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ON THE MORGAN-SHALEN COMPACTIFICATION OF THE SL$(2,\mathbb{C})$ CHARACTER VARIETIES OF SURFACE GROUPS

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1. Introduction. Let Σ be a closed, compact, oriented surface of genus $g \geq 2$ and fundamental group Γ. Let $\mathscr{X}(\Gamma)$ denote the $\mathrm{SL}(2,\mathbb{C})$ character variety of Γ, and $\mathscr{D}(\Gamma) \subset \mathscr{X}(\Gamma)$ the closed subset consisting of conjugacy classes of discrete, faithful representations. Then $\mathscr{X}(\Gamma)$ is an affine algebraic variety admitting a compactification $\overline{\mathscr{X}(\Gamma)}$ (due to Morgan and Shalen [MS1]), whose boundary points $\partial\mathscr{X}(\Gamma) = \overline{\mathscr{X}(\Gamma)} \setminus \mathscr{X}(\Gamma)$ correspond to elements of $\mathscr{PL}(\Gamma)$, the space of projective classes of length functions on Γ with the weak topology.

Choose a metric σ on Σ, and let $\mathscr{M}_{\mathrm{Higgs}}(\sigma)$ denote the moduli space of semistable rank-2 Higgs pairs on Σ (σ) with trivial determinant, as constructed by Hitchin [H]. Then $\mathscr{M}_{\mathrm{Higgs}}(\sigma)$ is an algebraic variety, depending on the complex structure defined by σ (cf. [Si]). By the theorem of Donaldson [D], $\mathscr{M}_{\mathrm{Higgs}}(\sigma)$ is homeomorphic to $\mathscr{X}(\Gamma)$, though not complex-analytically so. Let us denote this map $h : \mathscr{X}(\Gamma) \to \mathscr{M}_{\mathrm{Higgs}}$ (we henceforth assume the choice of base point σ).

We define a compactification of $\mathscr{M}_{\mathrm{Higgs}}$ as follows: Let QD (more precisely, $QD(\sigma)$) denote the finite-dimensional complex vector space of holomorphic quadratic differentials on Σ. Then there is a surjective, holomorphic map $\mathscr{M}_{\mathrm{Higgs}} \to QD$ taking the Higgs field Φ to $\varphi = \det \Phi$. We compose this with the map

$$\varphi \longrightarrow \frac{4\varphi}{1+4\|\varphi\|},$$

where $\|\varphi\| = \int_\Sigma |\varphi|$, and obtain

$$\widetilde{\det} : \mathscr{M}_{\mathrm{Higgs}} \longrightarrow BQD = \{\varphi \in QD : \|\varphi\| < 1\}.$$

Let $SQD = \{\varphi \in QD : \|\varphi\| = 1\}$ be the space of normalized holomorphic quadratic differentials. We then define $\overline{\mathscr{M}_{\mathrm{Higgs}}} = \mathscr{M}_{\mathrm{Higgs}} \cup SQD$ with the topology given via the

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map $\widetilde{\det}$. The aim of this paper is to compare the two compactifications $\overline{\mathscr{X}(\Gamma)}$ and $\overline{\mathcal{M}_{\text{Higgs}}}$.

The points of $\mathcal{PL}(\Gamma)$ may be regarded as arising from the translation lengths of minimal, nontrivial Γ actions on $\mathbb{R}$-trees. Modulo isometries and scalings, this correspondence is one-to-one, at least in the nonabelian case (cf. [CM] and our Section 2). The boundary $\partial\mathcal{D}(\Gamma)$ consists of *small actions*, that is, those for which the arc-stabilizer subgroups are all cyclic. With our choice of conformal structure σ, we can define a continuous, surjective map

$$H : \mathcal{PL}(\Gamma) \longrightarrow SQD. \tag{1.1}$$

When the length function $[\ell]$ is realized by the translation length function of a tree *dual* to the lift of a normalized holomorphic quadratic differential φ, then $H([\ell]) = \varphi$; the full map is a continuous extension of this (see Theorem 3.9) with the fibers of H corresponding more generally to *foldings* of dual trees.

Let $\mathcal{PMF}(\Gamma)$ denote the space of projective classes of measured foliations on Σ, modulo isotopy and Whitehead equivalence (cf. [FLP, exposé 5]). By the theorem of Hubbard-Masur [HM] we also have a homeomorphism $HM : \mathcal{PMF}(\Gamma) \xrightarrow{\sim} SQD$. It is not clear how to lift H to factor through $\mathcal{PMF}(\Gamma)$ in a manner independent of σ. However, it follows essentially by Skora's theorem [Sk] that if H is restricted to $\mathcal{PSL}(\Gamma)$, the small actions, then it factors through HM by a homeomorphism $\mathcal{PSL}(\Gamma) \xrightarrow{\sim} \mathcal{PMF}(\Gamma)$.

With this understood, we define a (set-theoretic) map

$$\bar{h} : \overline{\mathscr{X}(\Gamma)} \longrightarrow \overline{\mathcal{M}_{\text{Higgs}}} \tag{1.2}$$

by extending the map h to H on the boundary. We prove the following.

MAIN THEOREM. *The map $\bar{h}$ is continuous and surjective. Restricted to the compactification of the discrete, faithful representations $\overline{\mathcal{D}(\Gamma)}$, it is a homeomorphism onto its image.*

Note that the second statement follows from the first, since $\partial\mathcal{D}(\Gamma)$ consists of small actions, and therefore the restricted map is injective by the above-mentioned theorem of Skora. The full map is not bijective: For example, quadratic differentials that are squares of holomorphic 1-forms are images of the length functions of their dual trees, but they also appear as images of the limits of abelian representations (see Section 3). It would be interesting to determine the fibers of $\bar{h}$ in general; this question will be taken up elsewhere. We also remark that the $\mathrm{SL}(2,\mathbb{R})$ version of the above theorem leads to a harmonic-maps description of the Thurston compactification of Teichmüller space and was first proved by Wolf [W1]. Generalizing this result to $\mathrm{SL}(2,\mathbb{C})$ is one of the motivations for this paper.

This paper is organized as follows: In Section 2 we review the Morgan-Shalen compactification, the definition of the Higgs moduli space, and the notion of a harmonic map to an $\mathbb{R}$-tree. In Section 3, we define the boundary map H. The key point is that the nonuniqueness in the correspondence between abelian length functions and $\mathbb{R}$-trees alluded to above nevertheless leads, via harmonic maps, to a well-defined geometric object on Σ, in this case, a quadratic differential. The most important result here is Theorem 3.7. Along the way, we give a criterion, Theorem 3.3, for uniqueness of harmonic maps to trees, using the arguments in [W3]. The main theorem is then proven in Section 4 as a consequence of our previous work [DDW]. In the last section, a somewhat more concrete analysis of the behavior of high energy harmonic maps is outlined, illustrating previous ideas.

2. Definitions. Let Γ be a hyperbolic surface group as in the introduction. We denote by $\mathcal{R}(\Gamma)$ the set of representations of Γ into $\mathrm{SL}(2,\mathbb{C})$, and by $\mathcal{X}(\Gamma)$ the set of characters of representations. Recall that a representation $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ defines a character $\chi_\rho : \Gamma \to \mathbb{C}$ by $\chi_\rho(g) = \operatorname{Tr}\rho(g)$. Two representations ρ and ρ' are *equivalent* if $\chi_\rho = \chi_{\rho'}$. It is easily seen (cf. [CS]) that equivalent irreducible representations are conjugate. If ρ is a reducible representation, then we can write

$$\rho(g) = \begin{pmatrix} \lambda_\rho(g) & a(g) \\ 0 & \lambda_\rho(g)^{-1} \end{pmatrix}$$

for a representation $\lambda_\rho : \Gamma \to \mathbb{C}^*$. The character χ_ρ determines λ_ρ up to the inversion coming from the action of the Weyl group and is, in turn, completely determined by it. It is shown in [CS] that the set of characters $\mathcal{X}(\Gamma)$ has the structure of an affine algebraic variety.

In [MS1], a (nonalgebraic) compactification $\overline{\mathcal{X}(\Gamma)}$ of $\mathcal{X}(\Gamma)$ is defined as follows: Let C be the set of conjugacy classes of Γ, and let $\mathbb{P}(C) = \mathbb{P}(\mathbb{R}^C)$ be the (real) projective space of nonzero, positive functions on C. Define the map $\vartheta : \mathcal{X}(\Gamma) \to \mathbb{P}(C)$ by

$$\vartheta(\rho) = \left\{ \log\left(|\chi_\rho(\gamma)| + 2\right) : \gamma \in C \right\}$$

and let $\mathcal{X}(\Gamma)^+$ denote the 1-point compactification of $\mathcal{X}(\Gamma)$ with the inclusion map $\iota : \mathcal{X}(\Gamma) \to \mathcal{X}(\Gamma)^+$. Finally, $\overline{\mathcal{X}(\Gamma)}$ is defined to be the closure of the embedded image of $\mathcal{X}(\Gamma)$ in $\mathcal{X}(\Gamma)^+ \times \mathbb{P}(C)$ by the map $\iota \times \vartheta$. It is proved in [MS1] that $\overline{\mathcal{X}(\Gamma)}$ is compact and that the boundary points consist of *projective length functions* on Γ (see the definition below). Note that in its definition, $\vartheta(\rho)$ could be replaced by the function $\left\{\ell_\rho(\gamma)\right\}_{\gamma \in C}$, where ℓ_ρ denotes the translation length for the action of $\rho(\gamma)$ on $\mathbb{H}^3$:

$$\ell_\rho(\gamma) = \inf\left\{ \mathrm{dist}_{\mathbb{H}^3}(x, \rho(\gamma)x) : x \in \mathbb{H}^3 \right\}$$

(see [Cp]).

Recall that an $\mathbb{R}$-*tree* is a metric space (T, d_T) such that any two points $x, y \in T$ are connected by a *segment* $[x, y]$ (that is, a rectifiable arc isometric to a compact (possibly degenerate) interval in $\mathbb{R}$ whose length realizes $d_T(x, y)$) and that $[x, y]$ is the unique embedded path from x to y. We say that $x \in T$ is an *edge point* (resp., *vertex*) if $T \setminus \{x\}$ has two (resp., more than two) components. A Γ-*tree* is an $\mathbb{R}$-tree with an action of Γ by isometries, and it is called *minimal* if there is no proper Γ-invariant subtree. We say that Γ *fixes an end* of T (or more simply, that T has a fixed end) if there is a ray $R \subset T$ such that for every $\gamma \in \Gamma$, $\gamma(R) \cap R$ is a subray. When the action is understood, we often refer to "trees" instead of "Γ-trees."

Given an $\mathbb{R}$-tree (T, d_T), the associated length function $\ell_T : \Gamma \to \mathbb{R}^+$ is defined by $\ell_T(\gamma) = \inf_{x \in T} d_T(x, \gamma x)$. If $\ell_T \not\equiv 0$, which is equivalent to Γ having no fixed point in T (cf. [MS1, Prop. II.2.15]), then the class of ℓ_T in $\mathbb{P}(C)$ is called a projective length function. We denote by $\mathcal{PL}(\Gamma)$ the set of all projective length functions on Γ-trees. A length function is called *abelian* if it is given by $|\mu(\gamma)|$ for some homomorphism $\mu : \Gamma \to \mathbb{R}$. We use the following result.

THEOREM 2.1 [CM, Cor. 2.3 and Thm. 3.7]. *Let T be a minimal Γ-tree with nontrivial length function ℓ_T. Then ℓ_T is nonabelian if and only if Γ acts without fixed ends. Moreover, if T' is any other minimal Γ-tree with the same nonabelian length function, then there is a unique equivariant isometry $T \simeq T'$.*

It is a fact that abelian length functions, in general, no longer determine a unique minimal Γ-tree up to isometry (e.g., see [CM, Example 3.9]), and this presents one of the main difficulties dealt with in this paper.

We now give a quick review of the theory of Higgs bundles on Riemann surfaces and their relationship to representation varieties. Let Σ, Γ be as in the introduction. A *Higgs pair* is a pair (A, Φ), where A is an $\mathrm{SU}(2)$ connection on a rank-2 smooth vector bundle E over Σ; and $\Phi \in \Omega^{1,0}(\Sigma, \mathrm{End}_0(E))$, where $\mathrm{End}_0(E)$ denotes the bundle of traceless endomorphisms of E. The Hitchin equations are

$$F_A + [\Phi, \Phi^*] = 0, \qquad D_A''\Phi = 0. \tag{2.1}$$

The group $\mathfrak{G}$ of (real) gauge transformations acts on the space of Higgs pairs and preserves the set of solutions to (2.1). We denote by $\mathcal{M}_{\mathrm{Higgs}}$ the set of gauge equivalence classes of these solutions. Then $\mathcal{M}_{\mathrm{Higgs}}$ is a complex analytic variety of dimension $6g-6$ (the holomorphic structure depending upon the choice σ on Σ), which admits a holomorphic map (cf. [H])

$$\det : \mathcal{M}_{\mathrm{Higgs}} \longrightarrow QD = H^0(\Sigma, K_\Sigma^{\otimes 2}) : (A, \Phi) \mapsto \det \Phi = -\operatorname{Tr} \Phi^2. \tag{2.2}$$

By associating to $[(A, \Phi)] \in \mathcal{M}_{\mathrm{Higgs}}$ the character of the flat $\mathrm{SL}(2, \mathbb{C})$ connection $A + \Phi + \Phi^*$, one obtains a homeomorphism $h : \mathcal{M}_{\mathrm{Higgs}} \to \mathcal{X}(\Gamma)$ (cf. [D], [C]). Implicit in the definition of h is a Γ-equivariant harmonic map u from the universal cover $\mathbb{H}^2$ of Σ to $\mathbb{H}^3$. It is easily verified that the *Hopf differential* of u, $\mathrm{Hopf}(u) = \tilde{\varphi} = \langle u_z, u_z \rangle dz^2$, descends to a holomorphic quadratic differential φ on Σ equal to $\det \Phi$ (up to a universal nonzero constant).

Having introduced harmonic maps, we now give an alternative way to view the

Morgan-Shalen compactification. First, it follows by an easy application of the Bochner-Weitzenböck formula that a sequence of representations ρ_i diverges to the boundary only if the energies $E(u_{\rho_i})$ of the associated equivariant harmonic maps u_{ρ_i} are unbounded. Furthermore, given such a sequence, it is shown in [DDW] that if the ρ_i converge to a boundary point in the sense of Morgan-Shalen, then the harmonic maps u_{ρ_i} converge (perhaps after passing to a subsequence) in the sense of Korevaar-Schoen to a Γ-equivariant harmonic map $u : \mathbb{H}^2 \to (T, d_T)$, where (T, d_T) is a minimal Γ-tree having the same projective length function as the Morgan-Shalen limit of the ρ_i. As pointed out before, the tree is not necessarily uniquely defined, and even in the case where the tree is unique, uniqueness of the harmonic map is problematic.

Recall that a harmonic map to a tree means, by definition, an energy minimizer for the energy functional defined in [KS1]. Given such a map, its Hopf differential $\tilde{\varphi}$ can be defined almost everywhere, and by [S1, Lemma 1.1], which can be adapted to the singular case, one can show that the harmonicity of u implies that $\tilde{\varphi}$ is a holomorphic quadratic differential. The equivariance of u implies that $\tilde{\varphi}$ is the lift of a differential on Σ. Note also that if $u : \mathbb{H}^2 \to T$ is harmonic, then $\mathrm{Hopf}(u) \equiv 0$ if and only if u is constant. In the equivariant case, this in turn is equivalent to $\ell_T \equiv 0$ (cf. [DDW]). For the rest of the paper, we tacitly assume $\ell_T \not\equiv 0$.

A particular example is the following: Consider a nonzero holomorphic quadratic differential φ, and denote by $\tilde{\varphi}$ its lift to $\mathbb{H}^2$. Locally away from the zeros, $\tilde{\varphi}$ may be written as dz^2 with respect to a local conformal coordinate $z = \xi + i\eta$. The lines $\xi = \text{const}$ (the vertical leaf space) and transverse measure $|d\xi|$ give the structure of a metric space $T_{\tilde{\varphi}}$, which is independent of the choice of coordinate z and naturally extends past the zeros. According to [MS2] (and using the correspondence between measured foliations and geodesic laminations), $T_{\tilde{\varphi}}$ is an $\mathbb{R}$-tree with an action of Γ, and the projection $\pi : \mathbb{H}^2 \to T_{\tilde{\varphi}}$ is a Γ-equivariant continuous map. We note two important facts: (1) The vertices of $T_{\tilde{\varphi}}$ are precisely the image by π of the zeros of $\tilde{\varphi}$. (2) Since the action of Γ on $T_{\tilde{\varphi}}$ is small, $T_{\tilde{\varphi}}$ has no fixed ends (cf. [MO]).

PROPOSITION 2.2. *The map $\pi : \mathbb{H}^2 \to T_{\tilde{\varphi}}$ is harmonic with Hopf differential $\tilde{\varphi}$.*

Proof. Since $T_{\tilde{\varphi}}$ has no fixed ends, the existence of a harmonic map follows from [KS2, Cor. 2.3.2]. The fact that π is itself an energy minimizer seems to be well known. See, for example, [W2] and the introduction to [GS]: Although the definition of harmonic map in [W2] is a priori different from the notion of an energy minimizer, a proof follows easily. Indeed, for fixed $\varphi \neq 0$ and positive real numbers $t_i \to \infty$, we can find a sequence of hyperbolic metrics σ_i on Σ such that the unique harmonic maps $\Sigma(\sigma) \to \Sigma(\sigma_i)$ homotopic to the identity have Hopf differentials $t_i\varphi$ (cf. [W1] and [Wan]). Uniformizing the σ_i, we obtain a sequence ρ_i of discrete faithful $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{C})$ representations and ρ_i-equivariant harmonic maps $u_i : \mathbb{H}^2 \to \mathbb{H}^2$ with Hopf differentials $t_i\tilde{\varphi}$. Let d_i denote the pullback distance functions on $\mathbb{H}^2$ by the u_i, and let d_∞ denote the pseudometric obtained by pulling back the metric on $T_{\tilde{\varphi}}$ by the projection π. Extend all of these to pseudometrics, also denoted d_i and d_∞, on the space $\mathbb{H}^2_\infty$ constructed in [KS2]. Then the natural projection $\mathbb{H}^2 \to \mathbb{H}^2_\infty/d_\infty \simeq T_{\tilde{\varphi}}$ coincides with the map π. On the other hand, by [W2, Section 4.2], $d_i \to d_\infty$ pointwise, locally uniformly. Therefore, by [KS2, Thm. 3.9], π is an energy minimizer. □

Next, we consider Γ-trees that are not necessarily of the form $T_{\tilde{\varphi}}$. We need the following.

Definition 2.3. A *morphism* of $\mathbb{R}$-trees is a map $f : T \to T'$ such that every nondegenerate segment $[x, y]$ has a nondegenerate subsegment $[x, w]$ such that f restricted to $[x, w]$ is an isometry onto its image. The morphism f is said to *fold* at a point $x \in T$ if there are nondegenerate segments $[x, y_1]$ and $[x, y_2]$ with $[x, y_1] \cap [x, y_2] = \{x\}$ such that f maps each segment $[x, y_i]$ isometrically onto a common segment in T'.

It is a fact that a morphism $f : T \to T'$ is an isometric embedding unless it folds at some point (cf. [MO, Lemma I.1.1]). We also note that, in general, foldings $T \to T'$ may take vertices to edge points. Conversely, vertices in T' need not lie in the image of the vertex set of T.

PROPOSITION 2.4 (cf. [FW]). *Let T be an $\mathbb{R}$-tree with Γ action, and let $u : \mathbb{H}^2 \to T$ be an equivariant harmonic map with Hopf differential $\tilde{\varphi}$. Then u factors as $u = f \circ \pi$, where $\pi : \mathbb{H}^2 \to T_{\tilde{\varphi}}$ is as in Proposition 2.2 and $f : T_{\tilde{\varphi}} \to T$ is an equivariant morphism.*

Proof. Consider $f = u \circ \pi^{-1} : T_{\tilde{\varphi}} \to T$. We first show that f is well defined: Indeed, assume $z_1, z_2 \in \pi^{-1}(w)$. Then z_1 and z_2 may be connected by a vertical leaf e of the foliation of $\tilde{\varphi}$. Now, by the argument in [W3, p. 117], u must collapse e to a point, and so $u(z_1) = u(z_2)$. In order to show that f is a morphism, consider a segment $[x, z] \in T_{\tilde{\varphi}}$. We may lift x to a point $\tilde{x}$ away from the zeros of $\tilde{\varphi}$. Moreover, we may choose a small horizontal arc $\tilde{e}$ from $\tilde{x}$ to some $\tilde{y}$ projecting to $[x, y] \subset [x, z]$, still bounded away from the zeros. The analysis in [W3] again shows that this must map by u isometrically onto a segment in T. □

Remark. It is easily shown (cf. [DDW]) that images of equivariant harmonic maps to trees are always minimal subtrees; hence, throughout this paper we assume, without loss of generality, that our trees are minimal. Thus, for example, the factorization $f : T_{\tilde{\varphi}} \to T$ above either folds at some point or is an equivariant isometry.

3. The map H. The Hopf differential for a harmonic map to a given tree is uniquely determined, as shown by the following statement.

PROPOSITION 3.1. *Let T be a minimal $\mathbb{R}$-tree with a nontrivial Γ action. If u, v are equivariant harmonic maps $\mathbb{H}^2 \to T$, then* $\operatorname{Hopf}(u) = \operatorname{Hopf}(v)$.

Proof. This is proven in [KS1], where in fact the full pullback "metric tensor" is considered. In our situation, the result can also be seen as a direct consequence of the leaf structure of the Hopf differential. First, by [KS1, p. 633], the function $z \mapsto d_T^2(u(z), v(z))$ is subharmonic; hence, by the equivariance it must be equal to a constant c. We assume $c \neq 0$, since otherwise there is nothing to prove. Set $\tilde{\varphi} = \text{Hopf}(u)$, $\tilde{\psi} = \text{Hopf}(v)$. Suppose that $p \in \mathbb{H}^2$ is a zero of $\tilde{\varphi}$, and let Δ be a small neighborhood of p containing no other zeros of $\tilde{\varphi}$ and no zeros of $\tilde{\psi}$, except perhaps p itself. Then by Proposition 2.4 it follows that u is constant and equal to $u(p)$ on every arc $e \subset \Delta$ of the vertical foliation of $\tilde{\varphi}$ with endpoint p. On the other hand, $v(e)$ is a connected set satisfying $d_T(u(p), v(z)) = c$ for all $z \in e$. Since spheres are discrete in trees, v is constant and equal to $v(p)$ on e as well. Referring again to Proposition 2.4, this implies that e must be contained in a vertical leaf of $\tilde{\psi}$. In this way, one sees that the zeros of $\tilde{\varphi}$ and $\tilde{\psi}$ coincide with multiplicity in $\mathbb{H}^2$. Thus, the same is true for φ and ψ on Σ. Since the quadratic differentials are both normalized, they must be equal. $\square$

We also need the following restriction on the kinds of foldings that arise from harmonic maps.

LEMMA 3.2. *Let $T_{\tilde{\varphi}} \to T$ arise from a harmonic map as in Proposition 2.4. Then folding occurs only at vertices, that is, the images of zeros of $\tilde{\varphi}$. At the zeros of $\tilde{\varphi}$, adjacent edges may not be folded. In particular, folding cannot occur at simple zeros.*

Proof. The argument is similar to that in [W2, p. 587]. Suppose $p \in \mathbb{H}^2$ is a zero at which a folding occurs, and choose a neighborhood Δ of p contained in a fundamental domain and containing no other zeros. We can find distinct segments e, e' of the horizontal foliation of $\tilde{\varphi}$ with a common endpoint p that map to segments of $T_{\tilde{\varphi}}$. We may further assume that the folding $T_{\tilde{\varphi}} \to T$ carries each of e and e' isometrically onto a segment $\bar{e}$ of T. Suppose that e and e' are adjacent. Then there is a small disk $\Delta' \subset \mathbb{H}^2$ that, under the projection $\pi : \mathbb{H}^2 \to T_{\tilde{\varphi}}$, maps to $\pi(e) \cup \pi(e')$ and whose center maps to $\pi(p)$ (see Figure 1). Then the harmonic map $u : \mathbb{H}^2 \to T$ maps Δ' onto the segment $\bar{e}$ with the center mapping to an endpoint. Let q denote the other endpoint of $\bar{e}$. The function $z \mapsto (d_T(u(z), q))^2$ is subharmonic on Δ' with an interior maximum. It therefore must be constant, which contradicts $\varphi \not\equiv 0$. For the last statement, recall that the horizontal foliation is trivalent at a simple zero, so that any two edges are adjacent. $\square$

Though the following is not important in this paper, we find it interesting that a uniqueness result for equivariant harmonic maps to trees follows from these considerations, in certain cases.

THEOREM 3.3. *Let $u : \mathbb{H}^2 \to T$ be an equivariant harmonic map with $\tilde{\varphi} = \text{Hopf}(u)$. Suppose there is some vertex x of $T_{\tilde{\varphi}}$ such that the map $f : T_{\tilde{\varphi}} \to T$ from Proposition 2.4 does not fold at x. Then u is the unique equivariant harmonic map to T.*

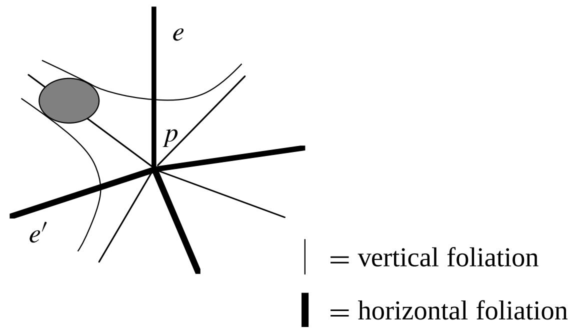


FIGURE 1

Proof. Let p be a zero of $\tilde{\varphi}$ projecting via π to x, and let v be another equivariant harmonic map to T. Choose a neighborhood Δ of p as in the proof of Proposition 3.1, and again suppose that the constant $c = d_T(u(z), v(z)) \neq 0$. Recall that x is a vertex of $T_{\tilde{\varphi}}$. By the assumption of no folding at x, there must be a segment e of the vertical foliation of $\tilde{\varphi}$ in Δ, with one endpoint being p, having the following property: For any $z \neq p$ in e there is a neighborhood $\Delta' \subset \Delta$ of z such that $u(\Delta') \cap [u(p), v(p)] = \{u(p)\}$. By Proposition 3.1 and Lemma 3.2, we see that for such Δ', $v(\Delta') \not\subset [u(p), v(p)]$. Thus, there is a $q \in \Delta$ such that $u(q) \notin [u(p), v(p)]$ and $v(q) \notin [u(p), v(p)]$. But then $d_T(u(q), v(q)) > d_T(u(p), v(p)) = c$, a contradiction. □

COROLLARY 3.4. *Let $\varphi \not\equiv 0$ be a holomorphic quadratic differential on Σ. Then the map $\pi : \mathbb{H}^2 \to T_{\tilde{\varphi}}$ in Proposition 2.2 is the unique equivariant harmonic map to $T_{\tilde{\varphi}}$. If $u : \mathbb{H}^2 \to T$ is an equivariant harmonic map and* $\mathrm{Hopf}(u)$ *has a zero of odd order, then u is unique.*

Proof. The first statement is clear from Theorem 3.3. For the second statement, notice that if p is a zero of odd order, we can still find a neighborhood Δ' as in the proof of Theorem 3.3. □

Proposition 3.1 allows us to associate a unique $\varphi \in SQD$ to any nonabelian length function.

PROPOSITION 3.5. *Let $[\ell] \in \mathcal{PL}(\Gamma)$ be nonabelian. Then there is a unique choice $\varphi \in SQD$ with the following property: If T is any minimal $\mathbb{R}$-tree with length function ℓ in the class $[\ell]$, and $u : \mathbb{H}^2 \to T$ is a Γ-equivariant harmonic map, then* $\mathrm{Hopf}(u) = \varphi$.

Proof. Let $\ell \in [\ell]$. By Theorem 2.1, there is a unique minimal tree T, up to isometry, with length function ℓ and no fixed ends. By Proposition 3.1, any two harmonic maps $u, v : \mathbb{H}^2 \to T$ have the same normalized Hopf differential. Furthermore, if T' is isometric to T and u' is a harmonic map to T', then, composing with the isometry, we see that u' has the same Hopf differential as any harmonic map to T. If the length function ℓ is scaled, then the normalized Hopf differential remains invariant. Finally, since T has no fixed ends, it follows from [KS2, Cor. 2.3.2] that there exists an equivariant harmonic map $u : \mathbb{H}^2 \to T$; so we set $\varphi = \mathrm{Hopf}(u)$. $\square$

We now turn our attention to the abelian length functions. These no longer determine a unique $\mathbb{R}$-tree in general; nevertheless, we see that there is still a uniquely defined quadratic differential associated to them.

PROPOSITION 3.6. *Let ℓ be an abelian length function, and let Γ act on $\mathbb{R}$ with translation length function equal to ℓ. Then there is an equivariant harmonic function $u : \mathbb{H}^2 \to \mathbb{R}$, unique up to translations of $\mathbb{R}$, with Hopf differential $\tilde{\varphi} = (\tilde{\omega})^2$, where $\tilde{\omega}$ is the lift to $\mathbb{H}^2$ of an abelian differential ω on Σ. Moreover, ℓ is determined by the periods of* $\mathrm{Re}(\omega)$.

Proof. The uniqueness statement is clear. By harmonic theory, there is a unique holomorphic one-form ω on Σ such that the real parts of its periods correspond to the homomorphism

$$\mu : \pi_1(\Sigma) \longrightarrow H_1(\Sigma, \mathbb{Z}) \longrightarrow \mathbb{R}.$$

Choosing any base point $*$ of $\mathbb{H}^2$, the desired equivariant harmonic function is the real part of the holomorphic function $f(z) = \int_*^z \tilde{\omega}$. The Hopf differential is $(f'(z))^2 = (\tilde{\omega})^2$. $\square$

It is generally true that harmonic maps to trees with abelian length functions have Hopf differentials with even-order vanishing and that the length functions are recovered from the periods of the associated abelian differential, as the next result demonstrates.

THEOREM 3.7. *Let $u : \mathbb{H}^2 \to T$ be an equivariant harmonic map to a minimal $\mathbb{R}$-tree with nontrivial abelian length function ℓ. Then* $\mathrm{Hopf}(u) = (\tilde{\omega})^2$, *where $\tilde{\omega}$ is the lift to $\mathbb{H}^2$ of an abelian differential ω on Σ. Moreover, ℓ is determined by the periods of* $\mathrm{Re}(\omega)$.

Proof. We first prove that the Hopf differential $\tilde{\varphi} = \mathrm{Hopf}(u)$ must be a square. It suffices to prove that the zeros of $\tilde{\varphi}$ are all of even order. Let p be such a zero, and choose a neighborhood Δ of p as above. Since T has an abelian length function, the action of Γ must fix an end E of T. Then, applying the construction of Section 5 of [DDW], we find a continuous family of equivariant harmonic maps u_ε obtained by "pushing" the image of u a distance ε in the direction of the fixed end. On the other hand, if $\tilde{\varphi}$ had a zero of odd order, this would violate Corollary 3.4.

We may therefore express $\tilde{\varphi} = (\tilde{\omega})^2$ for some abelian differential $\tilde{\omega}$ on $\mathbb{H}^2$. A priori, we can only conclude that $\tilde{\omega}$ descends to an abelian differential $\hat{\omega}$ on an unramified double cover $\widehat{\Sigma}$ of Σ determined by an index-2 subgroup $\widehat{\Gamma} \subset \Gamma$. Let L be a complete noncritical leaf of the horizontal foliation of $\tilde{\varphi}$. Choose a point $x_0 \in L$ and let $\bar{x}_0 = u(x_0)$. We assume that we have chosen x_0 so that $\bar{x}_0$ is an edge point.

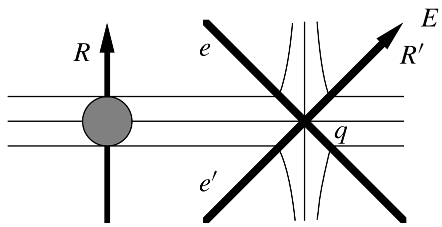


FIGURE 2

Then there is a unique ray $\bar{R}$ with end point $\bar{x}_0$ leading out to the fixed end E. Let R denote the half-leaf of L starting at x_0 and such that a small neighborhood of x_0 in R maps isometrically onto a small subsegment of $\bar{R}$.

We claim that R itself maps isometrically onto $\bar{R}$. For suppose to the contrary that there is a point $y \in R$ such that the portion $[x_0, y]$ of R from x to y maps isometrically onto a subsegment of $\bar{R}$, but that this is not true for any $y' \in R \setminus [x_0, y]$. Clearly, the image of y by u must be a vertex of T. Recall the factorization $f : T_{\tilde{\varphi}} \to T$ from Proposition 2.4. Since f is a surjective morphism of trees, the vertices of T are either images by f of vertices of $T_{\tilde{\varphi}}$ and, hence, images by u of zeros of $\tilde{\varphi}$, or they are vertices created by a folding of f. Thus, there are two cases to consider: (1) There is a point q such that y and q lie on the same vertical leaf and q is a zero of $\tilde{\varphi}$. Moreover, there is a critical horizontal leaf R' with one end point equal to q, a small subsegment of which maps isometrically onto a subsegment of $\bar{R}$ with end point $\bar{q} = u(q)$ (see Figure 2). (2) There is a point q such that y and q lie on the same vertical leaf, q is connected by a horizontal leaf to a zero p of $\tilde{\varphi}$, and the map f folds at $\pi(p)$, identifying the segment $[p, q]$ with a portion $[p, q']$ of another horizontal leaf R'. Moreover, $[p, q']$ maps isometrically onto a subsegment of the unique ray from $\bar{p} = u(p)$ to the end E (see Figure 3).

Consider case (1): As indicated in Figure 2, we can find a small neighborhood Δ of y and portions of horizontal leaves e and e' meeting at q that map isometrically onto segments of T intersecting the image $\bar{R}' = u(R')$ only in $\bar{q}$. Now, as above, by pushing the image of u in the direction of E and possibly choosing Δ smaller, we can find a harmonic map u_ε that maps Δ onto a segment with end point $\bar{q}$ and maps y to the opposite end point—a contradiction. The argument for case (2) is similar: We may find a disk Δ centered at y that maps to the union of segments $[\bar{p}, \bar{q}]$ and $[\bar{r}, \bar{q}]$, with y being mapped to $\bar{q}$. Then, pushing the map in the direction of E as above again leads to a contradiction (see Figure 3).

Next, we claim that for any $g \in \widehat{\Gamma}$, $\ell(g)$ is given by the period of $\mathrm{Re}(\hat{\omega})$ around a curve representing the class $[g]$. First, by definition of a fixed end, the intersection $\bar{R} \cap g(\bar{R})$ contains a subray of $\bar{R}$, and for all $\bar{x}$ in this subray, $\ell(g) = d_T(\bar{x}, g(\bar{x}))$ (cf. [CM, Thm. 2.2]). For simplicity then, we assume $g(\bar{R}) \subset \bar{R}$. Choose a lift of $\bar{x}$ to

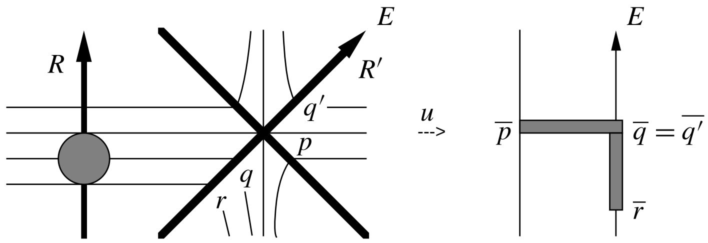


FIGURE 3

$x \in R$. Then $u(g(x)) = g(\bar{x}) \in \bar{R}$. Suppose $g(x)$ is connected by a (possibly empty) vertical leaf to a point x' on R. Then the curve $\tilde{\gamma}$ consisting of the portion $[x, x']$ of R from x to x' followed by the vertical leaf to $g(x)$ projects to a curve γ on Σ representing g. Moreover, since R maps isometrically onto $\bar{R}$, $\ell(g)$ is the length of $[x, x']$ with respect to the transverse measure determined by $\tilde{\varphi}$. Since R contains no zeros of $\tilde{\varphi}$, the latter is simply the absolute value of $\int_{[x,x']} \mathrm{Re}(\tilde{\omega})$. Futhermore, since the vertical direction lies in the kernel of $\mathrm{Re}(\hat{\omega})$, we also have

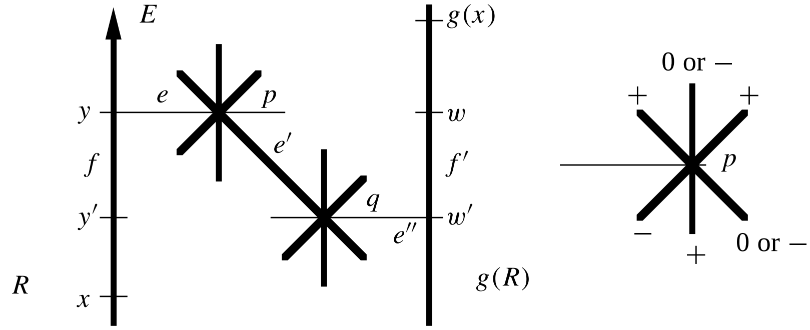
$$\ell(g) = \left| \int_{\tilde{\gamma}} \mathrm{Re}(\tilde{\omega}) \right| = \left| \int_{\gamma} \mathrm{Re}(\hat{\omega}) \right|$$

as desired.

Now consider the possibility that $g(x) \in g(R)$ is not connected to R by a vertical leaf. Since $g(\bar{x}) \in \bar{R}$, it follows from Proposition 2.4 and the fact that R maps onto $\bar{R}$ that there is an intervening folding of a subray of $g(R)$ onto R. Let $y \in R$ project to the vertex in $T_{\tilde{\varphi}}$ at which this occurs. The simplest case is where y is connected by a vertical leaf to a point $w \in g(R)$, and the folding identifies the subray of R starting at y isometrically with the subray of $g(R)$ starting at w. The same analysis as above then produces the closed curve γ.

A more complicated situation arises when there are intervening vertices (see Figure 4(a)): For example, there may be zeros p, q of $\tilde{\varphi}$, a point $w' \in g(R)$, and segments e, e', and e'' of the vertical, horizontal, and vertical foliations, respectively, with endpoints $\{y, p\}$, $\{p, q\}$, and $\{q, w'\}$, respectively. Moreover, the map u folds e' onto a subsegment f of R with endpoints y and y', and then it identifies the subray of R starting at y' isometrically with the subray of $g(R)$ starting at w'. In this way, we see that a subsegment f' of $g(R)$ with endpoints w' and w gets identified with f and e'; in particular, the transverse measures of these three segments are all equal. (Strictly speaking, y' need not lie on R as we choose it, but this does not affect the argument.)

Now consider the prongs at the zero p, for example. These project to distinct segments in $T_{\tilde{\varphi}}$, which are then either projected to segments in T intersecting $\bar{R}$ only in $\bar{y}$; or alternatively there may be a folding identifying them with subsegments of



(a) (b)

FIGURE 4

$\bar{R}$. Let us label the prongs with a $+$ sign if there is a folding onto a subsegment of $[\bar{y}, E)$, with a $-$ sign if there is a folding onto a subsegment of $[\bar{x}, \bar{y}]$, and with a 0 if no folding occurs or if the edge is folded along some other segment (see Figure 4(b)). Since p is connected by the vertical leaf e to R, we label the adjacent horizontal segments with $+$ and $-$ accordingly. Working our way around p in the clockwise direction, and repeatedly using the "pushing" argument from Section 5 of [DDW], we find that every second prong must be labeled $+$ while the intervening prongs may get either $-$ or 0 (recall Lemma 3.2). Therefore, there must be an odd number of prongs between e' and the one adjacent to e, which is identified in the leaf space with a portion of f. A similar argument applies to q, e'', and f'.

Let $\tilde{\gamma}'$ be the path from y' to w obtained by following f, e, e', e'', and then f'. Because of the odd sign to the folding of the prongs at p and q, one may easily verify that $\left|\int_{\tilde{\gamma}'} \operatorname{Re}(\tilde{\omega})\right|$ is the just the transverse measure of the segment f. Indeed, suppose $\tilde{\varphi}$ has a zero of order $2n$ at some point p, and choose a local conformal coordinate z such that $\tilde{\varphi}(z) = z^{2n} dz^2$. Then the foliation is determined by the leaves of $\xi = z^{n+1}/n+1$. If ζ is a primitive $2n+2$ root of unity, then $z \mapsto \zeta^k z$ takes one radial prong to another, with $k-1$ prongs in between (in the counterclockwise direction). The outward integrals of $\operatorname{Re}\sqrt{\tilde{\varphi}}$ along these prongs to a fixed radius differ by $(-1)^k$. Our analysis implies that $k-1$ is odd, so k is even, and we have the correct cancellation. If we extend $\tilde{\gamma}'$ along the horizontal leaves R and $g(R)$ to a path $\tilde{\gamma}$ from x to $g(x)$, then $\left|\int_{\tilde{\gamma}} \operatorname{Re}(\tilde{\omega})\right| = d_T(\bar{x}, g(\bar{x}))$ as required. In general, there are additional intervening zeros, and the procedure above applies to each of these with no further complication.

Thus, ℓ restricted to $\widehat{\Gamma}$ is given by the periods of $\operatorname{Re}(\hat{\omega})$. Since the real parts of the periods of an abelian differential determine the differential uniquely, $\hat{\omega}$ must agree with the pullback to $\widehat{\Sigma}$ of the form in Proposition 3.6; in particular, it descends to Σ. This completes the proof of Theorem 3.7. □

We immediately have the following.

COROLLARY 3.8. *Fix an abelian length function ℓ. Then for any tree T with length function ℓ and any equivariant harmonic map $v : \mathbb{H}^2 \to T$, we have* $\mathrm{Hopf}(v) = \mathrm{Hopf}(u)$ *where u is the equivariant harmonic function from Proposition 3.6 corresponding to ℓ.*

We are now prepared to define the map (1.1). Take a representative ℓ of $[\ell] \in \mathcal{PL}(\Gamma)$. There are two cases: If ℓ is nonabelian, use Proposition 3.5 to define $H([\ell]) = \varphi$. If ℓ is abelian, use Proposition 3.6. The main result of this section is the following.

THEOREM 3.9. *The map $H : \mathcal{PL}(\Gamma) \to SQD$ defined above is continuous.*

Proof. Suppose $[\ell_i] \to [\ell]$, and assume, to the contrary, that there is a subsequence, which we take to be the sequence itself, such that $H([\ell_i]) \to \varphi \neq H([\ell])$. Choose representatives $\ell_i \to \ell$. If there is a subsequence $\{i'\}$ consisting entirely of abelian length functions, then ℓ itself must be abelian, and from the construction of Proposition 3.6, $H(\ell_{i'}) \to H(\ell)$, a contradiction. Thus, we may assume all the ℓ_i's are nonabelian. There exist $\mathbb{R}$-trees T_i, unique up to isometry, and equivariant harmonic maps $u_i : \mathbb{H}^2 \to T_i$. We claim that the u_i have uniform modulus of continuity (cf. [KS2, Prop. 3.7]). Indeed, by [GS, Thm. 2.4], it suffices to show that $E(u_i)$ is uniformly bounded. If $E(u_i) \to \infty$, then the same argument as in [DDW, proof of Thm. 3.1] would give a contradiction. It follows by [KS2, Prop. 3.7] that there is a subsequence $\{i'\}$ (which we assume is the sequence itself) such that u_i converges in the pullback sense to an equivariant harmonic map $u : \mathbb{H}^2 \to T$, where T is a minimal $\mathbb{R}$-tree with length function equal to ℓ. In addition, by [KS2, Theorem 3.9], $\mathrm{Hopf}(u_i) \to \mathrm{Hopf}(u)$. If ℓ is nonabelian, we have a contradiction by Proposition 3.1; if ℓ is abelian, we have a contradiction by Corollary 3.8. □

4. Proof of the main theorem. We show how the results of the previous section, combined with those in [KS2] and [DDW], give a proof of the main theorem. We first reduce the proof of the continuity of $\bar{h}$ to the following.

CLAIM. *If $[\rho_i] \in \mathcal{X}(\Gamma)$ is a sequence of representations converging to $[\ell] \in \mathcal{PL}(\Gamma)$, then $h([\rho_i]) \to H([\ell])$.*

Suppose the claim holds and $\bar{h}$ is not continuous. Then we may find a sequence $x_i \in \mathcal{PL}(\Gamma) \cup \mathcal{X}(\Gamma)$ such that $x_i \to x$ but $\bar{h}(x_i) \to y \neq \bar{h}(x)$. If $x \in \mathcal{PL}(\Gamma)$ so that $\bar{h}(x) = H(x)$, the claim rules out the possibility that there is a subsequence of $\{x_i\}$ in $\mathcal{X}(\Gamma)$. In this case then, there must be a subsequence in $\mathcal{PL}(\Gamma)$. But this contradicts the continuity of H, by Theorem 3.9. Thus, x must be in $\mathcal{X}(\Gamma)$. But then we may assume that $\{x_i\} \subset \mathcal{X}(\Gamma)$, so that $\bar{h} = h$ on $\{x_i\}$. The continuity of the homeomorphism $h : \mathcal{X}(\Gamma) \to \mathcal{M}_{\mathrm{Higgs}}$ then provides the contradiction.

It remains to prove the claim. Again suppose to the contrary that $[\rho_i] \to [\ell]$ but $h([\rho_i]) \to \varphi \neq H([\ell])$ for $\varphi \in SQD$. First, suppose that there is a subsequence $[\rho_{i'}]$ with reducible representative representations $\rho_{i'} : \Gamma \to \mathrm{SL}(2, \mathbb{C})$. Up to conjugation, which amounts to changing the choice of representative, we may assume each $\rho_{i'}$ fixes a given vector $0 \neq v \in \mathbb{C}^2$, and that the action on the 1-dimensional line spanned by v is determined by a character $\chi_{i'} : \Gamma \to \mathbb{C}^*$. The associated translation length functions $\ell_{i'}$ are therefore all abelian, and so $[\ell]$ must be abelian. We may assume there is a representative ℓ such that $\ell_{i'} \to \ell$. By Proposition 3.6 there are harmonic functions

$$u, u_{i'} : \mathbb{H}^2 \longrightarrow \mathbb{R} \simeq \mathbb{C}^*/U(1) \hookrightarrow \mathbb{H}^3,$$

equivariant for the induced action of Γ on $\mathbb{C}^*$ by χ and $\chi_{i'}$, respectively. These converge (after rescaling) to a harmonic function $u : \mathbb{H} \to \mathbb{R}$, equivariant with respect to an action on $\mathbb{R}$ with translation length function ℓ. Since the length functions converge, it follows from the construction in Proposition 3.6 that $\mathrm{Hopf}(u_{i'}) \to \mathrm{Hopf}(u)$, and so by the definition of H, $h([\rho_{i'}]) \to H([\ell])$, a contradiction.

Second, suppose that there is a subsequence $[\rho_{i'}]$ of irreducibles. Then by the main result of [DDW] we can find a further subsequence (which we take to be the sequence itself) of $\rho_{i'}$-equivariant harmonic maps $u_{i'} : \mathbb{H}^2 \to \mathbb{H}^3$ converging in the sense of Korevaar-Schoen to a harmonic map $u : \mathbb{H}^2 \to T$, where T is a minimal $\mathbb{R}$-tree with an action of Γ by isometries and length function ℓ in the class $[\ell]$. As above, $\mathrm{Hopf}(u_{i'}) \to \mathrm{Hopf}(u)$, so by the definition of H, $h([\rho_{i'}]) \to H([\ell])$, a contradiction. Since we have accounted for both possible cases, this proves the claim.

5. Convergence of length functions. In this final section we briefly sketch an alternative argument for the convergence to the boundary in the main theorem, based on a direct analysis of length functions, more in the spirit of [W1]. The generalization of estimates for equivariant harmonic maps with target $\mathbb{H}^2$ to maps with target $\mathbb{H}^3$ has largely been carried out by Minsky [M]. We discuss this point of view, however, since it reveals how and why the folding of the dual tree $T_{\tilde{\varphi}}$ occurs.

The first step is to analyze the behavior of the induced metric for a harmonic map $u : \mathbb{H}^2 \to \mathbb{H}^3$ of high energy (at the points where u is an immersion). As usual we denote by $\tilde{\varphi}$ the Hopf differential for the map u. Because of equivariance, $\tilde{\varphi}$ is the lift of a holomorphic quadratic differential φ on Σ. Recall the norm $\|\varphi\|$ from the introduction, and let $Z(\varphi) \subset \Sigma$ denote the zero set of φ. We also set μ to be the Beltrami differential associated to the pullback metric $u^* ds^2_{\mathbb{H}^3}$.

LEMMA 5.1. *Fix $\delta, T > 0$. Then there are constants $B, \alpha > 0$ such that for all u, μ, and φ as above, $\|\varphi\| \geq T$, and all $p \in \Sigma$ satisfying* $\mathrm{dist}_\sigma(p, Z(\varphi)) \geq \delta$, *we have*

$$\log\left(\frac{1}{|\mu|}\right)(p) < B e^{-\alpha\|\varphi\|} .$$

Proof. This result is proven in [M, Lemma 3.4]. One needs only a statement concerning the uniformity of the constants appearing there. However, by using the compactness of *SQD*, one easily shows the following: For $\delta > 0$ there is a constant $c(\delta) > 0$ such that, for all $\varphi \in SQD$ and all $p \in \Sigma$ such that $\mathrm{dist}_\sigma(p, Z(\varphi)) \geq \delta$, the disk U of radius $\tilde{c}(\delta)$ (with respect to the singular flat metric $|\varphi|$) around p is embedded in Σ and contains no zeros of φ. Then the result cited above applies. $\square$

This estimate is all that is needed to prove convergence in the case where there cannot be a folding of the dual tree $T_{\tilde{\varphi}}$ such that the composition of projection to $T_{\tilde{\varphi}}$ with the folding is harmonic. From Lemma 3.2, this is guaranteed, for example, if φ has only simple zeros. For simplicity, in this section we assume all representations are irreducible.

THEOREM 5.2. *Given an unbounded sequence ρ_j of representations with Morgan-Shalen limit $[\ell]$, let $u_j : \mathbb{H}^2 \to \mathbb{H}^3$ be the associated ρ_j-equivariant harmonic maps. Suppose that for $\tilde{\varphi}_j = \mathrm{Hopf}(u_j)$ we have $\varphi_j/\|\varphi_j\| \to \varphi \in SQD$, where φ has only simple zeros. Then $[\ell] = [\ell_T]$, where $T = T_{\tilde{\varphi}}$.*

Proof. We prove the convergence of length functions in two steps. First, we compare the length of closed curves γ in the free isotopy class $[\gamma]$ with respect to the induced metric from u_j to the length with respect to the transverse measure. Second, we compare the length of the image by u_j of a lift $\tilde{\gamma}$ to $\mathbb{H}^2$ of γ to the translation length in $\mathbb{H}^3$ of the conjugacy class that $[\gamma]$ represents. The basic idea is that the image of $\tilde{\gamma}$ very nearly approximates a segment of the hyperbolic axis for $\rho_j([\gamma])$.

For φ and $[\gamma]$ as above, let $\ell_\varphi([\gamma])$ denote the infimum over all representatives γ of $[\gamma]$ of the length of γ with respect to the vertical measured foliation defined by φ. If $u : \mathbb{H}^2 \to \mathbb{H}^3$ is a differentiable equivariant map, we define $\ell_u([\gamma])$ as follows: For each representative γ of $[\gamma]$, where $[\gamma]$ corresponds to the conjugacy class of $g \in \Gamma$, lift γ to a curve $\tilde{\gamma}$ at a point $x \in \mathbb{H}^2$, terminating at gx. We then take the infimum over all such $\tilde{\gamma}$ of the length of $u(\tilde{\gamma})$. This is $\ell_u([\gamma])$, and by the equivariance of u it is independent of the choice of x. Finally, recall that the translation length $\ell_\rho([\gamma])$ for a representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{C})$ is defined in Section 2.

Given $\varepsilon > 0$, let $QD_\varepsilon \subset QD \backslash \{0\}$ denote the subset consisting of holomorphic quadratic differentials φ having only simple zeros, and such that the zeros are pairwise at least a σ-distance ε apart. Notice that for $t \neq 0$, $t\, QD_\varepsilon = QD_\varepsilon$. The next result is a consequence of Lemma 5.1.

PROPOSITION 5.3. *For all classes $[\gamma]$ and differentials $\varphi \in QD_\varepsilon$, there exist constants k and η depending on $\|\varphi\|$, $[\gamma]$, and ε, so that*

$$k\, \ell_\varphi([\gamma]) + \eta \geq \ell_u([\gamma]) \geq \ell_\varphi([\gamma])$$

where $k \to 1$ and $\eta \|\varphi\|^{-1/2} \to 0$ as $\|\varphi\| \to \infty$ in QD_ε.

Sketch of proof. We first need to choose an appropriate representative for the class of $[\gamma]$. Such a choice was explained in [W1]. Namely, for $\delta > 0$ and a given φ, we can find a representative γ consisting of alternating vertical and horizontal segments and having the transverse measure of the class $[\gamma]$. Moreover, because the zeros of φ are simple, for sufficiently small δ we can also guarantee that γ avoid a δ neighborhood of the zeros. Now the proof follows as in [W1, Lemma 4.6]. Note that along a *harmonic maps ray* (that is, a sequence u_i such that $\mathrm{Hopf}(u_i)$ is of the form $t_i\varphi$ for a fixed φ and an increasing unbounded sequence t_i), we no longer necessarily have monotonicity of the norm of the Beltrami differentials $|\mu(t_i)|$. The argument for the estimate still applies, however, since the representatives γ are uniformly supported away from the zeros. There, we apply the estimate Lemma 5.1. The details are omitted. □

Next, we compare ℓ_u with the translation length in $\mathbb{H}^3$.

PROPOSITION 5.4. *Let $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{C})$ and $u : \mathbb{H}^2 \to \mathbb{H}^3$ be the ρ-equivariant harmonic map with $\tilde{\varphi} = \mathrm{Hopf}(u)$. Suppose $\varphi \in QD_\varepsilon$. For all classes $[\gamma]$ there exist constants m and ζ depending on $\|\varphi\|$, $[\gamma]$, and ε, so that*

$$m\,\ell_\rho([\gamma]) + \zeta \geq \ell_u([\gamma]) \geq \ell_\rho([\gamma]),$$

where $m \to 1$ and $\zeta\,\|\varphi\|^{-1/2} \to 0$ as $\|\varphi\| \to \infty$ in QD_ε.

Combining Propositions 5.3 and 5.4 proves Theorem 5.2. □

Sketch of proof of Proposition 5.4. One observes that away from the zeros, the images of the horizontal leaves of the foliation of $\tilde{\varphi}$ closely approximate (long) geodesics in $\mathbb{H}^3$, while by Lemma 5.1 the images of vertical leaves collapse. More precisely, the following is proven in [M, Thm. 3.5].

LEMMA 5.5. *Fix $\delta > 0$, a representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{C})$, and let $u : \mathbb{H}^2 \to \mathbb{H}^3$ be the ρ-equivariant harmonic map with Hopf differential $\tilde{\varphi}$. Let $\tilde{\beta}$ be a segment of the horizontal foliation of $\tilde{\varphi}$ from x to y and suppose that, for all $\tilde{p} \in \tilde{\beta}$, $\mathrm{dist}_\sigma(p, Z(\varphi)) \geq \delta$. Then there is an ε, exponentially decaying in $\|\varphi\|$, such that the following hold:*

(1) *$u(\tilde{\beta})$ is uniformly within ε of the geodesic in $\mathbb{H}^3$ from $u(x)$ to $u(y)$.*

(2) *The length of $u(\tilde{\beta})$ is within ε of $\mathrm{dist}_{\mathbb{H}^3}(u(x), u(y))$.*

The following is a key result.

LEMMA 5.6. *Given $g \in \mathrm{SL}(2, \mathbb{C})$, let $\ell(g)$ denote the translation length for the action of g on $\mathbb{H}^3$. Suppose that $s \subset \mathbb{H}^3$ is a curve that is g invariant and satisfies the following property: For any two points $x, y \in s$, the segment of s from x to y is uniformly within a distance 1 of the geodesic in $\mathbb{H}^3$ joining x and y. Then there is a universal constant C such that*

$$\inf_{x\in s} \mathrm{dist}_{\mathbb{H}^3}\big(x, g(x)\big) \leq \ell(g) + C.$$

Proof. The intuition is clear; such an s must be an "approximate axis" for g. The proof proceeds as follows: Choose $x \in s$, and let c denote the geodesic in $\mathbb{H}^3$ from x to

$g(x)$. By [Cp, Lemma 2.4] there exists a universal constant D and a subgeodesic $\tilde{c}$ of c with the property that $|\text{length}(\tilde{c}) - \ell(g)| \leq D$. Let a and b be the endpoints of $\tilde{c}$ closest to x and $g(x)$, respectively. By the construction of $\tilde{c}$ in the reference cited above, it follows that $\text{dist}_{\mathbb{H}^3}(b, g(a)) \leq D$; hence, $\text{dist}_{\mathbb{H}^3}(b, g(b)) \leq \ell(g) + 2D$. Now by the assumption on c, there is a point $y \in s$ close to b, so that $\text{dist}_{\mathbb{H}^3}(y, g(y)) \leq \ell(g) + C$, where $C = 2(D+1)$. □

Proceeding with the proof of Proposition 5.4, choose the representative γ as discussed in Proposition 5.3. We may then lift to $\tilde{\gamma} \subset \mathbb{H}^2$ so that $\tilde{\gamma}$ is invariant under the action of g. Now $\tilde{\gamma}$ is written as a union of horizontal and vertical segments of the foliation of $\tilde{\varphi}$. Let $s = u(\tilde{\gamma})$.Then Lemmas 5.1 and 5.5 imply that s satisfies the hypothesis of Lemma 5.6. Moreover, using Lemma 5.5 again, along with some elementary hyperbolic geometry, one can show that $\inf_{x \in s} \text{dist}_{\mathbb{H}^3}(x, g(x))$ is approximated by the length of a segment of $u(\tilde{\gamma})$ from a point $u(x)$ to $u(gx)$. We leave the precise estimates to the reader. □

From Lemma 3.2, we see that foldings can only arise when the Hopf differentials converge in *SQD* to differentials with multiplicity at the zeros. From the point of view taken here, this corresponds to the fact that the representatives for closed curves γ chosen above may be forced to run into zeros of the Hopf differential where the estimate Lemma 5.1 fails. These may cause nontrivial angles to form in the image $u(\tilde{\gamma})$ which, in the limit, may fold the dual tree.

Consider again the situation along a harmonic maps ray with differential φ. Given $[\gamma]$ corresponding to the conjugacy class of an element $g \in \Gamma$, representatives γ still may be chosen as in the proof of Proposition 5.3 so that the horizontal segments remain bounded away from the zeros. However, it may happen that a vertical segment passes through a zero of order two or greater. For simplicity, assume this happens once. Divide γ into curves γ_1, γ_2, and γ_v, where γ_v is the offending vertical segment, and lift to segments $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $\tilde{\gamma}_v$ in $\mathbb{H}^2$. Note that one end point of each of the $\tilde{\gamma}_i$'s corresponds to either end point of $\tilde{\gamma}_v$, and the other end points of the $\tilde{\gamma}_i$'s are related by g. By the Lipschitz estimate for harmonic maps to nonpositively curved spaces, we have a bound on the distance in $\mathbb{H}^3$ between the end points of $u(\tilde{\gamma}_v)$ in terms of the length of γ_v and the energy $E(u)^{1/2}$ (cf. [S2]). Thus, the rescaled length is small; in fact, since the length of γ_v is arbitrary, the distance converges to zero. On the other hand, the previous argument applies to the segments $u(\tilde{\gamma}_1)$ and $u(\tilde{\gamma}_2)$, which are connected by $u(\tilde{\gamma}_v)$. Adding the geodesic in $\mathbb{H}^3$ joining the other end points of $u(\tilde{\gamma}_1)$ and $u(\tilde{\gamma}_2)$ forms an approximate geodesic quadrilateral, which, in the rescaled limit, converges either to an edge | (no folding) or a possibly degenerate tripod ⊣ (folding). In both cases, there is an edge that, by the same argument as in the proof of Proposition 5.4, approximates the axis of $\rho_j(g)$ for large j. At the same time, the rescaled length of this segment is approximated by the translation length of the element g acting on a folding of $T_{\tilde{\varphi}}$ at the zero.

An interesting question is whether this approach may be used to determine precisely the fibers of the map $\bar{h}$ in the main theorem. While the essential ideas are outlined here, a complete description is not yet available. We will return to this issue in a future work.

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