HIGGS BUNDLES AND LOCAL SYSTEMS ON RIEMANN SURFACES

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Contents

1. Preface

These notes are based on lectures given at the Third International School on Geometry and Physics at the Centre de Recerca Matemàtica in Barcelona, March 26–30, 2012. The aim of the School's four lecture series was to give a rapid introduction to Higgs bundles, representation varieties, and mathematical physics. While the scope of these subjects is very broad, that of these notes is far more modest. The main topics covered here are:

- The Hitchin-Kobayashi-Simpson correspondence for Higgs bundles on Riemann surfaces.
- The Corlette-Donaldson theorem relating the moduli spaces of Higgs bundles and semisimple representations of the fundamental group.
- A description of the oper moduli space and its relationship to systems of holomorphic differential equations, Higgs bundles, and the Eichler-Shimura isomorphism.

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These topics have been treated extensively in the literature. I have tried to condense the key ideas into a presentation that requires as little background as possible. With regard to the first item, I give a complete proof of the Hitchin-Simpson theorem (Theorem 2.17) that combines techniques that have emerged since Hitchin's seminal paper [36]. In the case of Riemann surfaces a direct proof for arbitrary rank which avoids introduction of the Donaldson functional can be modeled on Donaldson's proof of the Narasimhan-Seshadri theorem in [18] (such a proof was suggested in [57]). Moreover, the Yang-Mills-Higgs flow can be used to extract minimizing sequences with desirable properties. A similar idea is used in the Corlette-Donaldson proof of the existence of equivariant harmonic maps (Theorem 3.14). Indeed, I have sought in these notes to exhibit the parallel structure of the proofs of these two fundamental results. Continuity of the two flows is the key to the relationship between the equivariant cohomology of the moduli space of semistable Higgs bundles on the one hand and the moduli space of representations on the other. On first sight the last item in the list above is a rather different topic from the others, but it is nevertheless deeply related in ways that are perhaps still not completely understood. Opers [3] play an important role in the literature on the Geometric Langlands program [24]. My intention here is to give fairly complete proofs of the basic facts about opers and their relationship to differential equations and Higgs bundles (see also [63]).

Due to the limited amount of time for the lectures I have necessarily omitted many important aspects of this subject. Two in particular are worth mentioning. First, I deal only with vector bundles and do not consider principal bundles with more general structure groups. For example, there is no discussion of representations into the various real forms of a complex Lie group. Since some of the other lectures at this introductory school will treat this topic in great detail I hope this omission will not be serious. Second, I deal only with closed Riemann surfaces and do not consider extra "parabolic" structures at marked points. In some sense this ignores an important aspect at the heart of the classical literature on holomorphic differential equations (cf. [59, 7]). Nevertheless, for the purposes of introducing the global structure of moduli spaces, I feel it is better to first treat the case of closed surfaces. While much of the current research in the field is directed toward the two generalizations above, these topics are left for further reading.

I have tried to give references to essential results in these notes. Any omissions or incorrect attributions are due solely to my own ignorance of the extremely rich and vast literature, and for these I extend my sincere apologies. Also, there is no claim to originality of the proofs given here. A perusal of Carlos Simpson's foundational contributions to this subject is highly recommended for anyone wishing to learn about Higgs bundles (see [57, 58, 60, 61, 62]). In addition, the original articles of Corlette [11], Donaldson [18, 20], and of course Hitchin [36, 37, 39] are indispensable. Finally, I also mention more recent survey articles [10, 9, 29] which treat especially the case of representations to general Lie groups. I am grateful to the organizers, Luis Álvarez-Cónsul, Peter Gothen, and Ignasi Mundet i Riera, for inviting me to give these lectures, and to the CRM for its hospitality. Additional thanks to Bill Goldman, François Labourie, Andy Sanders, and Graeme

Wilkin for discussions related to the topics presented here, and to Benoît Cadorel for catching several typos. The anonymous referee also made very useful suggestions, for which I owe my gratitude.

Notation

- X = a compact Riemann surface of genus $g \geq 2$.
- $\pi = \pi_1(X, p)$ = the fundamental group of X.
- $\mathbb{H}$ = the upper half plane in $\mathbb{C}$.
- $\mathcal{O} = \mathcal{O}_X$ = the sheaf of germs of holomorphic functions on X.
- $\mathcal{K} = \mathcal{K}_X$ = the canonical sheaf of X.
- E = a complex vector bundle on X.
- H = a hermitian metric on E.
- ∇ = a connection on E.
- A (or d_A) = a unitary connection on (E, H).
- $\mathcal{C}_E$ = the space of connections on a rank n bundle E.
- $\mathcal{A}_E$ = the space of unitary connections on E.
- $\mathcal{B}_E$ = the space of Higgs bundles.
- $\mathcal{B}_E^{ss}$ = the space of semistable Higgs bundles.
- $\mathcal{G}_E$ (resp. $\mathcal{G}_E^{\mathbb{C}}$) = the unitary (resp. complex) gauge group.
- $\bar{\partial}_E$ = a Dolbeault operator on E, which is equivalent to a holomorphic structure.
- $(\bar{\partial}_E, H)$ = the Chern connection.
- $\mathcal{E}$ = sheaf of germs of holomorphic sections of a holomorphic bundle $(E, \bar{\partial}_E)$.
- $\mathfrak{g}_E$ = the bundle of skew-hermitian endomorphisms of E.
- $\mathrm{End}\, E = \mathfrak{g}_E^{\mathbb{C}}$ the endomorphism bundle of E.
- $\mathbf{V}$ = a local system on X.
- $\mathbf{V}_\rho$ = the local system associated to a representation $\rho : \pi \to \mathsf{GL}_n(\mathbb{C})$.
- $\underline{R}$ = the locally constant sheaf modeled on a ring R.
- L^p_k = the Sobolev space of functions/sections with k derivatives in L^p.
- $C^{k,\alpha}$ = the space of functions/sections with k derivatives being Hölder continuous with exponent α.

2. The Dolbeault Moduli Space

2.1. Higgs bundles.

2.1.1. *Holomorphic bundles and stability.* Throughout these notes, X will denote a closed Riemann surface of genus $g \geq 2$ and $E \to X$ a complex vector bundle. We begin with a discussion of the basic differential geometry of complex vector bundles. Good references for this material are Kobayashi's book [45] and Griffiths and Harris [26]. A holomorphic structure on E is equivalent to a choice of

$\bar\partial$-operator, i.e. a $\mathbb{C}$-linear map

$$\bar\partial_E : \Omega^0(X,E) \longrightarrow \Omega^{0,1}(X,E)$$

satisfying the Leibniz rule: $\bar\partial_E(fs) = \bar\partial f \otimes s + f\bar\partial_E s$, for a function f and a section s of E. Indeed, if $\{s_i\}$ is a local holomorphic frame of a holomorphic bundle, then the Leibniz rule uniquely determines the $\bar\partial$-operator on the underlying complex vector bundle. Conversely, since there is no integrability condition on Riemann surfaces, given a $\bar\partial$-operator as defined above one can always find local holomorphic frames (cf. [2, §5]). When we want to specify the holomorphic structure we write $(E, \bar\partial_E)$. We also introduce the notation $\mathcal{E}$ for a sheaf of germs of holomorphic sections of $(E, \bar\partial_E)$. We will sometimes confuse the terminology and call $\mathcal{E}$ a holomorphic bundle.

If $\mathcal{S} \subset \mathcal{E}$ is a holomorphic subbundle with quotient $\mathcal{Q}$, then a smooth splitting $E = S \oplus Q$ allows us to represent the $\bar\partial$-operators as

$$\bar\partial_E = \begin{pmatrix} \bar\partial_S & \beta \\ 0 & \bar\partial_Q \end{pmatrix} \tag{2.1}$$

where $\beta \in \Omega^{0,1}(X, \mathrm{Hom}(Q,S))$ is called the **second fundamental form**. A hermitian metric H on E gives an orthogonal splitting. In this case the subbundle S is determined by its orthogonal projection operator π, which is an endomorphism of E satisfying

(i) $\pi^2 = \pi$;
(ii) $\pi^* = \pi$;
(iii) $\mathrm{tr}\,\pi$ is constant.

The statement that $S \subset E$ be holomorphic is equivalent to the further condition

(iv) $(I - \pi)\bar\partial_E \pi = 0$.

Notice that (i) and (iv) imply (iii), and that $\beta = -\bar\partial_E\pi$. Hence, there is a 1-1 correspondence between holomorphic subbundles of $\mathcal{E}$ and endomorphisms π of the hermitian bundle E satisfying conditions (i), (ii), and (iv). I should point out that the generalization of this description of holomorphic subsheaves to higher dimensions is a key idea of Uhlenbeck and Yau [66].

A **connection** ∇ on E is a $\mathbb{C}$-linear map

$$\nabla : \Omega^0(X,E) \longrightarrow \Omega^1(X,E) \ ,$$

satisfying the Leibniz rule: $\nabla(fs) = df \otimes s + f\nabla s$, for a function f and a section s. Given a hermitian metric H, we call a connection **unitary** (and we will always then denote it by A or d_A) if it preserves H, i.e.

$$d\langle s_1, s_2\rangle_H = \langle d_A s_1, s_2\rangle_H + \langle s_1, d_A s_2\rangle_H \ . \tag{2.2}$$

The curvature of a connection ∇ is $F_\nabla = \nabla^2$ (perhaps more precise notation: $\nabla \wedge \nabla$). If $\mathfrak{g}_E$ denotes the bundle of skew-hermitian endomorphisms of E and $\mathfrak{g}_E^{\mathbb{C}}$ its complexification, then $F_A \in \Omega^2(X, \mathfrak{g}_E)$ for a unitary connection, and $F_\nabla \in \Omega^2(X, \mathfrak{g}_E^{\mathbb{C}})$ in general.

Remark 2.1. We will mostly be dealing with connections on bundles that induce a fixed connection on the determinant bundle. These will correspond, for example, to representations into SL_n as opposed to GL_n. In this case, the bundles $\mathfrak{g}_E$ and $\mathfrak{g}_E^{\mathbb{C}}$ should be taken to consist of traceless endomorphisms.

Finally, note that a connection always induces a $\bar\partial$-operator by taking its $(0,1)$ part. Conversely, a $\bar\partial$-operator gives a unique unitary connection, called the **Chern connection**, which we will sometimes denote by $d_A = (\bar\partial_E, H)$. The complex structure on X splits $\Omega^1(X)$ into $(1,0)$ and $(0,1)$ parts, and hence also splits the connections. We denote these by, for example, d'_A and d''_A, respectively. So for $d_A = (\bar\partial_E, H)$, $d''_A = \bar\partial_E$, and d'_A is determined by $\partial\langle s_1, s_2\rangle_H = \langle d'_A s_1, s_2\rangle_H$, for any pair of holomorphic sections s_1, s_2. Henceforth, I will mostly omit H from the notation if there is no chance of confusion.

Example 2.2. Let $\mathcal{L}$ be a holomorphic line bundle with hermitian metric H. For a local holomorphic frame s write $H_s = |s|^2$. Then $F_{(\bar\partial_L, H)} = \bar\partial\partial \log H_s$, and the right hand side is independent of the choice of frame.

The transition functions of a collection of local trivializations of a holomorphic line bundle on the open sets of a covering of X give a 1-cocycle with values in the sheaf $\mathcal{O}^*$ of germs of nowhere vanishing holomorphic functions. The set of isomorphism classes of line bundles is then $H^1(X, \mathcal{O}^*)$. Recall that on a compact Riemann surface every holomorphic line bundle has a meromorphic section. This gives an equivalence between the categories of holomorphic line bundles under tensor products and linear equivalence classes of divisors $\mathcal{D} = \sum_{x\in X} m_x x$ with their additive structure (here $m_x \in \mathbb{Z}$ is zero for all but finitely many $x \in X$). We shall denote by $\mathcal{O}(\mathcal{D})$ the line bundle thus associated to $\mathcal{D}$. Furthermore, a divisor has a **degree**, $\deg \mathcal{D} = \sum_{x\in X} m_x$. We define this to be the degree of $\mathcal{O}(\mathcal{D})$. Alternatively, from the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O} \xrightarrow{f\mapsto e^{2\pi i f}} \mathcal{O}^* \longrightarrow 0 \ ,$$

we have the long exact sequence in cohomology:

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow 0 \ .$$

The fundamental class of X identifies $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, and it is a standard exercise to show that under this identification: $\deg(\mathcal{D}) = c_1(\mathcal{O}(\mathcal{D}))$. For a holomorphic vector bundle $\mathcal{E}$, we declare the degree $\deg \mathcal{E} := \deg \det \mathcal{E}$. Notice that the degree is topological, i.e. it does not depend on the holomorphic structure, just on the underlying complex bundle E. By the Chern-Weil theory, for any hermitian metric H on $\mathcal{E}$ we have

$$c_1(E) = \left[\frac{\sqrt{-1}}{2\pi} \operatorname{tr} F_{(\bar\partial_E, H)}\right] = \left[\frac{\sqrt{-1}}{2\pi} F_{(\bar\partial_{\det E}, \det H)}\right] \ . \tag{2.3}$$

Complex vector bundles on Riemann surfaces are classified topologically by their rank and degree. We will also make use of the **slope** (or normalized degree) of a bundle, which is defined by the ratio $\mu(E) = \deg E / \operatorname{rank} E$.

If a line bundle $\mathcal{L} = \mathcal{O}(\mathcal{D})$ has a nonzero holomorphic section, then since $\mathcal{D}$ is linearly equivalent to an effective divisor (i.e. one with $m_x \geq 0$ for all x), $\deg \mathcal{L} \geq 0$. It follows that if $\mathcal{E}$ is a holomorphic vector bundle with a subsheaf $\mathcal{S} \subset \mathcal{E}$ and $\operatorname{rank} \mathcal{S} = \operatorname{rank} \mathcal{E}$, then $\deg \mathcal{S} \leq \deg \mathcal{E}$. Indeed, the assumption implies $\det \mathcal{E} \otimes (\det \mathcal{S})^*$ has a nonzero holomorphic section. We will use this fact later on. Notice that in the case above, $\mathcal{Q} = \mathcal{E}/\mathcal{S}$ is a torsion sheaf. In general, for any subsheaf $\mathcal{S} \subset \mathcal{E}$ of a holomorphic vector bundle, $\mathcal{S}$ is contained in a uniquely defined holomorphic subbundle $\mathcal{S}'$ of $\mathcal{E}$ called the **saturation** of $\mathcal{S}$. It is obtained by taking the kernel of the induced map $\mathcal{E} \to \mathcal{Q}/\mathrm{Tor}(\mathcal{Q}) \to 0$. From this discussion we conclude that $\deg \mathcal{S}$ is no greater than the degree $\deg \mathcal{S}'$ of its saturation.

Let ω be the Kähler form associated to a choice of conformal metric on X. This will be fixed throughout, and for convenience we normalize so that

$$\int_X \omega = 2\pi \ .$$

The contraction: $\Lambda : \Omega^2(X) \to \Omega^0(X)$, is defined by setting $\Lambda(f\omega) = f$ for any function f. For a holomorphic subbundle $\mathcal{S}$ of a hermitian holomorphic bundle $\mathcal{E}$ with projection operator π we have the following useful formula, which follows easily from direct calculation using (2.3).

$$\deg \mathcal{S} = \frac{1}{2\pi} \int_X \mathrm{tr}(\pi \sqrt{-1} \Lambda F_{(\bar\partial_E, H)})\, \omega - \frac{1}{2\pi} \int_X |\beta|^2 \,\omega \ . \tag{2.4}$$

Definition 2.3. We say that $\mathcal{E}$ is **stable** (resp. **semistable**) if for all holomorphic subbundles $\mathcal{S} \subset \mathcal{E}$, $0 < \operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$, we have $\mu(\mathcal{S}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{S}) \leq \mu(\mathcal{E})$). We call $\mathcal{E}$ **polystable** if it is a direct sum of stable bundles of the same slope.

Remark 2.4. Line bundles are trivially stable. If $\mathcal{E}$ is (semi)stable and $\mathcal{L}$ is a line bundle, then $\mathcal{E} \otimes \mathcal{L}$ is also (semi)stable.

Before giving an example, recall the notion of an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0 \ . \tag{2.5}$$

The **extension class** is the image of the identity endomorphism under the coboundary map of the long exact sequence associated to (2.5)

$$H^0(X, \mathcal{Q} \otimes \mathcal{Q}^*) \longrightarrow H^1(X, \mathcal{S} \otimes \mathcal{Q}^*) \ .$$

Notice that the isomorphism class of the bundle $\mathcal{E}$ is unchanged under scaling, so the extension class (if not zero) should be regarded as an element of the projective space $\mathbb{P}(H^1(X, \mathcal{S} \otimes \mathcal{Q}^*))$. It is then an exercise to see that in terms of the second fundamental form β, the extension class coincides (projectively) with the corresponding Dolbeault cohomology class $[\beta] \in H^{0,1}_{\bar\partial}(X, S \otimes Q^*)$. We say that (2.5) is **split** if the extension class is zero. Clearly, this occurs if and only there is an injection $\mathcal{Q} \to \mathcal{E}$ lifting the projection.

Example 2.5. Suppose $g \geq 1$. Consider extensions of the type

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(p) \longrightarrow 0 \ .$$

These are parametrized by $H^1(X, \mathcal{O}(-p)) \cong H^0(X, \mathcal{K}(p))^* \cong H^0(X, \mathcal{K})^*$, which has dimension g. Any non-split extension of this type is stable. Indeed, if $\mathcal{L} \hookrightarrow \mathcal{E}$ is a destabilizing line subbundle, then $\deg \mathcal{L} \geq 1$. The induced map $\mathcal{L} \to \mathcal{O}(p)$ cannot be zero, since then by the inclusion $\mathcal{L} \hookrightarrow \mathcal{E}$ it would lift to a nonzero map $\mathcal{L} \to \mathcal{O}$, which is impossible. Hence, $\mathcal{L} \to \mathcal{O}(p)$ must be an isomorphism. Such an $\mathcal{L}$ would therefore split the extension.

A connection is **flat** if its curvature vanishes. We say that ∇ is **projectively flat** if $\sqrt{-1}\Lambda F_\nabla = \mu$, where μ is a constant (multiple of the identity). Note that by our normalization of the area, $\mu = \mu(E)$. In Section 4, we will prove Weil's criterion for when a holomorphic bundle $\mathcal{E}$ admits a flat connection (i.e. $\nabla'' = \bar{\partial}_E$, $F_\nabla = 0$). Demanding that the connection be unitary imposes stronger conditions. This is the famous result of Narasimhan-Seshadri.

Theorem 2.6 (Narasimhan-Seshadri [50])**.** *A holomorphic bundle $\mathcal{E} \to X$ admits a projectively flat unitary connection if and only if $\mathcal{E}$ is polystable.*

In Section 2.3 we will prove Theorem 2.6 as a special case of the more general result on Higgs bundles (see Theorem 2.17).

2.1.2. *Higgs fields.* A **Higgs bundle** is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a holomorphic bundle and Φ is a holomorphic section of $\mathcal{K} \otimes \operatorname{End} \mathcal{E}$. We will sometimes regard Φ as a section of $\Omega^{1,0}(X, \mathfrak{g}_E^{\mathbb{C}})$ satisfying $\bar{\partial}_E \Phi = 0$.

Definition 2.7. We say that a pair $(\mathcal{E}, \Phi)$ is **stable** (resp. **semistable**) if for all Φ-invariant holomorphic subbundles $\mathcal{S} \subset \mathcal{E}$, $0 < \operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$, we have $\mu(\mathcal{S}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{S}) \leq \mu(\mathcal{E})$). It is **polystable** if it is a direct sum of Higgs bundles of the same slope.

The following is a simple but useful consequence of the definition and the additive properties of the slope on exact sequences.

Lemma 2.8. *Let $f : (\mathcal{E}_1, \Phi_1) \to (\mathcal{E}_2, \Phi_2)$ be a holomorphic homomorphism of Higgs bundles, $\Phi_2 f = f \Phi_1$. Suppose $(\mathcal{E}_i, \Phi_i)$ is semistable, $i = 1, 2$, and $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$. Then $f \equiv 0$. If $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$ and one of the two is stable, then either $f \equiv 0$ or f is an isomorphism.*

Proof. Consider the first statement. Then if $f \not\equiv 0$, the assumption $\Phi_2 f = f \Phi_1$ implies that the image of f is Φ_2-invariant, so by the condition on slopes f must have a kernel. But then $\ker f$ is Φ_1-invariant. So $\mu(\ker f) \leq \mu(\mathcal{E}_1) \leq \mu(\operatorname{coker} f) \leq \mu(\mathcal{E}_2)$; contradiction. The second statement follows similarly. □

A **Higgs subbundle** of $(\mathcal{E}, \Phi)$ is by definition a holomorphic subbundle $\mathcal{S} \subset \mathcal{E}$ that is Φ-invariant. The restriction $\Phi_{\mathcal{S}}$ of Φ to $\mathcal{S}$ then makes $(\mathcal{S}, \Phi_{\mathcal{S}})$ a Higgs bundle, where now the inclusion $\mathcal{S} \hookrightarrow \mathcal{E}$ gives a map of Higgs bundles. Similarly, Φ induces a Higgs bundle structure on the quotient $\mathcal{Q} = \mathcal{E}/\mathcal{S}$.

Given an arbitrary Higgs bundle, the **Harder-Narasimhan filtration of** $(\mathcal{E}, \Phi)$ is a filtration by Higgs subbundles

$$0 = (\mathcal{E}_0, \Phi_0) \subset (\mathcal{E}_1, \Phi_1) \subset \cdots \subset (\mathcal{E}_\ell, \Phi_\ell) = (\mathcal{E}, \Phi) \ ,$$

such that the quotients $(\mathcal{Q}_i, \Phi_{Q_i}) = (\mathcal{E}_i, \Phi_i)/(\mathcal{E}_{i-1}, \Phi_{i-1})$ are semistable (cf. [31]). The filtration is also required to satisfy $\mu(\mathcal{Q}_i) > \mu(\mathcal{Q}_{i+1})$, and one can show that the associated graded object $\mathrm{Gr}_{HN}(\mathcal{E}, \Phi) = \oplus_{i=1}^{\ell} (\mathcal{Q}_i, \Phi_{Q_i})$ is uniquely determined by the isomorphism class of $(\mathcal{E}, \Phi)$. The collection of slopes $\mu_i = \mu(\mathcal{Q}_i)$ is an important invariant of the isomorphism class of the Higgs bundle.

Remark 2.9. By construction, μ_i *is the maximal slope of a Higgs subbundle of* $\mathcal{E}/\mathcal{E}_{i-1}$ with its induced Higgs field. We can also interpret μ_i *as the minimal slope of a Higgs quotient of* $(\mathcal{E}_i, \Phi_i)$. Indeed, $(\mathcal{E}_1, \Phi_1)$ is semistable, so this is trivially true if $i = 1$. Suppose $\mathcal{E}_i \to \mathcal{Q} \to 0$ is a Higgs quotient for $i \geq 2$ and $\mu(\mathcal{Q}) \leq \mu_i$. If $\mathcal{Q}$ is the minimal such quotient, then it is semistable with respect to the induced Higgs field. It follows from Lemma 2.8 that the induced map $\mathcal{E}_1 \to \mathcal{Q}$ must vanish. Hence, the quotient passes to $\mathcal{E}/\mathcal{E}_1 \to \mathcal{Q} \to 0$. Now by the same argument, $\mathcal{E}_2/\mathcal{E}_1 \to \mathcal{Q}$ vanishes if $i \geq 3$. Continuing in this way, we obtain a quotient $\mathcal{Q}_i \to \mathcal{Q} \to 0$. Now since $(\mathcal{Q}_i, \Phi_{\mathcal{Q}_i})$ is semistable and the quotient is nonzero, applying Lemma 2.8 once again, we conclude that $\mu_i \leq \mu(\mathcal{Q})$.

Consider the n-tuple of numbers $\vec{\mu}(\mathcal{E}, \Phi) = (\mu_1, \ldots, \mu_n)$ obtained from the Harder-Narasimhan filtration by repeating each of the μ_i's according to the ranks of the $\mathcal{Q}_i$'s. We then get a vector $\vec{\mu}(\mathcal{E}, \Phi)$, called the **Harder-Narasimhan type** of $(\mathcal{E}, \Phi)$. There is a natural partial ordering on vectors of this type that is key to the stratification we desire. For a pair $\vec{\mu}$, $\vec{\lambda}$ of n-tuple's satisfying $\mu_1 \geq \cdots \geq \mu_n$, $\lambda_1 \geq \cdots \geq \lambda_n$, and $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i$, we define

$$\vec{\lambda} \leq \vec{\mu} \quad \Longleftrightarrow \quad \sum_{j \leq k} \lambda_j \leq \sum_{j \leq k} \mu_j \quad \text{for all } k = 1, \ldots, n \ .$$

The importance of this ordering is that it defines a stratification of the space of Higgs bundles. In particular, the Harder-Narasimhan type is upper semicontinuous. This is the direct analog of the Atiyah-Bott stratification for holomorphic bundles [2, §7].

There is a similar filtration of a semistable Higgs bundle $(\mathcal{E}, \Phi)$, where the successive quotients are stable, all with slope $= \mu(E)$. This is called the **Seshadri filtration** [55] and its associated graded $\mathrm{Gr}_S(\mathcal{E}, \Phi)$ is therefore polystable. When $\Phi \equiv 0$, we recover the usual Harder-Narasimhan and Seshadri filtrations of holomorphic bundles $\mathcal{E}$. We will denote these by $\mathrm{Gr}_{HN}(\mathcal{E})$ and $\mathrm{Gr}_S(\mathcal{E})$.

Example 2.10. Consider an extension (2.5) where $\operatorname{rank} \mathcal{S} = \operatorname{rank} \mathcal{Q} = 1$ and $\deg \mathcal{S} > \deg \mathcal{Q}$. Then the Harder-Narasimhan filtration of $\mathcal{E}$ is given by $0 \subset \mathcal{S} \subset \mathcal{E}$.

2.2. The moduli space.

2.2.1. *Gauge transformations.* Let $\mathcal{A}_E$ denote the space of unitary connections on a rank n hermitian vector bundle E. If $\mathfrak{g}_E$ denotes the associated bundle of skew-hermitian endomorphisms of E, then one observes from the Leibniz rule that $\mathcal{A}_E$ is an infinite dimensional affine space modeled on $\Omega^1(X, \mathfrak{g}_E)$. By the construction of the Chern connection discussed in Section 2.1.1, we also see that $\mathcal{A}_E$ can be identified with the space of holomorphic structures on E. We will most often be interested in the case of *fixed determinant*, i.e. where the induced holomorphic structure on $\det E$ is fixed.

The **gauge group** is defined by

$$\mathcal{G}_E = \{g \in \Omega^0(X, \operatorname{End} E) : gg^* = I\} \ .$$

In the fixed determinant case we also impose the condition that $\det g = 1$ (see Remark 2.1). The gauge group acts on $\mathcal{A}_E$ by pulling back connections: $d_{g(A)} = g \circ d_A \circ g^{-1}$. On the other hand, because of the identification with holomorphic structures we see that the complexification $\mathcal{G}_E^{\mathbb{C}}$, the **complex gauge group**, also acts on $\mathcal{A}_E$. Explicitly, if $\bar\partial_E = d''_A$, then $g(A)$ is the Chern connection of $g \circ \bar\partial_E \circ g^{-1}$.

The space of Higgs bundles is

$$\mathcal{B}_E = \{(A, \Phi) \in \mathcal{A}_E \times \Omega^0(X, K \otimes \mathfrak{g}_E^{\mathbb{C}}) : d''_A \Phi = 0\} \ .$$

Let $\mathcal{B}_E^{ss} \subset \mathcal{B}_E$ denote the subset of semistable Higgs bundles.

Definition 2.11. The moduli space of rank n semistable Higgs bundles (with fixed determinant) on X is $\mathfrak{M}_D^{(n)} = \mathcal{B}_E^{ss} /\!\!/ \mathcal{G}_E^{\mathbb{C}}$, where the double slash means that the orbits of $(\mathcal{E}, \Phi)$ and $\mathrm{Gr}_S(\mathcal{E}, \Phi)$ are identified.

We have not been careful about topologies. In fact, $\mathfrak{M}_D^{(n)}$ can be given the structure of a (possibly nonreduced) complex analytic space using the Kuranishi map (cf. [45]). An algebraic construction using geometric invariant theory is given in [61].

A second comment is that $\mathcal{G}_E^{\mathbb{C}}/\mathcal{G}_E$ may be identified with the space of hermitian metrics on E. This leads to an important interpretation when studying the behavior of functionals along $\mathcal{G}_E^{\mathbb{C}}$ orbits in $\mathcal{A}_E/\mathcal{G}_E$: we may either think of varying the complex structure $g(\bar\partial_E)$ with a fixed hermitian metric, or we may keep $\bar\partial_E$ fixed and vary the metric H by $\langle s_1, s_2\rangle_{g(H)} = \langle gs_1, gs_2\rangle_H$.

2.2.2. *Deformations of Higgs bundles.* Let $D'' = d''_A + \Phi$, $D' = d'_A + \Phi^*$. The metric ω on X and the hermitian metric on E define L^2-inner products on forms with values in E and $\operatorname{End} E$. We have the Kähler identities

$$\begin{aligned} (D'')^* &= -\sqrt{-1}[\Lambda, D'] \ ; \\ (D')^* &= \sqrt{-1}[\Lambda, D''] \ , \end{aligned} \tag{2.6}$$

(see [26, p. 111] for the case $\Phi = 0$; the case $\Phi \neq 0$ follows by direct computation).

The infinitesimal structure of the moduli space is governed by a deformation complex $C(A, \Phi)$, which is obtained by differentiating the condition $d''_A\Phi = 0$ and the action of the gauge group.

$$C(A, \Phi) \ : \ 0 \longrightarrow \Omega^0(X, \mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{1,0}(X, \mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{0,1}(X, \mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{1,1}(X, \mathfrak{g}_E^{\mathbb{C}}) \to 0 \ . \tag{2.7}$$

Note that the holomorphicity condition on Φ guarantees that $(D'')^2 = 0$. Serre duality gives an isomorphism $H^0(C(A,\Phi)) \simeq H^2(C(A,\Phi))$. We call a Higgs bundle **simple** if $H^0(C(A,\Phi)) \simeq \mathbb{C}$ (or $\{0\}$ in the fixed determinant case).

Remark 2.12. A stable Higgs bundle is necessarily simple. Indeed, if $\phi \in \ker D''$, then ϕ is a holomorphic endomorphism of $\mathcal{E}$ commuting with Φ. In particular, $\det\phi$ is a holomorphic function and is therefore constant. Also, $\ker\phi$ is Φ-invariant. If ϕ is nonzero but not an isomorphism

$$0 \longrightarrow \ker\phi \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\ker\phi \longrightarrow 0 \ .$$

Since $\mathcal{E}/\ker\phi$ is also a subsheaf of $\mathcal{E}$, stability implies both $\mu(\ker\phi)$ and $\mu(\mathcal{E}/\ker\phi)$ are both less than $\mu(E)$, which is a contradiction. Hence, ϕ is either zero or an isomorphism. But applying the same argument to $\phi-\lambda$ for any scalar λ, we conclude that ϕ is a multiple of the identity.

Proposition 2.13. *At a simple Higgs bundle $[A,\Phi]$, $\mathfrak{M}_D^{(n)}$ is smooth of complex dimension $(n^2-1)(2g-2)$, and the tangent space may be identified with*

$$H^1(C(A,\Phi)) \simeq \left\{(\varphi,\beta) : d''_A\varphi = -[\Phi,\beta]\ ,\ (d''_A)^*\beta = \sqrt{-1}\Lambda[\Phi^*,\varphi]\right\} \ . \tag{2.8}$$

Example 2.14. (cf. [36, 39]) We now give important examples of stable Higgs bundles; namely, the *Fuchsian* ones. First for rank 2. Fix a choice of square root $\mathcal{K}^{1/2}$ of the canonical bundle, and let $\mathcal{E} = \mathcal{K}^{1/2} \oplus \mathcal{K}^{-1/2}$. Then the part of the endomorphism bundle that sends $\mathcal{K}^{1/2} \to \mathcal{K}^{-1/2}$ is isomorphic to $\mathcal{K}^{-1}$. Tensoring by $\mathcal{K}$, it becomes trivial. Hence, the

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ ,$$

makes sense as a Higgs field, and it is clearly holomorphic. While $\mathcal{E}$ is unstable as a holomorphic vector bundle the Higgs bundle $(\mathcal{E},\Phi)$ is stable, since the only Φ-invariant sub-line bundle is $\mathcal{K}^{-1/2}$ which has negative degree. Let us remark in passing that if we consider a different holomorphic structure $\mathcal{V}$ on E given by the $\bar\partial$-operator

$$\bar\partial_E + \Phi^* = \begin{pmatrix} \bar\partial_{\mathcal{K}^{1/2}} & \omega \\ 0 & \bar\partial_{\mathcal{K}^{-1/2}} \end{pmatrix} \ ,$$

then $\mathcal{V}$ is the unique (up to isomorphism) non-split extension

$$0 \longrightarrow \mathcal{K}^{1/2} \longrightarrow \mathcal{V} \longrightarrow \mathcal{K}^{-1/2} \longrightarrow 0 \ .$$

We now compute the tangent space $\mathfrak{M}_D^{(2)}$ at $[(\mathcal{E},\Phi)]$. Write

$$\beta = \begin{pmatrix} b & b_1 \\ b_2 & -b \end{pmatrix} \qquad , \qquad \varphi = \begin{pmatrix} \phi & \phi_1 \\ \phi_2 & -\phi \end{pmatrix} \ ,$$

and compute

$$[\Phi,\beta] = \begin{pmatrix} -b_1 & 0 \\ 2b & b_1 \end{pmatrix} \qquad , \qquad \sqrt{-1}\Lambda[\Phi^*,\varphi] = \begin{pmatrix} \phi_2 & -2\phi \\ 0 & -\phi_2 \end{pmatrix} \ .$$

Then the conditions (2.8) that (β,φ) define a tangent vector are

$$\bar\partial_E\varphi = \begin{pmatrix} b_1 & 0 \\ -2b & -b_1 \end{pmatrix} \qquad , \qquad \bar\partial_E^*\beta = \begin{pmatrix} \phi_2 & -2\phi \\ 0 & -\phi_2 \end{pmatrix} \ .$$

In particular, $\phi_1 \in H^0(X, \mathcal{K}^2)$ and $b_2 \in H^{0,1}_{\bar\partial}(X, K^*) \simeq H^0(X, \mathcal{K}^2)^*$. I claim that the other entries vanish. Indeed, the equations for ϕ and b_1 are $\bar\partial\phi = b_1$, and $\bar\partial^* b_1 = -2\phi$. But this implies $(\bar\partial^*\bar\partial + 2)\phi = 0$. Hence, ϕ, and therefore also b_1, must vanish. The same argument works for ϕ_2 and b. We therefore have an isomorphism

$$T_{[\mathcal{E}_F, \Phi_F]}\mathfrak{M}_D^{(2)} \simeq H^0(X, \mathcal{K}^2) \oplus (H^0(X, \mathcal{K}^2))^* \ .$$

For $n \geq 2$, there is a similar argument. Here we take

$$\mathcal{E}_F = \mathcal{K}^{(n-1)/2} \oplus \mathcal{K}^{(n-3)/2} \oplus \cdots \oplus \mathcal{K}^{-(n-1)/2} \ ,$$

and

$$\Phi_F = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \vdots \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \ .$$

Notice that with respect to this splitting the (ij) entry of φ is a section of $\mathcal{K}^{j-i+1}$, and the (ij) entry of β is in $\Omega^{0,1}(X, K^{j-i})$. We obtain the following equations on the entries of a tangent vector (β, φ),

$$\begin{aligned} \bar\partial_E \varphi_{ij} &= \beta_{i-1,j} - \beta_{i,j+1} \ ; \\ \bar\partial_E^* \beta_{ij} &= \varphi_{i,j-1} - \varphi_{i+1,j} \ , \end{aligned} \tag{2.9}$$

where it is understood that terms with indices ≤ 0 or $\geq n+1$ are set to zero. Upon further differentiation as in the $n = 2$ case, we find

$$\begin{aligned} (L - \delta_{i1} - \delta_{jn})\varphi_{ij} &= \varphi_{i+1,j+1} + \varphi_{i-1,j-1} \ ; \\ (\widetilde{L} - \delta_{in} - \delta_{j1})\beta_{ij} &= \beta_{i+1,j+1} + \beta_{i-1,j-1} \ , \end{aligned} \tag{2.10}$$

where $L = \bar\partial_E^*\bar\partial_E + 2$ and $\widetilde{L} = \bar\partial_E\bar\partial_E^* + 2$. I claim that $\varphi_{ij} = 0$ (resp. $\beta_{ij} = 0$) for $i \geq j$ (resp. $i \leq j$). For example, by (2.10), $L\varphi_{n1} = 0$, and since L is a positive operator, φ_{n1} vanishes. More generally, fix $0 \leq p \leq n-2$. Then for $0 \leq \ell \leq n-p-1$, there are polynomials P_ℓ such that

$$\varphi_{p+\ell+1,\ell+1} = P_\ell(L)\varphi_{p+1,1} \ . \tag{2.11}$$

Indeed, let $P_0(L) = 1$, $P_1(L) = L$ if $p \neq 0$ and $P_1(L) = L - 1$ if $p = 0$. Suppose $P_k(L)$ has been defined for $0 \leq k \leq \ell$, where $0 < \ell < n-p-1$. Use (2.10) and (2.11) to find:

$$\begin{aligned} L\varphi_{p+\ell+1,\ell+1} &= \varphi_{p+\ell+2,\ell+2} + \varphi_{p+\ell,\ell} \\ LP_\ell(L)\varphi_{p+1,1} &= \varphi_{p+\ell+2,\ell+2} + P_{\ell-1}(L)\varphi_{p+1,1} \ . \end{aligned}$$

Hence, we let $P_{\ell+1}(L) = LP_\ell(L) - P_{\ell-1}(L)$. Since $L \geq 2$, we see from the recursive definition that $P_{\ell+1}(L) \geq P_\ell(L)$, and hence for all $\ell \geq 1$, $P_\ell(L) \geq P_1(L) \geq 1$, and ≥ 2 if $p \neq 0$. Taking $\ell = n-p-1$ in (2.11), we have

$$\varphi_{n,n-p} = P_{n-p-1}(L)\varphi_{p+1,1} \ . \tag{2.12}$$

On the other, a similar argument implies $\varphi_{p+1,1} = P_{n-p-1}(L)\varphi_{n,n-p}$, from which we obtain

$$0 = (P^2_{n-p-1}(L) - 1)\varphi_{p+1,1} = (P_{n-p-1}(L) + 1)(P_{n-p-1}(L) - 1)\varphi_{p+1,1} \ .$$

Hence, $\varphi_{p+1,1}$ is in the kernel of $P_{n-p-1}(L) - 1$. But then by the remark above, for $p \geq 1$, $\varphi_{p+1,1}$ must vanish. Since $p \geq 1$ is arbitrary, this implies by (2.11) that $\varphi_{ij} = 0$ for all $i > j$. In the case $p = 0$, notice that for all $\ell \geq 1$, $P_\ell(L)$ is a polynomial of positive degree in $\bar\partial_E^*\bar\partial_E$ with nonnegative coefficients and constant term $= 1$. Indeed, by the definition

$$P_{\ell+1}(L) - P_\ell(L) = (\bar\partial_E^*\bar\partial_E)P_\ell(L) + P_\ell(L) - P_{\ell-1}(L) \ ,$$

and so by induction $P_{\ell+1}(L) - P_\ell(L)$ has nonnegative coefficients and zero constant term. In this case, $(P_{n-1}(L) - 1)\varphi_{1,1} = 0$ implies that $\varphi_{1,1}$ is holomorphic. Using (2.11) again,

$$\varphi_{\ell+1,\ell+1} = P_\ell(L)\varphi_{1,1} = (P_\ell(L) - 1)\varphi_{1,1} + \varphi_{1,1} = \varphi_{1,1} \ ,$$

for all $\ell = 0, \ldots, n-1$. But since (φ_{ij}) is traceless, it follows that in fact $\varphi_{ii} = 0$ for all i. The proof for β_{ij} is exactly similar.

Going back to (2.9), we see that φ_{ij} (resp. β_{ji}) is holomorphic (resp. harmonic) if $i < j$. Moreover, for $p \geq 1$, (2.10) becomes

$$(2 - \delta_{i1} - \delta_{in-p})\varphi_{i,i+p} = \varphi_{i+1,i+1+p} + \varphi_{i-1,i-1+p} \ . \tag{2.13}$$

If $i = 1$ this implies $\varphi_{1,p+1} = \varphi_{2,p+2}$. Suppose by induction that $\varphi_{k,k+p} = \varphi_{1,p+1}$ for all $k \leq i$. Then if $i + p \neq n$, (2.13) implies

$$2\varphi_{i,i+p} = \varphi_{i+1,i+1+p} + \varphi_{i-1,i-1+p} \quad \Longrightarrow \quad \varphi_{1,p+1} = \varphi_{i+1,i+1+p} \ .$$

If $i + p = n$, we immediately get $\varphi_{in} = \varphi_{i-1,n-1} = \varphi_{1,p+1}$. Hence, all differentials φ_{ij}, $j - i = p$, are equal. The same argument applies to β_{ij}. From this we conclude that the map $(\varphi, \beta) \mapsto (\varphi_{12}, \ldots, \varphi_{1n}, \beta_{21}, \ldots, \beta_{n1})$ gives an isomorphism

$$T_{[\mathcal{E}_F,\Phi_F]}\mathfrak{M}_D^{(n)} \simeq \bigoplus_{j=2}^n H^0(X, \mathcal{K}^j) \oplus (H^0(X, \mathcal{K}^j))^* \ . \tag{2.14}$$

The rank n holomorphic vector bundle $\mathcal{V}$ whose $\bar\partial$-operator is $\bar\partial_E + \Phi_F^*$ is unstable and has a Harder-Narasimhan filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$, $\mathcal{V}_{j+1}/\mathcal{V}_j = \mathcal{K}^{-j+(n-1)/2}$, such that

$$0 \longrightarrow \mathcal{V}_j \longrightarrow \mathcal{V}_{j+1} \longrightarrow \mathcal{K}^{-j+(n-1)/2} \longrightarrow 0 \ .$$

is the (unique) non-split extension. This is an example of an *oper*. Opers will be discussed in Section 4.3.

2.2.3. *The Hitchin map.* Given a Higgs bundle $(\mathcal{E}, \Phi)$, the coefficient of λ^{n-i} in the expansion of $\det(\lambda + \Phi)$ is a holomorphic section of $\mathcal{K}^i$, $i = 1, \ldots, n$. In the case of fixed determinant that we will mostly be considering, $\operatorname{tr}\Phi = 0$, so the sections start with $i = 2$. These pluricanonical sections are clearly invariant under the action (by conjugation) of $\mathcal{G}_E^{\mathbb{C}}$, so we have a well-defined map, called the **Hitchin map**,

$$h : \mathfrak{M}_D^{(n)} \longrightarrow \bigoplus_{i=2}^{n} H^0(X, \mathcal{K}^i) \ . \tag{2.15}$$

The structure of this map and its fibers turns out be extremely rich (cf. [37]). In these notes, however, I will only discuss the following important fact which will be proven in the next section using Uhlenbeck compactness (for algebraic proofs, see [51, 60]).

Theorem 2.15. *The Hitchin map is proper.*

2.3. **The Hitchin-Kobayashi correspondence.**

2.3.1. *Stability and critical metrics.* **Hitchin's equations** for Higgs bundles on a trivial bundle are

$$F_A + [\Phi, \Phi^*] = 0 \ . \tag{2.16}$$

Here, Φ is regarded as an endomorphism valued $(1,0)$-form. It will also be convenient to consider the case of bundles of nonzero degree. In this case the equations become

$$f_{(A,\Phi)} := \sqrt{-1}\Lambda(F_A + [\Phi, \Phi^*]) = \mu \ . \tag{2.17}$$

Here we recall the normalization $\mathrm{vol}(X) = 2\pi$, and then on right hand side the scalar multiple of the identity endomorphism necessarily satisfies $\mu = \mu(E)$.

There are two ways of thinking of (2.17): for a Higgs bundle $(\mathcal{E}, \Phi)$ a choice of hermitian metric gives a Chern connection $A = (\bar\partial_E, H)$. Hence, we may either view (2.17) as an equation for a hermitian metric H, or alternatively (and equivalently) we may fix H and consider $f_{(A,\Phi)}$ for all (A, Φ) in a complex gauge orbit. We will often go back and forth between these equivalent points of view.

The solutions to the equations (2.17) may be regarded as the absolute minimum for the **Yang-Mills-Higgs functional** on the space of holomorphic pairs, defined as

$$\mathrm{YMH}(A, \Phi) = \int_X |F_A + [\Phi, \Phi^*]|^2 \ \omega \ . \tag{2.18}$$

The Euler-Lagrange equations for YMH are

$$d_A f_{(A,\Phi)} = 0 \ , \ [\Phi, f_{(A,\Phi)}] = 0 \ . \tag{2.19}$$

We call a metric **critical** if (2.19) is satisfied. In this case, it is easy to see the bundle $(\mathcal{E}, \Phi)$ splits holomorphically and isometrically as a direct sum of Higgs bundles that are solutions to (2.17) with possibly different slopes.

Proposition 2.16. *If a Higgs bundle* $(\mathcal{E}, \Phi)$ *admits a metric satisfying* (2.17)*, then* $(\mathcal{E}, \Phi)$ *is polystable.*

Proof. Let $\mathcal{S} \subset \mathcal{E}$ be a proper Φ-invariant subbundle. Let π denote the orthogonal projection to S and $\beta = -\bar{\partial}_E \pi$ the second fundamental form. Then since $\mathcal{S}$ is invariant, $(I - \pi)\Phi\pi = 0$, or

$$\Phi\pi = \pi\Phi\pi \ , \ \pi\Phi^* = \pi\Phi^*\pi \ .$$

In particular, this implies

$$\begin{aligned}
\operatorname{tr}(\pi[\Phi, \Phi^*]) &= \operatorname{tr}(\pi\Phi\Phi^*) + \operatorname{tr}(\pi\Phi^*\Phi) \\
&= \operatorname{tr}(\pi\Phi\Phi^*) - \operatorname{tr}(\Phi\pi\Phi^*) \\
&= \operatorname{tr}(\pi\Phi\Phi^*\pi) - \operatorname{tr}(\Phi\pi\Phi^*\pi) \\
&= \operatorname{tr}(\pi\Phi\Phi^*\pi) - \operatorname{tr}(\pi\Phi\pi\Phi^*\pi) \\
&= \operatorname{tr}(\pi\Phi(I - \pi)\Phi^*\pi) = \operatorname{tr}(\pi\Phi(I - \pi)(I - \pi)\Phi^*\pi) \\
&= \operatorname{tr}\left\{(\pi\Phi(I - \pi))(\pi\Phi(I - \pi))^*\right\} \ ;
\end{aligned}$$

$$\operatorname{tr}(\pi\sqrt{-1}\Lambda[\Phi, \Phi^*]) = |\pi\Phi(I - \pi)|^2 \ . \tag{2.20}$$

Plugging (2.17) into (2.4), and using (2.20), we have

$$\deg \mathcal{S} = \operatorname{rank}(\mathcal{S})\mu(\mathcal{E}) - \frac{1}{2\pi}(\|\pi\Phi(I - \pi)\|^2 + \|\beta\|^2) \ .$$

This proves $\mu(\mathcal{S}) \leq \mu(\mathcal{E})$. Moreover, equality holds if and only if the two terms on the right hand side above vanish; i.e. the holomorphic structure and Higgs field split. □

The main result we prove in this section is the converse to Proposition 2.16.

Theorem 2.17 (Hitchin [36], Simpson [58])**.** *If* $(\mathcal{E}, \Phi)$ *is polystable, then it admits a metric satisfying* (2.17)*.*

Remark 2.18. The result is straightforward in the case of line bundles $\mathcal{L}$. Indeed, in rank 1 the term $[\Phi, \Phi^*]$ vanishes, so (2.17) amounts to finding a constant curvature metric on L. If H is any metric, let $H_\varphi = e^\varphi H$ for a function φ. Then $F_{(\bar{\partial}_L, H_\varphi)} = F_{(\bar{\partial}_L, H)} + \bar{\partial}\partial\varphi$, and the problem is solved if we can find φ such that

$$\Delta\varphi = 2\sqrt{-1}\Lambda(F_{(\bar{\partial}_L, H)}) - 2\deg(L) \ .$$

By the Hodge theorem the only condition to finding a solution to this equation is that the integral of the right hand side vanish (cf. [26, p. 84]), which it does by (2.3).

In order to prove Theorem 2.17 in higher rank, it will be important to construct approximate critical metrics. Let $0 \subset (\mathcal{E}_1, \Phi_1) \subset \cdots \subset (\mathcal{E}_\ell, \Phi_\ell) = (\mathcal{E}, \Phi)$ be the Harder-Narasimhan filtration of the Higgs bundle $(\mathcal{E}, \Phi)$. We let $\mathcal{Q}_i = \mathcal{E}_i / \mathcal{E}_{i-1}$ and $\mu_i = \mu(\mathcal{Q}_i)$. Then there is a smooth splitting $E = \bigoplus_i Q_i$, and given a hermitian metric H we can make this splitting orthogonal. Hence, there is a well-defined endomorphism

$$\mu_{(\mathrm{Gr}(\mathcal{E}, \Phi), H)} = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_\ell \end{pmatrix} \ . \tag{2.21}$$

where the blocks μ_i have dimensions rank Q_i.

Definition 2.19. We say that a metric on $(\mathcal{E}, \Phi)$ is ε-approximate critical if

$$\sup \left| f_{((\bar\partial_E, H),\Phi)} - \mu_{(\mathrm{Gr}(\mathcal{E},\Phi),H)} \right| < \varepsilon \ .$$

Note that the $\bar\partial$-operator for $\mathcal{E}$ may be written in an upper triangular form with respect to this splitting, and the strictly upper triangular piece is determined by the extension classes. By acting with a complex gauge transformation that is block diagonal, the extension classes may be made arbitrarily small. If moreover the bundles $\mathcal{Q}_i$ with their induced Higgs fields admit Hermitian-Yang-Mills-Higgs connections, then we can sum these up and obtain the following (for more details, see [14]).

Lemma 2.20. *Let $(\mathcal{E}, \Phi)$ be an unstable Higgs bundle of rank n, and suppose that Theorem 2.17 has been proven for Higgs bundles of rank less than n. Then for any $\varepsilon > 0$ there is an ε-approximate critical metric on $(\mathcal{E}, \Phi)$.*

2.3.2. *Preliminary estimates.* Recall the map (2.15). A crucial point is the following a priori estimate.

Proposition 2.21. *Let $(\mathcal{E}, \Phi)$ be a Higgs bundle. There are constants $C_1, C_2 > 0$ depending only on the metrics on X and E, and on $\|h[\mathcal{E}, \Phi]\|$, such that*

$$\sup |\Phi|^2 \le C_1 + C_2 \sup \left|\sqrt{-1}\Lambda(F_A + [\Phi, \Phi^*])\right| \ .$$

We need the following

Lemma 2.22 (cf. [60, p. 27])**.** *For a matrix P there are constants $C_1, C_2 > 0$ depending only on the eigenvalues of P such that*

$$|[P, P^*]|^2 \ge C_1 |P|^4 - C_2(1 + |P|^2) \ .$$

Proof. Choose a unitary basis such that $P = S + N$, where S is diagonal and N is strictly upper triangular. By assumption, $|S|$ is bounded. It is easy to see that it then suffices to show there is $C > 0$ such that for all strictly upper triangular N, $|[N, N^*]| \ge C|N|^2$. Suppose not. Then by scaling we can find a sequence N_j, $|N_j| = 1$, and $[N_j, N_j^*] \to 0$. After passing to a subsequence, we may assume $N_j \to N$, with $[N, N^*] = 0$, $|N| = 1$. But this is a contradiction. Indeed, if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are the rows and columns of N, then reading off the diagonal of $NN^* = N^*N$ implies $|a_i|^2 = |b_i|^2$ for $i = 1, \ldots, n$. But $b_1 = 0$, which from this equality implies $a_1 = 0$. This in turn implies $b_2 = 0$, and hence $a_2 = 0$. Continuing in this way, we conclude $N = 0$; contradiction. □

We will also need the following computation.

$$
\begin{aligned}
[[P,P^*],P] &= (PP^*-P^*P)P - P(PP^*-P^*P) \\
&= 2PP^*P - P^*P^2 - P^2P^* \\
\langle [[P,P^*],P],P\rangle = \mathrm{tr}([[P,P^*],P]P^*) &= \mathrm{tr}((2PP^*P - P^*P^2 - P^2P^*)P^*) \\
&= 2\,\mathrm{tr}(PP^*)^2 - 2\,\mathrm{tr}(P^2(P^*)^2) \\
\langle \mathrm{ad}([P,P^*])P,P\rangle &= |[P,P^*]|^2\ . \qquad (2.22)
\end{aligned}
$$

Proof of Proposition 2.21. Regard Φ as a holomorphic section of $\mathcal{K}\otimes\mathrm{End}\,\mathcal{E}$. We also make use of three easy facts. First, if H is a hermitian metric on E and $\widehat{H}$ is the induced metric on $\mathrm{End}\,E$, then $F_{(\mathrm{End}\,\mathcal{E},\widehat{H})} = \mathrm{ad}\,F_{(\mathcal{E},H)}$, where the adjoint indicates that the curvature endomorphism acts by commutation. Second, if $\widehat{H}, h$ are hermitian metrics on $\mathrm{End}\,E$ and K, respectively, then

$$
F_{(\mathcal{K}\otimes\mathrm{End}\,\mathcal{E},h\otimes\widehat{H})} = F_{(\mathrm{End}\,\mathcal{E},\widehat{H})} + F_{(\mathcal{K},h)}\cdot I\ . \qquad (2.23)
$$

Third, if s is a holomorphic section of a vector bundle with unitary connection A and curvature F_A, then we have the following Weitzenböck formula:

$$
\Delta|s|^2 = 2|d_As|^2 - 2\langle\sqrt{-1}\Lambda F_As,s\rangle\ . \qquad (2.24)
$$

Indeed (cf. (2.6)),

$$
\begin{aligned}
\Delta|s|^2 = -2\bar\partial^*\bar\partial|s|^2 = 2\sqrt{-1}\Lambda\partial\bar\partial|s|^2 &= 2\sqrt{-1}\Lambda\partial\langle s,d'_As\rangle \\
&= 2\sqrt{-1}\Lambda\langle d'_As,d'_As\rangle + 2\sqrt{-1}\Lambda\langle s,d''_Ad'_As\rangle \\
&= 2|d'_As|^2 + 2\sqrt{-1}\Lambda\langle s,F_As\rangle \\
&= 2|d_As|^2 - 2\langle\sqrt{-1}\Lambda F_As,s\rangle\ .
\end{aligned}
$$

Now using eqs. (2.22), (2.23), and (2.24), along with Lemma 2.22, we have

$$
\begin{aligned}
\Delta|\Phi|^2 &\geq -2\langle\sqrt{-1}\Lambda F_{(\mathcal{K}\otimes\mathrm{End}\,\mathcal{E},h\otimes\widehat{H})}\Phi,\Phi\rangle \\
&\geq -2\langle\sqrt{-1}\Lambda F_{(\mathrm{End}\,\mathcal{E},\widehat{H})}\Phi,\Phi\rangle - C_3|\Phi|^2 \\
&= -2\langle\mathrm{ad}(\sqrt{-1}\Lambda F_{(\mathcal{E},H)})\Phi,\Phi\rangle - C_3|\Phi|^2 \\
&= 2\langle\mathrm{ad}(\sqrt{-1}\Lambda[\Phi,\Phi^*])\Phi,\Phi\rangle - 2\langle\mathrm{ad}(\sqrt{-1}\Lambda(F_{(\mathcal{E},H)}+[\Phi,\Phi^*])\Phi,\Phi\rangle - C_3|\Phi|^2 \\
&\geq C_1|\Phi|^4 - C_2(1+|\Phi|^2) - C_4\sup\left|\sqrt{-1}\Lambda(F_{(\mathcal{E},H)}+[\Phi,\Phi^*]\right|\,|\Phi|^2\ .
\end{aligned}
$$

Now at a maximum of $|\Phi|^2$ the left hand side is nonpositive. Since $C_1 > 0$, the proposition follows immediately. □

Remark 2.23. Notice that the sign in (2.16) is decisive for this argument (cf. [38]).

Finally, the existence proof will be based on Donaldson's elegant argument in [18]. This requires the introduction of the functional $J = J(A,\Phi)$, defined as follows. For a hermitian endomorphism

ϕ of E, let

$$\nu(\phi)=\sum_{i=1}^{n}|\lambda_i| \quad , \quad N^2(\phi)=\int_X \nu^2(\phi)\,\frac{\omega}{2\pi} \ ,$$

where the λ_i are the (pointwise) eigenvalues of ϕ. Then we define

$$J(A,\Phi)=N(f_{(A,\Phi)}-\mu(E)) \ . \tag{2.25}$$

We next prove the following two results of Donaldson (see [18, Lemmas 2 & 3]), adapted here to the case of Higgs bundles.

Lemma 2.24. *Let (A,Φ) be a Higgs bundle with underlying bundle $\mathcal{E}$. Suppose it fits into an extension of Higgs bundles $0\to\mathcal{M}\to\mathcal{E}\to\mathcal{N}\to 0$, and that $\mu(\mathcal{N})\leq\mu(\mathcal{E})\leq\mu(\mathcal{M})$. Then*

$$(\operatorname{rank}\mathcal{M})(\mu(\mathcal{M})-\mu(\mathcal{E}))+(\operatorname{rank}\mathcal{N})(\mu(\mathcal{E})-\mu(\mathcal{N}))\leq J(A,\Phi) \ .$$

Proof. With respect to the orthogonal splitting $E=M\oplus N$, and letting F_E, F_M, and F_N denote the curvature and induced curvatures of the Chern connection for $(\mathcal{E},H)$, we have

$$\sqrt{-1}\Lambda F_E=\begin{pmatrix}\sqrt{-1}\Lambda F_M+b_M & -(d''_A)^*\beta \\ -((d''_A)^*\beta)^* & \sqrt{-1}\Lambda F_N+b_N\end{pmatrix} \ ,$$

where β is the second fundamental form, and

$$b_M=-\sqrt{-1}\Lambda(\beta\wedge\beta^*) \ , \ b_N=-\sqrt{-1}\Lambda(\beta^*\wedge\beta) \ .$$

Notice that $\operatorname{tr} b_M=-\operatorname{tr} b_N=|\beta|^2$. Similarly, if we write $\Phi=\begin{pmatrix}\Phi_M & \varphi \\ 0 & \Phi_N\end{pmatrix}$, then

$$[\Phi,\Phi^*]=\begin{pmatrix}[\Phi_M,\Phi_M^*]+\varphi\wedge\varphi^* & \varphi\wedge\Phi_N^*+\Phi_M^*\wedge\varphi \\ \Phi_N\wedge\varphi^*+\varphi^*\wedge\Phi_M & [\Phi_N,\Phi_N^*]+\varphi^*\wedge\varphi\end{pmatrix} \ .$$

It follows that

$$f_{(A,\Phi)}=\begin{pmatrix}f_M+b_M+\sqrt{-1}\Lambda\varphi\wedge\varphi^* & \cdots \\ \cdots & f_N+b_N+\sqrt{-1}\Lambda\varphi^*\wedge\varphi\end{pmatrix} \ .$$

Hence, (cf. [18, p. 271]),

$$\begin{aligned}\nu(f_{(A,\Phi)}-\mu(E))\geq &\left|\operatorname{tr}(\sqrt{-1}\Lambda F_M)-(\operatorname{rank}\mathcal{M})\mu(\mathcal{E})+|\beta|^2+|\varphi|^2\right| \\ &+\left|\operatorname{tr}(\sqrt{-1}\Lambda F_N)-(\operatorname{rank}\mathcal{N})\mu(\mathcal{E})-|\beta|^2-|\varphi|^2\right| \ ,\end{aligned}$$

and therefore

$$\begin{aligned}J(A,\Phi)&\geq\int_X \nu(f_{(A,\Phi)}-\mu(\mathcal{E}))\frac{\omega}{2\pi} \\ &\geq\left|\int_X\left(\operatorname{tr}(\sqrt{-1}\Lambda F_M)-(\operatorname{rank}\mathcal{M})\mu(\mathcal{E})+|\beta|^2+|\varphi|^2\right)\frac{\omega}{2\pi}\right| \\ &\quad+\left|\int_X\left(\operatorname{tr}(\sqrt{-1}\Lambda F_N)-(\operatorname{rank}\mathcal{N})\mu(\mathcal{E})-|\beta|^2-|\varphi|^2\right)\frac{\omega}{2\pi}\right| \\ &\geq(\operatorname{rank}\mathcal{M})(\mu(\mathcal{M})-\mu(\mathcal{E}))+(\operatorname{rank}\mathcal{N})(\mu(\mathcal{E})-\mu(\mathcal{N})) \ .\end{aligned}$$

□

Lemma 2.25. *Let (A_0,Φ_0) be a stable Higgs bundle of rank n that fits into an extension of Higgs bundles $0\to\mathcal{S}\to\mathcal{E}\to\mathcal{Q}\to 0$. Assume Theorem 2.17 has been proven for Higgs bundles of rank less than n. Then we can choose a point (A,Φ) in the complex gauge orbit of (A_0,Φ_0) such that*

$$J(A,\Phi) < (\operatorname{rank}\mathcal{S})(\mu(\mathcal{E})-\mu(\mathcal{S})) + (\operatorname{rank}\mathcal{Q})(\mu(\mathcal{Q})-\mu(\mathcal{E}))\ .$$

Proof. First, consider the Harder-Narasimhan filtrations of $(\mathcal{S},\Phi_S)$ and $(\mathcal{Q},\Phi_Q)$. By applying Lemma 2.20 we may assume for any $\varepsilon>0$ that there is a metric on S such that

$$\sup\left|f_{((\bar\partial_S,H_S),\Phi_S)} - \mu_{(\mathrm{Gr}(\mathcal{S},\Phi_S),H_S)}\right| < \varepsilon\ ,$$

and similarly for Q. We endow $E=S\oplus Q$ with the sum of these two metrics. This is equivalent to a pair (A,Φ) in the orbit of (A_0,Φ_0). Next, since (A_0,Φ_0) (and hence also (A,Φ)) is simple we may further assume that

$$-\bar\partial^*_{A_0}\beta + \sqrt{-1}\Lambda\left(\varphi\wedge\Phi_Q^* + \Phi_S^*\wedge\varphi\right) = 0$$

(see (2.8)). This is accomplished via a complex gauge transformation of the form $g=\begin{pmatrix}1 & \phi\\ 0 & 1\end{pmatrix}$. In particular, the $\bar\partial$-operators on S and Q remain unchanged, and so the approximate critical structure still holds. With this understood, we perform a further gauge transformation so that (A,Φ) coincides with (A_0,Φ_0) but with β and φ scaled by t. Then $f_{(A,\Phi)}-\mu(E)$ is block diagonal with entries

$$\begin{aligned} &f_S - \mu_{(\mathrm{Gr}(\mathcal{S},\Phi_S),H_S)} + \mu_{(\mathrm{Gr}(\mathcal{S},\Phi_S),H_S)} - \mu(E) + t^2\left(b_S + \sqrt{-1}\Lambda\varphi\wedge\varphi^*\right)\ ;\\ &f_Q - \mu_{(\mathrm{Gr}(\mathcal{Q},\Phi_Q),H_Q)} + \mu_{(\mathrm{Gr}(\mathcal{Q},\Phi_Q),H_Q)} - \mu(E) + t^2\left(b_Q + \sqrt{-1}\Lambda\varphi^*\wedge\varphi\right)\ . \end{aligned} \tag{2.26}$$

Since $(\mathcal{E},\Phi)$ is stable, $\mu(\mathcal{E})$ is strictly bigger than the maximal slope of a subsheaf of $\mathcal{S}$, and strictly smaller than the minimal slope of a quotient of $\mathcal{Q}$. This says that for t and ε chosen sufficiently small, the first line in (2.26) is negative definite and the second is positive definite. It follows that

$$\nu(f_{(A,\Phi)}-\mu(E)) \le (\operatorname{rank}\mathcal{S})(\mu(E)-\mu(S)) + (\operatorname{rank}\mathcal{Q})(\mu(\mathcal{Q})-\mu(\mathcal{E})) - 2t^2\left(|\beta|^2+|\varphi|^2\right) + O(\varepsilon)\ .$$

Without loss of generality, assume that $\|\beta\|^2+\|\varphi\|^2=1$. By the argument in [18] we may also assume $|\beta|$, $|\varphi|$ are bounded uniformly in ε. The result now follows by fixing t and choosing ε sufficiently small. □

2.3.3. *The existence theorem.* We will prove the following in the next section where the Yang-Mills-Higgs flow will be introduced.

Lemma 2.26. *In any complex gauge orbit there exists a sequence (A_i,Φ_i) satisfying the following conditions:*

(i) *(A_i,Φ_i) is minimizing for J ;*
(ii) *if $f_{(A_j,\Phi_j)}=\sqrt{-1}\Lambda(F_{A_j}+[\Phi_j,\Phi_j^*])$, then $\sup|f_{(A_j,\Phi_j)}|$ is bounded uniformly in j;*
(iii) *$\|d_{A_j}f_{(A_j,\Phi_j)}\|_{L^2}\to 0$ and $\|[f_{(A_j,\Phi_j)},\Phi_i]\|_{L^2}\to 0$.*

Next, we will need one of the most fundamental results of gauge theory, stated here for the case of Riemann surfaces.

Proposition 2.27 (Uhlenbeck [65]). *Fix $p \geq 2$. Let $\{A_j\}$ be a sequence of L^p_1-connections with $\|F_{A_j}\|_{L^p}$ uniformly bounded. Then there exists a sequence of unitary gauge transformations $g_j \in L^p_2$ and a smooth unitary connection A_∞ such that (after passing to a subsequence) $g_j(A_j) \to A_\infty$ weakly in L^p_1 and strongly in L^p.*

Assuming these results, we now prove the existence theorem.

Proof of Theorem 2.17. It clearly suffices to assume $(\mathcal{E}, \Phi)$ is stable. Furthermore, by Remark 2.18, we may proceed by induction. Assume that the result has been proven for all bundles of rank $< n = \operatorname{rank} E$.

Step 1. *The limiting bundle* $(\mathcal{E}_\infty, \Phi_\infty)$. Choose a minimizing sequence for J as in Lemma 2.26. Since the sequence lies in a single complex gauge orbit, the image of the Hitchin map $h[A_i, \Phi_i]$ is unchanged. Hence, by Proposition 2.21 the Φ_i are uniformly bounded. By Lemma 2.26 (ii), this in turn implies that $\|F_{A_j}\|_{L^p}$ is bounded for any p. We therefore may assume by Proposition 2.27 that there is a smooth connection A_∞ so that if we write $\bar\partial_{A_j} = \bar\partial_{A_\infty} + a_j$, then $a_j \to 0$ weakly in L^p_1. By the Sobolev embedding theorem, we may assume in particular that the $a_j \to 0$ in some C^α. Notice that it follows that $F_{A_j} \to F_{A_\infty}$ weakly in L^p. From the holomorphicity condition

$$0 = \bar\partial_{A_j}\Phi_j = \bar\partial_{A_\infty}\Phi_j + [a_j, \Phi_j] \ .$$

Elliptic regularity for $\bar\partial_{A_\infty}$ implies a bound $\|\Phi_j\|_{L^2_1} \leq C\|\Phi_j\|_{L^2}$, say. Differentiating the previous equation gives

$$\bar\partial^*_{A_\infty}\bar\partial_{A_\infty}\Phi_j + \bar\partial^*_{A_\infty}[a_j, \Phi_j] = 0 \tag{2.27}$$

By the Cauchy-Schwarz inequality and the previous estimate we have

$$\|\bar\partial^*_{A_\infty}[a_j, \Phi_j]\|_{L^2} \leq C_1\|a_j\|_{L^4_1}\|\Phi_j\|_{L^4} + C_2\|\Phi_j\|_{L^2} \ . \tag{2.28}$$

Now we may assume $\{a_j\}$ is bounded in L^4_1, and using elliptic regularity for the Laplacian $\bar\partial^*_{A_\infty}\bar\partial_{A_\infty}$ along with the inclusions $L^2_1 \hookrightarrow L^4$, $L^2_2 \hookrightarrow C^\alpha$, by (2.27) and (2.28) we have an estimate $\|\Phi_j\|_{C^\alpha} \leq C\|\Phi_j\|_{L^2}$. Since the Φ_j are uniformly bounded their L^2 norms are bounded, so we may assume that Φ_j converges in C^α to some Φ_∞. Moreover, by holomorphicity of the Φ_j we can write

$$\bar\partial_{A_\infty}\Phi_\infty = \bar\partial_{A_\infty}(\Phi_\infty - \Phi_j) - [a_j, \Phi_j] \ ,$$

and since $[a_j, \Phi_j] \to 0$ in C^α we see that $\bar\partial_{A_\infty}\Phi_\infty = 0$ weakly. Hence, by Weyl's lemma Φ_∞ is actually holomorphic, and thus $(\mathcal{E}_\infty, \Phi_\infty)$ is a Higgs bundle.

Step 2. *Construction of a nonzero map* $\mathcal{E} \to \mathcal{E}_\infty$. Let g_j be complex gauge transformations such that $g_j(A) = A_j$. Holomorphicity of g_j implies $\bar\partial_{A_\infty} g_j + [a_j, g_j] = 0$. By the exact same argument as in Step 1, we have an estimate $\|g_j\|_{C^\alpha} \leq C\|g_j\|_{L^2}$. Now rescale g_j so that $\|g_j\|_{L^2} = 1$. The C^α-estimate above still holds for the rescaled map, so by compactness we may assume there is a continuous $g_\infty : \mathcal{E} \to \mathcal{E}_\infty$ such that $g_j \to g_\infty$ in C^α. Because of the normalization, we know that $g_\infty \not\equiv 0$. Moreover, it follows as in Step 1 that g_∞ is holomorphic. Finally, by the C^α convergence of g_j and Φ_j and the fact that $g_j\Phi = \Phi_j g_j$, we have $g_\infty\Phi = \Phi_\infty g_\infty$.

Step 3. *The map g_∞ is an isomorphism.* Suppose the contrary. Let $\mathcal{S} = \ker g_\infty$ and $\mathcal{Q} = \mathcal{E}/\mathcal{S}$. Then $\mathcal{Q}$ is a subsheaf of $\mathcal{E}_\infty$. Let $\mathcal{M}$ denote its saturation and $\mathcal{N} = \mathcal{E}_\infty/\mathcal{M}$. Since $\Phi_\infty g_\infty = g_\infty\Phi$, the subbundle $\mathcal{S}$ is Φ-invariant. Similarly, $\mathcal{M}$ is Φ_∞-invariant. Also, from the discussion in Section 2.1.1, we have

$$\begin{aligned} \mu(\mathcal{Q}) - \mu(\mathcal{E}) &\le \mu(\mathcal{M}) - \mu(\mathcal{E}) \ ; \\ \mu(\mathcal{E}) - \mu(\mathcal{S}) &\le \mu(\mathcal{E}) - \mu(\mathcal{N}) \ . \end{aligned} \tag{2.29}$$

Then we have the following extensions of Higgs bundles (see [18]):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle g_\infty} & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{N} & \longleftarrow & \mathcal{E}_\infty & \longleftarrow & \mathcal{M} & \longleftarrow & 0 \end{array} \tag{2.30}$$

Applying Lemma 2.24 to the bottom row of (2.30) and Lemma 2.25 to the top row implies

$$\begin{aligned} (\operatorname{rank}\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{E})) + (\operatorname{rank}\mathcal{N})(\mu(\mathcal{E}) - \mu(\mathcal{N})) &\le J(A_\infty, \Phi_\infty) \\ &\le \lim_{j\to\infty} J(A_j, \Phi_j) = \inf J(A, \Phi) \\ &< (\operatorname{rank}\mathcal{S})(\mu(\mathcal{E}) - \mu(\mathcal{S})) + (\operatorname{rank}\mathcal{Q})(\mu(\mathcal{Q}) - \mu(\mathcal{E})) \ , \end{aligned}$$

where for the second line we can use either the the lower semicontinuity of J (see [18]) or the argument in [14, Corollary 2.12 and Lemma 2.17]. But since $\operatorname{rank}\mathcal{M} = \operatorname{rank}\mathcal{Q}$ and $\operatorname{rank}\mathcal{S} = \operatorname{rank}\mathcal{N}$, this contradicts (2.29).

Step 4. *Solution to Hitchin's equations.* Finally, I claim that the Higgs bundle (A_∞, Φ_∞) is a solution to (2.16). Indeed, by the remark following eq. (2.19) this follows if we can show $d_{A_\infty} f_{(A_\infty,\Phi_\infty)} = 0$ and $[f_{(A_\infty,\Phi_\infty)}, \Phi_\infty] = 0$. The second fact holds, since $[f_{(A_j,\Phi_j)}, \Phi_j] \to 0$ in L^2 by assumption, and $f_{(A_j,\Phi_j)}$ (resp. Φ_j) converges weakly in L^p (resp. C^α). For the first claim, let B be a test form. Then

$$\begin{aligned} \langle d_{A_\infty} f_{(A_\infty,\Phi_\infty)}, B\rangle_{L^2} &= \langle f_{(A_\infty,\Phi_\infty)}, d^*_{A_\infty} B\rangle_{L^2} \\ &= \lim_{j\to\infty} \langle f_{(A_j,\Phi_j)}, d^*_{A_j} B\rangle_{L^2} + \lim_{j\to\infty} \int_X \operatorname{tr}\left\{ f_{(A_j,\Phi_j)}[a_j, B^*]\right\} \\ &= \lim_{j\to\infty} \langle d_{A_j} f_{(A_j,\Phi_j)}, B\rangle_{L^2} + \lim_{j\to\infty} \int_X \operatorname{tr}\left\{ f_{(A_j,\Phi_j)}[a_j, B^*]\right\} \ . \end{aligned}$$

The first term vanishes since $\|d_{A_j} f_{(A_j,\Phi_j)}\|_{L^2} \to 0$, and the second term vanishes since f_j is bounded and $a_j \to 0$ in C^α. Since B is arbitrary, $d_{A_\infty} f_{(A_\infty,\Phi_\infty)} = 0$, and this completes the proof. □

The same type of argument leads to the

Proof of Theorem 2.15. Let $[A_j, \Phi_j]$ be a sequence of polystable Higgs bundles with $h[A_j, \Phi_j]$ bounded. By Theorem 2.17 we may assume (A_j, Φ_j) satisfies (2.16). Since $h[A_j, \Phi_j]$ is bounded, the pointwise spectrum of Φ_j is uniformly bounded. Therefore, Proposition 2.21 provides uniform sup bounds on $|\Phi_j|$. Again using (2.16) we have uniform bounds on $|F_{A_j}|$. Now Uhlenbeck compactness can be used to extract a convergent subsequence which also satisfies (2.16) as in the proof of the existence theorem above. □

2.3.4. *The Yang-Mills-Higgs flow.* We define the **Yang-Mills-Higgs flow** for a pair (A, Φ) by the equations

$$\begin{aligned} \frac{\partial A}{\partial t} &= -d_A^*(F_A + [\Phi, \Phi^*]) \ ; \\ \frac{\partial \Phi}{\partial t} &= [\Phi, \sqrt{-1}\Lambda(F_A + [\Phi, \Phi^*])] \ . \end{aligned} \tag{2.31}$$

In the above, we only consider initial conditions where Φ is d_A''-holomorphic. Notice then that this holomorphicity condition is preserved along a solution to (2.31). Indeed, as in Donaldson [19], the flow is tangent to the complex gauge orbit and exists for all $0 \leq t < +\infty$. The flow equations may be regarded as the L^2-gradient flow of the YMH functional. They generalize the Yang-Mills flow equations. For more on this we refer to [40, 70] and the references therein. Here we limit ourselves to a discussion of a few key properties. In particular, we justify the assumptions in the previous section.

As in (2.17), set $f_{(A,\Phi)} = \sqrt{-1}\Lambda(F_A + [\Phi, \Phi^*])$.

Lemma 2.28. *For all $t \geq 0$,*

$$\frac{d}{dt}\,\mathrm{YMH}(A, \Phi) = -2\|d_A f_{(A,\Phi)}\|_{L^2}^2 - 4\|[\Phi, f_{(A,\Phi)}\|_{L^2}^2 \ .$$

Proof. We have

$$\frac{d}{dt}\,\mathrm{YMH}(A, \Phi) = 2\int_X \mathrm{tr}(f_{(A,\Phi)}\dot f_{(A,\Phi)})\omega \ .$$

Now using dots to denote time derivatives,

$$\begin{aligned} \dot f_{(A,\Phi)} &= \sqrt{-1}\Lambda\left(d_A\dot A + [\dot\Phi, \Phi^*] + [\Phi, \dot\Phi^*]\right) \\ &= \sqrt{-1}\Lambda\left(-d_A d_A^*(F_A + [\Phi, \Phi^*]) + [[\Phi, f], \Phi^*] + [\Phi, [\Phi, f]^*]\right) \\ &= -d_A^* d_A f_{(A,\Phi)} + \sqrt{-1}\Lambda\left([\Phi, f_{(A,\Phi)}]\Phi^* + \Phi^*[\Phi, f_{(A,\Phi)}] + \Phi[\Phi, f_{(A,\Phi)}]^* + [\Phi, f_{(A,\Phi)}]^*\Phi\right) \ . \end{aligned}$$

Taking traces we get

$$\mathrm{tr}(f_{(A,\Phi)}\dot f_{(A,\Phi)}) = -\,\mathrm{tr}(f_{(A,\Phi)} d_A^* d_A f_{(A,\Phi)}) - 2\sqrt{-1}\Lambda\,\mathrm{tr}\left([\Phi, f_{(A,\Phi)}][\Phi, f_{(A,\Phi)}]^*\right) \ , \tag{2.32}$$

and the result follows by integration by parts. □

As a consequence of Lemma 2.28, YMH decreases along the flow. Moreover, we have the following inequality

$$\int_0^\infty dt\left\{2\|d_A f_{(A,\Phi)}\|_{L^2}^2 + 4\|[\Phi, f_{(A,\Phi)}\|_{L^2}^2\right\} \leq \mathrm{YMH}(A_0, \Phi_0) \ .$$

It follows that if (A_j, Φ_j) is a sequence with $\mathrm{YMH}(A_j, \Phi_j)$ uniformly bounded, then we may replace it with another sequence $(\widetilde{A}_j, \widetilde{\Phi}_j)$ with $\mathrm{YMH}(\widetilde{A}_j, \widetilde{\Phi}_j)$ also uniformly bounded but such that $d_{A_j} f_{(\widetilde{A}_j, \widetilde{\Phi}_j)}$ and $[\Phi_j, f_{(\widetilde{A}_j, \widetilde{\Phi}_j)}]$ converge to 0 in L^2.

Now let's compute

$$\begin{aligned}
\Delta \left| f_{(A,\Phi)} \right|^2 &= -d^* d \left| f_{(A,\Phi)} \right|^2 = *d * d \,\mathrm{tr}\, f^2_{(A,\Phi)} \\
&= 2 * d * \mathrm{tr}(f_{(A,\Phi)} d_A f_{(A,\Phi)}) \\
&= 2 * \mathrm{tr}(df_{(A,\Phi)} \wedge *d_A f_{(A,\Phi)}) - 2\,\mathrm{tr}(f_{(A,\Phi)} d_A^* d_A f_{(A,\Phi)}) \\
&= 2 \left| df_{(A,\Phi)} \right|^2 + 4 \left| [\Phi, f_{(A,\Phi)}] \right|^2 + \frac{\partial}{\partial t} \left| f_{(A,\Phi)} \right|^2 \ ,
\end{aligned}$$

from (2.32). We have shown

Lemma 2.29. *For all $t \geq 0$,*

$$\frac{\partial}{\partial t} \left| f_{(A,\Phi)} \right|^2 - \Delta \left| f_{(A,\Phi)} \right|^2 = -2 \left| d_A f_{(A,\Phi)} \right|^2 - 4 \left| [\Phi, f_{(A,\Phi)}] \right|^2 \ .$$

In particular, $\left| f_{(A,\Phi)} \right|$ is a subsolution of the heat equation, and so $\sup \left| f_{(A,\Phi)} \right|$ is nonincreasing. In fact, one can use an explicit argument with the heat kernel to show that for $t \geq 1$, say, the $\sup \left| f_{(A_t,\Phi_t)} \right| \leq C\, \mathrm{YMH}(A_0, \Phi_0)$ for a fixed constant C. In particular, if (A_j, Φ_j) is a sequence with $\mathrm{YMH}(A_j, \Phi_j)$ uniformly bounded, then we may replace it with another sequence $(\widetilde{A}_j, \widetilde{\Phi}_j)$ with $f_{(\widetilde{A}_j, \widetilde{\Phi}_j)}$ uniformly bounded.

Proof of Lemma 2.26. Choose (A_j, Φ_j) a minimizing sequence for J in the complex gauge orbit of (A, Φ). Note that $\mathrm{YMH}(A_j, \Phi_j)$ is then uniformly bounded. In addition, by an argument similar to the one above (see [14]), J is also decreasing along the YMH-flow. Hence, replacing each (A_j, Φ_j) with a point along the YMH-flow with initial condition (A_j, Φ_j) also gives a J-minimizing sequence. On the other hand, by the discussion in this section, we can choose points along the flow where items (ii) and (iii) are also satisfied. This completes the proof. □

Let $\mathcal{B}_E^{min}$ be the set of all Higgs bundles satisfying the Hitchin equations (2.17). The YMH-flow sets up an infinite dimensional, singular Morse theory problem where $\mathcal{B}_E^{min}$ is the minimum of the functional, and Higgs bundles not in $\mathcal{B}_E^{min}$ but satisfying (2.19) play the role of higher critical points. This Morse theory picture can actually be shown to be more than just an analogy. In particular, we have the following

Theorem 2.30 (Wilkin [70])**.** *The* YMH*-flow gives a $\mathcal{G}_E$-equivariant deformation retraction of $\mathcal{B}_E^{ss}$ onto $\mathcal{B}_E^{min}$.*

3. The Betti Moduli Space

3.1. **Representation varieties.**

3.1.1. *Definition.* Fix a base point $p \in X$ and set $\pi = \pi_1(X, p)$. Let $\mathrm{Hom}(\pi, \mathsf{SL}_n(\mathbb{C}))$ denote the set of homomorphisms from π to $\mathsf{SL}_n(\mathbb{C})$. This has the structure of an affine algebraic variety. Let

$$\mathfrak{M}_B^{(n)} = \mathrm{Hom}(\pi, \mathsf{SL}_n(\mathbb{C})) /\!\!/ \mathsf{SL}_n(\mathbb{C}) \ ,$$

denote the representation variety, where the double slash indicates the invariant theoretic quotient by overall conjugation of $\mathsf{SL}_n(\mathbb{C})$. Then $\mathfrak{M}_B^{(n)}$ is an irreducible affine variety of complex dimension $(n^2-1)(2g-2)$. There is a surjective algebraic quotient map $\mathrm{Hom}(\pi, \mathsf{SL}_n(\mathbb{C})) \to \mathfrak{M}_B^{(n)}$, and this is a geometric quotient on the open set of irreducible (or simple) representations. Points of $\mathfrak{M}_B^{(n)}$ are in 1-1 correspondence with conjugacy classes of semisimple (or reductive) representations, and every $\mathsf{SL}_n(\mathbb{C})$ orbit in $\mathrm{Hom}(\pi, \mathsf{SL}_n(\mathbb{C}))$ contains a semisimple representation in its closure (for these results, see [49]). Following Simpson [61, 62] I will refer to $\mathfrak{M}_B^{(n)}$ as the **Betti moduli space** of rank n.

Let $E \to X$ be a trivial rank n complex vector bundle. A flat connection ∇ on E gives rise to a representation of π as follows. Recall that we have fixed a base point $p \in X$. We also fix a frame $\{\mathbf{e}_i\}$ of E_p. For each loop γ based at p, parallel translation of the frame $\{\mathbf{e}_i\}$ defines an element of $\mathsf{GL}_n(\mathbb{C})$. Since the connection is flat this is independent of the choice of path in the homotopy class. In this way we have defined an element $\mathrm{hol}(\nabla) \in \mathrm{Hom}(\pi, \mathsf{GL}_n(\mathbb{C}))$. If ∇ induces the trivial connection on $\det E$, the holonomy lies in $\mathsf{SL}_n(\mathbb{C})$, and we will assume this from now on. Conversely, given a representation $\rho : \pi_1(X, p) \to \mathsf{SL}_n(\mathbb{C})$, we obtain a holomorphic bundle $\mathcal{V}_\rho$ with a flat connection ∇ by the quotient $\mathcal{V}_\rho = \widetilde{X} \times \mathbb{C}^n/\pi$, where $\widetilde{X}$ is the universal cover of X, and the quotient identifies $(x, v) \sim (x\gamma, v\rho(\gamma))$. Let $\mathcal{C}_E$ denote the space of connections on E, and $\mathcal{C}_E^{flat} \subset \mathcal{C}_E$ the flat connections. Let $\mathcal{G}_E^{\mathbb{C}}(p)$ denote the space of complex gauge transformations that are the identity at p, acting on $\mathcal{C}_E$ by conjugation (**warning:** this is a different action of $\mathcal{G}_E^{\mathbb{C}}$ from the one on the space of *unitary* connections in Section 2.2.1).

Proposition 3.1. *The holonomy map gives an* $\mathsf{SL}_n(\mathbb{C})$*-equivariant homeomorphism*

$$\mathrm{hol} : \mathcal{C}_E^{flat}/\mathcal{G}_E^{\mathbb{C}}(p) \xrightarrow{\sim} \mathrm{Hom}(\pi, \mathsf{SL}_n(\mathbb{C})) \ .$$

In particular, $\mathcal{C}_E^{flat} /\!\!/ \mathcal{G}_E^{\mathbb{C}} \simeq \mathfrak{M}_B^{(n)}$.

3.2. **Local systems and holomorphic connections.**

3.2.1. *Definitions.*

Definition 3.2. A **complex n-dimensional local system** on X is a sheaf of abelian groups that is locally isomorphic to the constant sheaf $\underline{\mathbb{C}}^n$.

Here $\underline{\mathbb{C}}$ denotes the locally constant sheaf modeled on $\mathbb{C}$. Clearly a local system $\mathbf{V}$ is a sheaf of modules over $\underline{\mathbb{C}}$.

Definition 3.3. Let $\mathcal{V} \to X$ be a holomorphic bundle. A **holomorphic connection** on $\mathcal{V}$ is a $\mathbb{C}$-linear operator $\nabla : \mathcal{V} \to \mathcal{K} \otimes \mathcal{V}$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s \ , \tag{3.1}$$

for local sections $f \in \mathcal{O}$, $s \in \mathcal{V}$.

For a local system $\mathbf{V}$ let $\mathcal{V}$ be the holomorphic bundle $\mathcal{V} = \mathcal{O} \otimes_{\underline{\mathbb{C}}} \mathbf{V}$. Then $\mathcal{V}$ inherits a holomorphic connection as follows: choose a local parallel frame $\{\mathbf{v}_i\}$ for $\mathbf{V}$. Any local section of $\mathcal{V}$ may be written uniquely as $s = \sum_{i=1}^n f_i \otimes \mathbf{v}_i$, with $f_i \in \mathcal{O}$. Then *define* $\nabla s = \sum_{i=1}^n df_i \otimes \mathbf{v}_i$. Since the transition functions for $\mathbf{V}$ are constant this is well-defined independent of the choice of frame, and ∇ also immediately satisfies the Leibniz rule. Conversely, a holomorphic connection defines a *flat* connection on the underlying complex vector bundle, since in a local holomorphic frame the curvature F_∇ is necessarily of type $(2,0)$, and on a Riemann surface there are no $(2,0)$-forms. In particular, the $\underline{\mathbb{C}}$-subsheaf $\mathbf{V} \subset \mathcal{V}$ of locally parallel sections $\nabla s = 0$ defines a local system. This gives a categorical equivalence between local systems and holomorphic bundles with a holomorphic connection (see [16, Théorème 2.17]).

A local system has a **monodromy representation** $\rho : \pi \to \mathsf{GL}_n(\mathbb{C})$, obtained by developing local parallel frames. Conversely, given ρ we construct a local system as in the previous section. We will sometimes denote these $\mathbf{V}_\rho$ and $\mathcal{V}_\rho$. For simplicity, in these notes I will almost always assume the monodromy lies in $\mathsf{SL}_n(\mathbb{C})$, or in other words, $\det \mathcal{V}_\rho \simeq \mathcal{O}$ and the induced connection on $\det \mathcal{V}_\rho$ is trivial.

Not every holomorphic bundle $\mathcal{V}$ admits a holomorphic connection. In particular, such a connection is flat, and so by (2.3) a necessary condition is that $\deg \mathcal{V} = 0$. In fact, one can say more about the Harder-Narasimhan type of a bundle with a holomorphic connection.

Proposition 3.4 (cf. [23, 8])**.** *Suppose $\mathcal{V}$ is an unstable bundle with an irreducible holomorphic connection, and let $\mu_1 > \mu_2 > \cdots > \mu_\ell$ be the Harder-Narasimhan type. Then for each $i = 1, \ldots, \ell - 1$, $\mu_i - \mu_{i+1} \leq 2g - 2$.*

Proof. Let $0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_\ell = \mathcal{V}$ be the Harder-Narasimhan filtration of $\mathcal{V}$. Then since the connection is irreducible the $\mathcal{O}$-linear map $\mathcal{V}_i \xrightarrow{\nabla} \mathcal{V}/\mathcal{V}_i \otimes \mathcal{K}$ is nonzero for each $i = 1, \ldots, \ell - 1$. Let $j \leq i$ be the smallest integer such that $\mathcal{V}_j \to \mathcal{V}/\mathcal{V}_i \otimes \mathcal{K}$ is nonzero. Then it follows from the sequence

$$0 \longrightarrow \mathcal{V}_{j-1} \longrightarrow \mathcal{V}_j \longrightarrow \mathcal{Q}_j \longrightarrow 0$$

that there is a nonzero map $\mathcal{Q}_j \to \mathcal{V}/\mathcal{V}_i \otimes \mathcal{K}$. With this fixed j, let $k \geq i$ be the largest integer such that $\mathcal{Q}_j \to \mathcal{V}/\mathcal{V}_k \otimes \mathcal{K}$ is nonzero. It follows from

$$0 \longrightarrow \mathcal{Q}_{k+1} \longrightarrow \mathcal{V}/\mathcal{V}_k \longrightarrow \mathcal{V}/\mathcal{V}_{k+1} \longrightarrow 0$$

that $\mathcal{Q}_j \to \mathcal{Q}_{k+1} \otimes \mathcal{K}$ is nonzero. Since the $\mathcal{Q}_i$ are all semistable, we have by Lemma 2.8 that

$$\mu_j = \mu(\mathcal{Q}_j) \leq \mu(\mathcal{Q}_{k+1} \otimes \mathcal{K}) = \mu_{k+1} + 2g - 2 ,$$

and the result follows, since $\mu_i - \mu_{i+1} \leq \mu_j - \mu_{k+1}$. □

3.2.2. *The Weil-Atiyah theorem.* The goal of this section is to prove the following

Theorem 3.5 (Weil [67], Atiyah [1])**.** *A holomorphic bundle $\mathcal{V} \to X$ admits a holomorphic connection if and only if each indecomposable factor of $\mathcal{V}$ has degree zero.*

The proof I give here follows Atiyah. The following construction will be useful (see [1, p. 193]). Any holomorphic bundle $\mathcal{V} \to X$ gives rise to a counterpart $D(\mathcal{V})$ as follows. First, as a smooth bundle $D(\mathcal{V}) = (V \otimes K) \oplus V$. With respect to this splitting define the $\mathcal{O}$-module structure by

$$f(\varphi, s) = (f\varphi + s \otimes df, fs) \ , \qquad f \in \mathcal{O} \ , \ \varphi \in \mathcal{V} \otimes \mathcal{K} \ , \ s \in \mathcal{V} \ .$$

One checks that this gives $D(\mathcal{V})$ the structure of a locally free sheaf over $\mathcal{O}$. Then we have a compatible inclusion $\varphi \mapsto (\varphi, 0)$ and projection $(\varphi, s) \mapsto s$ making $D(\mathcal{V})$ into an extension

$$0 \longrightarrow \mathcal{V} \otimes \mathcal{K} \longrightarrow D(\mathcal{V}) \longrightarrow \mathcal{V} \longrightarrow 0 \ . \tag{3.2}$$

Observe that (3.2) *splits if and only if $\mathcal{V}$ admits a holomorphic connection.* Indeed, such a ∇ gives a splitting by $s \mapsto (\nabla s, s)$, and if (3.2) splits then there is a $\underline{\mathbb{C}}$-linear map $\mathcal{V} \to \mathcal{V} \otimes \mathcal{K}$ satisfying (3.1).

Remark 3.6. The construction is functorial with respect to subbundles. If $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_\ell = \mathcal{V}$ is a filtration of $\mathcal{V}$ by holomorphic subbundles, then there is a filtration

$$0 = D(\mathcal{V}_0) \subset D(\mathcal{V}_1) \subset \cdots \subset D(\mathcal{V}_\ell) = D(\mathcal{V}) \ .$$

Lemma 3.7. *Given a holomorphic bundle $\mathcal{V} \to X$, let*

$$[\beta] \in H^1(X, (\mathcal{V} \otimes \mathcal{K}) \otimes \mathcal{V}^*) \simeq H^{1,1}_{\bar\partial}(X, \operatorname{End} V) \ ,$$

denote the extension class. Then $[\operatorname{tr}\beta] = -2\pi\sqrt{-1}\, c_1(V)$.

Proof. Choose $s^{(i)}$ local holomorphic frames for $\mathcal{V}$ on U_i, and let ψ_{ij} denote the transition functions: $s^{(i)} = s^{(j)}\psi_{ij}$. We can define local splittings of (3.2) by $s^{(i)} f^{(i)} \mapsto s^{(i)} \otimes df^{(i)}$, for $f^{(i)}$ a vector of holomorphic functions on U_i. In particular,

$$f^{(j)} = \psi_{ij} f^{(i)} \ , \ \partial f^{(j)} = \psi_{ij}(\psi_{ij}^{-1}\partial\psi_{ij} f^{(i)} + \partial f^{(i)}) \ .$$

Since the extension class is given by the image of I under the map

$$H^0(X, \operatorname{End}\mathcal{V}) \to H^1(X, \operatorname{End}\mathcal{V} \otimes \mathcal{K}) \ ,$$

it follows from the local splitting above that $[\beta]$ is represented by the cocycle $[\psi_{ij}^{-1} d\psi_{ij}]$. Hence, $[\operatorname{tr}\beta] = [d\log\det\psi]$. On the other hand, if h is a hermitian metric on $\det\mathcal{V}$, then

$$h_i|s_1^{(i)} \wedge \cdots \wedge s_n^{(i)}|^2 = h_j|s_1^{(j)} \wedge \cdots \wedge s_n^{(j)}|^2 \ ,$$

so $h_i|\det\psi_{ij}|^2 = h_j$. This implies $d\log\det\psi_{ij} = \partial\log h_j - \partial\log h_i$. By the Dolbeault isomorphism $[\beta]$ is represented by $[\bar\partial\partial\log h_i] = [F_{(\bar\partial_{\det\mathcal{V}}, h)}] = -2\pi\sqrt{-1}\, c_1(V)$ (see Example 2.2 and (2.3)). □

Lemma 3.8. *If $\mathcal{V} \to X$ is an indecomposable holomorphic bundle and $\phi \in H^0(X, \operatorname{End}\mathcal{V})$, Then there is $\lambda \in \mathbb{C}$ such that $\phi - \lambda I$ is nilpotent.*

Proof. Since $\det(\phi-\lambda I)$ is holomorphic and X is closed, the eigenvalues of ϕ must be constant. So without loss of generality assume $\ker\phi\neq\{0\},\mathcal{V}$, and consider the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\phi & \longrightarrow & \mathcal{V} & \longrightarrow & \operatorname{coker}\phi & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathcal{S} & & & & \mathcal{Q} & & \end{array} \tag{3.3}$$

Write:

$$\bar\partial_E=\begin{pmatrix}\bar\partial_S & \beta\\ 0 & \bar\partial_Q\end{pmatrix}\qquad,\qquad \phi=\begin{pmatrix}0 & \phi_1\\ 0 & \phi_2\end{pmatrix}\ .$$

We wish to show $\phi_2=0$. First note that

$$0=\bar\partial_E\phi=\begin{pmatrix}0 & \bar\partial_E\phi_1+\beta\phi_2\\ 0 & \bar\partial_Q\phi_2\end{pmatrix}\ .$$

So ϕ_2 is holomorphic as an endomorphism of $\mathcal{Q}$. If $\phi_2\neq 0$, then it is an isomorphism. This is so because again the eigenvalues of ϕ_2 are constant, and by assumption 0 is not an eigenvalue. Hence, we can rewrite the upper right entry in the matrix equation above as: $\bar\partial_E(\phi_1\phi_2^{-1})+\beta=0$. But then the Dolbeault class of β vanishes and (3.3) splits, contradicting the assumption that $\mathcal{V}$ be indecomposable. □

Proof of Theorem 3.5. Suppose $\mathcal{V}$ has a holomorphic connection. Then by Remark 3.6, $D(\mathcal{V})$ splits. Moreover, since $D(\mathcal{V})$ is natural with respect to subbundles, $D(\mathcal{V}_i)$ splits for each indecomposable factor of $\mathcal{V}$. But then by Lemma 3.7, $\deg(\mathcal{V}_i)=0$ for all i. Conversely, suppose $\mathcal{V}$ is indecomposable and $\deg(\mathcal{V})=0$. It suffices to show $D(\mathcal{V})$ splits. Now by Serre duality the extension class

$$[\beta]\in H^1(X,\operatorname{End}(\mathcal{V})\otimes\mathcal{K})\simeq\left(H^0(X,\operatorname{End}(\mathcal{V}))\right)^*\ ,$$

and the perfect pairing is $(\beta,\phi)=\displaystyle\int_X\operatorname{tr}(\beta\phi)$. By Lemma 3.8 we may express $\phi=\lambda I+\phi_0$, where ϕ_0 is nilpotent. Then by Lemma 3.7,

$$(\beta,\phi)=(\beta,\phi_0)+\lambda(\beta,I)=(\beta,\phi_0)+\lambda\int_X\operatorname{tr}\beta=(\beta,\phi_0)-2\pi\sqrt{-1}\lambda\deg(E)=(\beta,\phi_0)\ . \tag{3.4}$$

Set $\mathcal{V}_\ell=\mathcal{V}$, and recursively define $\mathcal{V}_{i-1}$ to be the saturation of $\phi_0(\mathcal{V}_i)$. Note that $\mathcal{V}_{i-1}$ is a proper subbundle of $\mathcal{V}_i$, since otherwise the restriction of ϕ_0 would be almost everywhere an isomorphism. Eventually the process terminates. Adjust ℓ so that $\mathcal{V}_0=\{0\}$, $\mathcal{V}_1\neq\{0\}$. By Remark 3.6, β preserves the filtration $0=\mathcal{V}_0\subset\mathcal{V}_1\subset\cdots\subset\mathcal{V}_\ell=\mathcal{V}$. Choose a hermitian metric on V and let π_i be orthogonal projection to V_i. Note that

$$I=\sum_{i=1}^{\ell}(\pi_i-\pi_{i-1})=\sum_{i=1}^{\ell}(\pi_i-\pi_i\pi_{i-1})=\sum_{i=1}^{\ell}\pi_i(I-\pi_{i-1})\ ,$$

and $(I-\pi_i)\beta\pi_i = (I-\pi_{i-1})\phi\pi_i = 0$. Then

$$\begin{aligned}\mathrm{tr}(\beta\phi_0) = \mathrm{tr}(\phi_0\beta) &= \sum_{i=1}^{\ell} \mathrm{tr}(\phi_0\beta\pi_i(I-\pi_{i-1})) \\ &= \sum_{i=1}^{\ell} \mathrm{tr}((I-\pi_{i-1})\phi_0\beta\pi_i) \\ &= \sum_{i=1}^{\ell} \mathrm{tr}((I-\pi_{i-1})\phi_0\pi_i\beta\pi_i) \\ &= 0\ .\end{aligned}$$

So $(\beta, \phi_0) = 0$, and by (3.4) we conclude $[\beta] = 0$. The proof is complete. □

3.3. **The Corlette-Donaldson theorem.**

3.3.1. *Hermitian metrics and equivariant maps.* Let $D = \mathsf{SU}_n\backslash\mathsf{SL}_n(\mathbb{C})$ and $\rho : \pi \to \mathsf{SL}_n(\mathbb{C})$. Then π acts on the right on D via the representation ρ. Following Donaldson, we give a concrete description of D with its $\mathsf{SL}_n(\mathbb{C})$-action. Set

$$D = \{\text{positive hermitian } n \times n \text{ matrices } M \text{ with } \det M = 1\}\ .$$

Then the right SL_n action is given by $(M, g) \mapsto g^{-1}M(g^{-1})^*$. Note that the space D may be interpreted as the space of hermitian inner products on $\mathbb{C}^n$ which induce a fixed one on $\det \mathbb{C}^n$. The invariant metric on D is given by $|M^{-1}dM|^2 = \mathrm{tr}(M^{-1}dM)^2$.

Definition 3.9. A map $u : \widetilde{X} \to D$ is ρ-equivariant if $u(x\gamma) = u(x)\rho(\gamma)$ for all $x \in X$, $\gamma \in \pi$.

Let $E = \widetilde{X} \times \mathbb{C}^n/\pi$. We now claim that a hermitian metric on the bundle E is equivalent to a choice of ρ-equivariant map, up to the choice of basepoints. Indeed, suppose $u : \widetilde{X} \to D$ is ρ-equivariant. By definition, a section of E is a map $\sigma : \widetilde{X} \to \mathbb{C}^n$ such that $\sigma(x\gamma) = \sigma(x)\rho(\gamma)$. Hence, if we define $\|\sigma\|^2(x) = \langle\sigma(x), \sigma(x)u(x)\rangle_{\mathbb{C}^n}$, then

$$\|\sigma\|^2(x\gamma) = \langle\sigma(x)\rho(\gamma), \sigma(x)u(x)(\rho(\gamma)^{-1})^*\rangle_{\mathbb{C}^n} = \|\sigma\|^2(x)\ ,$$

and so this is a well-defined metric on E. In the other direction, given a metric H, if σ_i are sections, then write $\langle\sigma_i, \sigma_j\rangle_H(x) = \langle\sigma_i(x), \sigma_j(x)u(x)\rangle_{\mathbb{C}^n}$, for a hermitian matrix valued function $u(x)$. Then

$$\begin{aligned}\langle\sigma_i(x), \sigma_j(x)u(x)\rangle_{\mathbb{C}^n} &= \langle\sigma_i, \sigma_j\rangle_H(x) = \langle\sigma_i, \sigma_j\rangle_H(x\gamma) \\ &= \langle\sigma_i(x)\rho(\gamma), \sigma_j(x)\rho(\gamma)u(x\gamma)\rangle_{\mathbb{C}^n} \\ &= \langle\sigma_i(x), \sigma_j(x)\rho(\gamma)u(x\gamma)\rho(\gamma)^*\rangle_{\mathbb{C}^n}\end{aligned}$$

for all sections. Hence, $\rho(\gamma)u(x\gamma)\rho(\gamma)^* = u(x)$, and u is ρ-equivariant.

3.3.2. *Harmonic metrics.* If $u : \widetilde{X} \to D$ is a continuously differentiable ρ-equivariant map, we define its energy as follows. The derivative du is a section of $T^*\widetilde{X} \otimes u^*(TD)$. We have fixed an invariant metric on D, so the norm $e_u(x) = |du|^2(x)$. In fact, by equivariance, $e_u(x)$ is invariant under π, so it gives a well-defined function on X which is called the **energy density**. The **energy** of u is then by definition

$$E_\rho(u) = \int_X e_u(x)\,\omega \ . \tag{3.5}$$

Note that the energy only depends on the conformal structure on X and not the full metric.

The Euler-Lagrange equations for E_ρ are easy to write down. Define

$$\tau(u) = d_\nabla^* du \ . \tag{3.6}$$

In the above we note that the bundle $u^*(TD)$ has a connection ∇: the pull-back of the Levi-Civita connection on D. It is with respect to this connection that d_∇ is defined. The tensor $\tau(u)$ is called the **tension field**. It is a section of $u^*(TD)$.

Definition 3.10. A C^2 ρ-equivariant map u is called **harmonic** if it satisfies

$$\tau(u) = 0 \ . \tag{3.7}$$

Eq. (3.7) is a second order elliptic nonlinear partial differential equation in u. This statement is a slightly misleading because u is a mapping and not a collection of functions. This annoying fact makes defining weak solutions a little tricky. In the case of maps between compact manifolds (the non-equivariant problem) one way to circumvent this issue is to use a Nash isometric embedding of the target into a euclidean space and rewrite the equations in terms of coordinate functions (cf. [54]). A more sophisticated technique, better suited to the equivariant problem, is to define the Sobolev space theory intrinsically (cf. [46, 47, 42]). On the other hand, if we *assume* u is Lipschitz continuous, then we can introduce local coordinates $\{y^a\}$ on D and write (3.7) locally. By Rademacher's theorem the pull-backs $s_a = u^*(\partial/\partial y^a)$ give a local frame for $u^*(TD)$ almost everywhere, and the connection forms for ∇ in this frame are $\Gamma^c_{ab}(u)du^a \otimes s_c$, where $\Gamma^c_{ab}(u)$ are the Christoffel symbols on D evaluated along u. Writing $u = (u^1, \ldots, u^N)$ in terms of the coordinates on $\{y^a\}$, it is easy to see that the local expression of (3.7) becomes

$$-\tau(u)^a = \Delta u^a + \Gamma^a_{bc}(u)\nabla u^b \cdot \nabla u^c = 0 \ . \tag{3.8}$$

To be clear, the dot product in the second term refers to the metric on X, and Δ is the Laplace operator on X. Notice that this equation is conformally invariant with respect to the metric on X, a manifestation of the fact that the energy functional itself is conformally invariant.

In light of the previous section, ρ-equivariant maps are equivalent to choices of hermitian metrics. Given a flat connection ∇ and hermitian metric on E we can construct the equivariant map in a more intrinsic way. First, lift ∇ and E to obtain a flat connection on a trivial bundle on the universal cover $\widetilde{X}$. We will use the same notation to denote this lifted bundle and connection. If we choose a base point $\hat{p}$ covering the base point p for $\pi_1(X, p)$, and we choose a unitary frame

$\{\mathbf{e}_i(\hat{p})\}$ for the fiber $E_{\hat{p}}$, let $\{\mathbf{e}_i(x)\}$ be given by parallel transport with respect to ∇. Then the map $u : \widetilde{X} \to D$ is given by $x \mapsto \langle \mathbf{e}_i, \mathbf{e}_j\rangle(x)$. It is ρ-equivariant and uniquely determined up to the choice of $\hat{p}$ and the base point in D.

Conversely, if $u : \widetilde{X} \to D$ is any ρ-equivariant map such that $u(\hat{p}) = I$, then u defines a hermitian metric for which it is the equivariant map constructed above. Notice that there is an equivalence of the type we saw for Higgs bundles. If $g \in \mathcal{G}_E^{\mathbb{C}}(p)$ then the corresponding ρ-equivariant map obtained from the pair $(g(\nabla), H)$ is the same as that for (∇, Hg). Finally, if we act by a constant $g \in \mathsf{SL}_n(\mathbb{C})$, the same is true, but now the map is $(\rho \cdot g)$-equivariant. The moral of the story is that finding a harmonic metric is equivalent to finding a harmonic equivariant map in the $\mathcal{G}_E^{\mathbb{C}}$ orbit of ∇.

Given the data (∇, H), we may uniquely write $\nabla = d_A + \Psi$ where, d_A is a unitary connection on (E, H), and Ψ is a 1-form with values in the bundle $\sqrt{-1}\mathfrak{g}_E$ of hermitian endomorphisms. We can explicitly define Ψ with respect to a local frame $\{s_i\}$ by

$$\langle \Psi s_i, s_j\rangle = \frac{1}{2}\left\{\langle \nabla s_i, s_j\rangle + \langle s_i, \nabla s_j\rangle - d\langle s_i, s_j\rangle\right\}\ . \tag{3.9}$$

Lemma 3.11 (cf. [20])**.** *The energy of the map defined above is given by* $E_\rho(u) = 4\|\Psi\|^2$.

Proof. From the definition above and the fact that d_A is unitary,

$$du_{ij} = \langle d_A\mathbf{e}_i, \mathbf{e}_j\rangle + \langle \mathbf{e}_i, d_A\mathbf{e}_j\rangle\ .$$

On the other hand, the $\mathbf{e}_i$ are parallel with respect to ∇, so $d_A\mathbf{e}_j = -\Psi\mathbf{e}_j$. Hence, $u^{-1}du = -2\Psi$. □

Definition 3.12. We say that H is a **harmonic metric** if the map u defined above is a harmonic map.

Proposition 3.13 (Corlette [11])**.** *If ρ admits a harmonic metric then ρ is semisimple.*

Proof. Suppose that H is a critical metric but that ∇ is reducible. Let $V_1 \subset V$ be a subbundle invariant with respect to the connection ∇. Let V_2 be the orthogonal complement of V_1, and H_1, H_2 the induced metrics. We can express

$$\nabla = \begin{pmatrix} \nabla_1 & \beta \\ 0 & \nabla_2 \end{pmatrix} = \begin{pmatrix} d_{A_1} + \Psi_1 & \beta \\ 0 & d_{A_2} + \Psi_2 \end{pmatrix}\ ,$$

where $\beta \in \Omega^1(X, \mathrm{Hom}(V_2, V_1))$. It suffices to show that the connection splits, or in other words that $\beta \equiv 0$. The proposition then follows by induction. Now using (3.9) it follows that if s_1, s_2 are local sections of V_1, then $\langle \Psi s_1, s_2\rangle = \langle \Psi_1 s_1, s_2\rangle$. Similarly, $\langle \Psi s_1, s_2\rangle = \langle \Psi_1 s_1, s_2\rangle$ for local sections of V_2. On the other hand, if $s_i \in V_i$, then $\langle \Psi s_1, s_2\rangle = \frac{1}{2}\langle s_1, \beta s_2\rangle$. It follows that

$$\Psi = \begin{pmatrix} \Psi_1 & \frac{1}{2}\beta \\ \frac{1}{2}\beta^* & \Psi_2 \end{pmatrix}\ .$$

We now deform the metric H to a family H_t as follows: scale $H_1 \mapsto e^{-(\mathrm{rank}\, V_2)t}H_1$, and $H_2 \mapsto e^{+(\mathrm{rank}\, V_1)t}H_2$. This, of course, preserves the orthogonal splitting and the condition $\det H_t = 1$. But

H_t is a geodesic homotopy of ρ-equivariant maps, and so by a result of Hartman the energy $E_\rho(u_t)$ is convex [32]. On the other hand, by Lemma 3.11,

$$\frac{1}{4}E_\rho(u_t) = \|\Psi_1\|^2_{H_1} + \|\Psi_2\|^2_{H_2} + \|\beta\|^2_H \, e^{-(\mathrm{rank}\, V)t/2} \ .$$

In particular, $E_\rho(u_t)$ is bounded as $t \to \infty$. The only way $E_\rho(u_t)$ could have a critical point at $t = 0$ is if $E_\rho(u_t)$ is constant, which implies $\beta \equiv 0$. This completes the proof. □

3.3.3. *The Corlette-Donaldson Theorem.* In this section we prove the following

Theorem 3.14 (Corlette [11], Donaldson [20], Jost-Yau [43], Labourie [48])**.** *Let $\rho : \pi \to \mathsf{SL}_n(\mathbb{C})$ be semisimple. Then there exists a ρ-equivariant harmonic map $u : \widetilde{X} \to D$.*

The following result can be compared to Lemma 2.26. It will be proven when we discuss the harmonic map flow in the next section.

Lemma 3.15. *For any $\rho : \pi \to \mathsf{SL}_n(\mathbb{C})$ there is a sequence u_j of ρ-equivariant maps $u_j : \widetilde{X} \to D$ satisfying the conditions:*

(i) *u_j is energy minimizing.*
(ii) *The u_j have a uniformly bounded Lipschitz constant.*
(iii) *$\tau(u_j) \to 0$ in L^2.*

Lemma 3.16. *Let $\rho : \pi \to \mathsf{SL}_n(\mathbb{C})$ be irreducible, and let $u_j : \widetilde{X} \to D$ be a sequence of ρ-equivariant maps with a uniform Lipschitz constant. Then $u_j(\hat{p})$ is bounded.*

Proof. Suppose not. Set $h_j = u_j(\hat{p})$ and choose $\varepsilon_j \to 0$ such that (perhaps after passing to a subsequence) $\varepsilon_j h_j \to h_\infty \neq 0$. Notice that $\det h_\infty = 0$, so $V = \ker h_\infty$ is a proper subspace of $\mathbb{C}^n$. I claim $\rho(\pi)$ fixes V. Indeed, if $\rho(\gamma) = g^{-1}$ and $v \in V$, then since $d(u_j(\hat{p}), u_j(\hat{p}) \cdot g^{-1})$ is uniformly bounded we have

$$|\langle w, vh_j\rangle_{\mathbb{C}^n} - \langle w, vgh_jg^*\rangle_{\mathbb{C}^n}| \leq B \ ,$$

for a constant B independent of j, and all $w \in \mathbb{C}^n$. It follows that

$$|\langle w, v\varepsilon_j h_j\rangle_{\mathbb{C}^n} - \langle wg, vg\varepsilon_j h_j\rangle_{\mathbb{C}^n}| \longrightarrow 0 \ ,$$

and since $vh_\infty = 0$ we conclude that $\langle wg, vgh_\infty\rangle_{\mathbb{C}^n} = 0$. Since w was arbitrary, $vg \in V$. □

Theorem 3.14. By induction it suffices to prove the result for irreducible representations. Let u_j be a minimizing sequence as in Lemma 3.16, the existence of which is guaranteed by Lemma 3.15. It follows from Ascoli's theorem that there is a uniformly convergent subsequence, also denoted u_j, with the limit $u_j \to u_\infty$ a Lipschitz ρ-equivariant map. I claim that we may arrange for u_∞ to be a harmonic map. Indeed, since the convergence is uniform, we may choose local coordinates and write u^a. Then since $|du^a|$ is uniformly bounded, we may assume further that $u_j \to u_\infty$ weakly in $L^2_{1,loc.}$. By the condition in Lemma 3.15 (iii), the coordinates u^a_∞ are in $L^2_{1,loc.}$ and form a weak solution of (3.8). Since u_∞ is Lipschitz, elliptic regularity of the Laplace operator implies $u_\infty \in L^2_{2,loc.}$. By the remark following (3.8), we may assume that the local metric on X is euclidean. Now differentiate to obtain:

$$\Delta(\nabla u^a_\infty) + \nabla(\Gamma^a_{bc}(u_\infty)\nabla u^b_\infty \cdot \nabla u^c_\infty) = 0 \ ;$$
$$\Delta(\nabla^2 u^a_\infty) + \nabla^2(\Gamma^a_{bc}(u_\infty)\nabla u^b_\infty \cdot \nabla u^c_\infty) = 0 \ .$$

Notice that since u_∞ is Lipschitz the second term in the first equation is in L^2. It then follows that $u^a_\infty \in L^2_{3,loc.}$. Because of the inclusion $L^2_3 \hookrightarrow L^4_2$, the second term of the second equation above is then in L^2. This in turn implies $u^a_\infty \in L^2_{4,loc.}$. Finally, $L^2_4 \subset C^{2,\alpha}$, and so u_∞ is a strong solution to the harmonic map equations (3.7). This completes the proof. □

3.3.4. *The harmonic map flow.* The harmonic map flow is defined by

$$\dot{u} = -\tau(u) \ . \tag{3.10}$$

Here u_t is a family of ρ-equivariant maps. Since D has non-positive curvature, the flow is very well-behaved. Long time existence is proven in [21, 30].

The variation of the energy along the flow is given by

$$\frac{d}{dt}E(u_t) = 2\int_X \langle du, d\dot{u}\rangle = 2\int_X \langle d^*_\nabla du, \dot{u}\rangle\omega = -2\int_X |\tau(u)|^2\omega \ .$$

In particular, *energy decreases along the flow.* Moreover,

$$2\int_0^\infty dt \int_X |\tau(u_t)|^2\omega \leq E(u_0) \ . \tag{3.11}$$

We are now ready for the

Proof of Lemma 3.15. The proof is based on the famous Eells-Sampson-Bochner formula for the change of the energy density along the harmonic map flow [21]. Let $u = u(t,x)$ be a solution to (3.10), and $e = e_u(t,x)$. Then

$$-\frac{\partial e}{\partial t} + \Delta e = |\nabla du|^2 + \mathrm{Ric}_X(du,du) - \mathrm{Riem}_D(du,du,du,du)$$

Now since $\mathrm{Riem}_D \leq 0$ and Ric_X is bounded below a negative constant, we have

$$\frac{\partial e}{\partial t} - \Delta e \leq C\cdot e \ .$$

Using an explicit argument with the heat kernel, this inequality along with the fact that energy is decreasing imply an estimate of the following type

$$\sup e_{u_t} \leq C\cdot E_{u_0} \ , \tag{3.12}$$

for $t \geq 1$, say, where C is depends only on the geometry of X and D.

Now let $u^{(j)}$ be an energy minimizing sequence of ρ-equivariant maps. Let $u^{(j)}_t$ be the corresponding maps after the time t flow of (3.10). Then since energy is decreasing along the flow, $u^{(j)}_{t_j}$ is also energy minimizing for any choice of sequence t_j. On the other hand, the right hand of (3.12) is uniformly bounded, so if we choose each $t_j \geq 1$, say, then $u^{(j)}_{t_j}$ is also uniformly Lipschitz.

Finally, for each fixed initial condition u_0, (3.11) implies $\tau(u_{t_j}) \to 0$ in L^2 along some sequence. By a diagonalization argument we can arrange for $u^{(j)}_{t_j}$ to satisfy this property as well. □

3.4. **Hyperkähler reduction.**

3.4.1. *The moduli spaces are real isomorphic.* Using (3.9), given a hermitian metric we may identify the space of all connections

$$\mathcal{C}_E = \left\{(A, \Psi) \in \mathcal{A}_E \times \Omega^1(M, \sqrt{-1}\mathfrak{g}_E)\right\} \ .$$

Then $\mathcal{C}_E$ is a *hyperkähler manifold*, and the action of the gauge group $\mathcal{G}$ has associated moment maps

$$\mu_1(A, \Psi) = F_A + \tfrac{1}{2}[\Psi, \Psi] \ , \ \mu_2(A, \Psi) = d_A\Psi \ , \ \mu_3(A, \Psi) = d_A(*\Psi) \ . \tag{3.13}$$

Let $\mathbf{m} = (\mu_1, \mu_2, \mu_3)$. The hyperkähler quotient is by definition

$$\mathbf{m}^{-1}(0)/\mathcal{G} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathcal{G}_E \ .$$

The two pictures we have been discussing above are equivalent to a reduction of $\mathcal{C}_E$ in steps, but in two different ways. The first is the point of view of Hitchin and Simpson described in Section 2.3. Namely, the space of Higgs bundles is given by

$$\mathcal{B}_E = \mu_2^{-1}(0) \cap \mu_3^{-1}(0) \subset \mathcal{C}_E \ ,$$

where the relationship between Ψ is obtained from Φ by $\Psi = \Phi + \Phi^*$, and conversely Φ is the $(1, 0)$ part of Ψ. Just like for functions on surfaces, Ψ harmonic if and only if Φ is holomorphic. Now Theorem 2.17 guarantees that the orbit of every polystable Higgs bundle intersects locus $\mu_1^{-1}(0)$ in $\mathcal{B}^{ss}$. Hence, we have

$$\mathfrak{M}^{(n)}_D = \mathcal{B}^{ss}_E /\!\!/ \mathcal{G}^{\mathbb{C}}_E = \mathbf{m}^{-1}(0)/\mathcal{G}_E = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathcal{G}_E \ .$$

The second point of view (e.g. Corlette and Donaldson, Section 3.3) comes from the observation that the space of flat connections is

$$\mathcal{C}^{flat}_E = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \subset \mathcal{C}_E \ .$$

Given $\nabla \in \mathcal{C}^{flat}_E$, the condition that the associated hol(∇)-equivariant map be harmonic is precisely that $\nabla \in \mu_3^{-1}(0)$. Indeed, suppose $\delta\nabla$ is a variation of ∇. It follows from (3.9) that $\delta\Psi = \delta\nabla + (\delta\nabla)^*$. In the case of a complex gauge transformation with $g^{-1}\delta g = \phi$, $\delta\nabla = \nabla\phi$, and

$$\delta\Psi = d_A(\phi + \phi^*) + [\Psi, \phi - \phi^*] \ .$$

It is easy to see that the second term will not contribute in the variation $\mathrm{tr}(\delta\Psi \wedge *\Psi) + \mathrm{tr}(\Psi \wedge *\delta\Psi)$ (by direct computation, and also from the fact that unitary gauge transformations do not vary the associated equivariant map). So from Lemma 3.11 we have

$$\begin{aligned}\delta E(u) &= 4\int_X \operatorname{tr}(\delta\Psi\wedge *\Psi)+\operatorname{tr}(\Psi\wedge *\delta\Psi)\\ &= 4\int_X \operatorname{tr}(d_A(\phi+\phi^*)\wedge *\Psi)+\operatorname{tr}(\Psi\wedge *d_A(\phi+\phi^*))\\ &= -8\int_X \operatorname{tr}((\phi+\phi^*)d_A(*\Psi))\ .\end{aligned}$$

Since Ψ is hermitian and ϕ is arbitrary, Ψ is a critical point for the energy if and only if $d_A(*\Psi)=0$.

Now Theorem 3.14 guarantees that the orbit of every semisimple representation contains a harmonic map. It therefore follows that the holonomy map gives a homeomorphism

$$\mathfrak{M}_B^{(n)}\simeq \mathcal{C}_E^{flat}/\!\!/\mathcal{G}_E^{\mathbb{C}}\simeq \mu_1^{-1}(0)\cap\mu_2^{-1}(0)\cap\mu_3^{-1}(0)/\mathcal{G}_E\ .$$

So the Dolbeault and Betti moduli spaces coincide!

Theorem 3.17 ([61, 62])**.** *The identification above gives a homeomorphism* $\mathfrak{M}_D^{(n)}\simeq\mathfrak{M}_B^{(n)}$.

3.4.2. *Equivariant cohomology.* As in the case of the YMH-flow, the harmonic map flow actually has continuity properties as $t\to\infty$. To describe this, let $\mathcal{G}_E(p)\subset\mathcal{G}_E$ denote the subgroup of gauge transformations that are the identity at the point p. Now the holonomy map gives a proper embedding

$$\text{hol}:\mathbf{m}^{-1}(0)/\mathcal{G}_E(p)\hookrightarrow \operatorname{Hom}(\pi,\mathsf{SL}_n(\mathbb{C}))\ ,\tag{3.14}$$

which is SU_n-equivariant.

Theorem 3.18 (cf. [15])**.** *The inclusion* (3.14) *is an* SU_n*-equivariant deformation retract.*

An explicit retraction is defined using the harmonic map flow to define a flow on the space of representations. Fix a lift $\tilde p\in\widetilde X$ of p. Given $\rho\in\operatorname{Hom}(\pi,\mathsf{SL}_n(\mathbb{C}))$, choose $\nabla\in\mathcal{C}_E^{flat}$ with $\text{hol}(\nabla)=\rho$. The hermitian metric gives a unique ρ-equivariant lift $u:\widetilde X\to D$ with $u(\tilde p)=I$. Let u_t, $t\geq 0$, denote the solution to (3.10) with initial condition u. There is a unique continuous family $h_t\in\mathsf{SL}_n(\mathbb{C})$, $h_t^*=h_t$, such that $h_0=I$, and $h_tu_t(\tilde p)=z$. Notice that a different choice of flat connection $\widetilde\nabla$ with $\text{hol}(\widetilde\nabla)=\rho$ will be related to ∇ by a based gauge transformation g. The flow corresponding to $\widetilde\nabla$ is $\tilde u_t=g\cdot u_t$, and since $g(\tilde p)=I$, $\tilde h_t=h_t$. Hence, h_t is well-defined by ρ. The flow on $\operatorname{Hom}(\pi,\mathsf{SL}_n(\mathbb{C}))$ is then defined by $\rho_t=h_t\rho h_t^{-1}$. The result states that this flow defines a continuous retraction to $\text{hol}\left(\mathbf{m}^{-1}(0)/\mathcal{G}_E(p)\right)$. When ρ is not semisimple, the flow converges to a semisimplification.

This result has consequences for computing the equivariant cohomology of moduli space [2, 44, 12]. In particular, Theorem 3.18 implies

$$H^*_{\mathsf{SU}_n}(\mathbf{m}^{-1}(0)/\mathcal{G}_E(p))\simeq H^*_{\mathsf{SU}_n}(\operatorname{Hom}(\pi,\mathsf{SL}_n(\mathbb{C}))\ .$$

Note that since $\mathsf{SL}_n(\mathbb{C})/\mathsf{SU}_n$ is contractible, on the right hand side we may take equivariant cohomology with respect to $\mathsf{SL}_n(\mathbb{C})$. On the other hand, Theorem 2.30 implies

$$H^*_{\mathsf{SU}_n}(\mathbf{m}^{-1}(0)/\mathcal{G}_E(p)) = H^*_{\mathcal{G}_E}(\mathcal{B}_E^{min}) \simeq H^*_{\mathcal{G}_E}(\mathcal{B}_E^{ss}) \ .$$

It follows that the equivariant cohomology of the space of representations may be computed by studying the equivariant Morse theory of YMH on $\mathcal{B}_E$ in the spirit of [2]. This is complicated, since $\mathcal{B}_E$ is singular. Some progress has been made using this approach (see [13, 68].

Figure 1 gives a cartoon of $\mathcal{C}_E$ with the subspaces $\mathcal{C}_E^{flat}$ and $\mathcal{B}_E$, and the flows that have been defined.

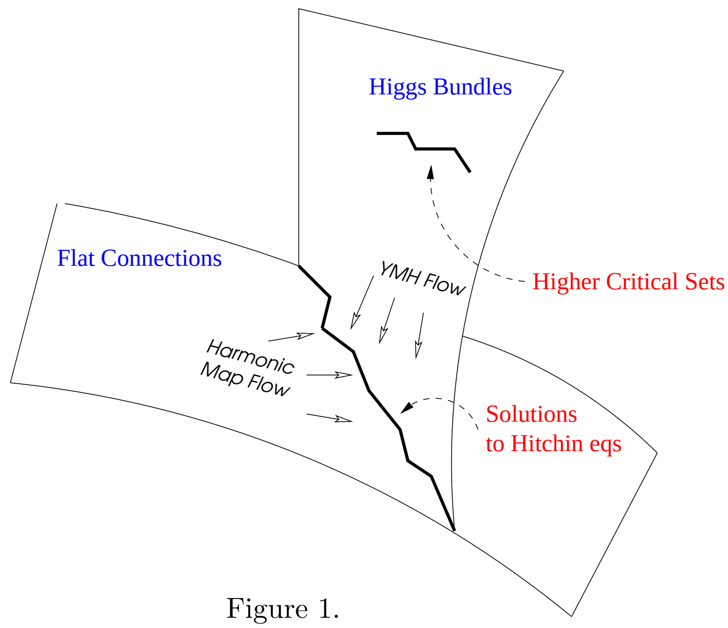

Figure 1.

4. Differential Equations

4.1. **Uniformization.** For more on the discussion in this section I refer to the classic text of Gunning [27].

Definition 4.1. The **Schwarzian derivative** of a univalent holomorphic function $f(z)$ defined on a domain in $\mathbb{C}$ is given by

$$S(f) = \{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \ .$$

By straightforward calculation one shows the following:

(i) $S(f \circ g) = (S(f) \circ g)(g')^2 + S(g)$;

(ii) $S(f) = 0 \iff f$ is the restriction of a Möbius transformation.

A particular consequence of (i) and (ii) is then

(iii) $S(f) = S(g) \Longrightarrow f = \phi \circ g$,

where ϕ is a Möbius transformation.

The Schwarzian derivative gives a link between uniformization and the monodromy of differential equations, as I briefly explain here. Let $Q(z)$, $y(z)$ be locally defined holomorphic functions, and consider the ODE

$$y''(z) + Q(z)y(z) = 0 \ . \tag{4.1}$$

If y_1, y_2 are independent solutions of (4.1) and $y_2 \neq 0$, then a calculation shows that $f = y_1/y_2$ satisfies $S(f) = 2Q$.

Note that for a univalent function f, $S = S(f)$ is not quite a tensor: rather, by (i) it transforms with respect to local coordinate changes as

$$S(w)(w')^2 = S(z) - \{w, z\} \ , \tag{4.2}$$

so S nearly transforms as a quadratic differential. A collection $\{S(z)\}$ of local holomorphic functions on X transforming as in (4.2) is called a **projective connection**. The space of projective connections on X is an affine space modeled on the space $H^0(X, \mathcal{K}^2)$ of holomorphic quadratic differentials.

Next, consider the transformation properties of the solutions y to (4.1), where $2Q = S$ is an arbitrary projective connection (cf. [33]). If we assume y is a local holomorphic section of $\mathcal{K}^{-1/2}$, then we have

$$y(z) = y(w)(w')^{-1/2} \ ;$$
$$y''(z) = y''(w)(w')^{3/2} - \tfrac{1}{2}y(z)\{w, z\} \ ,$$

and so

$$y''(z) + \tfrac{1}{2}S(z)y(z) = (y''(w) + \tfrac{1}{2}S(w)y(w))(w')^{3/2} \ .$$

We deduce that $Dy = y'' + \frac{1}{2}Sy$ gives a well-defined map of $\underline{\mathbb{C}}$-modules $D : \mathcal{K}^{-1/2} \to \mathcal{K}^{3/2}$. Therefore, given a projective connection S we have a rank 2 local system $\mathbf{V}$, defined by the solution space to (4.1), $2Q = S$. Moreover, there is an exact sequence of $\underline{\mathbb{C}}$-modules

$$0 \longrightarrow \mathbf{V} \longrightarrow \mathcal{K}^{-1/2} \longrightarrow \mathcal{K}^{3/2} \longrightarrow 0 \ .$$

Now assume X has a uniformization as a hyperbolic surface. So $\rho_F : \pi \to \mathsf{PSL}_2(\mathbb{R})$ is a discrete and faithful representation such that X is biholomorphic to $\mathbb{H}/\rho_F(\pi)$. Let u be a (multi-valued) inverse of the quotient map $\mathbb{H} \to X$. In other words, u is a univalent function $u : \widetilde{X} \to \mathbb{H}$ that is equivariant with respect to ρ_F. Set $S_F(z) = S(u)(z)$. Then by items (i) and (ii) above, for any $\gamma \in \pi$,

$$S_F(\gamma z) = S(u)(\gamma z) = S(\rho_F(\gamma) \circ u)(z) = S(u)(z) = S_F(z) \ .$$

So S_F is a well-defined projective connection on X.

Now the key point is the following: if y_1, y_2 are linearly independent solutions to (4.1) where $2Q = S_F$, then $S(y_1/y_2) = S(u)$ and so by (iii) above there is a Möbius transformation ϕ such that $y_1/y_2 = \phi \circ u$. It follows that the (projective) monodromy of the local system associated to (4.1) in the case $2Q = S_F$ is conjugate to ρ_F. If S is any fixed choice of projective connection, one may ask for the holomorphic quadratic differential Q such that $S_F = S + Q$. This is the famous problem of *accessory parameters* (cf. [53]).

Remark 4.2. I want to clarify the following issue: the bundle $\mathcal{K}^{1/2}$ involves a choice of square root of the canonical bundle (i.e. a **spin structure**), of which there are 2^{2g} possibilities. This choice is *precisely* equivalent to a lift of the corresponding monodromy ρ from $\mathsf{PSL}_2(\mathbb{C})$ to $\mathsf{SL}_2(\mathbb{C})$. To see this, let $\mathcal{V}_\rho = \mathcal{O} \otimes_{\underline{\mathbb{C}}} \mathbf{V}_\rho$, and notice that $\mathcal{V}_\rho$ fits into an exact sequence (now of $\mathcal{O}$-modules)

$$0 \longrightarrow \mathcal{K}_\rho^{1/2} \longrightarrow \mathcal{V}_\rho \longrightarrow \mathcal{K}_\rho^{-1/2} \longrightarrow 0 \ , \tag{4.3}$$

where now we also label the choice of spin structure by ρ. Since $\mathcal{V}_\rho$ has a holomorphic connection, by Theorem 3.5, (4.3) cannot split. On the other hand, the extensions are parametrized by the projective space of $H^1(X, \mathcal{K}) \simeq (H^0(X, \mathcal{O}))^* = \mathbb{C}$. So all the bundles $\mathcal{V}$ obtained in this way as ρ varies are isomorphic, modulo the choice of $\mathcal{K}^{1/2}$. Eq. (4.3) also implies that $\mathcal{V}_\rho^* \otimes \mathcal{K}_\rho^{-1/2}$ has a nonzero holomorphic section. Moreover, if we have such an exact sequence for one spin structure, then (4.3) cannot hold for any other choice $\mathcal{K}^{1/2}$. Indeed, the induced map $\mathcal{K}^{1/2} \to \mathcal{K}_\rho^{-1/2}$ would necessarily vanish, and so the inclusion $\mathcal{K}^{1/2} \to \mathcal{V}_\rho$ would lift to give an isomorphism $\mathcal{K}^{1/2} \simeq \mathcal{K}_\rho^{1/2}$. So $\mathcal{K}^{-1/2}$ is determined by ρ. Changing the lift of the projective monodromy ρ to $\mathsf{SL}_2(\mathbb{C})$ amounts to $\rho \mapsto \rho \otimes \chi$ for some character $\chi : \pi \to \mathbb{Z}/2$. This corresponds to tensoring $\mathcal{V}_\rho$ by a flat line bundle $\mathcal{L}_\chi$ whose square is trivial. It follows that from the condition that $H^0(X, \mathcal{V}_{\rho\otimes\chi}^* \otimes \mathcal{K}_{\rho\otimes\chi}^{-1/2}) \neq \{0\}$, and the argument given above, that $\mathcal{K}_{\rho\otimes\chi}^{1/2} = \mathcal{K}_\rho^{1/2} \otimes \mathcal{L}_\chi$.

4.2. **Higher order equations.**

4.2.1. *Invariance properties.* The structure outlined in the previous section for equations of the type (4.1) extends to higher order equations. We consider n-th order differential equations on $\mathbb{H}$:

$$y^{(n)} + Q_2 y^{(n-2)} + \cdots + Q_n y = 0 \ . \tag{4.4}$$

We would like an appropriate invariance property under coordinate changes in order to have solutions that are intrinsic to X. Motivated by the example of projective connections, we attempt to realize local solutions of (4.4) in the sheaf $\mathcal{K}^{1-q}$, where $n = 2q - 1$ and we have chosen a spin structure if q is a half-integer. Solutions to (4.4) are given by the kernel of an operator $\mathcal{K}^{1-q} \xrightarrow{D} \mathcal{K}^q$.

Theorem 4.3 (cf. [17], see also [69, 34])**.** *Let $D : \mathcal{K}^{1-q} \to \mathcal{K}^q$ be $\underline{\mathbb{C}}$-linear and locally of the form*

$$Dy = y^{(n)} + Q_2 y^{(n-2)} + \cdots + Q_n y \ .$$

Then $12Q_2/n(n^2-1)$ is a projective connection, and for $k \geq 3$, there exist w_k, linear combinations of Q_j, $j = 2, \ldots, k$ and derivatives, with coefficients polynomials in Q_2, such that w_k transform as a k-differentials. Conversely, given one such operator and k differentials w_k, $k = 2, \ldots, n$, these conditions uniquely determine an operator D.

The expressions for the w_k are quite complicated. For example, we reproduce some of [17, Table 1]:

$$
\begin{aligned}
w_2 &= Q_2 \\
w_3 &= Q_3 - \frac{n-2}{2} Q_2' \\
w_4 &= Q_4 - \frac{n-3}{2} Q_3' + \frac{(n-2)(n-3)}{10} Q_2'' - \frac{(n-2)(n-3)(5n+7)}{10n(n^2-1)} Q_2^2 \ .
\end{aligned}
\tag{4.5}
$$

It follows from Theorem 4.3 that the space of all such D is an affine space modeled on the Hitchin base $\bigoplus_{j=2}^n H^0(X, \mathcal{K}^j)$. The map $D : \mathcal{K}^{1-q} \to \mathcal{K}^q$ is clearly locally surjective. Moreover, the Wronskian of any fundamental set of solutions $Dy_i = 0$ is constant. We therefore obtain a local system $\mathbf{V}$ and an exact sequence of sheaves over $\underline{\mathbb{C}}$.

$$
0 \longrightarrow \mathbf{V} \xrightarrow{\ \varphi\ } \mathcal{K}^{1-q} \xrightarrow{\ D\ } \mathcal{K}^q \longrightarrow 0 \ . \tag{4.6}
$$

In this situation, we say that the local system **$\mathbf{V}$ is realized in** $\mathcal{K}^{1-q}$.

Remark 4.4. If we tensor by a line bundle with a holomorphic connection and replace derivatives $y^{(j)}$ with derivatives in a local parallel frame of the line bundle, then we can consider local systems realized in $\mathcal{L}$:

$$
0 \longrightarrow \mathbf{V} \xrightarrow{\ \varphi\ } \mathcal{L} \xrightarrow{\ D\ } \mathcal{L} \otimes \mathcal{K}^n \longrightarrow 0 \ , \tag{4.7}
$$

where $\deg \mathcal{L} = -(n-1)(g-1)$.

4.2.2. *The Riemann-Hilbert correspondence.* The goal of this section is to characterize which local systems can be realized as the monodromy of solutions to differential equations. To motivate the following, if $\mathbf{V}$ is a local system realized in $\mathcal{L}$, and $\mathcal{V} = \mathcal{O} \otimes_{\underline{\mathbb{C}}} \mathbf{V}$, notice that in (4.7) there is a surjective sheaf map $\mathcal{V} \to \mathcal{L}$ given by $f \otimes \mathbf{v} \mapsto f\varphi(\mathbf{v})$, for $f \in \mathcal{O}$, $\mathbf{v} \in \mathbf{V}$. In particular, $\mathcal{V}^* \otimes \mathcal{L}$ has a nonzero holomorphic section.

Theorem 4.5. *A representation $\rho : \pi \to \mathsf{SL}_n(\mathbb{C})$ can be realized in $\mathcal{L}$ if and only if ρ is irreducible, $H^0(X, \mathcal{V}_\rho^* \otimes \mathcal{L}) \neq \{0\}$, and $\mathcal{L}^n = \mathcal{K}^{-n(n-1)/2}$.*

Proof. According to Hejhal [34, Theorem 3], the monodromy representation arising from a differential operator D is necessarily irreducible. I shall give a proof of this fact below (see Proposition 4.8). Accepting this point for the time being, from the discussion above we also have a nonzero section of $\mathcal{V}_\rho^* \otimes \mathcal{L}$. Moreover, if $y_1, \ldots, y_n$ is an independent set of solutions $Dy_i = 0$ on $\mathbb{H}$, then the Wronskian

$$
W(y_1, \ldots y_n) = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \ ,
$$

is a well-defined nowhere vanishing global holomorphic section of $\mathcal{L}^n \otimes \mathcal{K}^{n(n-1)/2}$ on X. The latter is therefore trivial. This proves the necessity part of the assertion. For the converse, we follow a classical argument using the Wronskian (cf. [52]). Assume we have a nonzero holomorphic section φ of $\mathcal{V}_\rho^* \otimes \mathcal{L}$. This induces a map (also denoted by φ): $\mathbf{V}_\rho \to \mathcal{L}$. Because ρ is irreducible, φ is injective. Because $\mathcal{L}^n = \mathcal{K}^{-n(n-1)/2}$ we can write $\mathcal{L} = \mathcal{L}_0 \otimes \mathcal{K}^{-(n-1)/2}$, where $\mathcal{L}_0$ has a flat connection. If we express a section of $\mathcal{L}$ as $\mathbf{l} \otimes w$, where $\mathbf{l}$ is a parallel section of $\mathcal{L}_0$, then we define $y' = \mathbf{l} \otimes w'$. With this understood, choose a local frame $\{\mathbf{v}_i\}$ for $\mathbf{V}_\rho$, and set

$$Dy = \det \begin{pmatrix} \varphi(\mathbf{v}_1) & \cdots & \varphi(\mathbf{v}_n) & y \\ \varphi(\mathbf{v}_1)' & \cdots & \varphi(\mathbf{v}_n)' & y' \\ \vdots & & \vdots & \vdots \\ \varphi(\mathbf{v}_1)^{(n)} & \cdots & \varphi(\mathbf{v}_n)^{(n)} & y^{(n)} \end{pmatrix} .$$

Then if y is a local holomorphic section of $\mathcal{L}$, Dy is a well-defined local holomorphic section of $\mathcal{L}^{n+1} \otimes \mathcal{K}^{n(n+1)/2} = \mathcal{L} \otimes \mathcal{K}^n$. Clearly, the kernel of D is precisely $\mathbf{V}_\rho$. Moreover, since the monodromy of $\mathbf{V}_\rho$ is in $\mathsf{SL}_n(\mathbb{C})$, it is easy to see that Dy is actually globally defined on X. Finally, $\mathcal{L}^n = \mathcal{K}^{-n(n-1)/2}$, so

$$\det \begin{pmatrix} \varphi(\mathbf{v}_1) & \cdots & \varphi(\mathbf{v}_n) \\ \varphi(\mathbf{v}_1)' & \cdots & \varphi(\mathbf{v}_n)' \\ \vdots & & \vdots \\ \varphi(\mathbf{v}_1)^{(n-1)} & \cdots & \varphi(\mathbf{v}_n)^{(n-1)} \end{pmatrix} ,$$

is a nonzero holomorphic function on X, which may therefore be set equal to 1. Hence, Dy has the form (4.6). This completes the proof. □

Example 4.6. The lift of the monodromy of a projective connection defines a representation into $\mathsf{SL}_2(\mathbb{C})$ which, via the irreducible embedding $\mathsf{SL}_2 \hookrightarrow \mathsf{SL}_n$, gives a representation into $\mathsf{SL}_n(\mathbb{C})$. It is straightforward, if somewhat tedious, to calculate the differential equations associated to the local systems arising in this way. Below are some examples where we let $2Q$ to be a projective connection on X.

- $n = 2$: $y'' + Qy = 0$;
- $n = 3$: $y''' + 4Qy' + 2Q'y = 0$;
- $n = 4$: $y^{(4)} + 10Qy'' + 10Q'y' + (9Q^2 + 3Q'')y = 0$;
- $n = 5$: $y^{(5)} + 20Qy''' + 30Q'y'' + (64Q^2 + 18Q'')y' + (64QQ' + 4Q''')y = 0$;
- $n = 6$: $y^{(6)} + 35Qy^{(4)} + 70Q'y''' + (63Q'' + 259Q^2)y'' + (28Q''' + 518QQ')y' +$
$(130(Q')^2 + 155QQ'' + 5Q^{(4)} + 225Q^3)y = 0.$

Note that w_3, w_4 in (4.5) vanish for these examples.

4.3. **Opers.**

4.3.1. *Oper structures.* In this section we introduce opers. For more details consult [3, 4, 5, 6, 41, 63].

Definition 4.7 (Beilinson-Drinfeld [3])**.** An SL_n**-oper** is a holomorphic bundle $\mathcal{V} \to X$, a holomorphic connection ∇ inducing the trivial connection on $\det \mathcal{V}$, and a filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$ satisfying

(i) $\nabla \mathcal{V}_i \subset \mathcal{V}_{i+1} \otimes \mathcal{K}$;

(ii) the induced $\mathcal{O}$-linear map $\mathcal{V}_i/\mathcal{V}_{i-1} \xrightarrow{\nabla} \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K}$ is an isomorphism for $1 \leq i \leq n-1$.

There is an action of $\mathcal{G}^{\mathbb{C}}$ on the space of opers which pulls back connections and filtrations. Let Op_n denote the space of gauge equivalence classes of SL_n-opers on X. Given a holomorphic connection on a bundle $\mathcal{V}$, we shall call a filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$ satisfying (i) and (ii) an **oper structure**. Not every holomorphic connection admits an oper structure. For example, we have the following important

Proposition 4.8. *The holonomy representation of an oper is irreducible.*

First we have

Lemma 4.9. *For any SL_n-oper, $\det \mathcal{V}_j \simeq \mathcal{L}^j \otimes \mathcal{K}^{nj-j(j+1)/2}$, where $\mathcal{L} \simeq \mathcal{V}/\mathcal{V}_{n-1}$, and $\mathcal{L}^n \simeq \mathcal{K}^{-n(n-1)/2}$.*

Proof. To simplify notation, set $v_i = \det \mathcal{V}_i$, $\kappa = \mathcal{K}$, and use additive notation for line bundle tensor products. Then Definition 4.7 (ii) gives $v_i - v_{i-1} = v_{i+1} - v_i + \kappa$, and so

$$
\begin{aligned}
v_j = \sum_{i=1}^{j}(v_i - v_{i-1}) &= \sum_{i=1}^{j}(v_{i+1} - v_i + k) = v_{j+1} - v_1 + j\kappa \\
v_{j+1} - v_j &= v_1 - j\kappa \ .
\end{aligned}
$$

Now summing again

$$
\begin{aligned}
v_i - v_1 = \sum_{j=1}^{i-1}(v_{j+1} - v_j) &= (i-1)v_1 - \frac{i(i-1)}{2}\kappa \\
v_i &= iv_1 - \frac{i(i-1)}{2}\kappa \\
0 = v_n &= nv_1 - \frac{n(n-1)}{2}\kappa \ .
\end{aligned}
$$

Set $\mathcal{L} = v_1 - (n-1)\kappa$, and this completes the proof. □

Proof of Proposition 4.8. (cf. [41]) Suppose that $(\mathcal{V}, \nabla)$ has an oper structure and $0 \neq \mathcal{W} \subset \mathcal{V}$ is ∇-invariant. Let $\mathcal{W}_i = \mathcal{W} \cap \mathcal{V}_i$. I claim that the induced map

$$
\mathcal{W}_i/\mathcal{W}_{i-1} \longrightarrow \mathcal{W}_{i+1}/\mathcal{W}_i \otimes \mathcal{K} \ ,
$$

is an inclusion of sheaves for all $i = 1, \ldots, n-1$. Indeed, consider the commutative diagram of $\mathcal{O}$-modules:

$$\begin{array}{ccc}
\mathcal{W}_i/\mathcal{W}_{i-1} & \longrightarrow & \mathcal{W}_{i+1}/\mathcal{W}_i \otimes \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{V}_i/\mathcal{V}_{i-1} & \longrightarrow & \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K}
\end{array}$$

The vertical arrows are inclusions and the lower horizontal arrow is an isomorphism. This proves the claim. Set $r_i = \mathrm{rank}(\mathcal{W}_i/\mathcal{W}_{i-1})$. By the claim, if $r_i = 0$, then $r_j = 0$ for $j \leq i$. Let $1 \leq \ell \leq n$ be the smallest integer for which $r_\ell \neq 0$. It follows that $r_i = 1$ if and only if $\ell \leq i \leq n$.

Applying the inclusions recursively and using Lemma 4.9, we find

$$\mathcal{W}_i/\mathcal{W}_{i-1} \hookrightarrow \mathcal{V}/\mathcal{V}_{n-1} \otimes \mathcal{K}^{n-i} \cong \mathcal{K}^{(n-2i+1)/2} \ .$$

In particular (see Section 2.1.1),

$$\deg(\mathcal{W}_i/\mathcal{W}_{i-1}) \leq (n-2i+1)(g-1) \ ,$$

and so

$$\deg \mathcal{W} = \sum_{i=\ell}^{n} \deg(\mathcal{W}_i/\mathcal{W}_{i-1}) \leq \sum_{i=\ell}^{n} (n-2i+1)(g-1) = -(n-\ell+1)(\ell-1)(g-1) \ .$$

The right hand side is strictly negative unless $\ell = 1$. But since $\mathcal{W}$ has a holomorphic connection induced by ∇, $\deg \mathcal{W} = 0$. Hence, the only possibility is $\ell = 1$, which implies $\mathcal{W} = \mathcal{V}$. This completes the proof. □

We now show that if a holomorphic connection admits an oper structure, then that structure is unique up to gauge equivalence. For the next part of the discussion, it will be useful to have the following diagram in mind (cf. Lemma 4.9):

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{L} \otimes \mathcal{K}^{n-j} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{V}_{j-1} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{R}_{j-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & \mathcal{V}_j & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{R}_j & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{L} \otimes \mathcal{K}^{n-j} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{4.8}$$

Lemma 4.10. $H^1(X,(\mathcal{L}\otimes\mathcal{K}^{n-j})\otimes\mathcal{R}_i^*) = \begin{cases} 0 & i \geq j+1\ ; \\ H^1(X,\mathcal{K}) & i = j\ . \end{cases}$

Proof. Fix j and do induction on i. If $i = n-1$, then $\mathcal{R}_{n-1} = \mathcal{L}$ and

$$H^1(X,(\mathcal{L}\otimes\mathcal{K}^{n-j})\otimes\mathcal{R}_{n-1}^*) = H^1(X,\mathcal{K}^{n-j}) = \begin{cases} 0 & n-j-1 > 0\ ; \\ H^1(X,\mathcal{K}) & n = j+1\ . \end{cases}$$

Now the exact sequence $0 \to \mathcal{R}_i^* \to \mathcal{R}_{i-1}^* \to \mathcal{L}^*\otimes\mathcal{K}^{i-n} \to 0$ gives the following

$$H^1(X,(\mathcal{L}\otimes\mathcal{K}^{n-j})\otimes\mathcal{R}_i^*) \longrightarrow H^1(X,(\mathcal{L}\otimes\mathcal{K}^{n-j})\otimes\mathcal{R}_{i-1}^*) \longrightarrow H^1(X,\mathcal{K}^{i-j}) \longrightarrow 0\ .$$

By induction the first term vanishes and the last two terms are isomorphic. This proves the lemma. ☐

Lemma 4.11. $H^1(X,\mathcal{V}_j\otimes\mathcal{R}_i^*) = \begin{cases} 0 & i \geq j+1\ ; \\ H^1(X,\mathcal{K}) & i = j\ . \end{cases}$

Proof. Fix i and induct on j. Now $\mathcal{V}_1 \simeq \mathcal{L}\otimes\mathcal{K}^{n-1}$, so the result in this case follows from Lemma 4.10. Next consider the exact sequence

$$H^1(X,\mathcal{V}_{j-1}\otimes\mathcal{R}_i^*) \longrightarrow H^1(X,\mathcal{V}_j\otimes\mathcal{R}_i^*) \longrightarrow H^1(X,\mathcal{L}\otimes\mathcal{K}^{n-j}\otimes\mathcal{R}_i^*) \longrightarrow 0\ .$$

By induction, the first term vanishes and so the second and third terms are isomorphic. Again, the result follows from Lemma 4.10. ☐

Corollary 4.12. $H^1(X,\mathcal{V}_{j-1}\otimes(\mathcal{L}\otimes\mathcal{K}^{n-j})^*) = H^1(X,\mathcal{K})$

Proof. Consider the exact sequence

$$H^1(X,\mathcal{V}_{j-1}\otimes\mathcal{R}_j^*) \longrightarrow H^1(X,\mathcal{V}_{j-1}\otimes\mathcal{R}_{j-1}^*) \longrightarrow H^1(X,\mathcal{V}_{j-1}\otimes(\mathcal{L}_{j-1}\otimes\mathcal{K}^{n-j})^*) \longrightarrow 0\ .$$

By Lemma 4.11 the first term vanishes and the second is $\simeq H^1(X,\mathcal{K})$. ☐

Lemma 4.13. *The extension* $0 \to \mathcal{V}_{j-1} \to \mathcal{V}_j \to \mathcal{L}\otimes\mathcal{K}^{n-j} \to 0$, *is non-split.*

Proof. Consider the diagram:

$$\begin{array}{ccc}
 & & H^1(X,\mathcal{V}_{j-1}\otimes\mathcal{R}_j^*) \\
 & & \downarrow \\
H^0(X,\mathcal{R}_{j-1}\otimes\mathcal{R}_{j-1}^*) & \xrightarrow{I\mapsto[\beta]} & H^1(X,\mathcal{V}_{j-1}\otimes\mathcal{R}_{j-1}^*) \\
\downarrow & & \downarrow g \\
H^0(X,\mathcal{R}_{j-1}\otimes(\mathcal{L}\otimes\mathcal{K}^{n-j})^*) & \longrightarrow & H^1(X,\mathcal{V}_{j-1}\otimes(\mathcal{L}\otimes\mathcal{K}^{n-j})^*) \\
\uparrow & \nearrow_{I\mapsto[\alpha]} & \downarrow \\
H^0(X,(\mathcal{L}\otimes\mathcal{K}^{n-j})\otimes(\mathcal{L}\otimes\mathcal{K}^{n-j})^*) & & 0
\end{array} \tag{4.9}$$

By the comment following (2.5), $[\alpha]$ is the extension class of $0 \to \mathcal{V}_{j-1} \to \mathcal{V}_j \to \mathcal{L}\otimes\mathcal{K}^{n-j} \to 0$, and $[\beta]$ is the extension class of $0 \to \mathcal{V}_{j-1} \to \mathcal{V} \to \mathcal{R}_{j-1} \to 0$. By Lemma 4.11, g is injective. By tracing through the definition of the coboundary one has $[\alpha] = g[\beta]$. Finally, since $\mathcal{V}$ has a holomorphic connection and $\deg \mathcal{V}_{j-1} \neq 0$ by Lemma 4.9, it follows from Theorem 3.5 that $[\beta] \neq 0$. □

Finally, we can state the result on the uniqueness of the underlying holomorphic structures.

Proposition 4.14. *Let $(\mathcal{V}, \nabla)$ be an SL_n-oper. Then the oper structure on $\mathcal{V}$ is uniquely determined by $\mathcal{L} = \mathcal{V}/\mathcal{V}_{n-1}$. In particular, the isomorphism class of the bundle $\mathcal{V}$ is fixed on every connected component of* Op_n.

Proof. By Lemma 4.9, $\mathcal{V}_1 = \mathcal{L} \otimes \mathcal{K}^{n-1}$, and so is determined. By Corollary 4.12 and Lemma 4.13, each $\mathcal{V}_j$ is successively determined by $\mathcal{V}_{j-1}$ as the unique nonsplit extension of the sequence appearing in Lemma 4.13. Continuing in this way until $j = n$, this proves the first statement. The second statement follows as well, since by Lemma 4.9 we also have $\mathcal{L}^n \simeq \mathcal{K}^{-n(n-1)/2}$, and therefore the set of possible $\mathcal{L}$'s is discrete. □

Corollary 4.15. *The map sending an oper to its monodromy representation gives an embedding* $\mathrm{Op}_n \to \mathfrak{M}_B^{(n)}$.

Proof. Fix a representation $\rho : \pi \to \mathsf{SL}_n(\mathbb{C})$, and suppose that up to conjugation ρ is the monodromy of opers $(\mathcal{V}_\rho, \nabla_1)$ and $(\mathcal{V}_\rho, \nabla_2)$. In light of Proposition 4.14, it suffices to show that the line bundle $\mathcal{L}$ is uniquely determined by ρ. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles of degree $-(n-1)(g-1)$ such that $H^0(X, \mathcal{V}_\rho^* \otimes \mathcal{L}) \neq \{0\}$ and $H^0(X, \mathcal{V}_\rho^* \otimes \mathcal{M}) \neq \{0\}$. Let $\{\mathcal{V}_i\}$ be the oper structure for $(\mathcal{V}_\rho, \nabla_1)$, and assume $\mathcal{V}_\rho/\mathcal{V}_{n-1} = \mathcal{L}$. If $\mathcal{L}$ and $\mathcal{M}$ are not isomorphic, it follows from

$$0 \longrightarrow \mathcal{L}^* \otimes \mathcal{M} \longrightarrow \mathcal{V}_\rho^* \otimes \mathcal{M} \longrightarrow \mathcal{V}_{n-1}^* \otimes \mathcal{M} \longrightarrow 0 ,$$

that $H^0(X, \mathcal{V}_{n-1}^* \otimes \mathcal{M}) \neq \{0\}$. Now for $j \leq n-1$, $\deg \mathcal{R}_j^* \otimes \mathcal{M} < 0$, so by applying this argument successively we conclude that $H^0(X, \mathcal{V}_1^* \otimes \mathcal{M}) \neq \{0\}$. But $\mathcal{V}_1^* \otimes \mathcal{M} = \mathcal{L}^* \otimes \mathcal{M} \otimes \mathcal{K}^{1-n}$ also has negative degree, so we get a contradiction. □

Remark 4.16. There are precisely n^{2g} possibilities for the line bundle $\mathcal{L}$ in Proposition 4.14. These choices label the components of Op_n. As in Remark 4.2, these correspond precisely to the n^{2g} ways of lifting a monodromy representation in $\mathsf{PSL}_n(\mathbb{C})$ to $\mathsf{SL}_n(\mathbb{C})$. For simplicity, from now on we will always take $\mathcal{L} = \mathcal{K}^{-(n-1)/2}$ where if n is even we assume a fixed choice of $\mathcal{K}^{1/2}$.

4.3.2. *Opers and differential equations.* We first show how to obtain an oper from a local system that is realized in $\mathcal{K}^{1-q}$, $n = 2q-1$. So assume we are given the exact sequence (4.6), and set $\mathcal{V} = \mathcal{V}_n = \mathcal{O} \otimes_{\underline{\mathbb{C}}} \mathbf{V}$. For $k = 1, \ldots, n-1$, define

$$\mathcal{V}_{n-k} = \{\sum_{i=1}^n f_i \otimes \mathbf{v}_i : \sum_{i=1}^n f_i^{(j)} \varphi(\mathbf{v}_i) = 0 \ , \ j = 0, \ldots, k-1\} \ .$$

Then $\mathcal{V}_{n-k} \subset \mathcal{V}$ is a coherent subsheaf and we have exact sequences

$$0 \longrightarrow \mathcal{V}_{n-k-1} \longrightarrow \mathcal{V}_{n-k} \longrightarrow \mathcal{K}^{1-q+k} \longrightarrow 0 \ . \tag{4.10}$$

Property (i) of Definition 4.7 is clearly satisfied. Furthermore, in view of (4.10), the connection ∇ induces an $\mathcal{O}$-linear map $\mathcal{V}_{n-k-1} \to \mathcal{V}_{n-k}/\mathcal{V}_{n-k-1} \otimes \mathcal{K} \simeq \mathcal{K}^{2-q+k}$, by

$$\sum_{i=1}^{n} f_i \otimes \mathbf{v}_i \mapsto \sum_{i=1}^{n} f_i^{(k+1)} \varphi(\mathbf{v}_i) \ ,$$

and this is an isomorphism of sheaves. So property (ii) holds as well.

Conversely, suppose that $\mathcal{V}$ is a rank n holomorphic bundle with holomorphic connection ∇ that admits an oper structure. By Lemma 4.9 we have $\mathcal{V}/\mathcal{V}_{n-1} \simeq \mathcal{K}^{1-q}$. It follows that for any SL_n-oper (we continue to assume $\mathcal{L} = \mathcal{K}^{-(n-1)/2}$), $H^0(X, \mathcal{V}^* \otimes \mathcal{K}^{1-q}) \neq \{0\}$. Since the monodromy of an oper is irreducible by Proposition 4.8, the hypotheses of Theorem 4.5 are satisfied, and $(\mathcal{V}, \nabla)$ is realized in $\mathcal{K}^{1-q}$.

Theorem 4.17 (Beilinson-Drinfeld [3])**.** *The embedding above gives an isomorphism between the connected components of* Op_n *and the (affine) Hitchin base* $\bigoplus_{j=2}^{n} H^0(X, \mathcal{K}^j)$*.*

Corollary 4.18 (Teleman [64])**.** *The monodromy of a differential equation* (4.6) (*or* (4.7)) *is never unitary.*

Proof. If ρ is the monodromy, then from the correspondence above and Lemma 4.9 we see that $\mathcal{V}_\rho$ is an unstable bundle. But then from the easy direction of Theorem 2.6 (see Proposition 2.16), $\mathcal{V}_\rho$ cannot admit a flat unitary connection. □

4.3.3. *Opers and moduli space.* The main goal of this section is to prove the following

Theorem 4.19. *The map* $\mathrm{Op}_n \hookrightarrow \mathfrak{M}_B^{(n)}$ *is a proper embedding.*

By the upper semicontinuity of the Harder-Narasimhan type (see Section 2.1.2), this theorem is a direct consequence of the following

Proposition 4.20. *Among bundles with holomorphic connections, opers have strictly maximal Harder-Narasimhan type.*

We begin with

Lemma 4.21. *The Harder-Narasimhan filtration of a bundle* $\mathcal{V}$ *with an oper structure is given by the oper filtration itself.*

Proof. It suffices to show that for each $j = 0, \ldots, n-1$, $\mathcal{V}_{j+1}/\mathcal{V}_j$ is the maximally destabilizing subsheaf of $\mathcal{V}/\mathcal{V}_j$. In order to do this, let $\mu_{max}(\mathcal{V}/\mathcal{V}_j)$ denote the maximal slope of a subsheaf of $\mathcal{F} \subset \mathcal{V}/\mathcal{V}_j$, $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank}(\mathcal{V}/\mathcal{V}_j)$. We make the inductive hypothesis that

$$\mu_{max}(\mathcal{V}/\mathcal{V}_j) = \mu(\mathcal{V}_{j+1}/\mathcal{V}_j) = (n-1)(g-1) - j(2g-2) \ .$$

Note that this is trivially satisfied for $j = n-1$. Now suppose $j \leq n-2$ and let $\mathcal{F} \to \mathcal{V}/\mathcal{V}_j$ be the maximally destabilizing subsheaf. Then $\mathcal{F}$ is semistable, and from the sequence

$$0 \longrightarrow \mathcal{V}_{j+1}/\mathcal{V}_j \longrightarrow \mathcal{V}/\mathcal{V}_j \longrightarrow \mathcal{V}/\mathcal{V}_{j+1} \longrightarrow 0 \ ,$$

and the inductive hypothesis, we have

$$\mu(\mathcal{F}) \geq \mu(\mathcal{V}_{j+1}/\mathcal{V}_j) > \mu(\mathcal{V}_{j+2}/\mathcal{V}_{j+1}) = \mu_{max}(\mathcal{V}/\mathcal{V}_{j+1}) \ .$$

It follows that the induced map $\mathcal{F} \to \mathcal{V}/\mathcal{V}_{j+1}$ must vanish. Therefore, $\mathcal{F} \simeq \mathcal{V}_{j+1}/\mathcal{V}_j$, and moreover the inductive hypothesis is satisfied for j. This concludes the proof. □

Proof of Proposition 4.20. (cf. [41, Theorem 5.3.1]) Let $(\mathcal{V}, \nabla)$ be an unstable bundle with holomorphic connection. I claim that it suffices to assume that ∇ is irreducible. Indeed, in the case of rank 1 there is nothing to prove. Suppose the result has been proven for rank $< n$ and suppose $(\mathcal{V}, \nabla)$ is reducible. Since the Harder-Narasimhan type is upper semicontinuous, we may assume there is a splitting $(\mathcal{V}, \nabla) = (\mathcal{V}_1, \nabla_1) \oplus (\mathcal{V}_2, \nabla_2)$, with $n_i = \operatorname{rank} \mathcal{V}_i \geq 1$. Then by the induction hypothesis, it suffices to assume the $\mathcal{V}_i$ have the Harder-Narasimhan types of rank n_i-opers. Indeed, if not then we can change the Harder-Narasimhan types of $\mathcal{V}_i$, without changing the ordering of the slopes for $\mathcal{V}$, so that $\mathcal{V}$ has a larger Harder-Narasimhan type. Let

$$\mu_i = \mu_i^{(n)} = \mu(\mathcal{K}^{q-i}) = (n + 1 - 2i)(g - 1) \ , \tag{4.11}$$

be the Harder-Narasimhan type of a rank n-oper (see Lemmas 4.21 and 4.9). If λ_i is a reordering of the slopes $\{\mu_i^{(n_1)}, \mu_j^{(n_2)}\}$, we need to show

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i^{(n)} \ , \tag{4.12}$$

for all $k = 1, \ldots, n$, with strict inequality for some k. Assume $n_1 \geq n_2$. Without changing the ordering of the slopes we can sequentially subtract even integers from the leading entries $\mu_i^{(n)}$, $\lambda_i = \mu_i^{(n_1)}$ for $2i \leq n_1 - n_2$, and add the integers to last entries where $n_1 + n_2 + 2 \leq 2i$. Notice that the multiplicities of the resulting first and last slopes in $\{\mu_i\}$ and $\{\lambda_i\}$ are equal and will cancel in the sums, so it suffices to consider the intervening sums. This reduces the problem to one of two cases: $n_1 = n_2$ or $n_1 = n_2 + 1$ (and $n = n_1 + n_2$), where it is straightforward to verify (4.12).

With this understood, we may assume that $(\mathcal{V}, \nabla)$ is irreducible. The Harder-Narasimhan type of an oper is given by (4.11). Let $\mathcal{V}_{i-1} \subset \mathcal{V}_i$, $i = 1, \ldots, \ell$, be the Harder-Narasimhan filtration of $\mathcal{V}$, and $\lambda_i = \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$. Let $n_i = \operatorname{rank}(\mathcal{V}_i/\mathcal{V}_{i-1})$ and $d_i = n_i\lambda_i$. Then it suffices to show

$$\sum_{i=1}^{j} n_i\lambda_i \leq \sum_{i=1}^{\operatorname{rank}(\mathcal{V}_j)} \mu_i \ , \tag{4.13}$$

for $j = 1, \ldots, \ell$. The left hand side is just $\deg \mathcal{V}_j$ while the right hand side is

$$\sum_{i=1}^{\operatorname{rank}(\mathcal{V}_j)} (n + 1 - 2i)(g - 1) = (g - 1)\operatorname{rank}(\mathcal{V}_j)(n - \operatorname{rank}(\mathcal{V}_j)) \ .$$

Hence, (4.13) is equivalent to

$$\deg \mathcal{V}_j \leq (g - 1)\Big(\sum_{i=1}^{j} n_i\Big)\Big(n - \sum_{i=1}^{j} n_i\Big) \ . \tag{4.14}$$

Repeatedly apply Proposition 3.4 to find

$$\begin{aligned}
\lambda_j &\leq \lambda_{j+1} + 2g - 2 \\
\lambda_j &\leq \lambda_{j+2} + 2(2g-2) \\
\lambda_j &\leq \lambda_{j+i} + i(2g-2) \\
\lambda_j &\leq \lambda_\ell + (\ell - j)(2g-2) \ ,
\end{aligned}$$

for any $i \leq \ell - j$. This implies

$$\begin{aligned}
\frac{n_{j+1}}{n_j} d_j &\leq d_{j+1} + (2g-2)n_{j+1} \\
\frac{n_{j+i}}{n_j} d_j &\leq d_{j+i} + i(2g-2)n_{j+i} \\
\frac{n_\ell}{n_j} d_j &\leq d_\ell + (\ell - j)(2g-2)n_{j+1} \ ,
\end{aligned}$$

from which we have

$$\Big(\sum_{i=1}^{\ell-j} n_{i+j}\Big)\frac{d_j}{n_j} \leq \sum_{i=1}^{\ell-j} d_{i+j} + (2g-2)\sum_{i=1}^{\ell-j} i n_{i+j} \tag{4.15}$$

Consider first the case $j = 1$. Then (4.15) becomes

$$\begin{aligned}
\Big(\sum_{i=2}^{\ell} n_i\Big)\frac{d_1}{n_1} &\leq \sum_{i=2}^{\ell} d_i + (2g-2)\sum_{i=2}^{\ell}(i-1)n_i \\
(n - n_1)\frac{d_1}{n_1} &\leq -d_1 + (2g-2)\sum_{i=2}^{\ell}(i-1)n_i
\end{aligned}$$

$$d_1 \leq \frac{n_1}{n}(2g-2)\sum_{i=2}^{\ell}(i-1)n_i \ . \tag{4.16}$$

We claim that

$$\frac{2}{n}\sum_{i=2}^{\ell}(i-1)n_i \leq n - n_1 = \sum_{i=2}^{\ell} n_i \ . \tag{4.17}$$

Note that this combined with (4.16) proves (4.14) in the case $j = 1$. To prove the claim, let $r_i = n_i - 1 \geq 0$. Then (4.17) becomes

$$\begin{aligned}
2\sum_{i=2}^{\ell}(i-1)(r_i+1) &\leq n\sum_{i=2}^{\ell}(r_i+1) \\
2\sum_{i=2}^{\ell}(i-1)r_i + \ell(\ell-1) &\leq \left[\sum_{i=2}^{\ell}(r_i+1)\right]^2 + n_1\sum_{i=2}^{\ell}(r_i+1) \ ,
\end{aligned}$$

which holds if

$$2\sum_{i=2}^{\ell}(i-1)r_i + \ell(\ell-1) \leq \left[\sum_{i=2}^{\ell} r_i + (\ell-1)\right]^2 + (\ell-1) \ ,$$

which in turn, after canceling like terms from both sides, holds if

$$\sum_{i=2}^{\ell}(i-1)r_i \le \sum_{i=2}^{\ell}(\ell-1)r_i \ ,$$

and the latter is manifestly true since $r_i \ge 0$. Hence, (4.17) holds.

We now proceed by induction. So suppose that (4.14) holds for j. We show that it holds also for $j+1$. Adding (4.14) (for j) and (4.15) (for $j+1$) we have

$$\begin{aligned}
\deg \mathcal{V}_{j+1} = \deg \mathcal{V}_j + d_{j+1} \le (g-1)\sum_{i=1}^{j} n_i \sum_{i=1}^{\ell-j} n_{i+j} \\
- \frac{n_{j+1}}{\sum_{i=2}^{\ell-j} n_{i+j}} \sum_{i=1}^{j+1} d_i + \frac{n_{j+1}}{\sum_{i=2}^{\ell-j} n_{i+j}} (2g-2) \sum_{i=1}^{\ell-j-1} i n_{i+j+1} \\
\frac{\sum_{i=1}^{\ell-j} n_{i+j}}{\sum_{i=2}^{\ell-j} n_{i+j}} \deg \mathcal{V}_{j+1} \le (g-1)\sum_{i=1}^{j} n_i \sum_{i=1}^{\ell-j} n_{i+j} + \frac{n_{j+1}}{\sum_{i=2}^{\ell-j} n_{i+j}} (2g-2) \sum_{i=1}^{\ell-j-1} i n_{i+j+1} \\
\frac{\deg \mathcal{V}_{j+1}}{n - \sum_{i=1}^{j+1} n_i} \le (g-1)\sum_{i=1}^{j} n_i + \frac{n_{j+1}}{\sum_{i=1}^{\ell-j} n_{i+j} \sum_{i=2}^{\ell-j} n_{i+j}} (2g-2) \sum_{i=1}^{\ell-j-1} i n_{i+j+1} \ ,
\end{aligned}$$

where in going from the first inequality to the second we have used the fact that $\deg \mathcal{V}_{j+1} = \sum_{i=1}^{j+1} d_i$. Hence, it suffices to show

$$2\sum_{i=1}^{\ell-j-1} i n_{i+j+1} \le \sum_{i=1}^{\ell-j} n_{i+j} \sum_{i=2}^{\ell-j} n_{i+j} \ .$$

In terms of the r_i defined above, this becomes

$$\begin{aligned}
2\sum_{i=2}^{\ell-j}(i-1)(r_{i+j}+1) \le (r_{j+1}+1)\sum_{i=2}^{\ell-j}(r_{i+j}+1) + \left(\sum_{i=2}^{\ell-j}(r_{i+j}+1)\right)^2 \\
2\sum_{i=2}^{\ell-j}(i-1)r_{i+j} + (\ell-j)(\ell-j-1) \le (r_{j+1}+1)\sum_{i=2}^{\ell-j} r_{i+j} + (r_{j+1}+1)(\ell-j-1) \\
+ \left(\sum_{i=2}^{\ell-j} r_{i+j} + (\ell-j-1)\right)^2 \ .
\end{aligned}$$

But this is a consequence of

$$2\sum_{i=2}^{\ell-j}(i-1)r_{i+j} \le 2\sum_{i=2}^{\ell-j}(\ell-j-1)r_{i+j} \ ,$$

which obviously holds. This completes the proof of the maximality of the Harder-Narasimhan type.
We now show that if the Harder-Narasimhan type of $(\mathcal{V}, \nabla)$ is maximal then the filtration $\{\mathcal{V}_i\}$ is an oper structure. Indeed, consider the $\mathcal{O}$-linear map $\nabla : \mathcal{V}_i \to \mathcal{V}/\mathcal{V}_{i+1} \otimes \mathcal{K}$. By Remark 2.9, the minimal slope of a quotient of $\mathcal{V}_i$ is $\mu_i = \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$, whereas the maximal slope of a subsheaf of

$\mathcal{V}/\mathcal{V}_{i+1}\otimes\mathcal{K}$ is

$$\mu(\mathcal{V}_{i+2}/\mathcal{V}_{i+1}\otimes\mathcal{K}) = \mu_{i+2}+2g-2 = \mu_{i+1} = \mu_i-(2g-2) < \mu_i\ .$$

Hence, the map above must be zero, and $\nabla\mathcal{V}_i\subset\mathcal{V}_{i+1}\otimes\mathcal{K}$. By irreducibility of the connection, $\mathcal{V}_i/\mathcal{V}_{i-1}\to\mathcal{V}_{i+1}/\mathcal{V}_i\otimes\mathcal{K}$ is nonzero. Since these are line bundles with the same degree, this map is an isomorphism. Therefore, conditions (i) and (ii) in Definition 4.7 are satisfied. This completes the proof. □

4.4. **The Eichler-Shimura isomorphism.** Let us return in more detail to Example 4.6. For $q\in\frac{1}{2}\mathbb{Z}$, let V_q denote the $2q-1$ dimensional irreducible representation of $\mathsf{SL}_2(\mathbb{C})$. Let $\rho:\pi\to\mathsf{SL}_2(\mathbb{C})$ be the (lift of the) monodromy of a projective connection on X. We can realize the local system $\mathbf{V}_\rho$ in $\mathcal{K}^{-1/2}$ for some choice of spin structure. For $q\geq 3/2$, let $\mathbf{V}_q$ denote the local system obtained by composing ρ with the representation V_q:

$$\rho^{(n)}:\pi\longrightarrow\mathsf{SL}_2(\mathbb{C})\longrightarrow\mathsf{SL}(V_q)\ .$$

Then $\mathbf{V}_q$ is realized in $\mathcal{K}^{1-q}$, and we have

$$0\longrightarrow\mathbf{V}_q\longrightarrow\mathcal{K}^{1-q}\xrightarrow{\ D\ }\mathcal{K}^q\longrightarrow 0\ . \tag{4.18}$$

Since $q\geq 3/2$, $H^0(X,\mathcal{K}^{1-q})\simeq H^1(X,\mathcal{K}^q)^*=\{0\}$. This implies $H^0(X,\mathbf{V}_q)=H^2(X,\mathbf{V}_q)=\{0\}$, and the long exact sequence associated to (4.18) becomes

$$0\longrightarrow H^0(X,\mathcal{K}^q)\xrightarrow{\ \delta\ }H^1(X,\mathbf{V}_q)\longrightarrow H^1(X,\mathcal{K}^{1-q})\longrightarrow 0\ .$$

The coboundary map δ is called **Eichler integration**. The reason for the terminology is the following: if ω is a global holomorphic section of $\mathcal{K}^q$, then on sufficiently small open sets U_i we can solve the inhomogeneous equation $Dy_i=\omega\big|_{U_i}$. If we set $\mathbf{v}_{ij}=y_i-y_j$, then $\{\mathbf{v}_{ij}\}$ is a 1-cocycle with values in $\mathbf{V}_q$ which represents $\delta\omega$.

In any case, it follows that we have an isomorphism (cf. [22, 56, 28])

$$H^1(X,\mathbf{V}_q)\simeq H^0(X,\mathcal{K}^q)\oplus(H^0(X,\mathcal{K}^q))^*\ . \tag{4.19}$$

Eq. (4.19) can be used to describe the tangent space to the Betti moduli space at $[\rho^{(n)}]$ (this was explained to me by Bill Goldman [25]). From Weil's description of the tangent space,

$$T_{[\rho^{(n)}]}\mathfrak{M}_B^{(n)}\simeq H^1(X,\operatorname{End}\mathbf{V}_q)\ . \tag{4.20}$$

Now representations of $\mathsf{SL}_2(\mathbb{C})$ are self-dual: $V_q^*\simeq V_q$. By the Clebsch-Gordon rule for decomposition of tensor product representations, we have

$$\operatorname{End}V_q=(V_q\otimes V_q^*)_{\mathrm{tr}=0}\simeq(V_q\otimes V_q)_{\mathrm{tr}=0}=\bigoplus_{\substack{j=2\\ j\in\mathbb{Z}}}^{2q-1}V_j$$

(note that the trivial representation $V_{3/2}$ is eliminated by the traceless condition). This decomposition translates into one for the local system. It follows that

$$H^1(X, \mathrm{End}\,\mathbf{V}_q) = \bigoplus_{\substack{j=2 \\ j\in\mathbb{Z}}}^{2q-1} H^1(X, \mathbf{V}_j) \ .$$

Combining this with eqs. (4.19) and (4.20) we obtain

$$T_{[\rho^{(n)}]}\mathfrak{M}_B^{(n)} \simeq \bigoplus_{j=2}^{n} H^0(X, \mathcal{K}^j) \oplus (H^0(X, \mathcal{K}^j))^* \ .$$

This should be compared with (2.14)!

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