CONTINUITY OF THE YANG-MILLS FLOW ON THE SET OF SEMISTABLE BUNDLES

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Dedicated to Duong H. Phong, with admiration, on the occasion of his 65th birthday.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and $(E, h) \to X$ a C^∞ hermitian vector bundle on X. The celebrated theorem of Donaldson-Uhlenbeck-Yau states that if A is an integrable unitary connection on (E, h) that induces an ω-slope stable holomorphic structure on E, then there is a complex gauge transformation g such that $g(A)$ satisfies the Hermitian-Yang-Mills (HYM) equations. The proof in [40] uses the continuity method applied to a deformation of the Hermitian-Einstein equations for the metric h. The approach in [11, 12] deforms the metric using a nonlinear parabolic equation, the *Donaldson flow*. Deforming the metric is equivalent to acting by a complex gauge transformation modulo unitary ones, and in this context the Donaldson flow is equivalent (up to unitary gauge transformations) to the Yang-Mills flow on the space of integrable unitary connections. The proof in [12] assumes that X is a projective algebraic manifold (more precisely, that ω is a Hodge metric) whereas the argument in [40] does not. The methods of Uhlenbeck-Yau and Donaldson were combined by Simpson [35] to prove convergence of the Yang-Mills flow for stable bundles on all compact Kähler manifolds. The Yang-Mills flow thus defines a map $\mathcal{A}^s(E, h) \to M^*_{\mathrm{HYM}}(E, h)$ from the space of smooth integrable connections on (E, h) inducing stable holomorphic structures to the moduli space $M^*_{\mathrm{HYM}}(E, h)$ of irreducible HYM connections.[1] Continuity of this map follows by a comparison of Kuranishi slices (see [15, 31]).

When the holomorphic bundle $\mathcal{E}_A = (E, \bar\partial_A)$ is strictly semistable, then the Donaldson flow fails to converge unless $\mathcal{E}_A$ splits holomorphically into a sum of stable bundles (i.e. it is *polystable*). If $n = 1$ it is still true, however, that the Yang-Mills flow converges to a smooth HYM connection on E for any semistable initial condition. This was proven by Daskalopoulos and Råde [7, 32]. Moreover, the holomorphic structure of the limiting connection is isomorphic to the polystable holomorphic bundle $\mathrm{Gr}(\mathcal{E}_A)$ obtained from the

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[1] The notion of (semi)stability depends on the choice of Kähler class $[\omega]$; however, the class will remain fixed throughout, and we shall suppress this dependency from the notation.

associated gradation of the Jordan-Hölder filtration of $\mathcal{E}_A$. For $n \geq 2$, there is an obstruction to a smooth splitting into an associated graded bundle, and $\mathrm{Gr}(\mathcal{E}_A)$ may not be locally free. The new phenomenon of bubbling occurs, and one must talk of convergence *in the sense of Uhlenbeck*, that is, away from a singular set of complex codimension at least 2 (see Theorem 2.2 below). In [8] (see also [9]) it was shown for $n = 2$ that the Yang-Mills flow converges in the sense of Uhlenbeck to the reflexification $\mathrm{Gr}(\mathcal{E}_A)^{**}$, which is a polystable bundle. The bubbling locus, which in this case is a collection of points with multiplicities, is precisely the set where $\mathrm{Gr}(\mathcal{E}_A)$ fails to be locally free [10]. The extension of these results in higher dimensions was achieved in [33, 34]. Here, even the reflexified associated graded sheaf may fail to be locally free, and one must use the notion of an *admissible* HYM connection introduced by Bando and Siu [4]. Convergence of the flow to the associated graded sheaf for semistable bundles in higher dimensions was independently proven by Jacob [22].

In a different direction, a compactification of M^*_{HYM} was proposed by Tian in [37] and further studied in [38]. This may be viewed as a higher dimensional version of the Donaldson-Uhlenbeck compactification of ASD connections on a smooth manifold of real dimension 4 (cf. [14, 13]). It is based on a finer analysis of the bubbling locus for limits of HYM connections that is similar to the one carried out for harmonic maps by Fang-Hua Lin [27]. More precisely, Tian proves that the top dimensional stratum is rectifiable and calibrated by ω with integer multiplicities, and as a consequence of results of King [23] and Harvey-Shiffman [19], it represents an analytic cycle. The compactification is then defined by adding ideal points containing in addition to an admissible HYM connection the data of a codimension 2 cycle in an appropriate cohomology class (see Section 2). At least when X is projective, the space $\widehat{M}_{\mathrm{HYM}}$ of ideal HYM connections is a compact topological space (Hausdorff), and the compactification of M^*_{HYM} is obtained by taking its closure $\overline{M}_{\mathrm{HYM}} \subset \widehat{M}_{\mathrm{HYM}}$. Under this assumption, we recently showed, in collaboration with Daniel Greb and Matei Toma, that $\overline{M}_{\mathrm{HYM}}$ admits the structure of a seminormal complex algebraic space [16].

The purpose of this note is to point out the compatibility of this construction with the Yang-Mills flow. For example, in the case of a Riemann surface, the flow defines a continuous deformation retraction of the entire semistable stratum onto the moduli space of semistable bundles. This is precisely what is to be expected from Morse theory (see [2]). In higher dimensions, as mentioned above, bubbling along the flow needs to be accounted for. The result is the following.

Main Theorem. *Let (E, h) be a hermitian vector bundle over a compact Kähler manifold (X, ω) with $[\omega] \in H^2(X, \mathbb{Z})$. Let $\mathcal{A}^{1,1}(E, h)$ denote the set of integrable unitary connections on (E, h) with the smooth topology (see Section 2). Let $\mathcal{A}^{ss}(E, h) \subset \mathcal{A}^{1,1}(E, h)$ be the subset consisting of slope semistable holomorphic bundles. Then:*

(1) *$\mathcal{A}^{ss}(E, h)$ is an open subset of $\mathcal{A}^{1,1}(E, h)$;*

(2) *the Yang-Mills flow defines a continuous map*

$$\mathscr{F} : \mathcal{A}^{ss}(E, h) \to \widehat{M}_{\mathrm{HYM}}(E, h) \ . \tag{1.1}$$

In particular, the restriction of $\mathscr{F}$ *gives a continuous map* $\overline{\mathcal{A}^s}(E,h) \to \overline{M}_{\mathrm{HYM}}(E,h)$*, where* $\overline{\mathcal{A}^s}(E,h) \subset \mathcal{A}^{ss}(E,h)$ *is the closure of* $\mathcal{A}^s(E,h)$ *in the smooth topology.*

(3) *In fact, on the closure, the map* $\mathscr{F}$ *factors as follows:*

$$\begin{array}{ccc} \overline{\mathcal{A}^s}(E,h) & \xrightarrow{Q} & \overline{M}^{\mu}(E,h) \\ & \searrow^{\mathscr{F}} & \downarrow \overline{\Phi} \\ & & \overline{M}_{\mathrm{HYM}}(E,h) \end{array} \tag{1.2}$$

where $\overline{M}^{\mu}(E,h)$ *is a modular compactification of the moduli space of stable holomorphic structures on* E*,* $\overline{\Phi}$ *is a continuous comparison map between the two compactifications, and* Q *is a continuous map.*

The proof of item (1) of the Main Theorem relies on a theorem of Maruyama along with a comparison between $\mathcal{A}^{1,1}(E,h)$ and the Quot scheme (see Section 4 and Corollary 6.2 as well as 6.1). Part (2) is a consequence of the work in [16], with small modifications. Part (3) is also a corollary of the result of 6.1 in Section 6, combined with the continuity of the map $\overline{\Phi}$ proven in [16]. For the case of Kähler surfaces, the second statement of part (2) was claimed in [10, Thm. 2]. Unfortunately, there is an error in the proof of Lemma 8 of that paper, and hence also in the proof of Theorem 2. The Main Theorem above validates the statement in [10, Thm. 2], at least in the projective case. We do not know if the result holds when X is only a Kähler surface. The advantage of projectivity is that a twist of the bundle is generated by global holomorphic sections. These behave well with respect to Uhlenbeck limits and provide a link between the algebraic geometry of geometric invariant theory quotients and the analytic compactification. We review this in Section 4 below.

2. Uhlenbeck limits and admissible HYM connections

In this section we briefly review the compactification of $M^*_{\mathrm{HYM}}(E,h)$ by ideal HYM connections. As in the introduction, let (E,h) be a hermitian vector bundle on a compact Kähler manifold (X,ω) of dimension n, and let $\mathfrak{g}_E$ denote the bundle of skew-hermitian endomorphisms of E. The space $\mathcal{A}(E,h)$ of C^∞ unitary connections on E is an affine space over $\Omega^1(X,\mathfrak{g}_E)$, and we endow it with the smooth topology. A connection $A \in \mathcal{A}(E,h)$ is called *integrable* if its curvature form F_A is of type (1,1). Let $\mathcal{A}^{1,1}(E,h)$ denote the set of integrable unitary connections on (E,h). Then $\mathcal{A}^{1,1}(E,h) \subset \mathcal{A}(E,h)$ inherits a topology as a closed subset. The locus $\mathcal{A}^s(E,h)$ of *stable* holomorphic structures is open in $\mathcal{A}^{1,1}(E,h)$ (cf. [28, Thm. 5.1.1]). Under the assumption that ω is a Hodge metric we shall prove below that the subset $\mathcal{A}^{ss}(E,h)$ of *semistable* holomorphic structures is also open in $\mathcal{A}^{1,1}(E,h)$ (see Corollary 6.2).

We call the contraction $\sqrt{-1}\Lambda F_A$ of F_A with the Kähler metric the *Hermitian-Einstein tensor*. It is a hermitian endomorphism of E. The key definition is the following (cf. [4] and [37, Sect. 2.3]).

Definition 2.1. An *admissible connection* is a pair (A, S) where

(1) $S \subset X$ is a closed subset of finite Hausdorff $(2n-4)$-measure;
(2) A is a smooth integrable unitary connection on $E\big|_{X\backslash S}$;
(3) $\int_{X\backslash S} |F_A|^2\, dvol_X < +\infty$;
(4) $\sup_{X\backslash S} |\Lambda F_A| < +\infty$.

An admissible connection is called *admissible HYM* if there is a constant μ such that $\sqrt{-1}\Lambda F_A = \mu \cdot \mathbf{I}$ on $X\backslash S$.

The fundamental weak compactness result is the following.

Theorem 2.2 (Uhlenbeck [39])**.** *Let A_i be a sequence of smooth integrable connections on $(E, h) \to X$ with uniformly bounded Hermitian-Einstein tensors. Then for any $p > n$ there is*

(1) *a subsequence (still denoted A_i),*
(2) *a closed subset $S_\infty \subset X$ of finite $(2n-4)$-Hausdorff measure,*
(3) *a connection A_∞ on a hermitian bundle $E_\infty \to X\backslash S_\infty$, and*
(4) *local isometries $E_\infty \simeq E$ on compact subsets of $X\backslash S_\infty$*

such that with respect to the local isometries, and modulo unitary gauge equivalence, $A_i \to A_\infty$ weakly in $L^p_{1,loc}(X\backslash S_\infty)$.

Remark 2.3. If one further assumes that $||d_{A_i}\Lambda F_{A_i}||_{L^2(X,\omega)} \to 0$, then the limiting connection is Yang mills (see Section 3). For the proof of this see [8], Prop. 2.11.

We call the limiting connection A_∞ an *Uhlenbeck limit*. The set

$$S_\infty = \bigcap_{\sigma_0 \geq \sigma > 0} \left\{ x \in X \mid \liminf_{i\to\infty} \sigma^{4-2n} \int_{B_\sigma(x)} |F_{A_i}|^2 \frac{\omega^n}{n!} \geq \varepsilon_0 \right\} ,$$

where σ_0 and ε_0 are universal constants depending only on the geometry of X, is called the *(analytic) singular set*.

For the definition of a gauge theoretic compactification more structure is needed. This is provided by the following, which is a consequence of work of Tian [37] and Hong-Tian [20] (see for example [37],Thm 4.3.3).

Proposition 2.4. *The Uhlenbeck limit of a sequence of smooth HYM connections on (E, h) is an admissible HYM connection. Moreover, the corresponding singular set S_∞ is a holomorphic subvariety of codimension at least 2. The same is true for Uhlenbeck limits of sequences along the Yang-Mills flow, except that the limiting connection is merely Yang-Mills in general (note that the flow satisfies the condition of 2.3).*

For the definition of the flow see Section 3 below.

To be more precise, there is a decomposition $S_\infty = |\mathcal{C}_\infty| \cup S(A_\infty)$, where

$$S(A_\infty) := \left\{ x \in X \;\middle|\; \lim_{\sigma \downarrow 0} \sigma^{4-2n} \int_{B_\sigma(x)} |F_{A_\infty}|^2 \frac{\omega^n}{n!} \neq 0 \right\}. \tag{2.1}$$

has codimension ≥ 3, and $|\mathcal{C}_\infty|$ is the support of a codimension 2 cycle $\mathcal{C}_\infty$. The cycle appears as the limiting current of the Yang-Mills energy densities, just as in the classical approach of Donaldson-Uhlenbeck in real dimension 4. This structure motivates the following

Definition 2.5 ([16, Def. 3.15])**.** An *ideal HYM connection* is a triple $(A, \mathcal{C}, S(A))$ satisfying the following conditions:

(1) $\mathcal{C}$ is an $(n-2)$-cycle on X;
(2) the pair $(A, |\mathcal{C}| \cup S(A))$ is an admissible HYM connection on the hermitian vector bundle $(E, h) \to X$, where $S(A)$ is given as in eq. (2.1);
(3) $[\mathrm{ch}_2(A)] = \mathrm{ch}_2(E) + [\mathcal{C}]$, in $H^4(X, \mathbb{Q})$;

Note that here, instead of allowing arbitrary sets S as in Definition 2.1, we require the particular form in item (2) above. This gives better control of these higher codimensional sets in sequences. We also remark that since the set $S(A)$ is determined by A, we can also think of an ideal connection as simply a pair $(A, \mathcal{C})$, and we will make this abuse of notation in the sequel.

Here we have denoted by $\mathrm{ch}_2(A)$ the $(2,2)$-current given by

$$\mathrm{ch}_2(A)(\Omega) := -\frac{1}{8\pi^2} \int_X \mathrm{tr}(F_A \wedge F_A) \wedge \Omega \ ,$$

for smooth $(2n-4)$-forms Ω. This is well defined by Definition 2.1 (3), and in [37, Prop. 2.3.1] it is shown to be a closed current. It thus defines a cohomology class as above. By [4], there is a polystable reflexive sheaf $\mathcal{E}$ extending the holomorphic bundle $(E|_{X \setminus |\mathcal{C}| \cup S(A)}, \overline{\partial}_A)$. The singular set $\mathrm{sing}(\mathcal{E}_A)$ of $\mathcal{E}_A$, that is, the locus where $\mathcal{E}_A$ fails to be locally free, coincides with $S(A)$ (see [38, Thm. 1.4]). By the proof of [34, Prop. 3.3], $\mathrm{ch}_2(A)$ represents the class $\mathrm{ch}_2(\mathcal{E})$. Thus we may alternatively regard an ideal connection as a pair $(\mathcal{E}_A, \mathcal{C})$, where $\mathcal{E}_A$ is a reflexive sheaf, $\mathcal{C}$ is a codimension 2 cycle with $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(E) + [\mathcal{C}]$, and where the underlying smooth bundle of $\mathcal{E}_A$ on the complement of $|\mathcal{C}| \cup \mathrm{sing}(\mathcal{E})$ is isomorphic to E. See [16, Sec. 3.3] for more details. Moreover we consider two ideal connections $(\mathcal{E}_{A_1}, \mathcal{C}_1)$ and $(\mathcal{E}_{A_2}, \mathcal{C}_2)$ to be equivalent if $\mathcal{E}_{A_1}$ and $\mathcal{E}_{A_2}$ are isomorphic as sheaves (or equivalently A_1 and A_2 are gauge equivalent), and $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal.

Definition 2.6. We define $\widehat{M}_{\mathrm{HYM}}(E, h)$ to be the space of gauge equivalence classes of ideal HYM connections.

In what follows, we shall denote by $[A]$ the unitary gauge equivalance class of a connection $A \in \mathcal{A}^{1,1}(E, h)$, and by $[(A, \mathcal{C})]$ the equivalence class of an ideal HYM connection $(A, \mathcal{C})$. The main result is the following (see [16],Thm 3.17).

Theorem 2.7. *Assume ω is a Hodge metric. Let $[(A_i, \mathcal{C}_i)] \in \widehat{M}_{\mathrm{HYM}}(E,h)$. Then there is a subsequence (also denoted by $\{i\}$), and an ideal HYM connection $(A_\infty, \mathcal{C}_\infty, S(A_\infty))$ such that $\mathcal{C}_i$ converges to a subcycle of $\mathcal{C}_\infty$, and (up to gauge transformations) $A_i \to A_\infty$ in C^∞_{loc} on $X\backslash(|\mathcal{C}_\infty| \cup S(A_\infty))$. Moreover,*

$$\mathrm{ch}_2(A_i) - \mathcal{C}_i \longrightarrow \mathrm{ch}_2(A_\infty) - \mathcal{C}_\infty \tag{2.2}$$

in the mass norm; in particular, also in the sense of currents.

Using the result above one can define a compact Hausdorff topology on $\widehat{M}_{\mathrm{HYM}}(E,h)$, and from there a compactification $\overline{M}_{\mathrm{HYM}}(E,h)$ of $M_{\mathrm{HYM}}(E,h)$. For more details we refer to [37, 38] and [16, Thm. 1.1] which is proved in Section 3 of that paper.

3. The Yang-Mills flow

The *Yang-Mills flow* is a time dependent family of integrable connections $A(t)$ depending on $A_0 \in \mathcal{A}^{1,1}(E,h)$ satisfying the equations:

$$\frac{\partial A(t)}{\partial t} = -d_{A^*(t)} F_{A(t)} \ , \ A(0) = A_0 \tag{3.1}$$

Donaldson [11] shows that a solution to (3.1) exists (modulo gauge transformations) for all $0 \le t < +\infty$. Eq. (3.1) is formally the negative gradient flow for the Yang-Mills functional:

$$YM(A) = \frac{1}{2}\int_X |F_A|^2 \, dvol_\omega.$$

Critical points of YM are called *Yang-Mills connections* and satisfy $d_A^* F_A = 0$. A smooth integrable Yang-Mills connection A decomposes the bundle $\mathcal{E}_A$ holomorphically and isometrically into a direct sum of the (constant rank) eigenbundles of $\sqrt{-1}\Lambda_\omega F_A$, and the induced connections are Hermitian-Yang-Mills. Similarly, an admissible Yang-Mills connection on a reflexive sheaf gives a direct sum decomposition into reflexive sheaves admitting admissible HYM connections.

By Proposition 2.4, any sequence of times t_j along the flow has an Uhlenbeck limit, which is Yang-Mills. A priori this limit might depend on the choice of subsequence chosen to achieve convergence. It turns out that the limit is independent of the chosen subsequence however. There is a canonical ideal connection associated to the bundle $\mathcal{E}_{A_0}$ which is the putative limit for all subsequences.

Let $A \in \mathcal{A}^{ss}(E,h)$, i.e. the induced holomorphic bundle $\mathcal{E}_A = (E, \bar\partial_A)$ is semistable. Then there is a Seshadri filtration $\{0\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_\ell = \mathcal{E}_A$ such that the successive quotients $\mathcal{Q}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, $i = 1, \ldots, \ell$ are stable torsion-free sheaves all of equal slope to that of $\mathcal{E}_{A_0}$. Let $\mathrm{Gr}(\mathcal{E}_A) = \oplus_{i=1}^{\ell} \mathcal{Q}_i$ and $\mathcal{C}_\infty$ the cycle defined by the codimension 2 support of $\mathrm{Gr}(\mathcal{E}_A)^{**}/\,\mathrm{Gr}(\mathcal{E}_A)$ (see Section 5). By the result of Bando-Siu [4], there is an admissible HYM connection A_∞ on $\mathrm{Gr}(\mathcal{E}_A)^{**}$, such that $(A_\infty, \mathcal{C}_\infty)$ defines an ideal HYM connection in the sense of Definition 2.5. For a detailed description of this we refer to [16, Sec. 3.1].

Theorem 3.1 ([8, 10, 33, 34, 22, 20])**.** *Let $A_0 \in \mathcal{A}^{ss}(E,h)$. Then the Yang-Mills flow A_t with initial condition A_0 converges in the sense of Theorem 2.7 to an ideal connection $(A_\infty, \mathcal{C}_\infty)$, where A_∞ is the admissible HYM connection on $\mathrm{Gr}(\mathcal{E}_{A_0})^{**}$, and $\mathcal{C}_\infty$ is the codimension 2 cycle defined by the torsion-free sheaf $\mathrm{Gr}(\mathcal{E}_{A_0})$.*

Remark 3.2. Note that this says in particular that in the case that A_0 gives a semistable holomorphic structure, the Uhlenbeck limit is in fact HYM rather than merely Yang-Mills. A version of the theorem also holds when $\mathcal{E}_{A_0}$ is a general unstable bundle as well.

Remark 3.3. The theorem above states that there is a well-defined ideal HYM connection $\mathscr{F}([A_0])$ defined by the limit at ∞ of the Yang-Mills flow, and which is given purely in terms of the holomorphic initial data and the solution for admissible HYM connections on reflexive sheaves. In particular, if A and $\widetilde{A}$ are complex gauge equivalent (i.e. $\mathcal{E}_A \simeq \mathcal{E}_{\widetilde{A}}$), then $\mathscr{F}([A]) = \mathscr{F}([\widetilde{A}])$. Thus, the map $\mathscr{F}$ in (1.1) is alternatively defined by setting

$$\mathscr{F} : \mathcal{A}^{ss}(E,h) \longrightarrow \widehat{M}_{\mathrm{HYM}}(E,h) : [A] \mapsto [(A_\infty, \mathcal{C}_\infty)] \ . \tag{3.2}$$

4. The method of holomorphic sections

Admissibility of a connection is precisely the correct analytic notion to make contact with complex analysis. Bando [3] and Bando-Siu [4] show that bundles with admissible connections admit sufficiently many local holomorphic sections to prove coherence of the sheaf of L^2-holomorphic sections. This local statement only requires the Kähler condition. The key difference between the projective vs. Kähler case is, of course, the abundance of global holomorphic sections. These provide a link between the algebraic and analytic moduli. They are also well-behaved with respect to limits. The technique described here mimics that introduced by Jun Li in [26].

We henceforth assume $[\omega] \in H^2(X, \mathbb{Z})$. Let $L \to X$ be a complex line bundle with $c_1(L) = [\omega]$. Define the numerical invariant:

$$\tau_E(m) := \int_X \mathrm{ch}(E \otimes L^m)\mathrm{td}(X) \ . \tag{4.1}$$

Since ω is a $(1,1)$ class, L may be endowed with a holomorphic structure $\mathcal{L}$ making it the ample line bundle defining the polarization of X. We also fix a hermitian metric on L with respect to which the Chern connection of $\mathcal{L}$ has curvature $-2\pi i\omega$. Use the following notation: $\mathcal{E}(m) := \mathcal{E} \otimes \mathcal{L}^m$. The key property we exploit is the following, which is a consequence of Maruyama's boundedness result [30], as well as the Hirzebruch-Riemann-Roch theorem.

Proposition 4.1. *There is $M \geq 1$ such that for all $m \geq M$ and all $A \in \mathcal{A}^{ss}(E,h)$, if $\mathcal{E}_A = (E, \bar\partial_A)$ then the bundle $\mathcal{E}_A(m)$ is globally generated and all higher cohomology groups vanish. In particular, $\dim H^0(X, \mathcal{E}_A(m)) = \tau_E(m)$ for $m \geq M$.*

In the following, we shall assume m has been fixed sufficiently large. Fix a vector space V of dimension $\tau_E(m)$, and let

$$\mathcal{H} = V \otimes \mathcal{L}^{-m} \ . \tag{4.2}$$

The *Grothendieck Quot scheme* $\mathrm{Quot}(\mathcal{H}, \tau_E)$ is a projective scheme parametrizing isomorphism classes of quotients $\mathcal{H} \to \mathcal{F} \to 0$, where $\mathcal{F} \to X$ is a coherent sheaf with Hilbert polynomial τ_E [18, 1]. Proposition 4.1 states that there is a uniform m such that for every $A \in \mathcal{A}^{ss}(E,h)$ there is a quotient $\mathcal{H} \to \mathcal{E}_A \to 0$ in $\mathrm{Quot}(\mathcal{H}, \tau_E)$ with $\mathcal{E}_A \simeq (E, \bar{\partial}_A)$. The next result begins the comparison between Uhlenbeck limits and limits in $\mathrm{Quot}(\mathcal{H}, \tau_E)$.

Proposition 4.2. *Let $\{A_i\} \subset \mathcal{A}^{ss}(E,h)$, and suppose $A_i \to A_\infty$ in the sense of Uhlenbeck (Theorem 2.2), and assume uniform bounds on the Hermitian-Einstein tensors, and that A_∞ is Hermitian-Yang-Mills. Then there are quotients $\mathcal{H} \to \mathcal{F}_i$ in Quot($\mathcal{H}, \tau_E$), with $\mathcal{F}_i \cong \mathcal{E}_{A_i}$, converging to a semistable quotient $\mathcal{H} \to \mathcal{F}_\infty \to 0$ in* $\mathrm{Quot}(\mathcal{H}, \tau_E)$ *and an inclusion $\mathcal{F}_\infty \hookrightarrow \mathcal{E}_{A_\infty}$ such that $\mathcal{F}_\infty^{**} \simeq \mathcal{E}_{A_\infty}$.*

The proof of this result for sequences of HYM connections is in [16, Prop. 4.2], but the proof there works as well under the weaker assumption of a uniformly bounded Hermitian-Einstein tensor. Indeed, the first key point is the application of the Bochner formula to obtain uniform bounds on L^2-holomorphic sections. The precise statement is that if $s \in H^0(X, \mathcal{E}_{A_i}(m))$ then there is a constant C depending only on the geometry of X, m, and the uniform bound on the Hermitian-Einstein tensor, such that $\sup_X |s| \leq C\|s\|_{L^2}$. As noted above, we may realise the the bundles $\mathcal{E}_{A_i}$ as elements of $\mathrm{Quot}(\mathcal{H}, \tau_E)$. In fact, any choice of L^2-orthonormal basis for $H^0(X, \mathcal{E}_{A_i}(m))$ determines a specific representation $q_i : \mathcal{H} \to \mathcal{E}_{A_i} \to 0$. Using the sup-norm bound on sections, one can extract convergent subsequences for the elements of these orthonormal bases to obtain a map $q_\infty : \mathcal{H} \to \mathcal{E}_{A_\infty}$. The limiting sections may no longer form a basis of $H^0(X, \mathcal{E}_\infty(m))$, nor necessarily do they generate the fiber of $\mathcal{E}_{A_\infty}$. Remarkably, though, it is still the case that the rank of the image sheaf $\widetilde{\mathcal{E}}_{A_\infty} \subset \mathcal{E}_{A_\infty}$ of q_∞ agrees with $\mathrm{rank}(E)$ and has Hilbert polynomial τ_E (for this one may have to twist with a further power of $\mathcal{L}$). In fact, the quotient sheaf $\mathcal{T}_\infty = \mathcal{E}_\infty / \widetilde{\mathcal{E}}_\infty$ turns out to be supported in complex codimension 2 (the first Chern class is preserved under Uhlenbeck limits). Hence, in particular, $(\widetilde{\mathcal{E}}_{A_\infty})^{**} \simeq \mathcal{E}_{A_\infty}$. See [16, proof of Lemma 4.3] for more details.

The second ingredient in the proof is the fact that $\mathrm{Quot}(\mathcal{H}, \tau_E)$ is compact in the analytic topology. Hence, after passing to a subsequence, we may assume the q_i converge. Convergence in $\mathrm{Quot}(\mathcal{H}, \tau_E)$ means the following: there is a convergent sequence of quotients $\mathcal{F}_i \to \mathcal{F}_\infty$ and isomorphisms f_i making the following diagram commute

$$\begin{array}{ccccc} \mathcal{H} & \longrightarrow & \mathcal{F}_i & \longrightarrow & 0 \\ \| & & \downarrow {\scriptstyle f_i} & & \\ \mathcal{H} & \xrightarrow{q_i} & \mathcal{E}_{A_i} & \longrightarrow & 0 \end{array}$$

The proof is completed, as in [16, Lemma 4.4], by showing that $\mathcal{F}_\infty \simeq \widetilde{\mathcal{E}}_{A_\infty}$. The crucial point that is used in showing this is that the two sheaves are quotients of $\mathcal{H}$ with the same Hilbert polynomial.

5. Analytic cycles and the blow-up set

In the case of the stronger notion of convergence of Uhlenbeck-Tian, we go one step further and identify the cycle associated to the sheaf $\mathcal{F}_\infty$ with the cycle $\mathcal{C}_\infty$ that arises from bubbling of the connections. The candidate is the following: for any torsion-free sheaf $\mathcal{F} \to X$, define a codimension 2 cycle $\mathcal{C}_\mathcal{F}$ from the top dimensional stratum of the support of $\mathcal{F}^{**}/\mathcal{F}$. See for example [16, Sec. 2.5.3].

Proposition 5.1. *Let A_i be a sequence of connections as in Proposition 4.2, and suppose furthermore that they converge to an ideal HYM connection $(A_\infty, \mathcal{C}_\infty)$ in the sense of Theorem 2.7. Let $\mathcal{H} \to \mathcal{F}_\infty$ be as in the statement of Proposition 4.2. Then $\mathcal{C}_\infty = \mathcal{C}_{\mathcal{F}_\infty}$.*

The proof of this result follows from the discussion in [16, Sec. 4.3] (see in particular Prop. 4.7). Although the result there is stated for sequence of HYM connections, this is required only to obtain the same sup-norm inequality on the global sections of $\mathcal{E}(m)$ that was used to obtain Proposition 4.2. Thus, the uniform bound on the Hermitian-Einstein tensor suffices, we have first of all that $\mathcal{E}_{A_\infty} \cong \mathcal{F}_\infty^{**}$.

Let us sketch the argument. The first key point is that $[\mathcal{C}_{\mathcal{F}_\infty}] = [\ \mathcal{C}_\infty]$ in rational cohomology. Indeed, the connection A_∞ is defined on the smooth locus of the sheaf $\mathcal{F}_\infty^{**}$ and is smooth there, and $\mathrm{ch}_2(A_\infty)$ defines a closed current (see Section 2). It then follows as in the proof of [34, Prop. 3.3] that $[\mathrm{ch}_2(A_\infty)] = \mathrm{ch}_2(\mathcal{F}_\infty^{**})$. The exact sequence

$$0 \longrightarrow \mathcal{F}_\infty \longrightarrow \mathcal{F}_\infty^{**} \longrightarrow \mathcal{T}_\infty \longrightarrow 0\ ,$$

implies that

$$[\mathrm{ch}_2(A_\infty)] = \mathrm{ch}_2(\mathcal{F}_\infty) + \mathrm{ch}_2(\mathcal{T}_\infty) = \mathrm{ch}_2(E) + [\mathcal{C}_{\mathcal{F}_\infty}]\ ,$$

where in the second inequality we have used the fact that the Chern classes of $\mathcal{F}_\infty$ are the same as those of E, and Proposition 3.1 of [34] (see also the latest arxiv version of this reference). By the convergence of the currents in Theorem 2.7 and Chern-Weil theory, we have

$$\mathrm{ch}_2(E) + [\mathcal{C}_\infty] = [\mathrm{ch}_2(A_i)] + [\mathcal{C}_\infty] = [\mathrm{ch}_2(A_\infty)].$$

Combining these two equalities gives the statement.

What remains to be shown is that given any irreducible component $Z \subset \mathrm{supp}(\mathcal{T}_\infty)$, for the associated multiplicity m_Z as defined in [16, Sec. 2.5.3], we have an equality

$$m_Z = \lim_{i\to\infty} \frac{1}{8\pi^2} \int_\Sigma \mathrm{tr}(F_{A_i} \wedge F_{A_i}) - \mathrm{tr}(F_{A_\infty} \wedge F_{A_\infty})\ ,$$

where Σ is a generic real 4-dimensional slice intersecting Z transversely in a single smooth point. For Hermitian-Yang-Mills connections this is [16, Prop 4.9]. Again note that the proof provided there only uses the Hermitian-Yang-Mills condition to obtain the quotient $\mathcal{F}_\infty$, and so by the preceding discussion also applies here. With this in hand, the point is that if Z is contained in the support $|\mathcal{C}_\infty|$ then it must be equal to one of the irreducible components. In this case, the number on the right hand side of the equality above is exactly the multiplicity of this component in the cycle $\mathcal{C}_\infty$, and otherwise this number is zero, (see [16, Lemma 3.13] and [34, Lemma 4.1] and again note that the proof is completely general). If the equality holds, this number cannot be zero, since m_Z is strictly positive by definition, and therefore Z must be a component of $\mathcal{C}_\infty$, and the multiplicities agree. Since $\mathcal{C}_\infty$ and $\mathcal{C}_{\mathcal{F}_\infty}$ are equal in cohomology, there can be no other irreducible components of $\mathcal{C}_\infty$, and so $\mathcal{C}_\infty = \mathcal{C}_{\mathcal{F}_\infty}$. For more details, see the proof of [16, Prop. 4.7].

Remark 5.2. It should be emphasized that Proposition 5.1 does not claim that the support of $\mathcal{F}_\infty^{**}/\mathcal{F}_\infty$ coincides with the full bubbling locus $|\mathcal{C}_\infty| \cup S(A_\infty)$; only the top dimensional strata are necessarily equal. This differs from what occurs, for example, along the Yang-Mills flow (see [34, Thm. 1.1]). It would be interesting to understand the behavior of the higher codimensional pieces from this perspective. There are recent examples due to Chen-Sun indicating that this should be subtle (see [5, 6]).

6. Relation with the topology of the Quot scheme

In this section we consider the relationship between the Quot scheme $\mathrm{Quot}(\mathcal{H}, \tau_E)$ discussed in Section 4, and the infinite dimensional space $\mathcal{A}^{1,1}(E,h)$ of integrable connections. Recall that $\mathrm{Quot}(\mathcal{H}, \tau_E)$ has a $\mathsf{PGL}(V)$ action obtained by change of basis in the vector space V. We are interested in the points in $\mathrm{Quot}(\mathcal{H}, \tau_E)$ where the quotient sheaf is locally free and has underlying C^∞ bundle isomorphic to E. A $\mathsf{PGL}(V)$ orbit of such a point corresponds to an isomorphism class of holomorphic structures on E, or equivalently, to a complex gauge orbit in $\mathcal{A}^{1,1}(E,h)$. Conversely, a connection $A \in \mathcal{A}^{1,1}(E,h)$ gives a holomorphic bundle which, provided m is sufficiently large, can be realized as a quotient. Complex gauge equivalent connections give rise to different quotients in the same $\mathsf{PGL}(V)$ orbit. We wish to show that this correspondence between complex gauge orbits in $\mathcal{A}^{1,1}(E,h)$ and $\mathsf{PGL}(V)$-orbits in $\mathrm{Quot}(\mathcal{H}, \tau_E)$ can be made continuous in the respective topologies. Since the complex gauge orbit space in $\mathcal{A}^{1,1}(E,h)$ is non-Hausdorff in general (and similarly for $\mathrm{Quot}(\mathcal{H}, \tau_E)$), we will lift to a map from open sets in $\mathcal{A}^{1,1}(E,h)$ itself.

This discussion gives rise to the following notion. Let $U \subset \mathcal{A}^{1,1}(E,h)$. We call $\sigma : U \to \mathrm{Quot}(\mathcal{H}, \tau_E)$ a *classifying map* if the quotient $\sigma(A)$ is a holomorphic bundle isomorphic to $(E, \bar\partial_A)$. Recall from Section 4 that the bundle $\mathcal{H}$ depends on a sufficiently large choice of m, which we omit from the notation. Then the result is the following.

Theorem 6.1. *Fix $A_0 \in \mathcal{A}^{1,1}(E,h)$. Then for m sufficiently large (depending on A_0), there is an open neighborhood $U \subset \mathcal{A}^{1,1}(E,h)$ of A_0 and a continuous classifying map $\sigma : U \to \mathrm{Quot}(\mathcal{H}, \tau_E)$. On $\mathcal{A}^{ss}(E,h)$, the twist m may be chosen uniformly.*

Throughout the proof, as in Section 4, we fix a hermitian structure h_L on L such that the curvature of the Chern connection of $(\mathcal{L}, h_L)$ defines a Kähler metric ω on X.

Proof. Let $d(m,n) = \tau_E(m) \cdot \dim H^0(X, \mathcal{L}^n)$. For $n \gg 1$, $\mathrm{Quot}(\mathcal{H}, \tau_E)$ is embedded in the Grassmannian $G(d(m,n), \tau_E(m+n))$ of $\tau_E(m+n)$-dimensional quotients of $\mathbb{C}^{d(m,n)}$. More precisely, suppose $q : \mathcal{H} \to \mathcal{E}$ is a point in $\mathrm{Quot}(\mathcal{H}, \tau_E)$, and let $\mathcal{K} = \ker q$. There is a sufficiently large n (uniform over the whole Quot scheme) such that

$$H^i(X, \mathcal{K}(m+n)) = H^i(X, \mathcal{E}(m+n)) = \{0\}\ ,\ i \geq 1 \tag{6.1}$$

(cf. [21, Lemmata 1.7.2 and 1.7.6]). We therefore have a short exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{K}(m+n)) \longrightarrow H^0(X, \mathcal{H}(m+n)) \longrightarrow H^0(\mathcal{E}(m+n)) \longrightarrow 0 \tag{6.2}$$

Since the middle term has dimension $d(m,n)$, and by (6.1) the last term has dimension $\tau_E(m+n)$, we obtain a point in $G(d(m,n), \tau_E(m+n))$. For n sufficiently large, this correspondence between quotients and points in the Grassmannian gives an embedding of the Quot scheme.

Given $A_0 \in \mathcal{A}^{1,1}(E,h)$, $\mathcal{E}_{A_0} = (E, \bar\partial_{A_0})$, choose m such that $\mathcal{E}_{A_0}(m)$ is globally generated and has no higher cohomology. Set $V_0 = H^0(X, \mathcal{E}_{A_0}(m))$. Then $\dim V_0 = \tau_E(m)$. The map

$$\mathrm{ev} : V_0 \otimes \mathcal{O}_X \longrightarrow \mathcal{E}_{A_0}(m) : s \otimes_{\mathbb{C}} f \mapsto fs$$

realizes $\mathcal{E}_{A_0}(m)$ as a quotient of $V_0 \otimes \mathcal{O}_X$. After twisting back by $\mathcal{L}^{-m}$, we have a quotient $\mathcal{H} \to \mathcal{E}_{A_0} \to 0$.

For $A \in U$ (the open set U remains to be specified) in order to realize $\mathcal{E}_A = (E, \bar\partial_A)$ as a quotient of $\mathcal{H}$, it suffices to give an isomorphism of V_0 with $V_A = \ker \bar\partial_A \subset \Gamma(E \otimes L^m)$, for then $\mathcal{E}_A$ is obtained through this isomorphism followed by evaluation ev as above. Note here that we assume already that U has been chosen sufficiently small so that $\mathcal{E}_A(m)$ is globally generated and has no higher cohomology. This is the first condition on U, and it can be arranged by semicontinuity of cohomology (see [25, Ch. 7]).

On $\Gamma(E \otimes L^m)$ we have an L^2-inner product. Since V_A and V_0 are subspaces of $\Gamma(E \otimes L^m)$, we can define a map by orthogonal projection $\pi_A : V_A \to V_0$. Let us write this explicitly. For $s \in V_A$, let $\pi_A(s) = s_0 = s + u_s$, where $u_s \in V_0^\perp$. We require $\bar\partial_{A_0} s_0 = 0$, or $\bar\partial_{A_0}(s + u_s) = 0$. If we write $\bar\partial_A = \bar\partial_{A_0} + a$, $a \in \Omega^{0,1}(X, \mathfrak{g}_E)$, then the above is $\bar\partial_{A_0} u_s = as$. Let G_0 be the Green's operator for the $\bar\partial_{A_0}$ laplacian acting on $\Omega^{0,1}(E \otimes L^m)$. In general, the Green's operator inverts the laplacian up to projection onto the orthogonal complement of the harmonic forms in $\Omega^{0,1}(X, E \otimes L^m)$. We have assumed vanishing of $H^1(X, \mathcal{E}_{A_0}(m))$, so in our case G_0 is a genuine inverse. Set $u_s = \bar\partial^*_{A_0} G_0(as)$. Then

$$\bar\partial_{A_0} u_s = \bar\partial_{A_0} \bar\partial^*_{A_0} G_0(as) = \Box_{A_0} G_0(as) + \bar\partial^*_{A_0} \bar\partial_{A_0} G_0(as) = as\ ,$$

as desired. Here, we have used the fact that $\bar\partial_{A_0} G_0 = G_0 \bar\partial_{A_0}$, and that, by the integrability of $\bar\partial_A$ and $s \in V_A$, $\bar\partial_{A_0}(as) = 0$. Notice that this definition of u_s guarantees that it is orthogonal to V_0. Now, by Hodge theory $\bar\partial^*_{A_0} G_0$ is a bounded operator on L^2. More precisely, the image of $\bar\partial^*_{A_0}G_0$ lies in $(\ker\bar\partial_{A_0})^\perp$, $\bar\partial_{A_0}(\bar\partial^*_{A_0}G_0)=I$, and $\bar\partial^*_{A_0}(\bar\partial^*_{A_0}G_0)=0$ so boundedness follows from the elliptic estimate for $\bar\partial_{A_0}$. Namely, for any $\phi\in\Omega^{0,1}(E\otimes L^m)$ we have an estimate

$$||\bar\partial^*_{A_0}G_0(\phi)||_{L^2}\leq||\bar\partial^*_{A_0}G_0(\phi)||_{L^2_1}\leq B||\bar\partial_{A_0}(\bar\partial^*_{A_0}G_0)(\phi)||_{L^2}=B||\phi||_{L^2}. \tag{6.3}$$

We therefore obtain (setting $\phi=as$) the estimate

$$\|u_s\|_{L^2}\leq B\|as\|_{L^2}\leq B(\sup|a|)\|s\|_{L^2}\ . \tag{6.4}$$

In particular, for $\sup|a|$ sufficiently small, which we guarantee by shrinking U, we have $\|\pi_A(s)\|_{L^2}\geq(1/2)\|s\|_{L^2}$. Hence, π_A is injective and therefore an isomorphism. The classifying map is then defined by setting $\sigma(A)$ to be the quotient:

$$\mathcal{H}\xrightarrow{\pi_A^{-1}\otimes id}V_A\otimes\mathcal{L}^{-m}\xrightarrow{\text{ev}}\mathcal{E}_A\longrightarrow 0\ .$$

It remains to show that σ is continuous. We begin with a few preliminaries. For $s\in\Gamma(E\otimes L^m)$, let

$$\widetilde\pi_A:\Gamma(E\otimes L^m)\longrightarrow\Gamma(E\otimes L^m):s\mapsto s+\bar\partial^*_{A_0}G_0(as)\ ,$$

so that $\widetilde\pi_A$ restricted to V_A is π_A. Again using that $\bar\partial^*_{A_0}G_0$ is a bounded operator, for $A_1,A_2\in U$, and $\bar\partial_{A_i}=\bar\partial_{A_0}+a_i$, $i=1,2$, we have

$$\|(\widetilde\pi_{A_1}-\widetilde\pi_{A_2})s\|_{L^2}=\|\bar\partial^*_{A_0}G_0((a_1-a_2)s)\|_{L^2}\leq B\|(a_1-a_2)s\|_{L^2}\leq B\sup|a_1-a_2|\|s\|_{L^2}\ .$$

It follows that $\widetilde\pi_A$ is continuous in A. By the argument following (6.4), it is also uniformly invertible for $A\in U$, with

$$\|\widetilde\pi_A^{-1}\|\leq 2\ . \tag{6.5}$$

Hence,

$$\begin{aligned}(\widetilde\pi_{A_1}^{-1}-\widetilde\pi_{A_2}^{-1})s&=\widetilde\pi_{A_1}^{-1}(\widetilde\pi_{A_2}-\widetilde\pi_{A_1})\widetilde\pi_{A_2}^{-1}s\\ \|(\widetilde\pi_{A_1}^{-1}-\widetilde\pi_{A_2}^{-1})s\|_{L^2}&\leq 4\|\widetilde\pi_{A_1}-\widetilde\pi_{A_2}\|\cdot\|s\|_{L^2}\leq 4B\sup|a_1-a_2|\|s\|_{L^2}\ .\end{aligned}$$

We conclude that the map $\pi_A^{-1}:V_0\to\Gamma(E\otimes L^m)$, whose image is V_A, is continuous for $A\in U$. In fact, it satisfies an estimate:

$$\|(\pi_{A_1}^{-1}-\pi_{A_2}^{-1})s_0\|_{L^2}\leq 4B\sup|a_1-a_2|\|s_0\|_{L^2}\ , \tag{6.6}$$

for all $s_0\in V_0$.

The second ingredient we shall need is the following. Depending upon the choice of the set U there is a uniform bound on the Hermitian-Einstein tensors ΛF_A for each $A\in U$. It follows as in Section 4 that we have an estimate: $\sup|s|\leq C\|s\|_{L^2}$, for all $s\in V_A$. Therefore, using (6.5),

$$\sup|\pi_A^{-1}(s_0)|\leq C\|\pi_A^{-1}(s_0)\|_{L^2}\leq 2C\|s_0\|_{L^2}\ , \tag{6.7}$$

for all $s_0\in V_0$.

Finally, let $s_0 \in V_0$. The standard elliptic estimate states that there is a uniform constant C such that

$$\|f\|_{L^2_1} \leq C(\|f\|_{L^2} + \|\bar{\partial}_{L^n} f\|_{L^2})$$

for smooth sections f of $\mathcal{L}^n$. In case f is holomorphic we may bootstrap this estimate and use Sobolev embedding $L^2_k \subset L^4$, $k \geq n/2$ to deduce an estimate of the form $\|f\|_{L^4} \leq C\|f\|_{L^2}$ (for a possibly different constant). Using (6.6) and (6.7), there is a constant $C_1 > 0$ such that:

$$\begin{aligned} \|f(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s_0\|^2_{L^2} &\leq \|f\|^2_{L^4}\|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s_0\|^2_{L^4} \leq C_1\|f\|^2_{L^2}\|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s_0\|_{L^2}\|s_0\|_{L^2} \\ &\leq C_1(\sup|a_1 - a_2|)\|s_0\|^2_{L^2}\|f\|^2_{L^2} \ . \end{aligned} \tag{6.8}$$

To prove continuity of σ, we show that the corresponding quotients (6.2) vary continuously in the Grassmannian for $A \in U$. First, from the definition (4.2),

$$H^0(X, \mathcal{H}(m+n)) = H^0(X, V_0 \otimes \mathcal{L}^n) \simeq V_0 \otimes H^0(X, \mathcal{L}^n) \ .$$

On $V_0 \otimes H^0(X, \mathcal{L}^n)$, we choose the tensor product metric of the L^2 metrics on V_0 and $H^0(X, \mathcal{L}^n)$. The map induced by σ is described as follows: for each $A \in U$ we have

$$T_A : V_0 \otimes H^0(X, \mathcal{L}^m) \longrightarrow \Gamma(E \otimes L^{m+n}) : s_0 \otimes_{\mathbb{C}} f \mapsto \pi_A^{-1}(s_0) \otimes_{\mathcal{O}_X} f$$

with image $H^0(X, \mathcal{E}_A(m+n))$. Moreover, from (6.8) it follows that

$$\|(T_{A_1} - T_{A_2})(s_0 \otimes f)\|^2_{L^2} \leq C_1(\sup|a_1 - a_2|)\|s_0 \otimes f\|^2 \ .$$

Hence, T_A is continuous in A for $A \in U$.

Recall that a smooth model for the Grassmannian $G(N,k)$ of k-dimensional quotients of $\mathbb{C}^N$ is given by

$$G(N,k) = \left\{ P \in \operatorname{End}\mathbb{C}^N \mid P^* = P \ , \ P^2 = P \ , \ \operatorname{tr} P = N - k \right\} \ .$$

Indeed, a smooth transitive action of $\mathsf{U}(N)$ on the right hand side above is defined by $(g, P) \mapsto gPg^*$, and the stabilizer of the projection associated to the standard coordinate splitting $\mathbb{C}^N = \mathbb{C}^k \times \mathbb{C}^{N-k}$ is $\mathsf{U}(k) \times \mathsf{U}(N-k)$. Thus, the right hand side is indeed identified with the homogeneous space $\mathsf{U}(N)/\mathsf{U}(k) \times \mathsf{U}(N-k)$, which is the usual description of the Grassmannian.

With this understood, in the setting above let $P_A \in \operatorname{End}(V_0 \otimes H^0(X, \mathcal{L}^m))$ denote the orthogonal projection to $\ker T_A$, viewed as a point in $G(d(m,n), \tau_E(m+n))$. It suffices to show P_A is continuous in $A \in U$. Because the dimensions of the kernels of T_A are constant on U, this reduces to showing that for any sequence $A_j \to A$ and $s_j \in \ker T_{A_j} \subset V_0 \otimes H^0(X, \mathcal{L}^n)$, $\|s_j\| = 1$, there is a subsequence such that $s_j \to s \in \ker T_A$. Indeed, if this is the case we may choose an orthonormal basis of such sections, $\{s_j^\alpha\}$, so that for any $s \in V_0 \otimes H^0(X, \mathcal{L}^n)$,

$$P_{A_j} s = \sum_\alpha \langle s, s_j^\alpha \rangle s_j^\alpha \ ,$$

and the right hand side converges to $P_A s$, and so $\|P_{A_j} - P_A\| \to 0$.

By finite dimensionaliy of $V_0\otimes H^0(X,\mathcal{L}^n)$, we may assume $s_j\to s$ for some $s\in V_0\otimes H^0(X,\mathcal{L}^n)$. Let $s_j=s_j^0+s_j^1$ be the orthogonal decomposition with respect to the splitting $\ker T_A\oplus(\ker T_A)^\perp$. In particular, there is a constant $c>0$ such that

$$\|T_A s_j^1\|\geq c\|s_j^1\| \qquad \forall j\ . \tag{6.9}$$

But then

$$0=T_{A_j}s_j=(T_{A_j}-T_A)s_j+T_As_j^1$$

and so

$$(T_A-T_{A_j})s_j=T_As_j^1 \implies \|T_As_j^1\|\to 0\ .$$

The estimate (6.9) implies $s_j^1\to 0$. Hence, $s\in\ker T_A$, and continuity of σ is proven. The uniformity of m in the second statement follows from Proposition 4.1. □

By a theorem of Maruyama, the semistable quotients in $\mathrm{Quot}(\mathcal{H},\tau_E)$ form an open set [29, Thm. 2.8]. Combining this with Theorem 6.1 we obtain part (1) of the Main Theorem.

Corollary 6.2. *The set $\mathcal{A}^{ss}(E,h)$ is open in $\mathcal{A}^{1,1}(E,h)$.*

This is item (1) of the Main Theorem.

Remark 6.3. The result in Corollary 6.2 is straightforward for Riemann surfaces (cf. [2, p. 576]) based on the Shatz stratification. More generally, an analytic proof can be given for projective manifolds using the Yang-Mills flow, answering a question in [24, Rem. 7.3.38]. A direct analytic proof for Kähler or Hermitian manifolds seems more difficult (for a partial result, [36, Thm. 3]).

7. Proof of the Main Theorem

In this section we prove items (2) and (3) of the Main Theorem (in reverse order). As seen in Section 4, a consequence of the assumption that X be projective is a representation of holomorphic bundles and Uhlenbeck limits as quotients. The existence of many holomorphic sections also passes to certain line bundles on moduli spaces. This fact implies strong separation properties and will be used in this section to deduce the Main Theorem.

7.1. **Item (3).** We wish to prove that $\mathscr{F}$ is continuous on the closure of the stable locus. For this we invoke the moduli space construction of Greb-Toma [17]. Let $R^{\mu ss}\subset\mathrm{Quot}(\mathcal{H},\tau_E)$ denote the open subset consisting of quotients that are slope semistable torsion-free sheaves. Then there exists a (seminormal) projective variety $M^{\mu ss}$ and a morphism (in particular, continuous map) $R^{\mu ss}\to M^{\mu ss}:\mathcal{F}\mapsto[\mathcal{F}]$ with the following properties:

(1) If $\mathcal{F}_1\simeq\mathcal{F}_2$, then $[\mathcal{F}_1]=[\mathcal{F}_2]$ in $M^{\mu ss}$ (cf. the discussion preceding [16, Def. 2.20]);
(2) If $[\mathcal{F}_1]=[\mathcal{F}_2]$ in $M^{\mu ss}$, then $\mathcal{F}_1^{**}\simeq\mathcal{F}_2^{**}$ and $\mathcal{C}_{\mathcal{F}_1}=\mathcal{C}_{\mathcal{F}_2}$ [17, Thm. 5.5].

The association $[\mathcal{F}] \to (\mathcal{F}^{**}, \mathcal{C}_{\mathcal{F}})$ gives a well-defined map $\overline{\Phi} : \overline{M}^{\mu}(E,h) \to \overline{M}_{\mathrm{HYM}}(E,h)$, where $\overline{M}^{\mu}(E,h)$ is the closure of $M^{*}_{\mathrm{HYM}}(E,h)$ in $M^{\mu ss}$ (see Section 4.1 of [16]). The map $\overline{\mathcal{A}^{s}}(E,h) \xrightarrow{Q} \overline{M}^{\mu}(E,h)$ in the diagram (1.2) is defined by realizing a semistable bundle as a quotient in $R^{\mu ss}$ (see the discussion following Proposition 4.1), and sending this quotient to its equivalence class in $M^{\mu ss}$.

By construction, Q may be locally exhibited as the composition of the map $R^{\mu ss} \to M^{\mu ss}$ with a classifying map σ as discussed in the previous section. The former map is a morphism of complex spaces and is therefore continuous. By Theorem 6.1, σ is continuous as well. Since continuity is a local property, we deduce the continuity of Q. Now one of the main results of [16] is Theorem 4.11, which states that $\overline{\Phi}$ is also continuous. By Theorem 3.1, the diagram (1.2) commutes, and we therefore conclude that $\mathscr{F}$ is continuous on $\overline{\mathcal{A}^{s}}(E,h)$.

7.2. **Item (2).** To address the general situation, we first reduce the problem as in [10, Sec. 4]. Let $A_i \to A_0$ be a sequence in $\mathcal{A}^{ss}(E,h)$ converging in the C^∞ topology, and let $[A_\infty, \mathcal{C}^A_\infty, S(A_\infty)] = \mathscr{F}([A_0])$. By the compactness theorem [16, Thm. 3.23], we may assume that, after passing to a subsequence, there is an ideal connection $(B_\infty, \mathcal{C}^B_\infty)$ such that $\mathscr{F}([A_i]) \to [(B_\infty, \mathcal{C}^B_\infty)]$. We must show that the two limits agree.

Let $A_{i,t}$ denote the Yang-Mills flow at time t of A_i. Smooth dependence on initial conditions implies that for each fixed $T > 0$, $A_{i,t} \to A_t$ as $i \to +\infty$, uniformly for $t \in [0,T)$.

Lemma 7.1. *There is a subsequence (whose index set will be also denoted by $\{i\}$) and $t_i \to +\infty$, such that $[A_{i,t_i}] \to [(A_\infty, \mathcal{C}^A_\infty)]$ in the sense of Theorem 2.7.*

Proof. The proof relies on several properties. First, since $A_{i,t} \to A_t$ for every $t \geq 0$, by a diagonalization argument we may choose a sequence A_{i,t_i} so that (up to gauge), $A_{i,t_i} \to A_\infty$ weakly in $L^p_{1,loc}$ away from $|\mathcal{C}| \cup S(A_\infty)$. Next, by the result in [20, Thm. C], any sequence A_{i,t_i}, $t_i \to +\infty$, has a subsequence that converges to an ideal connection. This is shown in [20] for a sequence of times along a single flow, but the argument extends more generally. The key points are Theorem 8 and Proposition 9 of [20], and these hold uniformly for a smoothly convergent sequence of initial conditions. Note that there is a uniform bound on the Hermitian-Einstein tensor. Given this fact, we are exactly in the set-up of the proof of [16, Proposition 3.20] (the "boundedness" assumption of that result is guaranteed, since X is projective algebraic; see [16, Lemma 3.16]). The conclusion of that result is that we may choose the t_i such that the limiting ideal HYM connection of $\{A_{i,t_i}\}$ coincides with $[A_\infty, \mathcal{C}^A_\infty)]$. □

The Yang-Mills flow lies in a single complex gauge orbit: $A_{i,t}$ is complex gauge equivalent to A_i for all $t \in [0, +\infty)$. By Remark 3.3 it follows that and $\mathscr{F}([A_{i,t}]) = \mathscr{F}([A_i])$. Therefore, in the same way as above we may choose times s_i such that for $B_i = A_{i,s_i}$, $[B_i]$ is sufficiently close to $\mathscr{F}([A_i])$ so as to converge to $[(B_\infty, \mathcal{C}^B_\infty)]$. We state this as the following.

Lemma 7.2. *There are complex gauge transformations g_i such that if $B_i = g_i(A_{i,t_i})$, then after passing to a subsequence, $[B_i] \to [(B_\infty, \mathcal{C}^B_\infty)]$ in the sense of Theorem 2.7.*

As in Section 4, the holomorphic bundles $\mathcal{E}_{A_{i,t_i}}$ can be realized as a sequence of quotients $q_i^A : \mathcal{H} \to \mathcal{E}_{A_{i,t_i}}$, and the composition q_i^B with the g_i from the lemma above realizes $\mathcal{E}_{B_i}$ as quotients q_i^B. Note that by definition of the equivalence in the Quot scheme, since q_i^A and q_i^B have the same kernels in $\mathcal{H}$ they represent the same points in $\mathrm{Quot}(\mathcal{H}, \tau_E)$. We now apply Propositions 4.2 and 5.1 to both sequences A_{i,t_i} and B_i. One obtains quotients $q_i^A \to q_\infty^A : \mathcal{H} \to \mathcal{F}_\infty^A$ and $q_i^B \to q_\infty^B : \mathcal{H} \to \mathcal{F}_\infty^B$ in $\mathrm{Quot}(\mathcal{H}, \tau_E)$. Moreover, $(\mathcal{F}_\infty^A)^{**} \simeq \mathcal{E}_{A_\infty}$ and $(\mathcal{F}_\infty^B)^{**} \simeq \mathcal{E}_{B_\infty}$. In particular, since $\mathcal{E}_{A_\infty}$ and $\mathcal{E}_{B_\infty}$ have admissible Hermitian-Einstein metrics, the sheaves $\mathcal{F}_\infty^A$ and $\mathcal{F}_\infty^B$ are slope semistable, and so they lie in $R^{\mu ss}$. Also, $\mathcal{C}_\infty^A = \mathcal{C}_{\mathcal{F}_\infty^A}$ and $\mathcal{C}_\infty^B = \mathcal{C}_{\mathcal{F}_\infty^B}$ (see [16, Lemma 2.10]).

Now $[\mathcal{E}_{A_{i,t_i}}] = [\mathcal{E}_{B_i}]$ in $M^{\mu ss}$ for every i, since B_i is gauge equivalent to A_{i,t_i}. Because their limits are also semistable (in fact polystable), we conclude again from item (1) above and the continuity of the projection to $M^{\mu ss}$ that $[\mathcal{F}_\infty^A] = [\mathcal{F}_\infty^B]$. It then follows from item (2) that $\mathcal{E}_{A_\infty} \simeq \mathcal{E}_{B_\infty}$, and $\mathcal{C}_\infty^A = \mathcal{C}_{\mathcal{F}_\infty^A} = \mathcal{C}_{\mathcal{F}_\infty^B} = \mathcal{C}_\infty^B$. From the discussion following Definition 2.5, the isomorphism $\mathcal{E}_{A_\infty} \simeq \mathcal{E}_{B_\infty}$ implies that A_∞ and B_∞ are gauge equivalent on their common smooth locus. Moreover, since the singular set of the HYM connection agrees with the singular set of the Bando-Siu extension, we have

$$S(A_\infty) = \mathrm{sing}(\mathcal{E}_{A_\infty}) = \mathrm{sing}(\mathcal{E}_{B_\infty}) = S(B_\infty) \ .$$

Finally, since the codimension 2-cycles also agree, the ideal HYM connections are gauge equivalent. This completes the proof of the Main Theorem.

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