpha_i^\vee) = 0,\ \forall \alpha_i \in S_P\}.$$

In terms of the fundamental weights $\omega_1, \ldots, \omega_\ell$ of the Lie algebra $\mathfrak{g}$, we get that

$$\mathfrak{h}^\vee_{\mathbb{Z},P} := \bigoplus_{\alpha_i \neq S_P} \mathbb{Z}\omega_i.$$

Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \Phi_0^n$ be a choice of parabolic weight.

We further assume that each $\tau_i \in \Phi_0$ is *rational* , i.e., we can write $\tau = \overline{\tau}_i/d_i$ for some positive integers d_i and $\exp(2\pi\sqrt{-1}\overline{\tau}_i) = 1$, so $d_i \cdot \tau$ is in the coroot lattice (i.e. in the lattice spanned by the set of coroots $\Phi^\vee \subset \mathfrak{h}$). The integers d_i are not unique.

If $G = \mathsf{SL}_r$ and consider the standard representation of SL_r, then a choice of a rational $\tau \in \Phi_0$ via the normalized Killing form κ is the same as the choice of an integer $k \leq r$, a sequence of integers $\boldsymbol{r} := (r_1, \ldots, r_k)$ such that $\sum_{i=1}^k r_i = r$ and a nondecreasing sequence $0 \leq \alpha_1 < \cdots < \alpha_k < 1$. Hence a rational parabolic structure on a vector bundle $\mathscr{V}$ on a curve C associated to a parabolic SL_r-bundle at the points $p_1, \ldots, p_n$ is equivalent to the following:

(i) A choice of a flag of the fiber $\mathscr{V}_{|p_i}$ associated to the k_i-tuple $\boldsymbol{r}_i$ for each $1 \leq i \leq n$

$$\mathscr{F}_{\bullet,p_i} := \left(0 \subseteq F_{k_i+1}(\mathscr{V}_{|p_i}) \subseteq F_{k_i}(\mathscr{V}_{|p_i}) \subseteq \cdots \subseteq F_1(\mathscr{V}_{|p_i}) = \mathscr{V}_{|p_i}\right)$$

such that $\dim \mathrm{Gr}^j \mathscr{F}_{\bullet,p_i} = r_{j,p_i}$.

(ii) For each p_i, a sequence of rational numbers $\boldsymbol{\alpha}_{p_i}$

$$0 \leq \alpha_{1,i} < \cdots < \alpha_{k_i,i} < 1. \tag{51}$$

We refer the reader to Mehta-Seshadri [53] (for parabolic vector bundles), Ramanathan [62], Biswas ([20] and [19]), Balaji-Seshadri [9] and Balaji-Biswas-Nagaraj [8] for the notions of stability and semistability which is essential in defining the corresponding moduli spaces.

The following theorem is due to Mehta-Seshadri [53] for parabolic vector bundles of rank r and weight data $\boldsymbol{\alpha}$ and we will denote the moduli space by $M^{par,ss}_{\boldsymbol{\alpha},r}(C)$. It was proven for arbitrary semi-simple groups by Bhosle-Ramanathan [17]. Following the work of Seshadri ([66] and [67]), Balaji-Biswas-Nagaraj [8], Balaji-Seshadri [9], we will discuss an alternative realization in the following section.

Theorem A.7. *Let $(C,\vec{p})$ be a n-pointed smooth projective curve of genus g, and let $\boldsymbol{\tau} = (\tau_1,\dots,\tau_n)$ be a choice of rational parabolic weights in the fundamental alcove Φ_0. We further assume that $\theta(\tau_j) < 1$, where θ is the highest root of $\mathfrak{g}$. Then, the parabolic semistable G-bundles with a choice of rational parabolic weights $\boldsymbol{\tau}$ admit a coarse moduli space $M^{par,ss}_{G,\boldsymbol{\tau}}(C)$ which is a normal irreducible projective variety with rational singularities. Moreover, if $\iota : G \to G'$ is an embedding of connected simple, simply connected groups, then the corresponding map between the moduli space $M^{par,ss}_{G,\boldsymbol{\tau}} \to M^{par,ss}_{G',\boldsymbol{\tau}'}$ is finite. Here $\boldsymbol{\tau}' = \iota(\boldsymbol{\tau})$.*

For notational convenience, when the context is clear we will often suppress the subscript $\boldsymbol{\tau}$ and use $M^{par,ss}_G$ instead.

Definition A.8. *A parabolic G-bundle $\mathcal{P}$ with weights $\boldsymbol{\tau}$ is said to be regularly stable if it is stable and the automorphism group of $\mathcal{P}$ is the center $Z(G)$ of G.*

A.3. Line bundles on parabolic moduli spaces. In this section, we first recall the *determinant of cohomology* line bundle associated to a family of vector bundles $\mathcal{E}$ on a curve C parametrized by a connected Noetherian scheme T. Let $\pi_T : T \times X \to T$ be the projection to the Noetherian scheme, and consider $R\pi_{T,*}\mathcal{E}$ as an object of the bounded derived category $D^bCoh(T)$. We can represent $R\pi_{T,*}\mathcal{E}$ by a complex $\mathcal{E}_0 \to \mathcal{E}_1 \to 0$ of vector bundles on T. We define the determinant line bundle up to a unique isomorphism to be the following:

$$\operatorname{Det}\mathcal{E}_T := \bigwedge^{top}\mathcal{E}_1 \otimes \bigwedge^{top}\mathcal{E}_0^{\vee}.$$

We often drop T in the notation of $\operatorname{Det}\mathcal{E}_T$ when the context is clear. For any closed point $t \in T$, the fiber of $\operatorname{Det}\mathcal{E}_T$ over t is $\bigwedge^{top}\left(H^1(C,\mathcal{E}_t)\right) \otimes \bigwedge^{top}\left(H^0(C,\mathcal{E}_t)\right)^{\vee}$. The determinant bundle has the following important properties:

(i) For any morphism $f : T' \to T$, we have $\operatorname{Det}(f \times \mathrm{id})^*\mathcal{E})_{T'} = f^*\operatorname{Det}\mathcal{E}_T$.

(ii) For any line bundle $L \to T$, we have $\operatorname{Det}(\mathcal{E})_T \otimes L^{-\chi(\mathcal{E}_0)} = \operatorname{Det}(\mathcal{E} \otimes \pi_T^*L)_T$, where $\chi(\mathcal{E}_0)$ is the Euler characteristic of the vector bundle $\mathcal{E}_{|t\times C}$ for any point $t \in T$.

(iii) For any short exact sequence of bundles $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ on $T \times C$, we have $\operatorname{Det}\mathcal{E}_{1,T} \otimes \operatorname{Det}\mathcal{E}_{2,T} = \operatorname{Det}\mathcal{E}_T$.

Let $\mathscr{SU}_C(r,\xi)$ be the moduli space of semistable vector bundles of rank r on a curve C with determinant ξ of degree m. It was proved by [30] that the Picard group of $\mathscr{SU}_C(r,\xi)$ is $\mathbb{Z}\cdot\Theta$, where Θ is the ample generator. The following result of Drezet-Narasimhan [30] connects the determinant of cohomology with this Θ-line bundle.

Proposition A.9. *Let $\psi_{\mathcal{E}} : T \to \mathscr{SU}_C(r,\xi)$ be the morphism corresponding to a family $\mathcal{E}$ of semistable bundles of rank r and determinant ξ parametrized by a scheme T. Then the pullback of Θ via $\psi_{\mathcal{E}}$ is isomorphic to $(\operatorname{Det}\mathcal{E}_T)^{\frac{r}{(r,m)}} \otimes \left(\det\mathcal{E}_{|T\times p}\right)^{\frac{\chi}{(r,m)}}$, where p is any point on the curve C, m is the degree of the line bundle ξ, (r,m) is the greatest common divisor and $\chi = \chi(F_{|t\times C}) = m + r(1-g)$.*

Motivated by the above proposition, we define the following:

Definition A.10. *For any family $\mathcal{E}$ of vector bundles of rank r and determinant ξ of degree m, parametrized by a connected Noetherian scheme T, we define the theta-bundle*

$$\Theta(\mathcal{E}) := (\operatorname{Det} \mathcal{E}_T)^{\frac{r}{(r,m)}} \otimes \left(\det \mathcal{E}_{|T\times p}\right)^{\frac{\chi}{(r,m)}} , \tag{52}$$

where χ as in Proposition A.9 *is the Euler characteristic.*

Note that for any line bundle $\mathcal{L}$ over T, we have an isomorphism $\Theta(\mathcal{E}) \cong \Theta(\mathcal{E} \otimes \pi_T^* L)$. Similarly for any simple, simply connected algebraic group G and any family $\mathcal{E}$ of principal G-bundles on C parametrized by a scheme T, we can associate a natural line bundle on T as follows: Let (φ, V) be a representation of the group G. Then the associated vector bundle

$$\mathcal{E}(V) := \mathcal{E} \times^{\varphi} V$$

is a family of vector bundles on C parametrized by T. Observe that since G is simple, and hence G does not have any nontrivial character, it follows that $\mathcal{E}(V)$ has trivial determinant over $T \times C$. We define a line bundle on T

$$\operatorname{Det}(\mathcal{E}, \varphi)_T := \operatorname{Det}(\mathcal{E}(V))_T. \tag{53}$$

It follows from (52) that $\Theta(\mathcal{E}(V)) = \operatorname{Det}(\mathcal{E}, \varphi)_T$.

The parabolic determinant of cohomology in the SL_r case. We follow the notation and conventions as in [24]. Let $\mathcal{E}$ be a family of quasiparabolic SL_r bundles on a pointed curve $(C, \vec{p})$ parametrized by a scheme T considered as a parabolic vector bundle via the standard representation. Let $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_{p_1}, \ldots, \boldsymbol{\alpha}_{p_n})$ be a n-tuple of sequence of rational numbers as in (51) associated to each marked point p_i, $1 \le i \le n$. Consider the following element in $\operatorname{Pic}(T) \otimes \mathbb{Q}$,

$$\operatorname{Det} \mathcal{E}_T + \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{j,i} \det \operatorname{Gr}^j \left(\mathscr{F}_{\bullet, p_i}(\mathcal{E}_{|T\times p_i})\right) , \tag{54}$$

where the rational number $0 \le \alpha_{1,i} < \alpha_{2,i} < \cdots < \alpha_{k_i,i} < 1$ define $\boldsymbol{\alpha}_{p_i}$. Write $\alpha_{j,i} = b_{j,i}/q_{j,i}$, where $b_{j,i}$ and $q_{j,i}$ are relatively prime integers.

Definition A.11. *Let N be the least common multiple of all $\{q_{j,i}\}_{i,j}$, $1 \le i \le n$ and $1 \le j \le k_i$. We refer to N as the level of the weight $\boldsymbol{\alpha}$.*

Consider the integers $a_{j,i} := N \cdot \alpha_{j,i}$. Then for each $1 \le i \le n$ and $1 \le j \le k_i$,

$$0 \le a_{1,i} < a_{2,i} < \cdots < a_{k_i,i} \le N - 1.$$

Definition A.12. *Let $\mathcal{E}$ be a family of degree zero parabolic vector bundles on $T \times C$ with parabolic data $\{(\boldsymbol{r}_i, \boldsymbol{\alpha}_i)\}_{i=1}^n$. The parabolic determinant bundle on T is defined to be*

$$\operatorname{Det}_{par} \mathcal{E}_T(\boldsymbol{\alpha}) := (\operatorname{Det} \mathcal{E}_T)^{\otimes N} \bigotimes \left(\otimes_{i=1}^n \left(\otimes_{j=1}^{k_i} \det \operatorname{Gr}^j \mathscr{F}_{\bullet, p_i}(\mathcal{E}_{|T\times p_i})^{a_{j,i}}\right)\right) .$$

This is just eq. (54) multiplied by N. When the context is clear, we will simply denote $\operatorname{Det}_{par} \mathcal{E}_T(\boldsymbol{\alpha})$ by $\operatorname{Det}_{par} \mathcal{E}_T$. The line bundle $\operatorname{Det}_{par}(\mathcal{E}_T)$ may not descend to the moduli space, so we consider the following modification.

Definition A.13. *The parabolic Θ_{par}-line bundle on T is defined to be the following twist of parabolic determinant of cohomology*

$$\Theta_{par}(\mathcal{E}, \boldsymbol{\alpha}) := (\mathrm{Det}_{par}\, \mathcal{E}_T) \otimes \left(\det \mathcal{E}_{|T\times p_0}\right)^{\frac{N\cdot \chi_{par}}{r}}$$

(just as in the nonparabolic case), where $\chi_{par} = \chi - \sum_{i,j} \alpha_{j,i} r_{j,i}$ is the parabolic Euler characteristic (see [24, p. 60]*), χ is as in Proposition* A.9 *and p_0 is any point on the curve $C\backslash\vec{p}$.*

We remark that the definition of $\Theta_{par}(\mathcal{E}, \boldsymbol{\alpha})$ differs from the definition of parabolic determinant [24, Def. 4.8] by a multiplicative factor of r. The following proposition can be found in Biswas-Raghavendra [24], Pauly [58], and in Narasimhan-Ramadas [55] for $G = \mathsf{SL}_2$.

Proposition A.14. *Let $\psi_{\mathcal{E}} : T \to M^{par,ss}_{\boldsymbol{\alpha},r}(C)$ be a map from a scheme T to the Mehta-Seshadri moduli space $M^{par,ss}_{\boldsymbol{\alpha},r}(C)$ of parabolic bundles corresponding to a family $\mathcal{E}$ equipped with parabolic data $\boldsymbol{\alpha}$. Then there exists an ample line bundle $\Theta_{par}(\boldsymbol{\alpha})$ on $M^{par,ss}_{\boldsymbol{\alpha},r}(C)$ such that $\psi^*_{\mathcal{E}}\Theta_{par}(\boldsymbol{\alpha})$ is isomorphic to the line bundle* $\mathrm{Det}_{par}\, \mathcal{E}_T \otimes \left(\det \mathcal{E}_{|T\times p_0}\right)^{\frac{N\cdot\chi_{par}}{r}}$.

As discussed, the choice of the standard representation gives a map of the moduli stacks $\xi : M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\alpha}}(C) \to M^{par,ss}_{\boldsymbol{\alpha}}(C)$, the map $\psi_{\mathcal{E}}$ factors through $M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$ and we will use the notation $\Theta_{par}(\boldsymbol{\alpha})$ to also denote the pull back $\xi^*\Theta_{par}(\boldsymbol{\alpha})$.

The case of general groups. We first recall the notion of Dynkin index of an embedding. Let $\phi : \mathfrak{s}_1 \to \mathfrak{s}_2$ be a map of two simple Lie algebras, and let $\kappa_{\mathfrak{s}_1}$ (respectively, $\kappa_{\mathfrak{s}_2}$) be the normalized Killing form of $\mathfrak{s}_1$ (respectively, $\mathfrak{s}_2$).

Definition A.15. *The Dynkin index m_ϕ of a map of simple Lie algebras ϕ is the ratio of their normalized Killing forms, in other words, $\kappa_{\mathfrak{s}_2}(\ ,\ )_{|\mathfrak{s}_1} = m_\phi \kappa_{\mathfrak{s}_1}(\ ,\ )$.*

Let G be a simple, simply connected group, and let $\mathcal{E}$ be a principal G bundles on $T \times C$. Let (ϕ, V) be a representation of G, and consider the associated vector bundle $\mathcal{E}(V) := \mathcal{E} \times^G V$ on $T \times C$. Since G does not have any nontrivial character (it is simple), it follows that $\det \mathcal{E}(V) \cong \mathcal{O}_{T\times C}$. This implies $\Theta(\mathcal{E}(V)) = \mathrm{Det}(\mathcal{E}(V))_T$. Let $(\mathcal{E}, \vec{\sigma})$ be a family of quasi-parabolic G-bundles of type $\vec{P} = (P_1, \ldots, P_n)$ on a n-pointed curve $(C, \vec{p})$ parametrized by a scheme T.

Definition A.16. *For any positive integer d (usually it will be determined by the weights $\boldsymbol{\mu}$), a finite dimensional representation (ϕ, V) of the group G and a character μ_j of the parabolic P_j, define a line bundle on T by the following formula:*

$$\mathrm{Det}_{par}(\mathcal{E}(V), d, \boldsymbol{\mu}) := (\mathrm{Det}(\mathcal{E}(V))_T)^{\otimes d} \bigotimes \left(\otimes_{j=1}^n \sigma_j^*\left(\mathcal{E} \times^{P_j} \mathbb{C}_{\mu_j^{-1}}\right)\right) \tag{55}$$

(see [50]*), where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\mathbb{C}_{\mu_j^{-1}}$ is the one dimensional representation of the parabolic subgroup P_j corresponding to the character μ_j^{-1} of it. This line bundle will be called the* quasiparabolic determinant bundle. *We will refer to the integer d as the level of the quasiparabolic determinant bundle.*

Now let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ be an n-tuple of rational parabolic weights such that $\theta(\tau_i) < 1$ for all $1 \leq i \leq n$, where θ is the highest root of $\mathfrak{g}$. Consider a representation of G of V such that

(i) the representation (ϕ, V) is faithful;

(ii) the topological local type $\phi(\boldsymbol{\tau})$ of the associated bundle is rational;

(iii) $\theta_{\mathfrak{sl}(V)}(\phi(\tau_i)) < 1$ for all $1 \leq i \leq n$, where $\theta_{\mathfrak{sl}(V)}$ is the highest root of $\mathfrak{sl}(V)$.

We now recall the definition of the parabolic theta bundle for any simple group G. Using the Killing form $\kappa_{\mathfrak{g}}$ we will identify $\nu_{\mathfrak{g}} : \mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^\vee$ and realize $\boldsymbol{\tau}$ in the weight lattice of P of G. Let (ϕ, V) be a faithful representation of G satisfying the above conditions, and let d be any positive integer such that

$$\exp(2\pi\sqrt{-1}\nu_{\mathfrak{sl}(V)}(d \cdot \phi(\tau_i))) = 1 \tag{56}$$

for all $1 \leq i \leq n$. This d is not unique but usually one choose a minimal such d and denote it by N.

Definition A.17. *The parabolic theta bundle $\Theta_{par,G}(V, \boldsymbol{\tau}) \to M^{par,ss}_{G,\boldsymbol{\tau}}$ is defined to be the pull-back of $\Theta_{par,\mathsf{SL}(V)}(\phi(\boldsymbol{\tau})) \to M^{par,ss}_{\mathsf{SL}(V),\phi(\boldsymbol{\tau})}$ via the map $\overline{\phi} : M^{par,ss}_{G,\boldsymbol{\tau}} \to M^{par,ss}_{\mathsf{SL}(V),\phi(\boldsymbol{\tau})}$ induced by the representation (ϕ, V) of G, i.e., $\Theta_{par,G}(V, \boldsymbol{\tau}) := \overline{\phi}^* \Theta_{par,\mathsf{SL}(V)}(\phi(\boldsymbol{\tau}))$.*

The following well known result analogous to the SL_r case (cf. [47, Lemma 8.5.5]) relates the parabolic determinant of cohomology for arbitrary simple, simply connected groups G to the parabolic theta bundle.

Proposition A.18. *Let $\mathcal{E}$ be a family of parabolic G-bundles parametrized by a scheme T with parabolic data $\boldsymbol{\tau} \in \Phi_0^n$ satisfying the condition $\theta(\tau_i) < 1$ for all $1 \leq i \leq n$, and let $\psi_{\mathcal{E}} : T \to M^{par,ss}_{G,\boldsymbol{\tau}}$ be as before the map induced by $\mathcal{E}$.*

Further, let (ϕ, V) be a representation of G satisfying the above conditions. Then the pull-back $\psi^*_{\mathcal{E}}(\Theta_{par,G}(V, \boldsymbol{\tau}))$ equals $\mathrm{Det}_{par}(\mathcal{E}(V), N \cdot m_\phi \cdot \nu_{\mathfrak{g}}(\boldsymbol{\tau}))$, where m_ϕ is the Dynkin index of the map $\phi : \mathfrak{g} \to \mathfrak{sl}(V)$, $\mathrm{Det}_{par}(\mathcal{E}(V), N \cdot m_\phi \cdot \nu_{\mathfrak{g}}(\boldsymbol{\tau}))$ is as in Equation (55) *and N is the minimal positive integer satisfying* (56) *in Definition* A.17.

B. Γ-equivariant G-bundles

In this section, we recall the correspondence between parabolic bundles on a curve C and equivariant bundles on a ramified Galois cover $\widehat{C} \to C$ with Galois group Γ. Throughout this section G will be a simple, simply connected (or more generally simple but not simply connected) algebraic group. We start with the well-known genus computation of an orbifold curve. Let $\vec{p} = (p_1, \ldots, p_n)$ be points in C, and choose positive integers $\vec{d} = (d_1, \ldots, d_n)$, respectively.

Definition B.1. *The orbifold genus associated to $(C, \vec{p}, \vec{d})$ is*

$$g(\mathscr{C}) := g(C) + \frac{1}{2}\sum_{i=1}^{n}(1 - \frac{1}{d_i}) ,$$

where $g(C)$ be the genus of the curve C.

If C is a quotient of $\widehat{C}$ by Γ with ramification locus $p_1, \ldots, p_n$ of degrees $(d_1, \ldots, d_n)$, then by the Riemann-Hurwitz formula the genus of $\widehat{C}$ is given by the formula: $2 - 2g(\widehat{C}) = |\Gamma|\big(2 - 2g(C) + \sum_{i=1}^{n}(\frac{1}{d_i} - 1)\big)$. The genus of the quotient stack $\mathscr{C} := [\widehat{C}/\Gamma]$ is related to $g(\widehat{C})$ by the formula $g(\widehat{C}) - 1 = |\Gamma|(g(\mathscr{C}) - 1)$, and so we see that this is the orbifold genus defined above.

Conversely, given $\vec{p}$ and $\vec{d}$, then provided $g(\mathscr{C}) \geq 1$ we can find a branched cover $\widehat{C} \to C$ as above. For example, if $g(\mathscr{C}) > 1$ (we shall only be interested in this case), then C can be realized as a quotient of the upper half plane $\mathbb{H}$ by a Fuchsian group Π (cf. [73, Sec. 3.2]). The action of Π is not free: it contains elliptic elements of order d_i in the points over p_i. Applying the Selberg lemma to $\Pi \subset \mathrm{Aut}(\mathbb{H})$ (cf. [65]), we obtain a normal subgroup Π_0 of finite index that acts freely on $\mathbb{H}$. Let $\widehat{C} = \mathbb{H}/\Pi_0$. Since the action of Π_0 is free, we get that $\widehat{C}$ is a smooth projective curve. If we set $\Gamma = \Pi/\Pi_0$, then the natural map $\widehat{C} \to C$ is a ramified Galois cover with Galois group Γ.

Example B.2. *Assume that $g(C) = 0$, $d_1 = \cdots = d_n = d$ and d divides n. Then the super-elliptic curve $\widehat{C}$ given by the equation $y^d = \prod_{i=1}^{n}(x - p_i)$ is a ramified Galois covering of $C = \mathbb{P}^1$. The Galois group is $\mathbb{Z}/d\mathbb{Z}$ with ramifications of order d exactly at the points $p_1, \ldots, p_n$, and étale on the complement. Then we have $g(\mathscr{C}) = n(d-1)/2d$. Hence, $g(\mathscr{C}) \geq 1$ if $n \geq 2d/(d-1)$.*

Definition B.3. *Let $p : \widehat{C} \to C$ be a ramified Galois cover with Galois group Γ. A Γ-G-bundle $\widehat{E}$ on $\widehat{C}$ is a principal G bundle on $\widehat{C}$ together with a lift of the action of Γ on $\widehat{C}$ to an action of Γ on the total space of $\widehat{E}$ as bundles automorphism (meaning the actions of Γ and G on $\widehat{E}$ commute).*

Let $\mathcal{R}$ denote the set of branch points of C. For each point $p \in \mathcal{R}$, we choose a point $\widehat{p} \in \widehat{C}$ in the preimage of p, and let $\Gamma_{\widehat{p}} \subset \Gamma$ denote the stabilizer of the point $\widehat{p}$.

Definition B.4 (Balaji-Seshadri [9]). *The* type of a homomorphism $\rho : \Gamma \to G$ *is the set of isomorphism classes of the local representations $\rho_i : \Gamma_{\widehat{p}_i} \to G$, or equivalent, it is the set of conjugacy classes in G given by the images of $\rho_i(\gamma_i)$, where γ_i is a generator of the cyclic group $\Gamma_{\widehat{p}_i} = \langle \gamma_i \rangle$. The type of a homomorphism is denoted by $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$, where $n = |\mathcal{R}|$.*

Let $\widehat{p}_i$ be any branch point of $\widehat{C}$, and let $\widehat{t}_i$ be a *special formal parameter* at the point $\widehat{p}_i$, such that $\gamma \cdot \widehat{t}_i := (\exp(2\pi\sqrt{-1}/d_i)\widehat{t}_i$, where γ is a generator of the stabilizer $\Gamma_{\widehat{p}_i}$ and $d_i = |\Gamma_{\widehat{p}_i}|$. Any (Γ, G)-bundle $\widehat{E}$ is trivial as a G-bundle on a formal disk $D_{\widehat{p}_i} := \mathrm{Spec}[[\widehat{t}_i]]$, and in particular $\widehat{E}_{|D_{\widehat{p}_i}}$ is a $(\Gamma_{\widehat{p}_i}, G)$ bundle. So any $(\Gamma_{\widehat{p}_i}, G)$-bundle on $D_{\widehat{p}_i}$ is determined by a homomorphism $\rho_i : \Gamma_{\widehat{p}_i} \to G$ such that $\gamma \cdot (u, g) = (\gamma \cdot u, \rho_i(\gamma)g)$, where $u \in D_{\widehat{p}_i}$. Moreover such an homomorphism is unique up to conjugation. We refer the reader to [72, Lemma 2.5] and [47, Thm. 6.1.9].

Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ be the unique element of the Weyl alcove Φ_0 such that $\rho_i(\gamma_i)$ is conjugate to $\exp(2\pi\sqrt{-1}\tau_i)$ as described by Lemma A.1. We define the *local type* of a Γ-G-bundle $\widehat{E}$ to be the n-tuple $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and consider the following stack:

Definition B.5. *Let $\widehat{C}$ be a ramified Galois cover of C. Choose points $\widehat{p}_i$ for each point p_i in $\mathcal{R}$, and let $\boldsymbol{\tau}$ be an n-tuple of elements in Φ_0. We define the moduli stack $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ to be the groupoid parametrizing Γ-G-bundles on $\widehat{C}$ of local type $\boldsymbol{\tau}$.*

B.1. Uniformization of Γ-G-bundles of fixed local type. We will now discuss a uniformization theorem for $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ under the further assumption that $\theta(\tau_i) < 1$ for $1 \leq i \leq n$. We will show that the stack $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ is isomorphic to $\mathcal{P}ar_G(C, \vec{P})$, where $\vec{P} = (P_1, \dots, P_n)$ are standard parabolic subgroups of G determined by $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$.

As in the case of parabolic bundle we consider the functor $\mathscr{S}^{\boldsymbol{\tau}}_G : \mathcal{A}lg \to \mathcal{S}ets$ that assigns to a finitely generated k-algebra R the isomorphism classes of pairs $(\widehat{E}_R, \widehat{\sigma}_R)$, where

- $\widehat{E}_R$ is a (Γ, G) bundle over $\widehat{C} \times \operatorname{Spec} R$ of local type τ_i at the points $\widehat{p}_i$, and
- $\widehat{\sigma}_R$ is a Γ-equivariant section of $\widehat{E}_R$ over $p^{-1}(C \backslash \vec{p}) \times \operatorname{Spec} R$.

By [9, Prop. 3.1.1] and [47, Thm. 6.1.12], the functor $\mathscr{S}^{\boldsymbol{\tau}}_G$ is represented by the ind-scheme $\prod_{i=1}^n G((t_i))/\mathcal{P}_i$, where $t_i = (\widehat{t}_i)^{d_i}$ are local parameters at the points p_i and $\mathcal{P}_i$ are parabolic subgroups of the loop group $G((t_i))$. The following theorem is due to Balaji-Seshadri [9, Prop. 3.1.1] and it can also be found in Kumar [47, Thm. 6.1.15].

Theorem B.6. *Let $n \geq 1$ and $\boldsymbol{\tau}$ as above Then there is an isomorphism of the stacks $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ and the quotient stack $\mathrm{L}_{C,\vec{p}}(G) \backslash \left(\prod_{i=1}^n G((t_i))/\mathcal{P}_i\right)$.*

Remark B.7. We emphasize that Balaji-Seshadri [9] work without the assumption that $\theta(\tau_i) < 1$. In this general set-up the groups $\mathcal{P}_j \subset G((t_i))$ that appear in [9, Prop. 3.1.1] are not necessarily contained in $G[[t_i]]$.

B.2. Invariant direct image functor. Let $p : W \to T$ be a finite flat surjective morphism of Noetherian integral schemes (as in [9, Sec. 4]) such that the corresponding extension of function fields is Galois with Galois group Γ. It follows that Γ acts on W and $T = W/\Gamma$. Let $\mathscr{G}$ be a smooth affine group scheme on W. Following Balaji-Seshadri [9], Pappas-Rapoport [57], and Edixhoven [32], we define:

Definition B.8. *The invariant direct image of $\mathscr{G}$, namely $p^{\Gamma}_*(\mathscr{G}) := (p_*(\mathscr{G}))^{\Gamma}$, where $p_*\mathscr{G}$ is the group functor Weil restriction of scalars-$\mathrm{Res}_{W/T}(\mathscr{G})$ and $(p_*(\mathscr{G}))^{\Gamma}$ is the smooth closed fixed point subgroup scheme of the Γ-scheme $p_*(\mathscr{G})$. In particular for any T-scheme S, we get that $p^{\Gamma}_*(\mathscr{G})(S) := (\mathscr{G}(S \times_T W))^{\Gamma}$.*

In our present set-up we consider $\widehat{C} \to C$ to be a ramified Galois covering with Galois group Γ, and let $\mathcal{R}$ be the ramification locus. Let G be a connected, simple algebraic group and $\rho : \Gamma \to G$ and we fixed the local type $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ such that $\theta(\tau_i) < 1$ for all $1 \leq i \leq n$. Consider the invariant push forward $\mathscr{H} := p^{\Gamma}_*(\widehat{C} \times G)$ of the constant group scheme $\widehat{C} \times G$ to get a Bruhat-Tits type group scheme on C with the following property:

(i) The geometric fibers of $\mathscr{H}$ are connected.

(ii) On the punctured curve $C \backslash \mathcal{R}$, the group scheme $\mathscr{H}$ is split.

(iii) For $p_i \in \mathcal{R}$, the group scheme $\mathscr{H}(\widehat{\mathcal{O}}_{C,p_i})$ is the subgroup $\mathcal{P}_i := ev^{-1}_{p_i}(P_i) \subset G[[t_i]]$, where P_i is a standard parabolic subgroup in G given by τ_i.

Pappas-Rapoport [57] considered the moduli stack $\mathcal{B}un_{\mathscr{H}}(C)$ of $\mathscr{H}$-torsors on a curve C, where $\mathscr{H}$ is a parahoric Bruhat-Tits group scheme. A uniformization theorem for such torsors was proved by Heinloth [39]. Using [9, Thm. 4.1.6] and the discussion above, we can reformulate the correspondence in Theorem B.6 by the following:

Proposition B.9. *The stacks $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ and $\mathcal{P}ar_G(C,\vec{P})$ are isomorphic under the invariant push-forward functor. In particular if $\widehat{\mathcal{E}}$ is a family of Γ-G-bundles of type $\boldsymbol{\tau}$ on the curve $\widehat{C}$ parametrized by a schemes T, the $p^{\Gamma}_{*}(\widehat{\mathcal{E}})$ is a family of quasiparabolic G-bundles with parabolic structures at the ramification points determined by $\boldsymbol{\tau}$.*

Moreover, by Proposition A.6 *and Theorem* B.6*, both the stacks $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ and $\mathcal{P}ar_G(C,\vec{P})$ are isomorphic to $\mathrm{L}_{C,\vec{p}}(G)\backslash\left(\prod_{i=1}^{n} G((t_i))/\mathcal{P}_i\right)$, where $\vec{P}=(P_1,\dots,P_n)$ and $P_i=P(\tau_i)$ are Kempf parabolic subgroups determined by τ_i.*

Let $C_T \longrightarrow T$ be a family of smooth projective curves parametrized by T and $p_1,\dots,p_n$ are disjoint sections. Recall that given integers $d_1,\dots,d_n$ and a n-points curve $(C_0,p_1,\dots,p_n)$, we can find a Galois cover $(\widehat{C},\widehat{p}_1,\dots,\widehat{p}_n)$ with Galois group Γ and isotropy of order d_i at $\widehat{p_i}$. Fixing such a Γ, we can find a family of curves $\widehat{C}_T \longrightarrow T$ along with a finite map $p:\widehat{C}_T \longrightarrow C_T$ such that

- Γ acts on $\widehat{C}_T$ preserving p inducing a Galois covering $\pi:\widehat{C}_T \to C_T$.
- Section $\widehat{p}_1,\dots,\widehat{p}_n$ such that isotropy at $\widehat{p}_i$ is of order d_i for all i.
- The cover just depends on the choice of Γ and the integers $d_1,\dots,d_n$.

We refer the reader to [25, Sec. 4d] for the construction of such families. These covers are called pointed admissible covers, and a moduli stack for these objects has been constructed in [44].

Now given a Γ-Galois covering $\widehat{C}_T \to C_T$, the parabolic orbifold correspondence as described in Proposition B.9 works verbatim for families of parabolic and orbifold bundles parametrized by T.

B.3. Determinant of cohomology for $\widehat{C}$ and invariant pushforward. Let $\widehat{\mathcal{E}}$ be a family of Γ-G-bundles on $\widehat{C}$ of local type $\boldsymbol{\tau}$ parametrized by a scheme T. By Proposition B.9, we get a family $\mathcal{E}$ of quasiparabolic G-bundles on C with parabolic structures at the points $\vec{p}=(p_1,\dots,p_n)$ in the ramification locus. Observe that we have an n-tuple integers $\vec{d}=(d_1,\dots,d_n)$ which encodes the order of ramification at the points $(p_1,\dots,p_n)$. Moreover $\exp(2\pi\sqrt{-1}d_i\tau_i)=1$ for all $\leq i\leq n$. Now ignoring the Γ-action, we get a family of principal G-bundles on $\widehat{C}$ and hence by (53), we get a line bundle on T subject to the choice of a representation (ϕ,V) of G. On the other hand, we also get a line bundle on T by starting with a family of quasiparabolic bundles $\mathcal{E}$ obtained from the invariant push forward of $\widehat{\mathcal{E}}$ and then applying the construction in (55). The following proposition, which is minor variation of [24, Prop. 4.5], compares these two line bundles on T.

Proposition B.10. *Let $\phi: G\to \mathsf{SL}(V)$ be a representation of G. Choose a local-type $\boldsymbol{\tau}$ such that $\theta_{\mathfrak{sl}(V)}(\phi(\tau_i))<1$ for all $1\leq i\leq n$, where $\theta_{\mathfrak{sl}(V)}$ is the highest root. Then for any family $\widehat{\mathcal{E}}$ of Γ-G-bundles on $\widehat{C}$ parametrized by a scheme T of local type $\boldsymbol{\tau}$, we have:*

$$\mathrm{Det}(\widehat{\mathcal{E}}(V))\cong \mathrm{Det}((\mathrm{id}_T\times p)^*(\mathcal{E}(V)))\otimes\left(\otimes_{i=1}^{n}\left(\otimes_{j=1}^{k_i}\det \mathrm{Gr}^j\,\mathscr{F}_{\bullet,p_i}(\mathcal{E}_{|T\times p_i})^{\otimes N\alpha_{j,i}}\right)\right),$$

where

(i) *the filtration* $\mathscr{F}_{\bullet,p_i}$ *and the weights* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{p_1}, \ldots, \boldsymbol{\alpha}_{p_n})$ *are determined by the associated topological type* $\phi(\boldsymbol{\tau})$*,*

(ii) $N > 0$ *is the smallest integer such that* $N\alpha_{j,i}$ *are integers, and*

(iii) $\widehat{C} \to C$ *is a Galois* Γ*-cover such that the isotropy of order* N *at all points* p_i*.*

Proof. We will be done by [24, Prop. 4.5] once we can show that $p_*^{\Gamma}(\widehat{\mathcal{E}}) \times^{\phi} (V)$ equals $p_*^{\Gamma}\big(\widehat{\mathcal{E}} \times^{\phi} V\big)$ as a family parabolic vector bundles on C parametrized by T. This follows from the definition directly. □

Now following [24], we will construct a curve $\widehat{C}$ from the data $\boldsymbol{\tau}$ and compare the determinant of cohomology line bundle on $\mathcal{B}un^{\boldsymbol{\tau}}_{\Gamma,G}(\widehat{C})$ with the parabolic determinant of cohomology on C via the functor p_*^{Γ}. Mimicking the set-up of [24, Def. 4.10], given $\boldsymbol{\tau}$ in the Weyl alcove Φ_0 choose an integer N such that $\exp(2\pi\sqrt{-1}N\nu_{\mathfrak{sl}(V)}(\phi(\tau_i))) = 1$ for all $1 \leq i \leq n$. By the Selberg lemma, [65], we can find a ramified cover $p : \widehat{C} \to C$ with ramification exactly over the points p_i with cyclic isotropy group of order N at all the fixed points. Let Γ be the Galois group. With these assumptions, [24, Prop. 4.11] generalizes to the following:

Proposition B.11. *Let* $\mathcal{E} = p_*^{\Gamma}\widehat{\mathcal{E}}$ *be as in Proposition* B.10*. Then the line bundles* $\mathrm{Det}(\widehat{\mathcal{E}}(V))$ *and* $(\mathrm{Det}_{par}(\mathcal{E}(V), N \cdot m_{\phi} \cdot \boldsymbol{\tau}))^{\otimes\frac{|\Gamma|}{N}}$ *on* T *are canonically isomorphic.*

C. The properness condition and codimension estimates

In this section, we will show that the moduli space $M_G^{par,rs}$ of regularly stable parabolic G-bundles on a curve admits no nonconstant functions. This will imply Theorem 2.2 (iii). Throughout this section we assume that G is simple and simply connected (or more generally semisimple, but we do not need it for applications). We have the following key codimension estimate, which essentially follows from the same argument as in Faltings [33] and Laszlo [49]. Fix $n \geq 1$, and let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ be a n-tuple of weights in the Weyl alcove for a group G, and let d be the minimum positive integer such that $\exp(2\pi\sqrt{-1}d \cdot \nu_{\mathfrak{g}}(\tau_i)) = 1$ for all $1 \leq i \leq n$. Choose a curve $\widehat{C}$ that is a Galois cover over C ramified exactly over the points $p_1, \ldots, p_n$ with the same ramification order d and étale on the complement.

Lemma C.1. *Let* $\mathcal{P}ar_G(C, \vec{P})$ *(respectively,* $\mathcal{P}ar_G^{rs}(C, \vec{P})$*) be the moduli stack parametrizing parabolic G-bundles (respectively, regularly stable parabolic G-bundles) given by a choice of weights* $\boldsymbol{\tau}$ *on a* n*-pointed curve* C *of genus* $g(C)$*. Further assume that* $\mathcal{P}ar_G^{rs}(C, \vec{P})$ *is nonempty. Then the codimension of the complement* $\mathcal{P}ar_G(C, \vec{P}) \backslash \mathcal{P}ar_G^{rs}(C, \vec{P}) \subset \mathcal{P}ar_G(C, \vec{P})$ *is at least two provided* $g(\mathscr{C}) \geq 3$*, and* $g(\mathscr{C}) \geq 2$ *if G does not have an* SL_2 *factor. Moreover, if* $G = \mathsf{SL}_r$ *for* $r > 2$*, the codimension of the complement is at least* 3*.*

Proof. Let $\boldsymbol{\tau}$ be the choice of the weights determining the stability conditions and the parabolic subgroups $\vec{P} = (P_1, \ldots, P_n)$. Consider an n-tuple of Borel subgroups $\vec{B}$ and the the moduli stack of quasi parabolic bundles with full flags $\mathcal{P}ar_G(C, \vec{B})$. There is a natural forgetful map $\mathcal{P}ar_G(C, \vec{B}) \to \mathcal{P}ar_G(C, \vec{P})$ whose fibers are product of flag varieties. Now consider the substack $\mathcal{P}ar^{ss}_G(C,\vec{B}))$ (respectively, $\mathcal{P}ar^{s}_G(C,\vec{B})))$ parametrizing semistable (respectively, stable) parabolic bundles with respect to the same weight data $\boldsymbol{\tau}$. This preserves stability (hence also regular stability) and hence the forgetful map restricts to a map $\mathcal{P}ar^{ss}_G(C,\vec{B})) \to \mathcal{P}ar^{ss}_G(C,\vec{P}))$ that preserve both the stable and the regularly stable loci. Consequently, without loss of generality assume that we are in the case of full flags.

It is enough to show the following:

(i) *The codimension of the complement of $\mathcal{P}ar^{ss}_G(C,\vec{B})$ (respectively, $\mathcal{P}ar^{s}_G(C,\vec{B})))$ in $\mathcal{P}ar_G(C,\vec{B})$ is at least two:* We will freely use the parabolic orbifold correspondence. Let $\mathcal{E}$ be a parabolic G bundle admitting a reduction to parabolic bundle $\mathcal{E}_Q$ with structure group Q, where Q is a parabolic subgroup of G with its Levi subgroup L_Q. Consider the sheaf $\mathfrak{n}^{par}_Q(\mathrm{ad}\,\mathcal{E})$ given by the cokernel of map $\mathrm{SPar}(\mathcal{E}_{L_Q}) \hookrightarrow \mathrm{SPar}(\mathcal{E})$, where $\mathcal{E}_{L_Q}$ is the induced parabolic bundle with structure group L_Q. If $\mathcal{E}$ is in the complement of the $\mathcal{P}ar^{ss}_G(C,\vec{B})$ (respectively, $\mathcal{P}ar^{ss}_G(C,\vec{B})$), then $\deg \mathfrak{n}^{par}_Q(\mathrm{ad}\,\mathcal{E})$ is strictly positive (respectively, nonnegative). Let B_{L_Q} be the Borel of L_Q; then the complement has dimension

$$\begin{aligned}\dim \mathcal{P}ar_G(C,\vec{B}) - \big(\dim \mathcal{P}ar_{L_Q}(C,\vec{B}_{L_Q}) + h^1(C,\mathfrak{n}^{par}_Q(\mathrm{ad}\,\mathcal{E}))\big)\\ = (g(C)-1)(\dim G - \dim Q) + n(\dim G/B - \dim L_Q/B_{L_Q})\\ + \deg(\mathfrak{n}^{par}_Q(\mathrm{ad}\,\mathcal{E})) - h^0(C,\mathfrak{n}^{par}_Q(\mathrm{ad}\,\mathcal{E}))\\ \geq (g(C)+n-1)\dim \mathfrak{n}_Q - 1\ ,\end{aligned}$$

since we may assume $h^0(C,\mathfrak{n}^{par}_Q(\mathrm{ad}\,\mathcal{E}) \leq 1$. Now notice that $g(\mathscr{C}) \geq 2$ implies that $g(C)+n-1 \geq 2$, and $g(\mathscr{C}) \geq 3$ implies that $g(C)+n-1 \geq 3$. Further observe that if $G = \mathsf{SL}_r$, then $\dim \mathfrak{n}_Q > 1$ if $r > 2$.

(ii) *The codimension of the complement of $\mathcal{P}ar^{rs}_G(C,\vec{B})$ in $\mathcal{P}ar^{s}_G(C,\vec{B})$ is at least two:* Here we can assume $G \neq \mathsf{SL}_r$. If $\mathcal{P}$ is a stable orbifold bundle on $\mathscr{C}$ which has a noncentral automorphism, then by [49, Lemma 11.1] $\mathcal{P}$ has an L-structure where L is a reductive subgroup of G with Borel B_L. Then the required codimension is at least

$$\begin{aligned}&\dim \mathcal{P}ar_G(C,\vec{B}) - \dim \mathcal{P}ar_L(C,\vec{B}_L)\\ &= (g(C)-1)(\dim G - \dim L) + \sum_{i=1}^{n}(\dim G/B - \dim L/B_L)\\ &= (g(C)-1)(\dim G - \dim L) + n(\dim G/B - \dim L/B_L)\ .\end{aligned}$$

Now $\dim G/B - \dim L/B_L \geq 1$ and $\dim G - \dim L \geq 2$, so if $g(C) \geq 1$, then the codimension is at least $2g(C) - 2 + n \geq 3$, by the assumption that $g(\mathscr{C}) \geq 2$. Thus, we are left to consider the case where $g(C) = 0$.

Since $g(\mathscr{C}) \geq 2$, we have $n \geq 5$. Suppose first that L is not a torus. Then $n \dim L/B_L - \dim L$ is an increasing function of L. This implies that $n(\dim G/B - \dim L/B_L) - (\dim G - \dim L)$ is decreasing function of L. Hence, the codimension is at least

$$\min_{L=L_Q}\ (n(\dim G/B - \dim L/B_L) - (\dim G - \dim L))\ ,$$

where L ranges over the Levi subgroups L_Q of proper maximal parabolics Q in G. Thus we get that the codimension of the complement is at least

$$\min_{L=L_Q}\ (n(\dim G/B - \dim L/B_L) - (\dim G - \dim L)) = \min_Q((n-2)\dim \mathfrak{n}_Q) \geq 3\ .$$

Now suppose L is a torus. In this case, the codimension is simply

$$n(\dim G/B) - (\dim G - \dim L) \geq n(\dim G/B) - \dim G \geq (n-3)\dim \mathfrak{n}_B \geq 2\ .$$

This completes the proof of the Lemma. □

Let $M_G^{par,ss}$ (respectively, $\mathcal{P}ar_G^{ss}(C, \vec{P})$) be the moduli space (respectively, moduli stack) of semistable parabolic G-bundles on C with parabolic structures at n-marked points. It is well-known that $M_G^{par,ss}$ (respectively, the regularly stable part $M_G^{par,rs}$) is a GIT quotient (respectively, good quotient) of a smooth scheme $R_G^{par,ss}$ (respectively, $R_G^{par,rs}$) by a reductive group (cf. [8,9]). Moreover, $M_G^{par,ss}$ is a seminormal projective variety with rational singularities. Now Lemma C.1 implies that $\operatorname{codim}(R_G^{par,ss} \backslash R_G^{par,rs}) \geq 2$, provided $R_G^{par,rs}$ is nonempty. Hence, by Hartogs' theorem we get the following:

Corollary C.2. *The natural inclusion map $M_G^{par,rs} \to M_G^{par,ss}$ induces isomorphisms between $H^0(M_G^{par,rs}, \mathcal{O}_{M_G^{par,rs}})$ and $H^0(M_G^{par,ss}, \mathcal{O}_{M_G^{par,ss}})$.*

Recall $Y_G^{par,rs}$ from the proof of Proposition 5.2. Then we have the following lemma, the proof of which is analogous to that in [49, Prop. 11.6].

Lemma C.3. *The codimension of the complement of $Y_G^{par,rs}$ in $M_{G,\boldsymbol{\tau}}^{par,rs}$ is at least 3 if $g(\mathscr{C}) \geq 3$ for arbitrary $\mathfrak{g}$, or $g(\mathscr{C}) \geq 2$ when $\mathfrak{g}$ has no factor of type A_1 or C_2.*

Proof. Suppose $\mathcal{E}$ be a regularly stable Γ-G-bundle which is not stable as a G-bundle. Then we can realize it as the image of a rational map from the moduli space of Γ-L-bundles on $\widehat{C}$, where L is a Levi subgroup of a parabolic subgroup Q of G. If $\mathcal{E}$ is stable we can realize it as the image of rational map $M_L(\widehat{C})$, where L is a reductive subgroup ([49, Prop. 11.6]) of G. Thus, the complement of $Y_G^{par,rs}$ in $M_{G,\boldsymbol{\tau}}^{par,rs}$ is dominated by union of the moduli spaces of Γ-L-bundles on the curve $\widehat{C}$. of type $\boldsymbol{\tau}$, where L is a reductive subgroup. Now as in the proof of Lemma C.1, without loss of generality assume that $\boldsymbol{\tau}$ corresponds to a tuple of Borel subgroups. Then the required codimension is at least

$$(g(C)-1)(\dim G - \dim L) + n(\dim G/B - L/B_L) - \dim Z(L)\ .$$

Now $\dim G - \dim L$ is at least 4 unless $\mathfrak{g}$ has a factor of type A_1 or C_2. Thus, we are done by the assumptions on $g(\mathscr{C})$ and the calculations as in the proof of Lemma C.1. □

Acknowledgement. The authors warmly thank Johan Martens for enlightening discussions. They also thank the anonymous referee for numerous helpful suggestions.

References

[1] *D. Alfaya* and *T. L. Gómez*, Automorphism group of the moduli space of parabolic bundles over a curve, Adv. Math. **393** (2021), Paper No. 108070, 127, , URL `https://doi.org/10.1016/j.aim.2021.108070`.

[2] *J. E. Andersen*, Hitchin's connection, Toeplitz operators, and symmetry invariant deformation quantization, Quantum Topol. **3** (2012), no. 3-4, 293–325, URL `https://doi-org.proxy-um.researchport.umd.edu/10.4171/qt/30`.

[3] *J. E. Andersen* and *M. Bjerre*, The Hitchin connection for the quantization of the moduli space of parabolic bundles, preprint (2021), 43 pp.

[4] *J. E. Andersen*, *N. L. Gammelgaard* and *M. R. Lauridsen*, Hitchin's connection in metaplectic quantization, Quantum Topol. **3** (2012), no. 3-4, 327–357, URL `https://doi.org/10.4171/qt/31`.

[5] *M. Atiyah* and *R. Bott*, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond. A **308** (1982), 523–615.

[6] *L. Auslander* and *B. Kostant*, Quantization and representations of solvable Lie groups, Bull. Amer. Math. Soc. **73** (1967), 692–695, URL `https://doi.org/10.1090/S0002-9904-1967-11829-9`.

[7] *S. Axelrod*, *S. Della Pietra* and *E. Witten*, Geometric quantization of Chern-Simons gauge theory, J. Differential Geom. **33** (1991), no. 3, 787–902, URL `http://projecteuclid.org/euclid.jdg/1214446565`.

[8] *V. Balaji*, *I. Biswas* and *D. S. Nagaraj*, Ramified G-bundles as parabolic bundles, J. Ramanujan Math. Soc. **18** (2003), no. 2, 123–138.

[9] *V. Balaji* and *C. S. Seshadri*, Moduli of parahoric $\mathscr{G}$-torsors on a compact Riemann surface, J. Algebraic Geom. **24** (2015), no. 1, 1–49, URL `https://doi.org/10.1090/S1056-3911-2014-00626-3`.

[10] *D. Baraglia* and *M. Kamgarpour*, On the image of the parabolic Hitchin map, Q. J. Math. **69** (2018), no. 2, 681–708, , URL `https://doi.org/10.1093/qmath/hax055`.

[11] *D. Baraglia*, *M. Kamgarpour* and *R. Varma*, Complete integrability of the parahoric Hitchin system, Int. Math. Res. Not. IMRN (2019), no. 21, 6499–6528, URL `https://doi.org/10.1093/imrn/rnx313`.

[12] *A. Beauville* and *Y. Laszlo*, Conformal blocks and generalized theta functions, Comm. Math. Phys. **164** (1994), no. 2, 385–419, URL `http://projecteuclid.org/euclid.cmp/1104270837`.

[13] *A. Beauville*, *M. S. Narasimhan* and *S. Ramanan*, Spectral curves and the generalised theta divisor, J. Reine Angew. Math. **398** (1989), 169–179, URL `http://dx.doi.org/10.1515/crll.1989.398.169`.

[14] *P. Belkale*, The strange duality conjecture for generic curves, J. Amer. Math. Soc. **21** (2008), no. 1, 235–258 (electronic), URL `http://dx.doi.org/10.1090/S0894-0347-07-00569-3`.

[15] *P. Belkale*, Strange duality and the Hitchin/WZW connection, J. Differential Geom. **82** (2009), no. 2, 445–465, URL `http://projecteuclid.org/euclid.jdg/1246888491`.

[16] *A. A. Beĭlinson* and *V. V. Schechtman*, Determinant bundles and Virasoro algebras, Comm. Math. Phys. **118** (1988), no. 4, 651–701, URL `http://projecteuclid.org/euclid.cmp/1104162170`.

[17] *U. Bhosle* and *A. Ramanathan*, Moduli of parabolic G-bundles on curves, Math. Z. **202** (1989), no. 2, 161–180, URL `https://doi.org/10.1007/BF01215252`.

[18] *U. N. Bhosle*, Generalized parabolic bundles and applications. II, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), no. 4, 403–420, , URL `https://doi.org/10.1007/BF02837696`.

[19] *I. Biswas*, Parabolic bundles as orbifold bundles, Duke Math. J. **88** (1997), no. 2, 305–325, URL `https://doi.org/10.1215/S0012-7094-97-08812-8`.

[20] *I. Biswas*, Stable bundles and extension of structure group, Differential Geom. Appl. **23** (2005), no. 1, 67–78, URL `https://doi.org/10.1016/j.difgeo.2005.03.006`.

[21] *I. Biswas*, Parabolic principal Higgs bundles, J. Ramanujan Math. Soc. **23** (2008), no. 3, 311–325.

[22] *I. Biswas* and *N. Hoffmann*, Poincaré families of G-bundles on a curve, Math. Ann. **352** (2012), no. 1, 133–154, URL `https://doi.org/10.1007/s00208-010-0628-x`.

[23] *I. Biswas*, *S. Mukhopadhyay* and *R. Wentworth*, Ginzburg algebras and a parabolic generalization of a result of Beilinson and Schechtman, arXiv:2103.03792 (2021), 35 pp. To be split from arXiv:2103.03792.

[24] *I. Biswas* and *N. Raghavendra*, Determinants of parabolic bundles on Riemann surfaces, Proc. Indian Acad. Sci. Math. Sci. **103** (1993), no. 1, 41–71, URL `https://doi.org/10.1007/BF02837895`.

[25] *I. Biswas* and *N. Raghavendra*, Curvature of the determinant bundle and the Kähler form over the moduli of parabolic bundles for a family of pointed curves, Asian J. Math. **2** (1998), no. 2, 303–324, URL `https://doi.org/10.4310/AJM.1998.v2.n2.a4`.

[26] *M. Bjerre*, The Hitchin connection for the quantization of the moduli space of parabolic bundles on surfaces with marked points, PhD Thesis (2018), 101 pp. Aarhus University.

[27] *S. Bloch* and *H. Esnault*, Relative algebraic differential characters, in: Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press, Somerville, MA, 2002, *Int. Press Lect. Ser.*, volume 3, 47–73.

[28] *G. D. Daskalopoulos* and *R. A. Wentworth*, Geometric quantization for the moduli space of vector bundles with parabolic structure, in: Geometry, topology and physics (Campinas, 1996), de Gruyter, Berlin, 1997, 119–155.

[29] *R. Donagi* and *T. Pantev*, Langlands duality for Hitchin systems, Invent. Math. **189** (2012), no. 3, 653–735, URL `http://dx.doi.org/10.1007/s00222-012-0373-8`.

[30] *J.-M. Drezet* and *M. S. Narasimhan*, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. **97** (1989), no. 1, 53–94, URL `https://doi.org/10.1007/BF01850655`.

[31] *V. G. Drinfeld* and *C. Simpson*, B-structures on G-bundles and local triviality, Math. Res. Lett. **2** (1995), no. 6, 823–829, URL https://doi.org/10.4310/MRL.1995.v2.n6.a13.

[32] *B. Edixhoven*, Néron models and tame ramification, Compositio Math. **81** (1992), no. 3, 291–306, URL http://www.numdam.org/item?id=CM_1992__81_3_291_0.

[33] *G. Faltings*, Stable G-bundles and projective connections, J. Algebraic Geom. **2** (1993), no. 3, 507–568.

[34] *G. Faltings*, A proof for the Verlinde formula, J. Algebraic Geom. **3** (1994), no. 2, 347–374.

[35] *B. van Geemen* and *A. J. de Jong*, On Hitchin's connection, J. Amer. Math. Soc. **11** (1998), no. 1, 189–228, , URL https://doi-org.proxy-um.researchport.umd.edu/10.1090/S0894-0347-98-00252-5.

[36] *V. Ginzburg*, Resolution of diagonals and moduli spaces, in: The moduli space of curves (Texel Island, 1994), Birkhäuser Boston, Boston, MA, 1995, *Progr. Math.*, volume 129, 231–266, URL https://doi.org/10.1007/978-1-4612-4264-2_9.

[37] *T. L. Gómez* and *M. Logares*, A Torelli theorem for the moduli space of parabolic Higgs bundles, Adv. Geom. **11** (2011), no. 3, 429–444, , URL https://doi.org/10.1515/ADVGEOM.2011.010.

[38] *P. B. Gothen* and *A. G. Oliveira*, The singular fiber of the Hitchin map, Int. Math. Res. Not. IMRN (2013), no. 5, 1079–1121, , URL https://doi.org/10.1093/imrn/rns020.

[39] *J. Heinloth*, Uniformization of $\mathscr{G}$-bundles, Math. Ann. **347** (2010), no. 3, 499–528, URL https://doi.org/10.1007/s00208-009-0443-4.

[40] *J. Heinloth*, Hilbert-Mumford stability on algebraic stacks and applications to $\mathscr{G}$-bundles on curves, Épijournal Géom. Algébrique **1** (2017), Art. 11, 37.

[41] *S. Helgason*, Differential geometry, Lie groups, and symmetric spaces, *Pure and Applied Mathematics*, volume 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[42] *N. Hitchin*, Stable bundles and integrable systems, Duke Math. J. **54** (1987), no. 1, 91–114, URL https://doi-org.proxy-um.researchport.umd.edu/10.1215/S0012-7094-87-05408-1.

[43] *N. Hitchin*, Flat connections and geometric quantization, Comm. Math. Phys. **131** (1990), no. 2, 347–380, URL http://projecteuclid.org/euclid.cmp/1104200841.

[44] *T. J. Jarvis*, *R. Kaufmann* and *T. Kimura*, Pointed admissible G-covers and G-equivariant cohomological field theories, Compos. Math. **141** (2005), no. 4, 926–978, URL https://doi.org/10.1112/S0010437X05001284.

[45] *A. A. Kirillov*, Unitary representations of nilpotent Lie groups, Uspehi Mat. Nauk **17** (1962), no. 4 (106), 57–110.

[46] *B. Kostant*, Quantization and unitary representations. I. Prequantization, in: Lectures in modern analysis and applications, III, Springer, 1970, 87–208. Lecture Notes in Math., Vol. 170.

[47] *S. Kumar*, Conformal Blocks, Generalized Theta Functions and the Verlinde Formula, New Mathematical Monographs, Cambridge University Press, 2021.

[48] *S. Kumar*, *M. S. Narasimhan* and *A. Ramanathan*, Infinite Grassmannians and moduli spaces of G-bundles, Math. Ann. **300** (1994), no. 1, 41–75, URL http://dx.doi.org/10.1007/BF01450475.

[49] *Y. Laszlo*, Hitchin's and WZW connections are the same, J. Differential Geom. **49** (1998), no. 3, 547–576, URL http://projecteuclid.org/euclid.jdg/1214461110.

[50] *Y. Laszlo* and *C. Sorger*, The line bundles on the moduli of parabolic G-bundles over curves and their sections, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 4, 499–525, URL http://dx.doi.org/10.1016/S0012-9593(97)89929-6.

[51] *M. Logares* and *J. Martens*, Moduli of parabolic Higgs bundles and Atiyah algebroids, J. Reine Angew. Math. **649** (2010), 89–116, URL https://doi.org/10.1515/CRELLE.2010.090.

[52] *V. B. Mehta* and *T. R. Ramadas*, Moduli of vector bundles, Frobenius splitting, and invariant theory, Ann. of Math. (2) **144** (1996), no. 2, 269–313, , URL https://doi.org/10.2307/2118593.

[53] *V. B. Mehta* and *C. S. Seshadri*, Moduli of vector bundles on curves with parabolic structures, Math. Ann. **248** (1980), no. 3, 205–239, URL https://doi.org/10.1007/BF01420526.

[54] *M. S. Narasimhan*, Elliptic operators and differential geometry of moduli spaces of vector bundles on compact Riemann surfaces, in: Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), Univ. of Tokyo Press, Tokyo, 1970, 68–71.

[55] *M. S. Narasimhan* and *T. R. Ramadas*, Factorisation of generalised theta functions. I, Invent. Math. **114** (1993), no. 3, 565–623, , URL https://doi.org/10.1007/BF01232680.

[56] *Z. Ouaras*, Parabolic Hitchin Connection, Phd Thesis, Université Côte D'Azur (July, 2023).

[57] *G. Pappas* and *M. Rapoport*, Twisted loop groups and their affine flag varieties, Adv. Math. **219** (2008), no. 1, 118–198, URL https://doi.org/10.1016/j.aim.2008.04.006. With an appendix by T. Haines and M. Rapoport.

[58] *C. Pauly*, Espaces de modules de fibrés paraboliques et blocs conformes, Duke Math. J. **84** (1996), no. 1, 217–235, , URL http://dx.doi.org/10.1215/S0012-7094-96-08408-2.

[59] *C. Pauly*, *J. Martens*, *M. Bolognesi* and *T. Baier*, The Hitchin connection in arbitrary characteristic, J. Inst. Math. Jussieu **22** (2023), no. 1, 449–492, , URL `https://doi.org/10.1017/S1474748022000196`.

[60] *T. R. Ramadas*, Faltings' construction of the K-Z connection, Comm. Math. Phys. **196** (1998), no. 1, 133–143, URL `https://doi.org/10.1007/s002200050417`.

[61] *A. Ramanathan*, Stable principal bundles on a compact Riemann surface, Math. Ann. **213** (1975), 129–152, , URL `https://doi-org.proxy-um.researchport.umd.edu/10.1007/BF01343949`.

[62] *A. Ramanathan*, Moduli for principal bundles over algebraic curves. II, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), no. 4, 421–449, URL `http://dx.doi.org/10.1007/BF02837697`.

[63] *Z. Ran*, Jacobi cohomology, local geometry of moduli spaces, and Hitchin connections, Proc. London Math. Soc. (3) **92** (2006), no. 3, 545–580, URL `https://doi.org/10.1017/S0024611505015704`.

[64] *P. Scheinost* and *M. Schottenloher*, Metaplectic quantization of the moduli spaces of flat and parabolic bundles, J. Reine Angew. Math. **466** (1995), 145–219.

[65] *A. Selberg*, On discontinuous groups in higher-dimensional symmetric spaces, in: Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, 147–164.

[66] *C. S. Seshadri*, Moduli of π-vector bundles over an algebraic curve, in: Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), Edizioni Cremonese, Rome, 1970, 139–260.

[67] *C. S. Seshadri*, Remarks on parabolic structures, in: Vector bundles and complex geometry, Amer. Math. Soc., Providence, RI, 2010, *Contemp. Math.*, volume 522, 171–182, URL `https://doi.org/10.1090/conm/522/10299`.

[68] *J.-M. Souriau*, Quantification géométrique. Applications, Ann. Inst. H. Poincaré Sect. A (N.S.) **6** (1967), 311–341.

[69] *X. Su*, *B. Wang* and *X. Wen*, Parabolic Hitchin maps and their generic fibers, Math. Z. **301** (2022), no. 1, 343–372, URL `https://doi.org/10.1007/s00209-021-02896-3`.

[70] *X. Sun* and *I.-H. Tsai*, Hitchin's connection and differential operators with values in the determinant bundle, J. Differential Geom. **66** (2004), no. 2, 303–343, URL `http://projecteuclid.org/euclid.jdg/1102538613`.

[71] *X. Sun* and *M. Zhou*, Globally F-regular type of moduli spaces, Math. Ann. **378** (2020), no. 3-4, 1245–1270, URL `https://doi.org/10.1007/s00208-020-02077-3`.

[72] *C. Teleman* and *C. Woodward*, Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 3, 713–748, URL `http://aif.cedram.org/item?id=AIF_2003__53_3_713_0`.

[73] *M. Troyanov*, Prescribing curvature on compact surfaces with conical singularities, Trans. Amer. Math. Soc. **324** (1991), no. 2, 793–821, URL `https://doi-org.proxy-um.researchport.umd.edu/10.2307/2001742`.

[74] *A. Tsuchiya*, *K. Ueno* and *Y. Yamada*, Conformal field theory on universal family of stable curves with gauge symmetries, in: Integrable systems in quantum field theory and statistical mechanics, Academic Press, Boston, MA, 1989, *Adv. Stud. Pure Math.*, volume 19, 459–566.

[75] *B. Wang*, Generic fibers of parahoric Hitchin systems, preprint (2020), 30 pp. Arxiv:2008.02899.

[76] *G. E. Welters*, Polarized abelian varieties and the heat equations, Compositio Math. **49** (1983), no. 2, 173–194, URL `http://www.numdam.org/item?id=CM_1983__49_2_173_0`.

[77] *E. Witten*, Quantum field theory and the Jones polynomial, in: Braid group, knot theory and statistical mechanics, World Sci. Publ., Teaneck, NJ, 1989, *Adv. Ser. Math. Phys.*, volume 9, 239–329, URL `https://doi.org/10.1142/9789812798350_0009`.

[78] *K. Yokogawa*, Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves, J. Math. Kyoto Univ. **33** (1993), no. 2, 451–504, , URL `https://doi.org/10.1215/kjm/1250519269`.

Indranil Biswas, Mathematics Department, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, Uttar Pradesh 201314, India
e-mail: indranil.biswas@snu.edu.in, indranil29@gmail.com

Swarnava Mukhopadhyay, School of Mathematics, Tata Institute of Fundamental Research, Mumbai-400005, India
e-mail: swarnava@math.tifr.res.in

Richard Wentworth, Department of Mathematics, University of Maryland, College Park, MD 20742, USA
e-mail: raw@umd.edu

Eingegangen XXX, in revidierter Fassung XXX