GEOMETRIZATION OF THE TUY/WZW/KZ CONNECTION

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Abstract. Given a simple, simply connected, complex algebraic group G, a flat projective connection on the bundle of nonabelian theta functions on the moduli space of semistable parabolic G-bundles over any family of smooth projective curves with marked points was constructed by the authors in an earlier paper. Here, it is shown that the identification between the bundle of nonabelian theta functions and the bundle of WZW conformal blocks is flat with respect to this connection and the one constructed by Tsuchiya-Ueno-Yamada. As an application, we give a geometric construction of the Knizhnik-Zamolodchikov connection on the trivial bundle over the configuration space of points in the projective line whose typical fiber is the space of invariants of tensor product of representations.

Contents

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1. Introduction

The Wess-Zumino-Witten (WZW) model [48, 69] is a cornerstone of two dimensional rational conformal field theories [10, 44]. The WZW conformal blocks were constructed mathematically by Tsuchiya-Ueno-Yamada [66]. Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra and $(C, \boldsymbol{p})$ a smooth curve C with n-distinct marked points $\boldsymbol{p} = (p_1, \cdots, p_n)$. Choose formal coordinates $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ around $\boldsymbol{p}$, and using these coordinates assign a copy of $\widehat{\mathfrak{g}}$ to each point p_i. Fix a positive integer ℓ. Then for any choice of n-tuple of integrable highest weights $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ of level ℓ, the construction in [66] associates a finite dimensional vector space $\mathcal{V}^{\dagger}_{\vec{\lambda}}(C, \boldsymbol{p}, \boldsymbol{\xi}, \mathfrak{g}, \ell)$ to the data $(C, \boldsymbol{p}, \boldsymbol{\xi})$. For a family of smooth curves $\pi : \mathcal{C} \to S$ with n-distinct sections $\boldsymbol{p}$, these vector spaces patch together to produce a coherent sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell) \to S$. The Sugawara construction [66] endows this sheaf with the structure of a twisted $\mathcal{D}$-module, and hence $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ is actually a holomorphic vector bundle. The authors of [66] show that this vector bundle extends to the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}'_{g,n}$ of the moduli spaces of n-pointed curves $\mathcal{M}'_{g,n}$ with chosen formal coordinates. Moreover, the flat projective connection on the interior $\mathcal{M}'_{g,n}$ extends to a flat projective connection with logarithmic singularities over $\overline{\mathcal{M}}'_{g,n}$. The bundle $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell) \to S$ of conformal blocks is sometimes called the *Friedan-Shenker bundle*. We refer to the above mentioned flat projective connection on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell) \to S$ as the *WZW/TUY connection*.

Later, Tsuchimoto [64] gave a coordinate free construction of the bundle of conformal blocks and showed that it descends to a vector bundle $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ on the Deligne-Mumford-Knudsen moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable nodal curves (cf. Fakhruddin [25]). The flat projective connection also descends to a projective connection with logarithmic singularities. In other words, there is a projectively flat isomorphism between the conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ and the pullback $F^{*}\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ under the natural forgetful map $F : \overline{\mathcal{M}}'_{g,n} \to \overline{\mathcal{M}}_{g,n}$. We refer the reader to Section 2 for a construction of conformal blocks and to Section 3 for the construction of the WZW/TUY connection.

We now discuss how conformal blocks are related to moduli spaces of bundles on curves. The moduli space $M_G(C)$ of principal bundles, with a reductive structure group G, on a smooth projective curve C, provides a natural nonabelian generalization of the Jacobian variety $J(C)$, which parametrizes line bundles of degree zero on C. The moduli space of (semistable) principal

G-bundles on a smooth projective algebraic curve is itself a projective variety. It was originally constructed through Geometric Invariant Theory. Its smooth locus parametrizes isomorphism classes of stable bundles with minimal automorphism groups (see [18]), also known as the regularly stable loci. There are various important variations on this construction. One can choose marked points $\boldsymbol{p} = (p_1, \cdots, p_n)$ on the algebraic curve C and decorate a principal G-bundle $\mathcal{P}$ with a generalized flag structure over $\boldsymbol{p}$, leading to the notion of quasi-parabolic bundles. Additionally, one can choose weight data $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ in the Weyl alcoves, or equivalently weights $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$, and use them to define a suitable notion of stability and semistablity. The corresponding moduli spaces $M^{par,ss}_{G,\boldsymbol{\tau}}(C, \boldsymbol{p})$ can, in turn, be understood as the space of representations of the fundamental group of the corresponding punctured surface $C \setminus \{\boldsymbol{p}\}$, where the loops around the marked points go to fixed conjugacy classes determined by $\boldsymbol{\tau}$ [43, 57]. This generalizes the classical results of Narasimhan-Seshadri [47] and Ramanathan [51], proved in the non-parabolic case.

The moduli space $M^{par,ss}_{G,\boldsymbol{\tau}}(C, \boldsymbol{p})$ is equipped with a natural ample *determinant of cohomology* line bundle $\mathrm{Det}_{par,\phi}(\boldsymbol{\tau})$ associated to a choice of faithful linear representation ϕ of G. This generalizes the theta line bundle on the Jacobian variety $J(C)$. Therefore, the global sections of this line bundle on $M^{par,ss}_{G,\boldsymbol{\tau}}(C, \boldsymbol{p})$ can thus be thought of as a nonabelian generalization of the classical theta functions. We refer the reader to Section A.1 for more details on the constructions of the moduli space and the parabolic determinant line bundle on it.

Via the uniformization theorems of Harder and Drinfeld-Simpson [24, 31], moduli spaces of parabolic bundles also have an adèlic description that directly connects to the representation theory of affine Lie algebras via the work of several authors (see [8, 27, 37, 40, 49, 59]). Using this, the corresponding moduli stack of principal G-bundles and its parabolic analog $\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$ can be expressed as a double quotient

$$\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau}) \;=\; G(\Gamma(C, \mathcal{O}_C(*\boldsymbol{p}))\backslash \prod_{i=1}^{n} G(\mathbb{C}((\xi_i)))/\mathcal{P}_i \;,$$

where the $\mathcal{P}_i$ are parahoric subgroups of $G(\mathbb{C}[[\xi_i]])$ determined by the weights τ_i. The weights also determine a homogeneous $G(\Gamma(C, \mathcal{O}_C(*\boldsymbol{p})))$-equivariant line bundle $\mathscr{L}_{\vec{\lambda}}$ on $\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$. The line bundle $\mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau})$ coincides with $\mathscr{L}_{\vec{\lambda}}$, where a is a rational number determined by the Dynkin index of the representation ϕ. Generalizations (see Kumar [36], Mathieu [42]) of the Borel-Weil theorems (see (B.3)) for affine flag varieties $G(\mathbb{C}((\xi)))/\mathcal{P}_i$, coupled with the adèlic description, give a canonical isomorphism (see (B.2)) with conformal blocks

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(C, \boldsymbol{p}, \boldsymbol{\xi}, \mathfrak{g}, \ell) \;\cong\; H^0(M^{par,ss}_{G,\boldsymbol{\tau}}(C, \boldsymbol{p}),\, \mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau})).$$

This isomorphism can be reinterpreted as the Chern-Simon/WZW correspondence. More details are given in Section B.1.

Using differential geometric methods, Hitchin, [32], generalizes a construction of Mumford-Welters [68] to obtain a flat projective connection on the Friedan-Shenker bundle with fibers $H^0(M_G(C), \mathrm{Det}^{\otimes \ell})$, from the viewpoint of *geometric quantization* in the sense of Kostant-Souriau. This connection also appears in Witten's [70] interpretation of Jones polynomial link invariants as 3-manifold invariants. Hitchin's construction was reinterpreted by van Geemen-de Jong [67] sheaf theoretically in terms of the existence of a "heat operator", which in the relative setting is a differential operator that is a combination of a first order operator with one that is second order on the fibers (see Section 4). We recall the details of the general methods of Hitchin-van Geemen-de Jong [67] in Section 4. We also refer to the several complementary approaches of Andersen [1], Axelrod-Witten-della Pietra [3], Baier-Bolognesi-Martens-Pauly [4], Faltings [26], Ginzburg [30], Ran [52], Ramadas [50], Sun-Tsai [61] and for generalizations to reductive groups, Belkale [11].

In [39], Laszlo showed that the connection constructed by Hitchin and the one in [66] coincide under the natural identification of $H^0(M_G(C), \mathrm{Det}^{\otimes \ell})$ with $\mathcal{V}_0^\dagger(C, \mathfrak{g}, \ell)$. A similar result for twisted Spin groups was also proved by Mukhopadhyay-Wentworth [45]. The following questions are natural in the context of parabolic moduli spaces:

(1) Is there a projective heat operator (see Section 4 and Definition 4.1) on the line bundle $\mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau})$) that induces a flat projective connection on the vector bundle over $\mathcal{M}_{g,n}$ with fibers $H^0(M^{par,ss}_{G,\boldsymbol{\tau}}(C,\boldsymbol{p}), \mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau}))$?
(2) If such a connection exists, is the identification of conformal blocks with nonabelian parabolic theta functions flat with respect to this connection and the WZW/TUY connection?

For $g(C) \geq 2$, Scheinost-Schottenloher, [55], constructed a parabolic Hitchin connection for $G = \mathsf{SL}_r$ under the assumption that the canonical bundle of $M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\tau}}(C,\boldsymbol{p})$ admits a square-root. Bjerre [21] removed the "restriction" in [55] for $G = \mathsf{SL}_r$ by working on a different parabolic moduli space with full flags. In both [21, 55], the authors construct a connection on the push-forward "metaplectically corrected" line bundles of the form $\mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau}) \otimes K^{1/2}_{M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\tau}}(C,\boldsymbol{p})}$. We also refer the reader to Remark 6.3. In [19], we constructed a projective heat operator on $\mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau})$ in general. This was produced from a candidate parabolic Hitchin symbol (see (5.6)) satisfying a Hitchin-van Geemen-de Jong type equation (see (5.2))[1], [2].

The following result answers the above question (2) and thus generalizes the result of Laszlo, proved in the non-parabolic case.

Theorem 1.1 (Main Theorem)**.** *Let S parametrize a smooth family of n-pointed curves. Let $\pi_e : M^{par,rs}_{G,\boldsymbol{\tau}} \to S$ be the relative moduli space parametrizing regularly stable parabolic G bundles, i.e., stable parabolic bundles with minimal automorphisms. The natural isomorphism*

$$\mathbb{P}\mathbb{V}^\dagger_{\vec{\lambda}}(\mathfrak{g}, \ell) \xrightarrow{\sim} \mathbb{P}\pi_{e*}\,\mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau}) ,$$

constructed via the uniformization theorem, between the projectivizations of the bundles of conformal blocks and nonabelian parabolic theta functions, is flat for the WZW/TUY connection on $\mathbb{P}\mathbb{V}^\dagger_{\vec{\lambda}}(\mathfrak{g}, \ell)$ and the parabolic Hitchin connection on $\mathbb{P}\pi_{e*}\,\mathrm{Det}^{\otimes a}_{par,\phi}(\boldsymbol{\tau})$.

We are guided by a fundamental observation that if an algebraic group G acts on a smooth variety X and $\mathscr{L}$ is a G-equivariant line bundle on X, then the map induced by the Beilinson-Bernstein localization functor $\mathrm{Loc} : \mathcal{U}\mathfrak{g} \to \Gamma(X, \mathcal{D}(\mathscr{L}))$ is a quantum analog of the moment map for the G action on X, and the corresponding graded map $\mathrm{Sym}\,\mathfrak{g} \to \Gamma(X, gr(\mathcal{D}(\mathscr{L})))$ is dual to the moment map. Hence, it is "independent" of the line bundle $\mathscr{L}$. Thus, an essential point in the proof of this theorem is the fact that the symbols of the Sugawara operators coming from affine Lie algebras do not depend on the highest weights. This is checked via a direct calculation generalizing the non-parabolic counterparts in the works on Laszlo [39] and Tsuchiya-Ueno-Yamada [66]. The counterpart of this statement on the moduli of parabolic bundles side for the parabolic Hitchin symbol is therefore the crux of the argument. This is carried out in Proposition

[1]Subsequent to the submission of our paper [19], in May 2023 a draft of the thesis of Zakaria Ouaras appeared in which the author proves the existence of a unique flat projective connection in the case of moduli spaces of parabolic vector bundles with arbitrary fixed determinant and genus $g \geq 2$

[2]We have been informed [2] that in the case of genus zero, SL_2, and equal weights λ sufficiently small so that conformal blocks are invariants, the Hitchin connection constructed in [1] agrees with the KZ equation.

5.6 using Corollary 5.5. These are the key new features/differences in the proof of Theorem 1.1 to that in non-parabolic case considered by Laszlo [39].

Application. Now we discuss an application of the parabolic generalization of Theorem 1.1 by giving a geometric reconstruction of the Knizhnik-Zamolodchikov (KZ) equation. Let us now focus on the genus zero case. Since $\mathbb{P}^1$ has a global coordinate and a global meromorphic two form on $\mathbb{P}^1 \times \mathbb{P}^1$ with second order poles along the diagonal, the WZW/TUY connection gives a flat (honest) connection on the bundle of conformal blocks. The equations for the flat sections are known as *Knizhnik-Zamolodchikov equations* [34]. Thus, the KZ equations constitute a system of first order differential equations, arising from the conformal Ward identities, that determines n-point correlation functions in the Wess-Zumino-Witten-Novikov model of two dimensional conformal field theory. The KZ equations have remarkable realizations in many other areas. For example higher dimensional generalizations of hypergeometric functions are known to be solutions of these equations [54]. The KZ equations can also be regarded as quantizations of the isomonodromy problem for differential equations of Fuchsian type [53]. The Kohno-Drinfeld [23, 35] theorem relates the monodromy representation of the braid group induced by the KZ connection with solutions of the Yang-Baxter equation.

Here, we consider the KZ equations as equations for flat sections of the trivial vector bundle $\mathbb{A}_{\vec{\lambda}}$ over the configuration space X_n of n-points in $\mathbb{C}$ with fibers

$$A_{\vec{\lambda}} := \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, \mathbb{C}) .$$

We restrict the projective heat operator constructed in [19] to the open substack $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{p}, \vec{\lambda})$ of quasiparabolic bundles in case of genus zero, where the underlying principal G-bundle is trivial. This turns out to be the quotient stack

$$\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{p}, \vec{\lambda}) = [(G/P_{\lambda_1} \times \cdots \times G/P_{\lambda_n})/G],$$

where P_{λ_i} are parabolic subgroup determined by λ_i and the global sections of the homogeneous line bundle $\mathscr{L}_{\vec{\lambda}}$ are just the invariants $A_{\vec{\lambda}}$. Thus, we obtain a flat connection on the vector bundle $\mathbb{A}_{\vec{\lambda}}$ over X_n. Finally using Theorem 1.1, we identify this connection with the KZ connection. This gives an alternative geometric construction of the KZ equations. We refer the reader to Section 7 and Corollary 7.1 for more details and precise statements.

The outline of the paper is as follows. In Sections 2 and 3 we review in some detail the construction of the WZW/TUY connection in the parabolic setting. In Section 4, we review the construction from [19] of the Hitchin connection in the parabolic setting. This involves the metaplectic correction in a central way. An important step in this section is the re-expression of one of the "controlling equations" of van Geemen-de Jong for the existence of a projective heat operator on elements of the rational Picard group (see Section 4.2). Finally, in Section 5.2 we state the fundamental result, Theorem 5.3, which provides a simplification of the expression for parabolic Hitchin symbol. This is the geometric reflection of the aforementioned fact that the Sugawara operators do not depend on the highest weights.

Finally, in Section 6, using this result we directly relate the symbols of the Sugawara tensor and the parabolic Hitchin connection, thus proving Theorem 1.1 (see Section 6.4). In the last Section 7 we elaborate on the special case of genus zero curves. In Appendix A, we define the line bundles over moduli spaces of parabolic G-bundles whose sections give rise to the Friedan-Shenker bundles. We also relate these line bundles to the determinant of cohomology in the relative setting.

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2. Conformal Blocks

In this section we recall the basic notions of conformal blocks, following Tsuchiya-Ueno-Yamada [66]. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let Δ be a system of roots and $\kappa_{\mathfrak{g}}$ the Cartan-Killing form, normalized so that $\kappa_{\mathfrak{g}}(\theta_{\mathfrak{g}}, \theta_{\mathfrak{g}}) = 2$ for the longest root $\theta_{\mathfrak{g}}$. We let $\nu_{\mathfrak{g}} : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ denote the isomorphism induced by $\kappa_{\mathfrak{g}}$.

2.1. Affine Lie algebras and integrable modules.

Let ξ be a formal parameter. The affine Lie algebra
$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c$$
is a central extension of the loop algebra $\mathbb{C}((\xi))$ by c. The Lie bracket operation on $\widehat{\mathfrak{g}}$ is given by the formula
$$[X \otimes f, Y \otimes g] \quad := [X, Y] \otimes fg + \kappa_{\mathfrak{g}}(X, Y) \operatorname{Res}_{t=0}(gdf) \cdot c, \tag{2.1}$$
where $X,\ Y \in \mathfrak{g}$ and $f,\ g$ are elements of $\mathbb{C}((\xi))$.

We now briefly recall the basic objects in the representation theory of $\widehat{\mathfrak{g}}$. It is well-known that the finite dimensional $\mathfrak{g}$ modules are parametrized by the subset $P_+(\mathfrak{g}) \subset \mathfrak{h}^*$ consisting of dominant integral weights. The representation corresponding to a weight λ will be denoted by V_λ. Let $\ell > 0$ be a positive integer, and consider the set $P_\ell(\mathfrak{g}) := \{\lambda \in P_+(\mathfrak{g}) \mid \kappa_{\mathfrak{g}}(\lambda, \theta_{\mathfrak{g}}) \leq \ell\}$. The highest weight irreducible integrable representations of $\widehat{\mathfrak{g}}$ at level ℓ are classified by the set $P_\ell(\mathfrak{g})$ defined above. The $\widehat{\mathfrak{g}}$-module corresponding to λ will be denoted by $\mathcal{H}_\lambda$.

2.2. Sheaf of Conformal Blocks.

Integrable representations of affine Lie algebras were used by Tsuchiya-Ueno-Yamada [66] and Tsuchiya-Kanie [65] to define conformal blocks. In this paper, we will restrict ourselves to conformal blocks associated to smooth curves. Let $\pi : \mathcal{C} \to S$ be a family of smooth projective curves, and let $\boldsymbol{p} = (p_1, \cdots, p_n)$ be n non-intersecting sections of the map π such that the complement $\mathcal{C} \backslash \cup_{i=1}^n p_i(S)$ is an affine scheme.

Consider formal coordinates $\xi_1, \cdots, \xi_n$ around the sections $\boldsymbol{p} = (p_1, \cdots, p_n)$ giving isomorphisms $\lim_m \mathcal{O}_{\mathcal{C}}/\mathcal{I}_{p_i}^m \cong \mathcal{O}_S[[\xi_i]]$, where $\mathcal{I}_{p_i}$ is the ideal sheaf of the section p_i. Let $\mathfrak{g}$ be a simple Lie algebra. As before, we get a sheaf of $\mathcal{O}_S$ Lie algebras defined by
$$\widehat{\mathfrak{g}}_n(S) := \mathfrak{g} \otimes \left(\bigoplus_{i=1}^n \mathcal{O}_S((\xi_i)) \right) \oplus \mathcal{O}_S \cdot c \tag{2.2}$$
The Lie algebra $\widehat{\mathfrak{g}}_n(S)$ in (2.2) contains a natural subsheaf of Lie algebras $\mathfrak{g} \otimes_{\mathbb{C}} \pi_* \mathcal{O}_{\mathcal{C}}(*D)$, where $D = \sum_{i=1}^n p_i(S)$. That it satisfies the Lie algebra condition is actually guaranteed by the residue theorem.

For any choice of n-tuple of weights $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ in $P_\ell(\mathfrak{g})$, consider the $\widehat{\mathfrak{g}}_n(S)$-module (hence it is also a $\mathfrak{g} \otimes_{\mathbb{C}} \pi_* \mathcal{O}_{\mathcal{C}}(*D)$-module)
$$\mathcal{H}_{\vec{\lambda}}(S) := \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes_{\mathbb{C}} \mathcal{O}_S.$$

Definition 2.1. The sheaf of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \mathcal{C}/S, \boldsymbol{p}, \ell)$ at level ℓ is defined to be the largest quotient of $\mathcal{H}_{\vec{\lambda}}(S)$ on which $\mathfrak{g} \otimes_{\mathbb{C}} \pi_* \mathcal{O}_{\mathcal{C}}(*D)$ acts trivially. The sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \mathcal{C}/S, \boldsymbol{p}, \ell)$ is defined to be the $\mathcal{O}_S$ dual of the sheaf of covacua.

Since the above definition uses only the fact that the formal coordinates identify the completed local ring with the Laurent series ring, the definition of sheaf of covacua and the sheaf of conformal blocks can be extended in a straightforward manner to families of nodal curves with Deligne-Mumford stability property.

2.3. **Coordinate free construction.** The sheaf $\widehat{\mathfrak{g}}_n$ and its integrable modules can be defined without the choice of formal coordinates $\boldsymbol{\xi}$; we recall this from [64]. Let $\pi : \mathcal{C} \to S$ be as above. Consider the sheaf of formal meromorphic functions on $\mathcal{C}$ with poles along the marked sections

$$\mathscr{K}_{\mathcal{C}/p_i} := \lim_{a\to+\infty} \lim_{m\to+\infty} \mathcal{O}(ap_i(S))/\mathcal{I}^m_{p_i}.$$

Using this, the *coordinate free affine Lie algebra* is defined as follows:

$$\widehat{\mathfrak{g}}(\mathcal{C}/p_i) := \mathfrak{g}\otimes \mathscr{K}_{\mathcal{C}/p_i} \oplus \mathcal{O}_S\cdot c. \tag{2.3}$$

Its n-pointed analog is defined to be

$$\widehat{\mathfrak{g}}_n(\mathcal{C}/S) := \left(\mathfrak{g}\otimes \bigoplus_{i=1}^n \mathscr{K}_{\mathcal{C}/p_i}\right) \oplus \mathcal{O}_S\cdot c.$$

The Lie bracket operation is defined as in the previous section.

The Verma module $\mathcal{M}_\lambda(\mathcal{C}/S)$ (resp.the coordinate free highest weight integrable module $\mathbb{H}_\lambda(\mathcal{C}/S)$) can be defined similarly by inducing representations (and taking quotients) using a parabolic subalgebra

$$\widehat{\mathfrak{p}}_{p_i} := \mathfrak{g}\otimes \widehat{\mathcal{O}}_{\mathcal{C}/p_i}(S) \oplus \mathcal{O}_S\cdot c$$

on the finite dimensional module V_λ via evaluation. More precisely $\mathcal{M}_{\lambda_i}(\mathcal{C}/S) = \mathrm{Ind}^{\widehat{\mathfrak{g}}(\mathcal{C}/p_i)}_{\widehat{\mathfrak{p}}_{p_i}} V_{\lambda_i}$ refer the reader to [64] for more details.

Definition 2.2. The coordinate free sheaf of covacua $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g},\, \mathcal{C}/S,\, \ell)$ is defined to be the sheaf of coinvariants $\mathbb{H}_{\vec{\lambda}}(\mathcal{C}/S)/\left(\mathfrak{g}\otimes_{\mathbb{C}} \pi_*\mathcal{O}_{\mathcal{C}}(*D)\mathbb{H}_{\vec{\lambda}}(\mathcal{C}/S)\right)$. As before, the sheaf of coordinate free conformal blocks $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g},\, \mathcal{C}/S,\, \ell)$ is defined to be the $\mathcal{O}_S$-module dual of $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g},\, \mathcal{C}/S,\, \ell)$.

Observe that a choice of formal coordinates around the marked points actually induces isomorphisms between $\widehat{\mathfrak{g}}(\mathcal{C}/p_i)$ and the sheaf $\widehat{\mathfrak{g}}_{\xi_i} := \mathfrak{g}\otimes \mathcal{O}_S((\xi_i))\oplus \mathcal{O}_S\cdot c$. This identifies the coordinate free conformal blocks and the sheaf of covacua with those obtained by a choice of coordinates.

We now recall some important properties of the sheaf of conformal blocks. The reader is referred to [66], [64], [25], and [58] for proofs and further exposition.

Theorem 2.3. *Let $\mathfrak{g}$ be a simple Lie algebra and $\ell > 0$ a positive integer. Then the following statements hold:*

1. *The sheaf of conformal blocks $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \mathcal{C}/S, \ell)$ carries a flat projective connection and hence it is locally free* [64]*.The Verlinde formula,* [27, 62]*, computes the rank of the vector bundle $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g},\, \ell)$.*
2. *Let $\overline{\mathcal{M}'}_{g,n}$ be the moduli stack of stable curves along with a choice of formal coordinates around the marked points. Then the sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g},\, \mathcal{C}/S,\, \ell)$ gives a vector bundle with a flat projective connection with logarithmic singularities along the boundary.*
3. *The sheaf of conformal blocks descends to a vector bundle $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ with a flat projective connection on the moduli stack $\mathcal{M}_{g,n}$. Moreover, the vector bundle $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ extends to a vector bundle over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$, and the projective connection extends to a projective connection with logarithmic singularities along the boundary.*
4. *The natural forgetful map form $F : \overline{\mathcal{M}}'_{g,n} \to \overline{\mathcal{M}}_{g,n}$ identifies the vector bundle $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g},\, \ell)$ with the pull-back $F^*\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g},\, \ell)$.*

We will refer to the connections in Theorem 2.3 as the *TUY/WZW* connections; their construction is recalled in Section 3.

Remark 2.4. There are several other important properties — e.g. "propagation of vacua" and "factorization theorems" — exhibited by conformal blocks. We refer the reader to [66] for more details.

We also point out that $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ and $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ do not agree as twisted $\mathcal{D}$-modules. This issue does not appear in [39], as the weights there are zero. We refer the reader to [64] for the computation of the Atiyah algebra of $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$.

3. Energy momentum tensor and the Sugawara construction

In this section, following the discussion in [66] we recall the definition of the Sugawara tensor, which will be used in defining the connections on the sheaf of covacua and conformal blocks.

For any $X \in \mathfrak{g}$, the element $X \otimes \xi^m \in \widehat{\mathfrak{g}}$ will be denoted by $X(m)$. The energy momentum tensor $T(z)$ at level ℓ is defined by the formula

$$T(z) \;=\; \frac{1}{2(\ell + h^{\vee}(\mathfrak{g}))} \sum_{a=1}^{\dim \mathfrak{g}} :J^a(z)J^a(z):\,, \tag{3.1}$$

where : : is the normal ordering (cf. [66, p. 467]), $h^{\vee}(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$, and $\{J^1, \cdots, J^{\dim \mathfrak{g}}\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the normalized Cartan-Killing form. Also, define

$$X(z) \;:=\; \sum_{n\in\mathbb{Z}} X(n) z^{-n-1}$$

for any element $X \in \mathfrak{g}$. The n-th Virasoro operator L_n is defined by the formula (see [33]):

$$L_n \;:=\; \frac{1}{2(\ell + h^{\vee}(\mathfrak{g}))} \sum_{m\in\mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} :J^a(m)J^a(n-m):\;. \tag{3.2}$$

The operators L_n act on the module $\mathcal{H}_{\lambda}$ defined in Section 2.1.

For $X \in \mathfrak{g}$, $f(z) \in \mathbb{C}((z))$ and $\underline{\ell} = \ell(z)\frac{d}{dz}$, define (as in [66])

$$X[f] \;:=\; \mathrm{Res}_{z=0}\left(X(z)f(z)\right)dz, \quad T[\underline{\ell}] \;:=\; \mathrm{Res}_{z=0}\left(T(z)\ell(z)\right)dz.$$

3.1. **Construction of the WZW/TUY connection.** Let C be an irreducible smooth projective curve with n-marked points $\boldsymbol{p} = (p_1, \cdots, p_n)$. For every $1 \le i \le n$, we choose a formal coordinate ξ_i around the marked point p_i on the curve C. Following [66], let us briefly recall the construction of a flat projective connection.

Let $\pi : \mathcal{C} \to S$ be a versal family of n-pointed stable curves equipped with n non-intersecting sections $p_i : S \to \mathcal{C}$. Let $D = \sum_{i=1}^{n} p_i(S)$ be the corresponding divisor on $\mathcal{C}$. We have a short exact sequence of sheaves

$$0 \longrightarrow \pi_*\mathcal{T}_{\mathcal{C}/S}(*D) \longrightarrow \pi_*\mathcal{T}_{\mathcal{C},\pi}(*D) \longrightarrow \mathcal{T}_S \longrightarrow 0 \tag{3.3}$$

on S. On the other hand, we also have the short exact sequence

$$0 \longrightarrow \pi_*\mathcal{T}_{\mathcal{C}/S}(*D) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_S[\xi_i^{-1}]\frac{d}{d\xi_i} \longrightarrow R^1\pi_*\mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow 0. \tag{3.4}$$

obtained from pushing forward the following short exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{C}/S}(-D) \longrightarrow \mathcal{T}_{\mathcal{C}/S}(*D) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_S[\xi_i^{-1}]\frac{d}{d\xi_i} \longrightarrow 0. \tag{3.5}$$

Since the family of n-pointed curves $\mathcal{C}$ is versal, we have the Kodaira-Spencer isomorphism $\mathcal{T}_S \xrightarrow{\sim} R^1\pi_*\mathcal{T}_{\mathcal{C}/S}(-D)$. Combining (3.3) and (3.4), the following commutative diagram of homomorphisms is obtained:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_*\mathcal{T}_{\mathcal{C}/S}(*D) & \longrightarrow & \pi_*\mathcal{T}_{\mathcal{C},\pi}(*D) & \longrightarrow & \mathcal{T}_S & \longrightarrow & 0 \\ & & \| & & \downarrow \mathbf{t} & & \downarrow KS & & \\ 0 & \longrightarrow & \pi_*\mathcal{T}_{\mathcal{C}/S}(*D) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_S[\xi_i^{-1}]\frac{d}{d\xi_i} & \xrightarrow{\theta} & R^1\pi_*\mathcal{T}_{\mathcal{C}/S}(-D) & \longrightarrow & 0, \end{array} \tag{3.6}$$

where $\mathbf{t}$ is the projection to the polar part of the Laurent expansion of sections in terms of the given coordinates ξ_i around the divisors $p_i(S)$. This map $\mathbf{t}$ is an isomorphism because the family $\mathcal{C}$ is versal.

Let $\vec{\ell} = (l_1, \cdots, l_n)$ and $\vec{m} = (m_1, \cdots, m_n)$ be two formal vector fields; both l_i and m_i are defined on a formal neighborhood of $p_i(S)$. The Lie bracket $[\vec{\ell}, \vec{m}]_d$ is given by the formula

$$[\vec{\ell}, \vec{m}]_d := [\vec{\ell}, \vec{m}]_0 + \theta(\vec{\ell})(\vec{m}) - \theta(\vec{m})(\vec{\ell}), \tag{3.7}$$

where $[\ ,\ ]_0$ is the usual Lie bracket of formal vector fields and $\theta(\vec{\ell})$ acts componentwise using the formal parameters ξ_i. Now for any formal vector field $\vec{\ell}$, define the operator $\mathscr{D}(\vec{\ell})$ on $\mathcal{H}_{\vec{\lambda}}$ by the formula

$$\mathscr{D}(\vec{\ell})(F \otimes |\Phi\rangle) := \theta(\vec{\ell})(F) \otimes |\Phi\rangle - F \cdot \left(\sum_{j=1}^n \rho_j(T[\underline{l}_j])\right)|\Phi\rangle, \tag{3.8}$$

where ρ_j is the action on $\mathcal{H}_{\lambda_j}$ defined on [66, p. 475].

3.2. **WZW/TUY connection.**

After possibly shrinking S, we can find a symmetric bidifferential ω on $\mathcal{C} \times_S \mathcal{C}$ with a pole of order two on the diagonal such that the biresidue is 1. For any formal vector field $\vec{\ell}$ define

$$a_\omega(\vec{\ell}) := -\frac{c_v}{12}\sum_{i=1}^n \mathrm{Res}_{\xi_i=0}\left(\ell_i(\xi_i)S_\omega(\xi_i)d\xi_i\right),$$

where S_ω is the projective connection associated to ω, and $c_v = \dfrac{\ell \dim \mathfrak{g}}{\ell + h^\vee(\mathfrak{g})}$ is the central charge.

Now let τ be a vector field on S. Take a lift of τ to $\pi_*\mathcal{T}_{\mathcal{C}/S}(*D)$ and denote it by $\vec{\ell}$. With the choice of a bidifferential ω as above, we define the following operator on the sheaf of conformal blocks:

$$\nabla_\tau^{(\omega)}(\langle\Psi|) := \mathscr{D}(\vec{\ell})(\langle\Psi|) + a_\omega(\vec{\ell})(\langle\Psi|). \tag{3.9}$$

We recall the following result (see [66, Thm. 5.3.3]).

Proposition 3.1. *The operator $\nabla_\tau^{(\omega)}$ in (3.9) defines a flat projective connection on the sheaf of conformal blocks $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{g}, \ell)$.*

Remark 3.2. Consider the natural forgetful map $F : \overline{\mathcal{M}}'_{g,n} \to \overline{\mathcal{M}}_{g,n}$ constructed by forgetting the choice of formal parameters at the n-marked points. Then the natural identification between $F^*\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ as locally free sheaves intertwines, up to a first order operator, the pull-back of the coordinate free TUY connection on $F^*\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ and the TUY connection on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$.

4. Projective connections via heat operators

4.1. **Heat operators and the Hitchin-van Geemen-de Jong equation.** Let $\pi : M \to S$ be a smooth map of smooth varieties, and let $\mathcal{L}$ be a line bundle on M. The Kodaira-Spencer infinitesimal deformation map is given by:

$$KS_{M/S} : \mathcal{T}_S \longrightarrow R^1\pi_*\mathcal{T}_{M/S}.$$

On the other hand, we have the coboundary map

$$\mu_{\mathcal{L}} : \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow R^1\pi_*\mathcal{T}_{M/S},$$

occurring in the long exact sequence obtained from the push forward π_* of the fundamental short exact sequence of differential operators

$$0 \longrightarrow \mathcal{T}_{M/S} \cong \mathcal{D}^{\leq 1}_{M/S}(\mathcal{L})/\mathcal{O}_M \longrightarrow \mathcal{D}^{\leq 2}_{M/S}(\mathcal{L})/\mathcal{O}_M \xrightarrow{s_2} \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow 0 \, ,$$

where s_2 is the symbol map and $\mathcal{D}^{\leq i}_{M/S}(\mathcal{L})$ is the sheaf of relative differential operators of order atmost i. Following [67], consider the sheaf $\mathcal{W}(\mathcal{L})$ defined by

$$\mathcal{W}(\mathcal{L}) := \mathcal{D}^{\leq 1}_{M}(\mathcal{L}) + \mathcal{D}^{\leq 2}_{M/S}(\mathcal{L}). \tag{4.1}$$

There is a natural short exact sequence

$$0 \longrightarrow \mathcal{D}^{\leq 1}_{M/S}(\mathcal{L}) \longrightarrow \mathcal{W}(\mathcal{L}) \xrightarrow{q_0} \pi^*\mathcal{T}_S \oplus \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow 0. \tag{4.2}$$

Definition 4.1. A *heat operator* $\mathbb{D} : \pi^*\mathcal{T}_S \to \mathcal{W}(\mathcal{L})$ is a section of the natural projection map $\mathcal{W}(\mathcal{L}) \to \pi^*\mathcal{T}_S$. A *projective heat operator* is a section of $\mathcal{W}(\mathcal{L})/\mathcal{O}_M \to \pi^*\mathcal{T}_S$.

A projective heat operator evidently lifts, locally, to a heat operator. Given a homomorphism $\rho : \pi^*\mathcal{T}_S \to \operatorname{Sym}^2 \mathcal{T}_{M/S}$, one can ask whether there is a canonical *projective heat operator* $\mathbb{D} : \pi^*\mathcal{T}_S \to \mathcal{W}(\mathcal{L})/\mathcal{O}_M$ such that $q_1 \circ \mathbb{D} = \rho$. The following theorems of Hitchin [32] and van Geemen-de Jong [67], answer a fundamental question on existence of projective heat operators.

Theorem 4.2 ([32, 67])**.** *Assume that the following conditions hold:*

- $KS_{M/S} + \mu_{\mathcal{L}} \circ \rho = 0$*;*
- $\cup[\mathcal{L}] : \pi_*\mathcal{T}_{M/S} \to R^1\pi_*\mathcal{O}_M$ *is an isomorphism;*
- $\pi_*\mathcal{O}_M = \mathcal{O}_S$*.*

Then there exists a unique projective heat operator $\overline{\mathbb{D}}$ *lifting any candidate symbol* $\rho : \pi^*\mathcal{T}_S \to \operatorname{Sym}^2 \mathcal{T}_{M/S}$*. Moreover the coherent sheaf* $\pi_*\mathcal{L}$ *carries a projective connection.*

Remark 4.3. We can take $\mathcal{L}$ to be an object in the rational Picard group $\operatorname{Pic}(M) \otimes \mathbb{Q}$. All the sheaves that appear in the statement of Theorem 4.2 are well-defined, and the proof in [67] does not require the assumption that $\mathcal{L}$ be a line bundle.

4.2. **Heat operators and metaplectic quantization.** We are interested in the case where the Kodaira-Spencer map $KS_{M/S}$, the candidate symbol $\rho : \mathcal{T}_S \to \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S}$ and the class of $\mathcal{L}$ are intertwined by the equation:

$$KS_{M/S} + \cup[\mathcal{L}] \circ \rho = 0. \tag{4.3}$$

We refer to (4.3) as the equation *controlling the deformations*. Recall that the connecting homomorphism

$$\mu_{\mathcal{L}^{\otimes k}} : \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow R^1\pi_*\mathcal{T}_{M/S}$$

is given by the formula (see [4])

$$\mu_{\mathcal{L}^{\otimes k}} = \cup \Big([k \cdot \mathcal{L}] - \frac{1}{2}[K_{M/S}] \Big) . \tag{4.4}$$

4.2.1. *Rewriting the deformation equation.* We will now rewrite (4.3) to produce a projective heat operator on $\mathcal{L}$. Throughout this subsection we assume that for any positive $k \in \mathbb{Q}$, the connecting homomorphism $\mu_{\mathcal{L}^{\otimes k}}$ in (4.4) is an isomorphism. This condition holds, for example, in the case of moduli spaces of parabolic bundles.

Now

$$\begin{aligned} KS_{M/S} + \cup[\mathcal{L}] \circ \rho \quad &= KS_{M/S} + \frac{1}{k}\left(\cup\left(k[\mathcal{L}] - \frac{1}{2}K_{M/S}\right) + \cup\frac{1}{2}[K_{M/S}]\right) \circ \rho \\ &= KS_{M/S} + \mu_{\mathcal{L}^{\otimes k}} \circ \left(1 + \mu^{-1}_{\mathcal{L}^{\otimes k}} \circ (\cup\frac{1}{2}[K_{M/S}])\right) \circ \frac{1}{k}\rho \\ &= KS_{M/S} + \mu_{\mathcal{L}^{\otimes k}} \circ \widetilde{\rho}_k, \end{aligned}$$

where $\widetilde{\rho}_k = \left(1 + \mu^{-1}_{\mathcal{L}^{\otimes k}} \circ \left(\cup\frac{1}{2}[K_{M/S}]\right)\right) \circ \frac{1}{k}\rho$ is the symbol map. Again assuming that the conditions of Theorem 4.2 are satisfied, we get a projective heat operator $\overline{\mathscr{D}}$ on $\mathcal{L}$ with symbol $\widetilde{\rho}_k$ such that the following diagram commutes

$$\begin{array}{ccc} & & \pi_*\mathcal{D}^{\leq 2}_M(\mathcal{L}^{\otimes k}) \\ & \nearrow & \downarrow \scriptstyle{symb} \\ \mathcal{T}_S & \xrightarrow{\rho_k} & \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S} \end{array} \tag{4.5}$$

This induces a projective connection on $\pi_*\mathcal{L}^{\otimes k}$ for any positive $k \in \mathbb{Z}$ with symbol $\widetilde{\rho}_k$

4.2.2. *Metaplectic correction d'après Scheinost-Schottenloher.* We can rewrite the left-hand side of (4.3) as follows:

$$\begin{aligned} KS_{M/S} + \cup[\mathcal{L}] \circ \rho \quad &= KS_{M/S} + \frac{1}{k}\left(\cup\left([k \cdot \mathcal{L}] + \frac{1}{2}[K_{M/S}]\right) - \cup\frac{1}{2}[K_{M/S}]\right) \circ \rho \\ &= KS_{M/S} + \mu_{\mathcal{L}^{\otimes k} \otimes K^{\frac{1}{2}}_{M/S}} \circ \rho_k, \end{aligned}$$

where $\rho_k := \frac{1}{k}\rho$ and $\mathcal{L}^{\otimes k} \otimes K^{1/2}_{M/S}$ is considered as an element of the rational Picard group. Thus, from (4.3) the following equation is obtained:

$$KS_{M/S} + \mu_{\mathcal{L}^{\otimes k} \otimes K^{\frac{1}{2}}_{M/S}} \circ \rho_k = 0. \tag{4.6}$$

Assume that the other conditions of the Hitchin-van Geemen-de Jong existence theorem are satisfied. Then Theorem 4.2 tells us that there exists a unique projective heat operator $\widehat{\mathbb{D}}_k$ with symbol ρ_k and a connection on $\pi_*(\mathcal{L}^k \otimes K^{\frac{1}{2}}_{M/S})$. As pointed out in Remark 4.3, the projective heat operator makes sense even if the square-root of $K_{M/S}$ does not exist.

It is easy to see that for any candidate symbol, there exists a second order projective operator $\widehat{\mathbb{D}}$ on $K^{\frac{1}{2}}$ with the same given symbol. However, this operator is not a projective heat operator, since the natural projection of it to $\pi^*\mathcal{T}_S$ is zero. On the other hand, we have a projective heat operator $\overline{\mathbb{D}}$ on $\mathcal{L}^{\otimes k}$ with the same symbol $\widetilde{\rho}_k$. The following is then a natural question.

Question 4.4. Using the projective heat operator $\overline{\mathbb{D}}$ and the projective operator $\widehat{\mathbb{D}}$ with symbol ρ_k, can we construct a projective heat operator $\widetilde{\mathbb{D}}$ on $\mathcal{L}^k \otimes K^{\frac{1}{2}}_{M/S}$?

Remark 4.5. Observe that the equations in Theorem 4.2 imply that there exists at most one heat operator provided

$$\mu_{\mathcal{L}^{\otimes k} \otimes K^{\frac{1}{2}}_{M/S}} : \pi_* \operatorname{Sym}^2 \mathcal{T}_{M/S} \longrightarrow R^1\pi_*\mathcal{T}_{M/S}$$

is an isomorphism. A positive answer to Question 4.4 would immediately imply that the symbol of $\widetilde{\mathbb{D}}$ satisfies the equation for ρ_k given in (4.6). This will provide a necessary relation that the linear maps $\cup[\mathcal{L}]$ and $\cup[K_{M/S}]$ are scalar multiples of each other. This would give an alternative, more conceptual understanding of Theorem 4.1 in [19]

5. Parabolic Hitchin symbol as in Biswas-Mukhopadhyay-Wentworth

In this section we first briefly recall the construction of the Hitchin connection for the moduli space of parabolic bundles $M^{par,rs}_{G,\boldsymbol{\tau}}$ obtained in [19]. We refer the reader to Appendices A.1 and A.3 for a brief review of the moduli stack of parabolic bundles and the parabolic determinant of cohomology line bundles. We will freely use the correspondence between parabolic G bundles on a curve C and equivariant $(\Gamma$-$G)$-bundles on a Galois cover $\widehat{C}$ of the curve C with Galois group Γ. This is recalled in Appendix A.2.

In particular, we focus on the parabolic Hitchin symbol defined in the paper [19]. Using restriction to fibers of the Hitchin map, we give a simplification of the expression for the symbol that enables us to compare the parabolic Hitchin symbol with the symbol of the Sugawara operators as constructed in [66]. The main result of this section is Proposition 5.6. This is a key new feature and one of the main technical difficulties in the parabolic set-up that does not appear in [39].

5.1. **The Hitchin symbol.**

In [19] (recalled in Appendix A.2), we identified the moduli space $M^{par}_{G,\boldsymbol{\tau}}$ of parabolic bundles with the moduli space $M^{\boldsymbol{\tau},ss}_{G}$ of $(\Gamma,\, G)$-bundles on a Galois cover $\widehat{C}$ of the curve C via the invariant pushforward functor [5, 6].

This includes the following identifications: Let $\mathcal{P}$ be a regularly stable parabolic bundle and $\widehat{\mathcal{P}}$ the corresponding $(\Gamma,\, G)$ bundle (see Appendix A.2). Let $\mathrm{Par}(\mathcal{P})$ (resp. $\mathrm{Spar}(\mathcal{P})$) denote the bundle of parabolic (resp. strictly parabolic) automorphisms of $\mathcal{P}$. Then

$$H^0(C,\, \mathrm{Spar}(\mathcal{P})\otimes K_C(D)) \;\cong\; H^0(\widehat{C},\, \mathrm{ad}\,\mathcal{P}\otimes K_{\widehat{C}})^{\Gamma} \tag{5.1}$$

and

$$H^1(C,\, \mathrm{Par}(\mathcal{P})) \;\cong\; H^1(\widehat{C},\, \mathrm{ad}\,\mathcal{P})^{\Gamma}. \tag{5.2}$$

The Hitchin symbol was defined using the natural map

$$H^0(\widehat{C},\, \mathrm{ad}\,\mathcal{P}\otimes K_{\widehat{C}})^{\Gamma}\otimes H^1(\widehat{C},\, \mathrm{ad}\,\mathcal{P})^{\Gamma} \xrightarrow{\kappa_{\mathfrak{g}}} H^1(\widehat{C},\, K_{\widehat{C}})^{\Gamma} \;\cong\; \mathbb{C}; \tag{5.3}$$

where the last isomorphism is given by the Serre duality on $\widehat{C}$. As in [32, Prop. 2.16], this induces a natural map

$$\rho_{sym}\;:\; R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D) \;\longrightarrow\; \pi_{e*}\,\mathrm{Sym}^2\,\mathcal{T}_{M^{\boldsymbol{\tau},rs}_{G}/S}\,, \tag{5.4}$$

where $\pi_e\,:\, M^{\boldsymbol{\tau},rs}_{G} \to S$ and $\pi_s\,:\, \mathcal{C}\to S$ are the projections. On the other hand we also have the pairing

$$\begin{array}{c} H^0(C,\, \mathrm{Spar}(\mathcal{P})\otimes K_C(D))\otimes H^1(C,\, \mathrm{Par}(\mathcal{P})) \\ \big\downarrow \\ H^1(C,\, \mathrm{Spar}(\mathcal{P})(D)\otimes \mathrm{Par}(\mathcal{P})\otimes K_C) \xrightarrow{\quad\kappa_{\mathfrak{g}}\quad} H^1(C,\, K_C) \;\cong\; \mathbb{C}. \end{array} \tag{5.5}$$

This also induces a map

$$\widetilde{\rho}_{sym}\;:\; R^1\pi_{s*}\mathcal{T}_{\mathcal{C}/S}(-D) \;\longrightarrow\; \pi_{e*}\,\mathrm{Sym}^2\,\mathcal{T}_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}. \tag{5.6}$$

The isomorphisms $H^1(C,\, K_C) \cong \mathbb{C}$ in (5.5) and $H^1(\widehat{C},\, K_{\widehat{C}}) \cong \mathbb{C}$ are both given by the residue theorem and Serre duality, but for different curves. Hence, the map on Hitchin symbols ρ_{sym} and $\widetilde{\rho}_{sym}$ do not commute under the identifications given by (5.1) and (5.2). However they are related as follows:

Lemma 5.1. *Let ρ_{sym} and $\widetilde{\rho}_{sym}$ be as above. Then $|\Gamma| \cdot \rho_{sym} = \widetilde{\rho}_{sym}$.*

Proof. This is immediate from the commutativity of

$$\begin{array}{ccc} H^1(C, K_C) & \xrightarrow{\sim} & H^1(\widehat{C},\, K_{\widehat{C}})^{\Gamma} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times|\Gamma|} & \mathbb{C} \end{array}$$

□

5.2. **Parabolic Hitchin connection via heat operators.** Let $\phi : G \to \mathsf{SL}(V)$ be a linear representation satisfying the hypothesis of Section A.3. Let $\mathcal{L}_\phi$ be the pullback of the determinant of cohomology line bundle to $M_G^{\boldsymbol{\tau},ss}$. Via the identification of parabolic G-bundles as equivariant bundles and Proposition A.7, we have identified it with the parabolic determinant of cohomology $\mathrm{Det}_{par}(\nu_{\mathfrak{sl}(V)}(\phi(\boldsymbol{\tau})))$. For notational convenience, we will drop ϕ when denoting the line bundle $\mathrm{Det}_{par,\phi}(\mathcal{P}, \boldsymbol{\tau})$ and simply write $\mathrm{Det}_{par}(\mathcal{P}, \boldsymbol{\tau})$.

In [19], the authors produced a projective heat operator on the line bundle $\mathcal{L}_\phi^{\otimes k}$ whose symbol satisfies the following Hitchin-van Geemen-de Jong equation:

$$KS_{M^{par,rs}_{G,\boldsymbol{\tau}}/S} + \mu_{\mathcal{L}^k_\phi} \circ \left(\frac{1}{m_\phi k} \,\mathrm{Id} + \mu^{-1}_{\mathcal{L}^k_\phi} \circ \left(\cup \frac{1}{2 m_\phi k} [K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] \right) \right) \circ \rho_{sym} \circ KS_{\mathcal{C}/S} = 0. \tag{5.7}$$

Setting $\rho = \rho_{sym} \circ KS_{\mathcal{C}/S}$ gives

$$KS_{M^{par,rs}_{G,\boldsymbol{\tau}}/S} + \mu_{\mathcal{L}^k_\phi} \circ \left(\mathrm{Id} + \mu^{-1}_{\mathcal{L}^k_\phi} \circ \left(\cup \frac{1}{2} [K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] \right) \right) \circ \frac{1}{m_\phi \cdot k} \rho = 0, \tag{5.8}$$

where k is a rational number.

Let $k = \ell/|\Gamma|$. Using the identification $\mathcal{L}_\phi \cong (\mathrm{Det}_{par}(\mathcal{P}, \boldsymbol{\tau}))^{\frac{|\Gamma|}{\ell}}$ in Proposition A.7, from (5.8) we get that

$$KS_{M^{par,rs}_{G,\boldsymbol{\tau}}/S} + \mu_{\mathrm{Det}_{par}(\mathcal{P},\boldsymbol{\tau}) \otimes K^{1/2}_{M^{par,ss}_{G,\boldsymbol{\tau}}/S}} \circ \frac{|\Gamma|}{m_\phi \cdot \ell} \rho \;=\; 0.$$

Definition 5.2. For any rational number a, we will denote the projective heat operator on $\mathrm{Det}^{\otimes a}_{par,\phi}(\mathcal{P}, \boldsymbol{\tau})$ obtained in [19] by $\mathbb{D}(\mathfrak{g}, a \cdot m_\phi \cdot \ell)$.

The following is one of the main results of [19, Theorem 4.1].

Theorem 5.3. *Let $\mathbb{L}$ be an element of $\mathrm{Pic}(M^{par,rs}_{G,\boldsymbol{\tau}}) \otimes \mathbb{Q}$ of level a. Then there is an equality $\cup[\mathbb{L}] = \cup a[\mathrm{Det}]$ as linear maps $\pi_{e*} \operatorname{Sym}^2 \mathcal{T}_{M^{par,rs}_{G,\boldsymbol{\tau}}/S} \to R^1\pi_{e*}\mathcal{T}_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}$, where* Det *is the determinant of cohomology (non-parabolic) line bundle.*

Remark 5.4. The above result should be put in the more general context of deformation theory of the moduli space of the parabolic bundles as studied in Boden-Yokogawa [22], and the birational variation of these moduli spaces as the weights vary.

Since line bundles on $M^{par,rs}_{G,\boldsymbol{\tau}}$ are pull-backs of rational powers of line bundles from $M^{par,s}_{\mathsf{SL}(V),\boldsymbol{\alpha}}$ for an appropriate choice of representation $(\phi,\, V)$ of G, the following is an immediate consequence of Theorem 5.3.

Corollary 5.5. *Let $\widehat{M}_G$ and $M^{par,rs}_{G,\boldsymbol{\tau}}$ be as in Section A.3. Let (ϕ, V) be a representation of G. Then the line bundle $K_{\widehat{M}_G/S}$ restricted to $M^{par,rs}_{G,\boldsymbol{\tau}}$ is of level $-\frac{2h^\vee(\mathfrak{g}).|\Gamma|}{m_\phi}$ with respect to $\mathsf{SL}(V)$, and hence*

$$\cup[K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] \;=\; \cup\frac{[K_{\widehat{M}_G/S}]}{|\Gamma|}$$

as linear maps from $\pi_*\operatorname{Sym}^2\mathfrak{T}_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}$ to $R^1\pi_*\mathfrak{T}_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}$.

Proof. On the one hand, we have the fact from [37] that $K_{\widehat{M}_G/S}$ is $-2h^\vee(\mathfrak{g})\mathcal{L}_\phi$, where $\mathcal{L}_\phi$ is the ample generator of the Picard group of the moduli stack of $\mathsf{SL}(V)$ bundles on $\widehat{C}$, where $\widehat{C}\to C$ is a Γ-cover. Moreover, $\mathcal{L}_\phi$ restricted to $M^{par,rs}_{G,\boldsymbol{\tau}}/S$ is of level $|\Gamma|$ (see Appendix A.3). On the other hand, the canonical bundle $K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}$ has a component that is $-2h^\vee(\mathfrak{g})$ times the ample generator of the Picard group of the moduli stack of G bundles on C, which in turn is of level $\frac{1}{m_\phi}$ with respect to the $\mathfrak{sl}(V)$ embedding of $\mathfrak{g}$. This proves the result. ☐

Corollary 5.5 further simplifies the parabolic Hitchin symbol, as shown by the following.

Proposition 5.6. *Assume that $m_\phi\cdot\ell+h^\vee(\mathfrak{g})\neq 0$. Then*

$$\frac{1}{m_\phi\ell}\left(1+\mu^{-1}_{\mathrm{Det}_{par}(\boldsymbol{\tau})}\circ\left(\frac{1}{2}\cup[K_{M^{par,rs}_{\boldsymbol{\tau},G}/S}]\right)\right)\circ\widetilde{\rho}_{sym} \;=\; \frac{\widetilde{\rho}_{sym}}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}. \tag{5.9}$$

Proof. Since $\widetilde{\rho}_{sym}$ is invertible, we need to show that

$$\frac{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}{m_\phi\cdot\ell}\left(1+\mu^{-1}_{\mathrm{Det}_{par}(\boldsymbol{\tau})}\left(\frac{1}{2}\cup[K_{M^{par,rs}_{\vec{\alpha},\mathsf{SL}_r}/S}]\right)\right) \;=\; \mathrm{Id}\,.$$

So it suffices to prove that

$$\mu^{-1}_{\mathrm{Det}_{par}}(\boldsymbol{\tau})\circ\left(\frac{1}{2}\cup[K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}]\right) \;=\; \left(-1+\frac{m_\phi\cdot\ell}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}\right)\mathrm{Id}\,.$$

By multiplying with $\mu_{\mathrm{Det}_{par}(\boldsymbol{\tau})}$, it suffices to show that

$$\cup\,[K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] \;=\; \left(\frac{-2h^\vee(\mathfrak{g})}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}\right)\mu_{\mathrm{Det}_{par}(\boldsymbol{\tau})}. \tag{5.10}$$

Now by [37, 38], applied to the moduli space $\widehat{M}_G$ of principal G bundle on $\widehat{C}$, we get that

$$[K_{\widehat{M}_G}] \;=\; -2h^\vee(\mathfrak{g}).m_\phi.[\mathcal{L}_\phi]. \tag{5.11}$$

By Corollary 5.5,

$$\cup[K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] = \cup\frac{-2h^\vee(\mathfrak{g})[\mathcal{L}_\phi]}{m_\phi\cdot|\Gamma|} = \cup\frac{-2h^\vee(\mathfrak{g})}{m_\phi\cdot|\Gamma|}\left(\frac{|\Gamma|[\mathrm{Det}_{par}(\boldsymbol{\tau})]}{\ell}\right).$$

Rewriting the above equation, we find

$$\begin{aligned}
\cup m_\phi\cdot\ell[K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] &= \cup\left(\frac{-2h^\vee(\mathfrak{g})\cdot m_\phi\cdot\ell}{|\Gamma|\cdot m_\phi}\right)\frac{|\Gamma|[\mathrm{Det}_{par}(\boldsymbol{\tau})]}{\ell},\\
\cup\frac{m_\phi\cdot\ell}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}[K_{M^{par}_{G,\boldsymbol{\tau}}/S}] &= \left(\cup\frac{1}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}\left(-2h^\vee(\mathfrak{g})[\mathrm{Det}_{par}]\right)\right),\\
\cup[K_{M^{par,rs}_{G,\boldsymbol{\tau}}/S}] &= \left(\cup\frac{-2h^\vee(\mathfrak{g})}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}\left([\mathrm{Det}_{par}(\boldsymbol{\tau})]-\frac{1}{2}[K_{M^{par,rs}_{\boldsymbol{\tau},G}/S}]\right)\right),\\
\cup\frac{1}{2}[K_{M^{par}_{\boldsymbol{\tau},G}/S}] &= \left(\frac{-h^\vee(\mathfrak{g})}{m_\phi\cdot\ell+h^\vee(\mathfrak{g})}\right)\cdot\mu_{\mathrm{Det}_{par}(\boldsymbol{\tau})} \qquad \text{(from (4.4))}\,.
\end{aligned}$$

This completes the proof. □

6. Proof of Theorem 1.1

In this section, we give a proof of the main theorem in this article by comparing the Sugawara tensor and the parabolic analog of the heat operator with a given symbol constructed by the authors in [19].

Let $\mathcal{P}ar_G^{rs}(C, \boldsymbol{p}, \boldsymbol{\tau})$ be the open substack parametrizing regularly stable parabolic bundles of parabolic type $\boldsymbol{\tau}$. Then the natural map $\mathcal{P}ar_G^{rs}(C, \boldsymbol{p}, \boldsymbol{\tau}) \rightarrow M_{G,\boldsymbol{\tau}}^{par,rs}(C, \boldsymbol{p})$ is a gerbe banded by the center $Z(G)$ of the group G.

Similarly, let $\mathcal{Q}_{\boldsymbol{\tau}}^{rs}$ be the open ind-subscheme of $\mathcal{Q}_{\boldsymbol{\tau}}$ parametrizing the regularly stable bundles. The natural map π^{reg} given by the composition

$$\pi^{reg} : \mathcal{Q}_{\boldsymbol{\tau}}^{rs} \longrightarrow \mathcal{P}ar_G^{rs}(\boldsymbol{\tau}) \longrightarrow M_{G,\boldsymbol{\tau}}^{par,rs}$$

is a $L_{\mathcal{C}',G}/Z(G)$ torsor which is étale locally trivial. Here, $L_{\mathcal{C}',G}$ is the loop group associated to a punctured curve.

6.1. **Twisted $\mathcal{D}$-modules via quasi-section of Drinfeld-Simpson.** Let $\pi_s : \mathcal{C} \rightarrow S$ be a versal family of n-pointed smooth curves of genus g. We choose formal coordinates $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ along the sections $\boldsymbol{p}$.

Let $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ be as in Section B.1, and let $\mathcal{Q}_{\tau_i}$ be the affine flag variety associated to τ_i. The above choice of coordinates gives an identification of $\mathcal{Q}_{\boldsymbol{\tau}}$ with $\prod_{i=1}^n L_G/\mathcal{P}_{\tau_i}$. By the discussion in [9, Secs. 5.2.9-5.2.12], the infinitesimal action, of the central extension $\widehat{L}_G$ of the loop group, on $\mathcal{Q}_{\tau_i}$ gives a map

$$\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp} \longrightarrow H^0(\mathcal{Q}_{\tau_i}, \mathcal{D}_{\mathcal{Q}_{\tau_i}/S}(\mathscr{L}_{\vec{\lambda}})).$$

Here $\mathcal{D}_{\mathcal{Q}_{\tau_i}/S}(\mathscr{L}_{\lambda_i})$ is the ring of relative $\mathscr{L}_{\lambda_i}$-twisted differential operators on $\mathcal{Q}_{\tau_i}$, and $\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})$ is a suitable completion of the universal enveloping algebra of $\widehat{\mathfrak{g}}_{\xi_i}$, and $\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}$ is the opposite algebra. Summing over all the coordinates, we get a map

$$\bigoplus_{i=1}^{n} \left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right) \longrightarrow H^0(\mathcal{Q}_{\boldsymbol{\tau}}, \mathcal{D}_{\mathcal{Q}_{\boldsymbol{\tau}}/S}(\mathscr{L}_{\vec{\lambda}})) \tag{6.1}$$

which via further restriction gives a map $\bigoplus_{i=1}^n \left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right) \rightarrow H^0(\mathcal{Q}_{\boldsymbol{\tau}}^{rs}, \mathcal{D}_{\mathcal{Q}_{\boldsymbol{\tau}}/S}(\mathscr{L}_{\vec{\lambda}}))$. Both sides of (6.1) carry natural filtrations and the map in (6.1) is a map of filtered sheaves of algebras.

As in [39], we consider a quasi-section of π^{rs}. The result of [24] implies that the natural étale locally trivial torsor $\pi^{rs} : \mathcal{Q}_{\boldsymbol{\tau}}^{rs} \rightarrow M_{G,\boldsymbol{\tau}}^{rs}$ has a quasi-section $N_{G,\boldsymbol{\tau}}^{par,rs} \xrightarrow{r} M_{G,\boldsymbol{\tau}}^{par,rs}$ such that r is an étale epimorphism, and there is a map $\sigma : N_{G,\tau}^{par,rs} \rightarrow \mathcal{Q}_{\boldsymbol{\tau}}^{rs}$ such that the following diagram commutes

$$\begin{array}{ccc} & & \mathcal{Q}_{\boldsymbol{\tau}}^{rs} \\ & \overset{\sigma}{\nearrow} & \downarrow \pi^{rs} \\ N_{G,\boldsymbol{\tau}}^{par,rs} & \xrightarrow{r} & M_{G,\boldsymbol{\tau}}^{par,rs} \\ & \searrow & \downarrow \pi_e \\ & & S \end{array} \tag{6.2}$$

Now since the map r is étale, we get an isomorphism

$$H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, r^*\mathcal{T}_{M_{G,\boldsymbol{\tau}}^{par,rs}/S}) = H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathcal{T}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S}).$$

Given any relative differential operator $\mathfrak{D}$ on the line bundle $\mathscr{L}_{\vec{\lambda}}$, we can pull it back via σ (see Section 8.1 and 8.7 in [39]) to a differential operator on the line bundle $\sigma^*\mathscr{L}_{\vec{\lambda}}$ which, by an abuse of notation, is again denote by $\mathscr{L}_{\vec{\lambda}}$. Thus, (6.1) gives the following map of filtered sheaves of algebras

$$\hbar_{\mathscr{L}_{\vec{\lambda}}} : \bigoplus_{i=1}^{n} \left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right) \longrightarrow H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathcal{D}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S}(\mathscr{L}_{\vec{\lambda}})). \tag{6.3}$$

The sheaf of Lie algebras $\bigoplus_{i=1}^{n}\left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right)$ carries a natural PBW filtration and we let $\left(\bigoplus_{i=1}^{n}\left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right)\right)^{\leq m}$ be the m-th part of the filtration. Then the following diagram is commutative

(6.4)

$$\left(\bigoplus_{i=1}^{n}\left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right)\right)^{\leq m} \xrightarrow{\hbar^{\leq m}_{\mathscr{L}_{\vec{\lambda}}}} H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathcal{D}^{\leq m}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S}(\mathscr{L}_{\vec{\lambda}})) \xrightarrow{symb^{\leq m}} H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathrm{Sym}^m\, \mathcal{T}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S})$$

$$\left(\bigoplus_{i=1}^{n}\left(\overline{U}(\widehat{\mathfrak{g}}_{\xi_i})^{opp}\right)\right)^{\leq m} \xrightarrow{\hbar^{\leq m}_{\mathcal{O}}} H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathcal{D}^{\leq m}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S}(\mathcal{O}_{N_{G,\boldsymbol{\tau}}^{par,rs}})) \xrightarrow{symb^{\leq m}} H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathrm{Sym}^m\, \mathcal{T}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S})$$

where $symb^{\leq m}$ denotes the principal m-th order symbol map of a differential operator.

6.2. **Projective heat operator from Sugawara.** We now give a local description of the map $\hbar_{\mathscr{L}_{\vec{\lambda}}}$. Let $\mathcal{P}$ be a regularly stable parabolic G-bundle in the moduli space of parabolic bundles of parabolic weights $\vec{\lambda}$ on a curve C with parabolic structure over $\boldsymbol{p}$. We consider it as a point in $N_{G,\boldsymbol{\tau}}^{par,rs}$. The tangent space at $\mathcal{P}$ is given by $H^1(C, \mathrm{Par}(\mathcal{P}))$, where $\mathrm{Par}(\mathcal{P})$ is the sheaf of Lie algebras given by parabolic endomorphisms of the bundle $\mathcal{P}$.

Let $P_i \subset G$ be the parabolic subgroup determined by the weight λ_i attached to the point $p_i \in \boldsymbol{p}$, and let $\mathfrak{p}_i$ be the corresponding Lie algebra. We denote by $\mathfrak{p}_i^-$ the opposite parabolic and by $\mathfrak{n}_i^-$ the nilpotent radical of $\mathfrak{p}_i^-$. We have a short exact sequence of sheaves

$$0 \longrightarrow \mathrm{Par}(\mathcal{P}) \longrightarrow \mathrm{Par}(\mathcal{P})\left(\sum_{i=1}^{n} m_i p_i\right) \longrightarrow \bigoplus_{i=1}^{n}\left(\mathfrak{n}_i^- \oplus \bigoplus_{j=1}^{m_i} \mathfrak{g}\otimes \xi_i^j\right) \longrightarrow 0, \tag{6.5}$$

where $m_1, \cdots, m_n$ are nonnegative integers. Taking the long exact sequence of cohomologies associated to (6.5), we get a homomorphism

$$\bigoplus_{i=1}^{n}\left(\mathfrak{n}_i^- \oplus \mathfrak{g}\otimes \mathbb{C}[\xi_i^{-1}]\xi_i^{-1}\right) \longrightarrow H^1(C, \mathrm{Par}(\mathcal{P})). \tag{6.6}$$

Combining this with the natural projection $\mathfrak{g}\otimes\mathbb{C}((\xi_i)) \to \mathfrak{n}_i^- \oplus \mathfrak{g}\otimes\mathbb{C}[\xi_i^{-1}]\xi_i^{-1}$ for each $1 \leq i \leq n$, we get a homomorphism

$$\rho_i : \mathfrak{g}\otimes\mathbb{C}((\xi_i)) \longrightarrow \mathfrak{n}_i^- \oplus \mathfrak{g}\otimes\mathbb{C}[\xi_i^{-1}]\xi_i^{-1} \longrightarrow H^1(C, \mathrm{Par}(\mathcal{P})). \tag{6.7}$$

The composition of maps ρ_i in (6.7) is the local description of $\hbar_{\mathcal{O}}$ (defined in (6.3))

$$\left(\overline{U}(\widehat{\mathfrak{g}}_{p_i})^{opp}\right)^{\oplus n} \longrightarrow H^0(N_{G,\boldsymbol{\tau}}^{par,rs}, \mathcal{D}_{N_{G,\boldsymbol{\tau}}^{par,rs}/S}(\mathcal{O})). \tag{6.8}$$

The operator $\mathscr{D}(\vec{\ell})$ defined in (3.8) gives a relative second order differential operator $\mathfrak{D}$ on $N^{par,rs}_{G,\boldsymbol{\tau}}$ which acts on the i-th factor by $(T[\underline{l}_i])$ (see (3.3) and (6.7)). Thus we have the following diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^n \mathcal{O}_S((\xi_i))\frac{d}{d\xi_i} & \xrightarrow{\ \mathfrak{D}\ } & H^0(N^{par,rs}_{G,\boldsymbol{\tau}},\, \mathcal{D}^{\leq 2}_{N^{par,rs}_{G,\boldsymbol{\tau}}/S}(\mathscr{L}_{\vec{\lambda}})) \\ \big\downarrow \theta & & \\ H^0(S,\, T_S). & & \end{array} \tag{6.9}$$

We can realize $\mathfrak{D}$ as a projective heat operator by taking a lift of a vector field on S to an element of $\bigoplus_{i=1}^n \mathcal{O}_S((\xi_i))\frac{d}{d\xi_i}$. Now as described in the previous section, the difference between two lifts can be understood as a $\mathcal{O}_S$-module homomorphism a_ω. Thus the map $\mathfrak{D}$ descends to a projective heat operator, and we will also denote the descended operator by $\mathfrak{D}$. In the rest of this section we show that the symbol of $\mathfrak{D}$ is the Hitchin symbol $\widetilde{\rho}_{sym}$, which will complete the proof of Theorem 1.1.

6.3. **The parabolic duality pairing and the Hitchin symbol.** Recall that the Cartan-Killing form induces a nondegenerate bilinear form between the sheaves

$$\kappa_{\mathfrak{g}} \,:\, \mathrm{Spar}(\mathcal{P})(D) \otimes \mathrm{Par}(\mathcal{P}) \longrightarrow \mathcal{O}_C.$$

Let D_{p_i} be a formal disc around each marked point p_i in C, and let $C^* = C\backslash\{p_1, \cdots, p_n\}$ be the complement. Consider the following open covering:

$$C = C^* \cup (\sqcup_{i=1}^n D_{p_i}).$$

A section of $\mathrm{Spar}(\mathcal{P})$ restricted to D_{p_i} consists of an element of $\mathfrak{g}\otimes\mathbb{C}[[\xi_i]]$ whose image under the natural evaluation map

$$\mathrm{ev}_{p_i} \,:\, \mathfrak{g}\otimes\mathbb{C}[[\xi_i]] \longrightarrow \mathfrak{g}$$

is contained in the nilradical $\mathfrak{n}_i$ of the parabolic subalgebra $\mathfrak{p}_i$. Similarly, $\mathrm{Par}(\mathcal{P})$ consists of sections whose restriction to any formal disc D_{p_i} has the property that the image of the evaluation map is in $\mathfrak{p}_i$.

Let $\{\overline{P}_i\}$ be a Čech cocycle representative in $\prod_{i=1}^n \left(\mathrm{Par}(\mathcal{P})(D^*_{p_i})\right)$ of a cohomology class of $H^1(C, \mathrm{Par}(\mathcal{P}))$ with respect to the covering $C = C^* \cup (\sqcup D_{p_i})$. Here we have $\overline{P}_i \in \mathfrak{g}\otimes\mathbb{C}((\xi_i))$, under a trivialization of $\mathcal{P}$ restricted to D_{p_i}. Similarly we let $\{\phi_i d\xi_i\} \in \mathfrak{g}\otimes\mathbb{C}((\xi_i))d\xi_i$ denote the restriction of an element of $H^0(C, \mathrm{Spar}(\mathcal{P})\otimes K_C(D))$ to $\sqcup D^*_{p_i}$.

The natural pairing in (5.5) takes the form

$$H^0(C, \mathrm{Spar}(\mathcal{P})\otimes K_C(D)) \otimes H^1(C, \mathrm{Par}(\mathcal{P})) \longrightarrow \mathbb{C} \tag{6.10}$$

$$\{\phi_i d_{\xi_i}\} \times \{\overline{P}_i\} \longmapsto \sum_{i=1}^n \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(\phi_i, \overline{P}_i) d\xi_i. \tag{6.11}$$

Now consider a Čech representative $\vec{\ell} = \{\underline{l}_i\} \in \oplus_{i=1}^n \mathbb{C}((\xi_i))\frac{d}{d\xi_i}$ of a cohomology class in $H^1(C, T_C(-D))$. Let ϕ be a global section of the sheaf $\mathrm{Spar}(\mathcal{P})\otimes K_C(D)$. For each i, we have

$$\underline{l}_i = \sum_{m=-m_i}^{\infty} l_{i,m}\xi_i^{m+1}\frac{d}{d\xi_i},$$

and ϕ restricted to $D^*_{p_i}$ is of the form

$$\phi_i d\xi_i = \sum_{m\in\mathbb{Z}} X_{i,m}\xi_i^{-m-1}d\xi_i \in \mathfrak{g}\otimes\mathbb{C}((\xi_i))d\xi_i.$$

Since the diagram in (6.4) commutes, we can evaluate the symbol of $\mathfrak{D}$ by computing the following:

$$\begin{aligned}
\langle \phi\otimes\phi, \sum_{i=1}^n T[\underline{l}_i]\rangle &= \sum_{i=1}^n \langle \phi_i\otimes\phi_i d\xi_i^2, \sum_{m=-m_i}^{\infty} l_{i,m}L_m\rangle\\
&= \sum_{i=1}^n \sum_{m=-m_i}^{\infty} l_{i,m}\langle \phi_i\otimes\phi_i d\xi_i^2, L_m\rangle\\
&= \sum_{i=1}^n \sum_{m=-m_i}^{\infty} l_{i,m}\langle \phi_i\otimes\phi_i d\xi_i^2, \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{k\in\mathbb{Z}}\sum_{a=1}^{\dim\mathfrak{g}} {:}J^a(k)J^a(m-k){:}\rangle.
\end{aligned}$$

Now if $\underline{l}_i = \xi_i^{n_i+1}\frac{d}{d\xi_i}$, we get that

$$\begin{aligned}
\langle \phi\otimes\phi, \sum_{i=1}^n T[\xi_i^{n_i+1}\frac{d}{d\xi_i}]\rangle &= \sum_{i=1}^n \langle \phi_i\otimes\phi_i d\xi_i^2, L_{n_i}\rangle\\
&= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{i=1}^n\sum_{k\in\mathbb{Z}}\sum_{a=1}^{\dim\mathfrak{g}} \langle \phi_i\otimes\phi_i d\xi_i^2, {:}J^a(k)J^a(n_i-k){:}\rangle.
\end{aligned}$$

If $n_i \neq 0$, then we get that

$$\begin{aligned}
\langle \phi_i\otimes\phi_i d\xi_i^2, L_{n_i}\rangle &= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{k\in\mathbb{Z}}\sum_{a=1}^{\dim\mathfrak{g}} \langle \phi_i\otimes\phi_i d\xi_i^2, J^a(k)J^a(n_i-k)\rangle\\
&= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{k\in\mathbb{Z}}\sum_{a=1}^{\dim\mathfrak{g}} \mathrm{Res}_{\xi_i=0}\langle \phi_i, J^a(k)\rangle d\xi_i.\,\mathrm{Res}_{\xi_i=0}\langle \phi_i, J^a(n_i-k)\rangle d\xi_i\\
&= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{k\in\mathbb{Z}}\sum_{a=1}^{\dim\mathfrak{g}} \left(\sum_{m\in\mathbb{Z}} \mathrm{Res}_{\xi_i=0}\,\kappa_{\mathfrak{g}}(X_{i,m}, J^a)\xi^{k-m-1}d\xi_i\right)\\
&\qquad\times\left(\sum_{m\in\mathbb{Z}} \mathrm{Res}_{\xi_i=0}\,\kappa_{\mathfrak{g}}(X_{i,m}, J^a)\xi^{n_i-k-m-1}d\xi_i\right)\\
&= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{k\in\mathbb{Z}}\sum_{a=1}^{\dim\mathfrak{g}} \kappa_{\mathfrak{g}}(X_{i,k}, J^a)\kappa_{\mathfrak{g}}(X_{i,n_i-k}, J^a)\\
&= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{k\in\mathbb{Z}} \kappa_{\mathfrak{g}}(X_{i,k}, X_{i,n_i-k}).
\end{aligned}$$

The zero-th Virasoro operator L_0 can be rewritten without normal ordering as follows:

$$L_0 = \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{a=1}^{\dim\mathfrak{g}} J^aJ^a + \frac{1}{(\ell+h^\vee(\mathfrak{g}))}\sum_{k=1}^{\infty} J^a(-k)J^a(k).$$

Thus we get the following:

$$\begin{aligned}
&\langle \phi_i\otimes\phi_i d\xi_i^2, L_0\rangle\\
&= \frac{1}{2(\ell+h^\vee(\mathfrak{g}))}\sum_{a=1}^{\dim\mathfrak{g}} \langle \phi_i\otimes\phi_i d\xi_i^2, J^aJ^a\rangle + \frac{1}{(\ell+h^\vee(\mathfrak{g}))}\sum_{k=1}^{\infty}\langle \phi_i\phi_i d\xi_i^2, J^a(-k)J^a(k)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \sum_{a=1}^{\dim \mathfrak{g}} \left(\sum_{m\in\mathbb{Z}} \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(X_{i,m},\, J^a)\xi_i^{-m-1} d\xi_i \right)^2 \\
&\quad + \frac{1}{(\ell + h^\vee(\mathfrak{g}))} \sum_{k=1}^{\infty} \sum_{a=1}^{\dim \mathfrak{g}} \left(\left(\sum_{m\in\mathbb{Z}} \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(X_{i,m},\, J^a)\xi_i^{-k-m-1} d\xi_i \right) \right. \\
&\qquad \times \left. \left(\sum_{m\in\mathbb{Z}} \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(X_{i,m},\, J^a)\xi_i^{k-m-1} d\xi_i \right) \right) \\
&= \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \sum_{a=1}^{\dim \mathfrak{g}} \kappa_{\mathfrak{g}}(X_{i,0},\, J^a)\kappa_{\mathfrak{g}}(X_{i,0},\, J^a). \\
&\quad + \frac{1}{(\ell + h^\vee(\mathfrak{g}))} \sum_{k=1}^{\infty} \sum_{a=1}^{\dim \mathfrak{g}} \kappa_{\mathfrak{g}}(X_{i,-k},\, J^a)\kappa_{\mathfrak{g}}(X_{i,k},\, J^a) \\
&= \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \kappa_{\mathfrak{g}}(X_{i,0},\, X_{i,0}) + \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \sum_{k\in\mathbb{Z}\setminus\{0\}} \kappa_{\mathfrak{g}}(X_{i,-k},\, X_{i,k}) \\
&= \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \sum_{k\in\mathbb{Z}} \kappa_{\mathfrak{g}}(X_{i,k},\, X_{i,-k}).
\end{aligned}
$$

We summarize the above calculations in the following proposition.

Proposition 6.1. *For any $1 \leq i \neq n$, and for any $m_i \in \mathbb{Z}$,*

$$
\langle \phi_i \otimes \phi_i d\xi_i^2,\, L_{m_i} \rangle \;=\; \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \sum_{k\in\mathbb{Z}} \kappa_{\mathfrak{g}}(X_{i,k},\, X_{i,m_i-k}).
$$

6.4. **Proof of the Main theorem (Theorem 1.1).** Recall that the product

$$
H^1(C,\, T_C(-D)) \otimes H^0(C,\, \mathrm{SPar}(\mathcal{P}) \otimes K_C(D)) \longrightarrow H^1(C,\, \mathrm{Par}\,\mathcal{P})
$$

induces a homomorphism

$$
\widetilde{\rho}_{sym} \,:\, R^1\pi_{n*}\mathcal{T}_{\mathfrak{X}_G^{par}/M_{G,\boldsymbol{\tau}}^{par,rs}}(-D) \longrightarrow \pi_{n*}\,\mathrm{Sym}^2\,\mathcal{T}_{M_{G,\boldsymbol{\tau}}^{par,rs}/S}. \tag{6.12}
$$

Consider the Čech cover of C given by $C = C^* \cup (\sqcup_{i=1}^n D_{p_i})$. In particular, given any Čech cohomology class $\{\xi_i^{n_i+1}\frac{d}{d\xi_i}\}$ in $H^1(C,\, T_C(-D))$, using Serre duality and the identification of $\mathrm{SPar}(\mathcal{P})(D)$ with $\mathrm{Par}(\mathcal{P})^\vee$, we get a symmetric bilinear form on $H^0(C,\, \mathrm{SPar}(\mathcal{P}) \otimes K_C(D))$.

As in the previous section, consider a section $\phi \in H^0(C,\, \mathrm{SPar}(\mathcal{P}) \otimes K_C(D)$. For each i, the section ϕ restricted to $D^*_{p_i}$ is of the form

$$
\phi_i d\xi_i \;=\; \sum_{m\in\mathbb{Z}} X_{i,m}\xi_i^{-m-1} d\xi_i \in \mathfrak{g} \otimes \mathbb{C}((\xi_i))d\xi_i.
$$

Thus evaluating a cocycle class $\{\xi_i^{n_i+1}\frac{d}{d\xi_i}\}$ against a section ϕ written in the Čech cover as $\{\phi_i d\xi_i\}$, we get that

$$
\begin{aligned}
\{\xi_i^{n_i+1}\frac{d}{d\xi_i}\}(\phi) \;&:=\; \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(\phi_i d\xi_i \otimes \langle \xi_i^{n_i+1}\frac{d}{d\xi_i},\, \phi_i d\xi_i\rangle) \\
&=\; \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(\phi_i d\xi_i,\, \sum_{m\in\mathbb{Z}} X_{i,m}\xi_i^{n_i-m}) \\
&=\; \mathrm{Res}_{\xi_i=0}\, \kappa_{\mathfrak{g}}(\sum_{k\in\mathbb{Z}} X_{i,k}\xi_i^{-k-1},\, \sum_{m\in\mathbb{Z}} X_{i,m}\xi_i^{n_i-m}) d\xi_i
\end{aligned}
$$

$$= \sum_{k\in\mathbb{Z}} \kappa_{\mathfrak{g}}(X_{i,k},\, X_{i,n_i-k}).$$

We summarize the discussion in this subsection in the following proposition which completes the proof the Theorem 1.1.

Proposition 6.2. *Let a be any rational number and ϕ be a faithful representation with Dynkin index m_ϕ. Then the symbol of the projective heat operator $\mathfrak{D}$ acting on $\mathscr{L}_{\vec{\lambda}}^{\otimes a.m_\phi} \cong \mathrm{Det}_{par,\phi}^{\otimes a}(\boldsymbol{\tau})$ constructed from the Sugawara tensor and uniformization (see Section 6.2) coincides with*

$$\frac{1}{2(a\cdot m_\phi\cdot \ell + h^\vee(\mathfrak{g}))}\widetilde{\rho}_{sym}.$$

Hence, the projective heat operator $\mathfrak{D}$ and the projective heat operator constructed in [19] via (5.2) coincide.

Remark 6.3. If a square-root of $K_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$ exists, then it follows that the push-forward of the line bundle $\mathrm{Det}_{par}(\boldsymbol{\alpha})\otimes K^{\frac{1}{2}}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$ produces conformal blocks of level $\ell - h^\vee(\mathfrak{sl}(r))$, where $h^\vee(\mathfrak{g})$ is the dual Coxeter number of a Lie algebra $\mathfrak{g}$. From the calculations in this section, and the fact that the tangent and cotangent spaces of the moduli space $M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$ are only dependent on the flag type of $\boldsymbol{\alpha}$, it follows that the symbol ρ_{sym}/ℓ equals the symbol of the differential operator that induces the TUY connection. However, it should be mentioned that even if $K_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}$ has a square-root, the pushforward $\pi_*\left(\mathrm{Det}_{par}(\boldsymbol{\alpha})\otimes K^{\frac{1}{2}}_{M^{par,s}_{\mathsf{SL}_r,\boldsymbol{\alpha}}/S}\right)$ may not have any sections.

7. Geometrization of the KZ equation on invariants

In this section, we show a geometric construction of the Knizhnik-Zamolodchikov connection (KZ). This question was suggested to us by Professor P. Belkale. Let us first recall the classical construction of the KZ connection [34].

7.1. **KZ connection.** Let $\mathfrak{g}$ be a fixed semisimple Lie algebra, and let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ be an n-tuple of highest weights. Consider the vector space of invariants of tensor product of representations

$$A_{\vec{\lambda}}(\mathfrak{g}) := \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}, \mathbb{C}).$$

The space of invariants sits inside the zero weight space $(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})^*_0$ of the dual of the tensor product of representations.

Let $X_n = \{\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$, and consider the trivial vector bundle $\mathbb{A}_{\vec{\lambda}}$ on the configuration space of points X_n whose fiber is $A_{\vec{\lambda}}(\mathfrak{g})$. It is well known [29, 28, 66] that the space of conformal blocks $\mathcal{V}^\dagger_{\vec{\lambda}}(C, \mathfrak{g}, \ell, \boldsymbol{z})$ on $\mathbb{P}^1 = \mathbb{C}\sqcup\{\infty\}$ with n marked points $(z_1, \cdots, z_n)$ for $\mathfrak{g}$ at level ℓ and weights $\vec{\lambda}$ injects into $A_{\vec{\lambda}}(\mathfrak{g})$:

$$\iota : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{g}, \ell, \boldsymbol{z}) \hookrightarrow A_{\vec{\lambda}}(\mathfrak{g}). \tag{7.1}$$

This map is actually an isomorphism for $\ell \gg 0$. Specific bounds for ℓ are given in Belkale-Gibney-Mukhopadhyay ([14, 15]).

As in Section 3, consider an orthonormal basis $J^1, \cdots, J^{\dim\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ for the normalized Cartan-Killing form. Define the Casimir operator $\Omega = \sum_{a=1}^{\dim\mathfrak{g}} J^aJ^a$. For pairs of integers $1 \le i \neq j \le n$, and vectors $v_1\otimes\cdots\otimes v_n \in V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$, let

$$\Omega_{i,j}(v_1\otimes\cdots\otimes v_n) := \sum_{a=1}^{\dim\mathfrak{g}} v_1\otimes\cdots\otimes J^av_i\otimes\cdots\otimes J^av_j\otimes\cdots\otimes v_n.$$

For any complex number $\kappa \neq 0$, the formula

$$\left(\nabla^{(\kappa)}_{\frac{\partial}{\partial z_i}}(f \otimes \langle \Psi|\right)(|\Phi\rangle) := \frac{\partial f}{\partial z_i}\langle \Psi|\Phi\rangle - \frac{f}{\kappa}\sum_{j\neq i}\frac{\langle \Psi|\Omega_{i,j}(\phi)\rangle}{z_i - z_j}, \tag{7.2}$$

defines a flat connection on $(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})_0^* \otimes \mathcal{O}_{X_n}$ over X_n that preserves the subbundle $\mathbb{A}_{\vec{\lambda}}$. Hence, its monodromy gives a representation of the pure braid group $\pi_1(X_n, \boldsymbol{z})$.

In this discussion, we restrict ourselves to the case where $\kappa = \ell + h^\vee(\mathfrak{g})$, and $\kappa_{\mathfrak{g}}(\lambda_i, \theta_{\mathfrak{g}}) < 1$ for all i. In this case it is known that the connection $\nabla^{(\ell+h^\vee(\mathfrak{g}))}$ preserves the bundle $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{g}, \ell)$ of conformal blocks and it is equal to the TUY/WZW connection [28, 29, 66].

7.2. **Invariants as global sections.** As in Section A.1 consider the moduli stack of quasi-parabolic bundles $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ of local type $\boldsymbol{\tau}$ on $\mathbb{P}^1$, where $\boldsymbol{\tau}$ and $\vec{\lambda}$ are related by the usual exponential map as before. Consider the open substack $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \tau)$ of $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ parametrizing quasi-parabolic bundle on $\mathbb{P}^1$ whose underlying bundle is trivial. By construction, we have an isomorphism of $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ with the quotient stack

$$[(G/P_1 \times \cdots \times G/P_n)/G], \tag{7.3}$$

where $P_1, \cdots, P_n$ are the parabolics determined by $\tau_1, \cdots, \tau_n$ and G acts diagonally on the product of partial flag varieties.

Let $\mathscr{L}_{\vec{\lambda}}$ be the Borel-Weil-Bott line bundle on $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$, and consider the restriction of $\mathscr{L}_{\vec{\lambda}}$ to $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$. We get a natural map

$$H^0(\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau}), \mathscr{L}_{\vec{\lambda}}) \longrightarrow H^0(\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau}), \mathscr{L}_{\vec{\lambda}}). \tag{7.4}$$

Now the restriction of $\mathscr{L}_{\vec{\lambda}}$ to $[(G/P_1 \times \cdots \times G/P_n)/G]$ is $L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_n}$, where the L_{λ_i} are the natural homogeneous line bundles on G/P_i determined by the weights λ_i. Moreover, by the Borel-Weil theorem, we have $H^0(G/P_i, L_{\lambda_i}) = V^*_{\lambda_i}$. Thus, from the restriction we get the natural commutative diagram

(7.5)

$$\begin{array}{ccccc}
H^0(\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau}), \mathscr{L}_{\vec{\lambda}}) & \xrightarrow{\text{res}} & H^0(\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau}), \mathscr{L}_{\vec{\lambda}}) & = & \left(\bigotimes_{i=1}^n H^0(G/P_i, L_{\lambda_i})\right)^{\mathfrak{g}} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathcal{V}^\dagger_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{g}, \ell, \boldsymbol{z}) & \xhookrightarrow{\iota} & \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}, \mathbb{C}) & = & \left(V^*_{\lambda_1}\otimes\cdots\otimes V^*_{\lambda_n}\right)^{\mathfrak{g}}.
\end{array}$$

Here the left vertical isomorphism is due to Laszlo-Sorger [40]; the diagram was used in [14]. Now it follows that the complement of $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ in $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ is just the ordinary theta divisor.

7.3. **Differential operators.** Recall the notion of a *good stack* from Beilinson-Drinfeld [9]: an equidimensional algebraic stack $\mathcal{Y}$ over complex numbers is *good* if the dimension of $\mathcal{Y}$ is half the dimension of the cotangent stack $\mathcal{T}^\vee_{\mathcal{Y}}$. Let $\mathcal{Y}_{sm}$ be the smooth topology of $\mathcal{Y}$. For any object $S \in \mathcal{Y}_{sm}$ and a smooth 1-morphism $\pi_S \in \mathcal{Y}$, we have the exact sequence

$$\mathcal{T}_{S/\mathcal{Y}} \longrightarrow \mathcal{T}_S \longrightarrow \pi_S^*\mathcal{T}_Y \longrightarrow 0.$$

Consider the sheaf of differential operators $\mathcal{D}_S$ on S and the left ideal $I = \mathcal{D}_S\mathcal{T}_{S/\mathcal{Y}} \subset \mathcal{D}_S$. Set $\mathcal{D}_{\mathcal{Y}}(S) := \mathcal{D}_S/I,$. This $\mathcal{D}_{\mathcal{Y}}$ is an $\mathcal{O}_{\mathcal{Y}}$ module along with a natural filtration such that $\mathrm{Sym}\,\mathcal{T}_{\mathcal{Y}} \cong \mathrm{gr}\,\mathcal{D}_{\mathcal{Y}}$. The above also works for differential operators twisted by a line bundle.

Since the nilpotent cone of the moduli space of parabolic Higgs bundles is isotropic of exactly half the dimension [7, 26, 41], it follows that the stack $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ is good. Moreover, since both $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ and $\mathcal{P}ar^{rs}_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ are quotients of a smooth scheme by a reductive group, they are also good.

Now we know that the line bundle $\mathscr{L}_{\vec{\lambda}}$ descends to a line bundle on $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$. The construction of the projective heat operator (cf. Definition 5.2) with symbol (5.2) gives a second order differential operator $\mathbb{D}$ on $\mathscr{L}_{\vec{\lambda}}$ over the moduli stack $\mathcal{P}ar^{rs}_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$. Since the sheaf $\mathcal{D}^{\leq 2}(\mathscr{L}_{\vec{\lambda}})$ is coherent and $\mathcal{P}ar^{rs}_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ has complement of dimension at least two (provided $\boldsymbol{\tau}$ satisfies the conditions in the statement of Theorem B.1) applying Hartogs theorem, we get a differential operator on $\mathscr{L}_{\vec{\lambda}}$ over the entire stack $\mathcal{P}ar_G(\mathbb{P}^1, \boldsymbol{z}, \tau)$, which we will still denote by $\mathbb{D}$.

Recall that via the uniformization theorem and the Sugawara construction, we have a degree two differential operator $\mathscr{D}$ on $\mathscr{L}_{\vec{\lambda}}$, which by Theorem 1.1 agrees with $\mathbb{D}$. Since the Sugawara construction restricted to the open substack $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$ induces the KZ connection, we have the following corollary obtained by restricting $\mathbb{D}$ to $\mathcal{P}ar^c_G(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{\tau})$.

Corollary 7.1. *Let $\pi^c : \mathcal{P}ar^c_G(\boldsymbol{\tau}) \to X_n$ be the relative open substack of quasi-parabolic bundles whose underlying bundle is trivial. Then the heat operator $\mathbb{D}$ induces a flat connection on the vector bundle $\pi^c_*\mathscr{L}_{\vec{\lambda}}$ over X_n whose fiber at a point $\boldsymbol{z}$ is $H^0(\mathcal{P}ar^c_G(C, \boldsymbol{z}, \boldsymbol{\tau}), \mathscr{L}_{\vec{\lambda}})$. Moreover, the natural identification of $\pi^c_*\mathscr{L}_{\vec{\lambda}}$ with $\mathbb{A}_{\vec{\lambda}}$ is flat for the geometric connection on $\pi^c_*\mathscr{L}_{\vec{\lambda}}$ and the Knizhnik-Zamolodchikov connection on $\mathbb{A}_{\vec{\lambda}}$.*

Appendix A. Moduli spaces of parabolic bundles

In this section, we briefly recall the basic notion of parabolic bundles and the natural line bundles on their moduli spaces.

Let $C, \boldsymbol{p}$ be as in Section 3.1. Let E be a vector bundles on C. A *quasi-parabolic structure* on E at a point $p \in \boldsymbol{p}$ is a strictly decreasing flag

$$E_p = F^1E_p \supset F^2E_p \cdots \supset \cdots \supset F^{k_p}E_p \supset F^{k_p+1}E_x = 0.$$

of linear subspaces in E_p. The above integer k_p is the length of the flag at p, and the tuple

$$(r_1(F^\bullet E_p), \cdots, r_{k_p}(F^\bullet E_p))$$

records the jumps in the dimension of the subspaces and is defined by

$$r_j(F^\bullet E_p) := \dim F^jE_p - \dim F^{j+1}E_p.$$

A *parabolic structure* on E at p is a quasi-parabolic structure as above together with a sequence of rational numbers

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{k_p} < 1$$

known as the *weights*. A parabolic bundle $(E, \boldsymbol{\alpha}, \boldsymbol{r})$ on C with parabolic divisor $\boldsymbol{p}$ is a vector bundle E on C along with parabolic structure over the points in $\boldsymbol{p}$. Using the weights $\boldsymbol{\alpha}$, the *parabolic degree* of E is defined to be

$$\mathrm{pdeg}(E) := \deg(E) + \sum_{i=1}^{n}\sum_{j=1}^{k_{p_i}} r_j(F^\bullet(E_{p_i}))\alpha_j(F^\bullet(E_{p_i})).$$

Stable and semistable parabolic bundles are defined using the parabolic degree (see [43]). Mehta and Seshadri constructed the moduli space $M^{par}_{\boldsymbol{\alpha}}$ of semistable parabolic bundles [43].

We now discuss some natural ample line bundles on $M^{par}_{\boldsymbol{\alpha}}$, following [20]. Let $\boldsymbol{\alpha}$ be a fixed set of weights for fixed flag type $\boldsymbol{r}$, and let $(E, \boldsymbol{\alpha}, \boldsymbol{r})$ be a parabolic bundle on $(C, \boldsymbol{p})$. Define the parabolic Euler characteristic

$$\chi_{\boldsymbol{p}}(E) := \chi(E) - \sum_{i=1}^{n}\sum_{j=1}^{k_{p_i}} r_j(F^{\bullet}(E_{p_i}))\alpha_j(F^{\bullet}(E_{p_i})).$$

Let $\mathcal{E}$ be a family of parabolic bundles on C, parametrized by a scheme T, of rank r, weight $\boldsymbol{\alpha}$ and flag type $\boldsymbol{r}$. For each point p_i, we have a string of rational numbers

$$\boldsymbol{\alpha}_{p_i} = (0 \leq \alpha_1(p_i) < \cdots < \alpha_j(p_i) < \cdots < \alpha_{k_{p_i}}(p_i) < 1)$$

which are the parabolic weights. Observe that the parabolic Euler characteristic $\chi_{\boldsymbol{p}}$ remains constant in a connected family.

Let ℓ be the least common multiple of all denominators of all the rational numbers appearing in $\boldsymbol{\alpha}$.

Definition A.1. The *parabolic determinant bundle of level* ℓ on $M^{par}_{\boldsymbol{\alpha}}$ is the element of the rational Picard group $\mathrm{Pic}(T)_{\mathbb{Q}}$ given by
(A.1)

$$\mathrm{ParDet}(\mathcal{E},\boldsymbol{\alpha}) := \mathrm{Det}(\mathcal{E})^{\ell}\bigotimes\left(\bigotimes_{i=1}^{n}\left(\bigotimes_{j=1}^{k_{p_i}}\det \mathrm{Gr}^j\,\mathscr{F}_{\bullet,p_i}(\mathcal{E}_{|T\times p_i})\right)^{\ell.\alpha_j(p_i)}\right)\otimes\left(\det \mathcal{E}_{|T\times p_0}\right)^{\frac{\ell\cdot\chi_{\boldsymbol{p}}(\mathcal{E})}{r}},$$

where p_0 is a fixed point of C, and Gr^j denotes the j-th graded piece of the filtration $\mathscr{F}_{\bullet,p_i}$ on $\mathcal{E}_{|T\times p_i}$ (cf. [20, Prop. 4.5]).

Let $M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$ be the moduli space of semistable parabolic SL_r bundles or equivalently parabolic bundles with trivialized determinant. Then $\mathrm{ParDet}(\mathcal{E},\boldsymbol{\alpha})$ descends to a line bundle on $M^{par,ss}_{\mathsf{SL}_r,\boldsymbol{\alpha}}$, which will be denoted by $\mathrm{Det}_{par}(\boldsymbol{\alpha})$.

A.1. **Parabolic G bundles.** We shall follow the notation in [19, App. A] and refer the reader there for more details. Consider the fundamental alcove Φ_0, and let $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ be a choice of n-tuple of weights in Φ_0 which will be referred to as *parabolic weights*.

Definition A.2. Let G be a connected complex reductive group. A parabolic structure on a principal G–bundle $E \to C$ with parabolic structures at the points $\boldsymbol{p} = (p_1, \cdots, p_n)$ is a choice of parabolic weights $\boldsymbol{\tau}$ along with a section σ_i of the homogeneous space $E_{p_i}/P(\tau_i)$, for each $1 \leq i \leq n$, where $P(\tau_i)$ is the standard parabolic associated to τ_i. Throughout this paper we will assume that $\theta_{\mathfrak{g}}(\tau_i) < 1$ for all $1 \leq i \leq n$.

We observe that when $G = \mathsf{GL}_r$, the associated bundle constructed via the standard representation of GL_r recovers the notion of parabolic bundles and parabolic weights as in the beginning of the present section. The notions of stability and semistability for parabolic G-bundles appear in the work of Bhosle-Ramanathan [16]; for $G = \mathsf{GL}_r$ they coincide with the notions of stable and semistable parabolic vector bundles.

Let $\boldsymbol{\tau}$ be an n-tuple of parabolic weights in the interior of the Weyl alcove of G. The corresponding moduli space $M^{par,ss}_{G,\boldsymbol{\tau}}$ (respectively, $M^{par,s}_{G,\boldsymbol{\tau}}$) of semistable (respectively, stable) parabolic G-bundles was constructed in [5]. These moduli spaces are normal irreducible quasi-projective varieties. The smooth locus of $M^{par,ss}_{G,\boldsymbol{\tau}}$ is denoted by $M^{par,rs}_{G,\boldsymbol{\tau}}$ and it parametrizes regularly stable parabolic bundles [18] or equivalently stable parabolic bundles with minimal automorphisms.

Let $\iota : G \to G'$ be an embedding of connected semisimple groups. This homomorphism ι produces a map $M^{par,ss}_{G,\boldsymbol{\tau}} \to M^{par,ss}_{G',\boldsymbol{\tau}'}$ which is a finite morphism. The weights $\boldsymbol{\tau}'$ and $\boldsymbol{\tau}$ are related by ι. This plays a key role in construction of the moduli spaces. In fact, choosing an appropriate representation of the group G, one can reduce the question of construction to the corresponding question on parabolic vector bundles.

Remark A.3. Let $\mathcal{C} \to S$ be a family of smooth curves with n disjoint sections. We will denote the corresponding semistable and regularly stable moduli spaces also by $M_{G,\boldsymbol{\tau}}^{par,ss}$ and $M_{G,\boldsymbol{\tau}}^{par,rs}$ respectively. When there is a scope of confusion, for any n-pointed smooth curve $(C, \boldsymbol{p})$, we will use the notation $M_{G,\boldsymbol{\tau}}^{par,ss}(C, \boldsymbol{p})$ and $M_{G,\boldsymbol{\tau}}^{par,rs}(C, \boldsymbol{p})$ respectively.

A.2. **Parabolic bundles as equivariant bundles.** We now discuss parabolic bundles from the point of view of equivariant bundles. We refer the reader to [6], [17], [57], and [19, App. B] for more details. This was used in [19] to construct a Hitchin type connection for parabolic bundles and it will be crucial here as well.

Definition A.4. Let $p : \widehat{C} \to C$ be a Galois cover of curves with Galois group Γ. A (Γ, G)-bundle is a principal G-bundle $\widehat{E}$ on $\widehat{C}$ together with a lift of the action of Γ on $\widehat{E}$ as bundle automorphism that commutes with the action of G on $\widehat{E}$.

Assume that the map $p : \widehat{C} \to C$ is ramified over $p_i \in C, 1 \leq i \leq n$. Let $\Gamma_{q_i} \subset \Gamma = \mathrm{Gal}(p)$ be the isotropy subgroup for some q_i over p_i. A (Γ, G)–bundle on a formal disc around q_i is uniquely determined by the conjugacy class of a homomorphism $\rho_i : \Gamma_{q_i} \to G$ given by the action of Γ_{q_i} on the fiber of the principal G–bundle over the point q_i (see [6, 63]). Fix a generator γ_i of the cyclic group Γ_{q_i}. Now consider a string of parabolic weights $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ such that $\rho_i(\gamma_i)$ is conjugate to τ_i for each $1 \leq i \leq n$. We will refer to this $\boldsymbol{\tau}$ as the local type of a (Γ, G)–bundle.

The notions of stability and semistability for (Γ, G)–bundles are similar to those for the usual principal G-bundles; more precisely, the inequality is checked only for the Γ equivariant reductions of the structure group to a parabolic subgroup of G ([5, 51]). Let $M_G^{\boldsymbol{\tau},ss}$ (respectively, $M_G^{\boldsymbol{\tau},s}$) denote the moduli spaces of semistable (respectively, stable) (Γ, G) bundles of local type $\boldsymbol{\tau}$.

Recall the isomorphism $\nu_{\mathfrak{g}} : \mathfrak{g}^\vee \xrightarrow{\sim} \mathfrak{g}$ from the Killing form. Given a string of parabolic weights $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$, choose a minimal integer ℓ such that $\exp\left(2\pi\sqrt{-1}\,(\ell \cdot \nu_{\mathfrak{g}}(\tau_i))\right) = 1$. Then by [46, 56], we can find a ramified Galois cover $p : \widehat{C} \to C$ with ramification exactly over n-points $\{p_i\}_{i=1}^n$ whose isotropy at any ramification point is a cyclic group of order ℓ. From now on we will restrict ourselves only to such Galois covers. The following theorem is due to [5, 6, 17].

Theorem A.5. *Consider the moduli stack $\mathcal{B}un_{\Gamma,G}^{\boldsymbol{\tau}}(\widehat{C})$ of (Γ, G)–bundles of fixed local type $\boldsymbol{\tau}$. The invariant direct image functor identifies the stack $\mathcal{B}un_{\Gamma,G}^{\boldsymbol{\tau}}(\widehat{C})$ with the moduli stack $\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$ of quasi-parabolic bundles of flag type $\boldsymbol{\tau}$. Moreover, the invariant push-forward functor also induces an isomorphism between the moduli spaces $M_{G,\boldsymbol{\tau}}^{par,ss}$ (respectively, $M_{G,\boldsymbol{\tau}}^{par,s}$) and $M_G^{\boldsymbol{\tau},ss}$ (respectively, $M_G^{\boldsymbol{\tau},s}$).*

A.3. **Parabolic determinants as equivariant determinants.** Consider the moduli space $M_G^{\boldsymbol{\tau},ss}$ of (Γ, G) bundles associated to a Galois cover $p : \widehat{C} \to C$ with Galois group Γ. Let $\widehat{M}_G$ be the moduli space of semistable principal G-bundles on the curve $\widehat{C}$. There is a natural forgetful map $M_{G,\boldsymbol{\tau}}^{par,ss} \to \widehat{M}_G$ that simply forgets the action of Γ.

Given a representation $\phi : G \to \mathsf{SL}_r$, consider the associated morphism $\overline{\phi} : \widehat{M}_G \to \widehat{M}_{\mathsf{SL}_r}$ between the corresponding moduli spaces. Let $\mathcal{L}$ be the determinant of cohomology line bundle on $\widehat{M}_{\mathsf{SL}_r}$. Let

$$\mathcal{L}_\phi := \overline{\phi}^* \mathcal{L}$$

be its pullback to $\widehat{M}_G$. If $G = \mathsf{SL}_r$, then ϕ can be taken to be the standard representation. Now Theorem A.5 realizes the moduli space $M^{par,ss}_{G,\boldsymbol{\tau}}$ of parabolic bundles as a moduli space $M^{\boldsymbol{\tau},ss}_G$ of (Γ, G)–bundles on $\widehat{C}$, which maps further into $\widehat{M}_G$ by forgetting the action of Γ. Thus using the identification between $M^{\boldsymbol{\tau},ss}_G$ and $M^{par,ss}_{G,\boldsymbol{\tau}}$, we get a natural line bundle $\mathcal{L}_\phi$ on $M^{par,ss}_{G,\boldsymbol{\tau}}$.

On the other hand, using the parabolic determinant of cohomology, one can construct natural line bundles on $M^{par,ss}_{G,\boldsymbol{\tau}}$ as follows:

Let $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ be a string of parabolic weights such that $\theta_{\mathfrak{g}}(\tau_i) < 1$ for all $1 \leq i \leq n$. Take a faithful representation (ϕ, V) of the group G satisfying the following condition:

- The local type $\phi(\boldsymbol{\tau}) = (\phi(\tau_1), \cdots, \phi(\tau_n))$ is rational, and $\theta_{\mathfrak{sl}(V)}(\phi(\tau_i)) < 1$.

Here, $\theta_{\mathfrak{g}}$ and $\theta_{\mathfrak{sl}(V)}$ are the highest roots of the Lie algebras $\mathfrak{g}$ and $\mathfrak{sl}(V)$, respectively. We now recall the definition of the parabolic determinant of cohomology for G–bundles.

Definition A.6. Let $\mathcal{E}$ be a family of parabolic G–bundles on a curve C with n-marked points, and let $\phi : \mathrm{G} \to \mathsf{SL}(V)$ be a faithful representation. Then the parabolic G-determinant bundle $\mathrm{Det}_{par,\phi}(\boldsymbol{\tau})$ with weight $\boldsymbol{\tau}$ is defined to be the line bundle $\mathrm{Det}_{par}(\nu_{\mathfrak{sl}(V)}(\phi(\boldsymbol{\tau})))$.

The following is recalled from [20].

Proposition A.7. *Let ℓ be the order of the stabilizer at each ramification point of the Galois cover $p : \widehat{C} \to C$ with Galois group Γ, then under the isomorphism in Theorem A.5, the parabolic determinant of cohomology is related to $\mathcal{L}_\phi$ by the formula*

$$\mathcal{L}_\phi \cong (\mathrm{Det}_{par,\phi}(\boldsymbol{\tau}))^{\frac{|\Gamma|}{\ell}},$$

where the Γ cover $\widehat{C}$ is determined by the parabolic weight data $\nu_{\mathfrak{sl}(V)}(\phi(\tau))$.

Appendix B. Uniformization of moduli spaces and conformal blocks

In this section, following the work of Belkale-Fakhruddin [12], Laszlo [39], and Laszlo-Sorger [40], we discuss the universal isomorphism between the sections of the parabolic determinant of cohomology bundle and the spaces of conformal blocks. If $(C, \vec{p})$ is a fixed smooth n-pointed curve, this identification is due to Beauville-Laszlo [8] ($G = \mathsf{SL}_r$ and $n = 0$), Faltings [27], Kumar-Narasimhan-Ramanathan [37] (for $n = 0$), Pauly [49] (for $G = \mathsf{SL}_r$) and Laszlo-Sorger [40]. The result has been extended to nodal curves by Belkale-Fakhruddin [12]. All of the results use a key uniformization theorem of Harder [31] and Drinfeld-Simpson [24] in the smooth case and its generalization in [12, 13] for the nodal case. We mostly follow the discussion in [12, Sec. 6].

B.1. **The line bundle on the universal moduli stack.** Consider the moduli stack $\mathcal{M}_{g,n}$ parametrizing smooth n-pointed curves of genus g. Recall from Section A.1 that given a tuple $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ in the fundamental Weyl alcove Φ of a simple Lie algebra $\mathfrak{g}$, we have the moduli stack $\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$ of quasi-parabolic G bundles of type $\boldsymbol{\tau}$ on a smooth curve C. This construction for families of smooth n-pointed curves gives relative moduli stacks $\pi_e : \mathcal{P}ar_G(\boldsymbol{\tau}) \to \mathcal{M}_{g,n}$ such that for any smooth curve $(C, \vec{p})$ we have $\pi_e^{-1}(C, \boldsymbol{p}) = \mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$. Throughout this discussion, it is assumed that $\theta_{\mathfrak{g}}(\tau_i) < 1$ for all $1 \leq i \leq n$.

Following [12] and [39], we construct a line bundle $\mathscr{L}_{\vec{\lambda}} \to \mathcal{P}ar_G(\boldsymbol{\tau})$, such that $\pi_* \mathscr{L}_{\vec{\lambda}} = \mathbb{V}^*_{\vec{\lambda}}(\mathfrak{g}, \ell)$, where $\vec{\lambda}$ and ℓ are related to $\boldsymbol{\tau}$ by the exponential map. The construction in [12] extends to the stable nodal curves.

B.1.1. *The relative affine flag varieties.* Let $\mathcal{C} \to S$ be a family of smooth n-pointed curves, and let $S = \operatorname{Spec} R$. Consider the affine curve $\mathcal{C}' = \mathcal{C} - \sqcup_{i=1}^{n} p_i(S)$. Let $\mathcal{C}_A = \mathcal{C} \times_R \operatorname{Spec}(A)$ for an R algebra A and similarly define $\mathcal{C}'_A$. Let $\widehat{\mathcal{C}}_A$ denote the completion of $\mathcal{C}_A$ along the sections $\boldsymbol{p}$. The sections $\boldsymbol{p}$ induce sections of $\widehat{\mathcal{C}}_A$, and $\widehat{\mathcal{C}}'_A$ denotes its complement.

Consider the following:

(1) $L_{\mathcal{C}',G}(A) = \operatorname{Mor}_k(\mathcal{C}'_A, G)$.
(2) $\mathcal{L}\mathcal{G}(A) = G(\Gamma(\widehat{\mathcal{C}}'_A, \mathcal{O}))$.

Each τ_i determines a parabolic subgroup $P(\tau_i) \subset G$, and we consider the standard parahoric subgroup $\mathcal{P}_{\boldsymbol{\tau}}$ given by the inverse image of $\prod_{i=1}^{n} P(\tau_i)$ under the natural evaluation map $L_G(A) \longrightarrow G^n$. Proposition 6.3 of [12] shows that the R group $L_{\mathcal{C}',G}$ is relatively ind-affine and formally smooth with connected integral geometric fibers over $\operatorname{Spec}(A)$. Observe that if $n = 1$ and $\lambda = 0$, and if t is a formal coordinate at the marked point p_1, then $\mathcal{L}\mathcal{G}$ gets identified with the loop group L_G and $\mathcal{P}_{\boldsymbol{\tau}}$ is the group of positive loops L_G^+.

B.1.2. *The central extension.* Faltings ([27], and also [8, Lemma 8.3], [40]) constructed a projective representation of $\mathcal{L}\mathcal{G}$ on $\mathbb{H}_{\vec{\lambda}} = R \otimes (\bigotimes \mathcal{H}_{\lambda_i}(\mathfrak{g}, \ell))$ whose derivative coincides with the natural projective action of the Lie algebra of $\mathcal{L}\mathcal{G}$. This gives us a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{\mathcal{L}\mathcal{G}} \longrightarrow \mathcal{L}\mathcal{G} \longrightarrow 1. \tag{B.1}$$

The extension $\widehat{\mathcal{L}\mathcal{G}}$ splits over $\mathcal{P}_{\boldsymbol{\tau}}$ (see [60, Lemma 7.3.5]), and the central extension $\widehat{\mathcal{L}\mathcal{G}}$ is independent of the chosen representations $\vec{\lambda}$. Moreover the extension (B.1) splits over $L_{\mathcal{C}',G}$ ([59], [12, Lemma 6.5]).

B.2. **The relative uniformization and parabolic theta functions.** Let $\widehat{\mathcal{P}}_{\boldsymbol{\tau}} := \mathcal{P}_{\boldsymbol{\tau}} \times \mathbb{G}_m$. The weight vectors $\vec{\lambda}$ give natural characters on $\widehat{\mathcal{P}}_{\boldsymbol{\tau}}$ and the product of characters induces a line bundle

$$\mathscr{L}_{\vec{\lambda}} \longrightarrow \mathcal{Q}_{\boldsymbol{\tau}} := \widehat{\mathcal{L}\mathcal{G}}/\widehat{\mathcal{P}}_{\boldsymbol{\tau}}.$$

Moreover, from the uniformization theorems [12, 8, 24, 31], it follows that the quotient of $\mathcal{Q}_{\boldsymbol{\tau}}$ by $L_{\mathcal{C}',G}$ is isomorphic to the pullback $\mathcal{P}ar_G(\boldsymbol{\tau})_S$ of the stack $\mathcal{P}ar_G(\boldsymbol{\tau})$ to S. Now since the extension in (B.1) splits over $L_{\mathcal{C}',G}$, the line bundle $\mathscr{L}_{\vec{\lambda}}$ descends to a line bundle over the stack $\mathcal{P}ar_G(\boldsymbol{\tau})$ which we will also denote by $\mathscr{L}_{\vec{\lambda}}$. Observe that the line bundle $\mathscr{L}_{\vec{\lambda}}$ is trivialized along the trivial section of $\mathcal{P}ar_G(\boldsymbol{\tau})$ over S, and such data determine the line bundle up to canonical isomorphism. We will refer to the line bundle $\mathscr{L}_{\vec{\lambda}}$ as the *Borel-Weil-Bott* line bundle.

B.2.1. *Parabolic determinant as the Borel-Weil-Bott line bundle.* We now compare the parabolic determinant of cohomology of the universal bundle with the line bundle $\mathscr{L}_{\vec{\lambda}}$.

Recall from Definition A.6 the notion of the parabolic determinant $\operatorname{Det}_{par,\phi}(\boldsymbol{\tau})$ of cohomology associated to a family of parabolic G bundles on $\mathcal{C} \to S$ and a suitable representation $\phi : G \to \mathsf{SL}(V)$. Now for the fixed n-pointed curve $(C, \boldsymbol{p})$, it is known that the line bundles $\mathscr{L}_{\vec{\lambda}}^{\otimes m_\phi}$ and $\operatorname{Det}_{par,\phi}(\boldsymbol{\tau})$ on $\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$ are isomorphic, where m_ϕ is the Dynkin index of the embedding ϕ. Since these line bundles are determined up to a normalizing factor, it follows that the corresponding projective bundles are identified as

$$\mathbb{P}\pi_{e*}\left(\operatorname{Det}_{par,\phi}(\boldsymbol{\tau})\right) \cong \mathbb{P}\pi_{e*}\left(\mathscr{L}_{\vec{\lambda}}^{\otimes m_\phi}\right), \tag{B.2}$$

where $\pi_e : \mathcal{P}ar_G(\boldsymbol{\tau}) \to \mathcal{M}_{g,n}$ is the natural projection.

B.2.2. *Parabolic theta functions and conformal blocks.* For any choice of formal parameters, the ind-scheme $\mathcal{Q}_{\boldsymbol{\tau}}$ can be identified with the product of affine flag varieties $\prod_{i=1}^{n} L_G/\mathcal{P}_{\tau_i}$ and the line bundle $\mathscr{L}_{\vec{\lambda}}$ pulls back to the corresponding line bundle on $L_G/\mathcal{P}_{\tau_i}$ given by the character λ_i. Now by Kumar [36] and Mathieu [42], we get that

$$H^0(\mathcal{Q}_{\boldsymbol{\tau}},\ \mathscr{L}_{\vec{\lambda}}) = \mathbb{H}^*_{\vec{\lambda}}. \tag{B.3}$$

We end this discussion with the following theorem (see [12, Theorem 1.7] and [39, Sec. 5.7]) which we will refer to as the universal identification of the parabolic theta functions and the conformal blocks. In the case when S is a point, the result can be found in [8, 27, 37, 40].

Theorem B.1. *The push-forward of $\mathscr{L}_{\vec{\lambda}}$ along the map $\pi_e : \mathcal{P}ar_G(\boldsymbol{\tau}) \to \mathcal{M}_{g,n}$ can be identified canonically with the bundle of coordinate free conformal blocks $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$. Moreover, $\mathscr{L}_{\vec{\lambda}}$ descends to a line bundle on $M^{par,rs}_{G,\boldsymbol{\tau}}$, and $(\pi_{e|M^{par,rs}_{G,\boldsymbol{\tau}}})_*\mathscr{L}_{\vec{\lambda}}$ is isomorphic to $\mathbb{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ provided the following conditions hold:*

- *The genus of the orbifold curve determined by $\boldsymbol{\tau}$ is at least 2, if G is not SL_2.*
- *The genus of the orbifold curve is at least 3, if $G = \mathsf{SL}_2$.*

Remark B.2. The last conditions ensure that for any smooth pointed curve $(C, \boldsymbol{p})$, the codimension of the moduli space $M^{par,rs}_{G,\boldsymbol{\tau}}(C, \boldsymbol{p})$ in the moduli stack $\mathcal{P}ar_G(C, \boldsymbol{p}, \boldsymbol{\tau})$ is at least two. We refer the reader to [19, App. C].

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