ON THE BRILL-NOETHER PROBLEM FOR VECTOR BUNDLES

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Abstract. On an arbitrary compact Riemann surface, necessary and sufficient conditions are found for the existence of semistable vector bundles with slope between zero and one and a prescribed number of linearly independent holomorphic sections. Existence is achieved by minimizing the Yang-Mills-Higgs functional.

1. Introduction

In this note we exhibit the existence of semistable vector bundles on compact Riemann surfaces with a prescribed number of linearly independent holomorphic sections. This may be regarded as a higher rank version of the classical Brill-Noether problem for line bundles.

Fix a compact Riemann surface Σ of genus $g \geq 2$ and integers r and d satisfying

$$0 \leq d \leq r \ , \quad r \geq 2 \ . \tag{1.1}$$

Then the main result may be stated as follows:

Main Theorem . *Let k be a positive integer and suppose that r and d satisfy* (1.1)*. Then the necessary and sufficient conditions for the existence of a semistable bundle of rank r and degree d on Σ with at least k linearly independent holomorphic sections are $k \leq r$ and if $d \neq 0$, $r \leq d + (r-k)g$.*

That such a criterion should hold was originally conjectured by Newstead. By analogy with the classical situation of special divisors (cf. [1, 7]) one can define the higher rank version of the Brill-Noether number:

$$\rho^{k-1}_{r,d} = r^2(g-1) + 1 - k(k-d+r(g-1)) \ . \tag{1.2}$$

Then $\rho^{k-1}_{r,d}$ is the formal dimension of the locus $W^{k-1}_{r,d}$ in the moduli space of semistable bundles of rank r and degree d. $W^{k-1}_{r,d}$ is defined as the closure of the set of stable bundles with at least k linearly independent sections. Note that the condition in the Main Theorem implies that $\rho^{k-1}_{r,d} \geq 1$, except in the trivial case $d = 0$ where $W^{k-1}_{r,d}$ is necessarily empty. The converse, in

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general, is not true. Thus, unlike the case of divisors, there are situations where $\rho^{k-1}_{r,d} \geq 0$ and $W^{k-1}_{r,d} = \emptyset$.

By a dimension counting argument, we can also give a statement, first proved by Brambila Paz, et. al., concerning the existence of stable bundles:

Corollary . *(see [5]) For $0 < d < r$ and $r \leq d + (r-k)g$, there exists a stable bundle of rank r and degree d with at least k linearly independent holomorphic sections.*

Instead of the constructive approach to theorems of this type taken in references [9, 10], we use a variational method. More precisely, we study the Morse theory of the Yang-Mills-Higgs functional (cf. [3]). The idea is simply the following: Let $(A^i, \vec{\varphi}^i)$ be a minimizing sequence with respect to the Yang-Mill-Higgs functional (2.1). Here, $\vec{\varphi}^i = (\varphi^i_1, \dots, \varphi^i_k)$ is a k-tuple of linearly independent holomorphic sections with respect to A^i. If the sequence converges to a solution to the k-τ-vortex equation, then for an appropriate choice of τ the limiting holomorphic structure is semistable (cf. [2]). Otherwise, we show that under the assumptions of the Main Theorem there exist "negative directions" which contradict the fact that the sequence is minimizing.

The energy estimates used closely follow [6]. However, an extra combinatorial argument is needed to ensure that the bundles constructed have the correct number of holomorphic sections, and this is where the assumption $r \leq d + (r-k)g$ is needed.

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2. The Yang-Mills-Higgs Functional

Let Σ, d, and r be as in the Introduction, and let k be a positive integer. Let E be a fixed hermitian vector bundle on Σ of rank r and degree d. Let $\mathcal{A}$ denote the space of hermitian connections on E, $\Omega^0(E)$ the space of smooth sections of E, and $\mathcal{H} \subset \mathcal{A} \times \Omega^0(E)^{\oplus k}$ the subspace consisting of holomorphic k-pairs. Thus,

$$\mathcal{H} = \left\{ (A, \vec{\varphi} = (\varphi_1, \dots, \varphi_k)) : D''_A \varphi_i = 0,\ i = 1, \dots, k \right\} .$$

We also set $\mathfrak{G}$ (resp. $\mathfrak{G}^{\mathbb{C}}$) to be the real (resp. complex) gauge groups, i.e. $\mathfrak{G}$ is the group of unitary automorphisms of E and $\mathfrak{G}^{\mathbb{C}}$ is its complexification. The groups $\mathfrak{G}$ and $\mathfrak{G}^{\mathbb{C}}$ then act on $\mathcal{H}$.

Given a real parameter τ, we define the Yang-Mills-Higgs functional:

$$f_\tau : \mathcal{A} \times \Omega^0(E)^{\oplus k} \longrightarrow \mathbb{R}$$

$$f_\tau(A, \vec{\varphi}) = \|F_A\|^2 + \sum_{i=1}^{k} \|D_A\varphi_i\|^2 + \frac{1}{4}\left\| \sum_{i=1}^{k} \varphi_i \otimes \varphi_i^* - \tau \mathbf{I} \right\|^2 - 2\pi\tau d \tag{2.1}$$

In the above, the $\|\cdot\|$ denotes L^2 norms. Notice that f_τ is invariant with respect to the action of $\mathfrak{G}$ on $\mathcal{H}$. Using a Weitzenböck formula we obtain (cf. [3, Theorem 4.2])

$$f_\tau(A, \vec{\varphi}) = 2\sum_{i=1}^{k} \|D''_A\varphi_i\|^2 + \left\| \sqrt{-1}\Lambda F_A + \frac{1}{2}\sum_{i=1}^{k} \varphi_i \otimes \varphi_i^* - \frac{\tau}{2}\mathbf{I} \right\|^2 ,$$

and therefore the zero set of f_τ consists of holomorphic k-pairs satisfying the k-τ-vortex equations (cf. [2]):

$$\sqrt{-1}\Lambda F_A + \frac{1}{2}\sum_{i=1}^{k} \varphi_i \otimes \varphi_i^* - \frac{\tau}{2}\mathbf{I} = 0 .$$

Proposition 2.1. (i) *The L^2-gradient of f_τ is given by*

$$\left(\nabla_{(A,\vec{\varphi})} f_\tau\right)_1 = D_A^* F_A + \frac{1}{2}\sum_{j=1}^{k} \left(D_A\varphi_j \otimes \varphi_j^* - \varphi_j \otimes D_A\varphi_j^*\right)$$

$$\left(\nabla_{(A,\vec{\varphi})} f_\tau\right)_{2,i} = \Delta_A\varphi_i - \frac{\tau}{2}\varphi_i + \frac{1}{2}\sum_{j=1}^{k} \langle\varphi_i, \varphi_j\rangle\varphi_j$$

(ii) *If $(A, \vec{\varphi}) \in \mathcal{H}$, then under the usual identification $\Omega^1(\Sigma, \operatorname{ad} E) \simeq \Omega^{0,1}(\Sigma, \operatorname{End} E)$, we have*

$$\left(\nabla_{(A,\vec{\varphi})} f_\tau\right)_1 = -D''_A \left(\sqrt{-1}\Lambda F_A + \frac{1}{2}\sum_{j=1}^{k} \varphi_j \otimes \varphi_j^* \right)$$

$$\left(\nabla_{(A,\vec{\varphi})} f_\tau\right)_{2,i} = \sqrt{-1}\Lambda F_A(\varphi_i) - \frac{\tau}{2}\varphi_i + \frac{1}{2}\sum_{j=1}^{k} \langle\varphi_i, \varphi_j\rangle\varphi_j$$

(iii) *If $(A, \vec{\varphi}) \in \mathcal{H}$ is a critical point of f_τ, then either* (I) *$\vec{\varphi} \equiv 0$ and A is a direct sum of Hermitian-Yang-Mills connections (not necessarily of the same slope), or* (II) *A splits as $A = A' \oplus A_Q$ on $E = E' \oplus E_Q$, where $(A', \vec{\varphi})$ solves the k-τ-vortex equations and A_Q is a direct sum of Hermitian-Yang-Mills connections (not necessarily of the same slope).*

Proof. (i) is a standard calculation, and (ii) follows from (i) via the Kähler identities. We are going to prove (iii). If $(A, \vec{\varphi})$ is critical, then since $\sqrt{-1}\Lambda F_A + \frac{1}{2}\sum_{j=1}^{k} \varphi_j \otimes \varphi_j^*$ is a self-adjoint holomorphic endomorphism, it must give a splitting $A = A_0 \oplus \cdots \oplus A_\ell$ according to its distinct (constant) eigenvalues $\sigma_0, \ldots, \sigma_\ell$. Write

$$\sqrt{-1}\Lambda F_A = \begin{pmatrix} -\frac{1}{2}\sum_{j=1}^k \varphi_j \otimes \varphi_j^* + \sigma_0\,\mathbf{I} & 0 & \cdots & 0 \\ 0 & \sigma_1\,\mathbf{I} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_\ell\,\mathbf{I} \end{pmatrix} .$$

Thus,

$$\begin{aligned} 0 &= \sqrt{-1}\Lambda F_A(\varphi_i) - \frac{\tau}{2}\varphi_i + \frac{1}{2}\sum_{j=1}^k \langle \varphi_i, \varphi_j\rangle \varphi_j \\ &= -\frac{1}{2}\sum_{j=1}^k \langle \varphi_i, \varphi_j\rangle \varphi_j + \sigma_0\varphi_i - \frac{\tau}{2}\varphi_i + \frac{1}{2}\sum_{j=1}^k \langle \varphi_i, \varphi_j\rangle \varphi_j \\ &= \left(\sigma_0 - \frac{\tau}{2}\right)\varphi_i \ , \end{aligned}$$

from which we obtain either Case I or Case II, depending upon whether $\vec{\varphi} \equiv 0$. □

Next, recall (cf. [2, 4]) that $\mathcal{H}$ is an infinite dimensional complex analytic variety whose tangent space at a point $(A, \vec{\varphi})$ is given by the kernel of the differential defined by

$$\begin{aligned} &d_2 : \Omega^{0,1}(\Sigma, \operatorname{End} E) \oplus \Omega^0(E)^{\oplus k} \longrightarrow \Omega^{0,1}(E)^{\oplus k} \\ &d_2(\alpha, \eta_1, \ldots, \eta_k) = (D_A''\eta_1 + \alpha\phi_1, \ldots, D_A''\eta_k + \alpha\phi_k) \end{aligned}$$

As already noted, $\mathcal{H}$ is preserved by the action of the complex gauge group $\mathfrak{G}^{\mathbb{C}}$, and the tangent space at $(A, \vec{\varphi})$ to the $\mathfrak{G}^{\mathbb{C}}$ orbit is given by the image of d_1, where

$$\begin{aligned} &d_1 : \Omega^0(\Sigma, \operatorname{End} E) \longrightarrow \Omega^{0,1}(\Sigma, \operatorname{End} E) \oplus \Omega^0(E)^{\oplus k} \\ &d_1(u) = (-D_A''u, u\phi_1, \ldots, u\phi_k) \end{aligned}$$

With this preparation, we have the following:

Proposition 2.2. *If $(A, \vec{\varphi}) \in \mathcal{H}$, then $\nabla_{(A,\vec{\varphi})} f_\tau$ is tangent to the orbits of $\mathfrak{G}^{\mathbb{C}}$. In particular, $\nabla_{(A,\vec{\varphi})} f_\tau$ is tangent to $\mathcal{H}$ itself.*

Proof. Set $u = \sqrt{-1}\Lambda F_A + \frac{1}{2}\sum_{j=1}^k \varphi_j \otimes \varphi_j^* - \frac{\tau}{2}\mathbf{I}$. By Proposition 2.1 (ii) we have that $\nabla_{(A,\vec{\varphi})} f_\tau = d_1(u)$, where d_1 is the differential defined above. The Proposition follows. □

Because of Proposition 2.2, the critical points of the functional f_τ restricted to $\mathcal{H}$ are characterized by Proposition 2.1 (iii).

A solution $(A(t), \vec{\varphi}(t))$, $t \in [0, t_0)$ to the initial value problem

$$\left(\frac{\partial A}{\partial t}, \frac{\partial \vec{\varphi}}{\partial t}\right) = -\nabla_{(A,\vec{\varphi})} f_\tau \ , \quad (A(0), \vec{\varphi}(0)) = (A_0, \vec{\varphi}_0) \ , \tag{2.2}$$

is called the L^2-gradient flow of f_τ starting at $(A_0, \vec{\varphi}_0)$. Notice that

$$\frac{d}{dt} f_\tau(A(t), \vec{\varphi}(t)) = -\left\|\nabla_{(A(t),\vec{\varphi}(t))} f_\tau\right\|^2 \ , \tag{2.3}$$

and so the energy decreases along the L^2-gradient flow.

Proposition 2.3. *Given $(A_0, \vec{\varphi}_0) \in \mathcal{H}$, there is a $t_0 > 0$ such that the L^2-gradient flow exists for $0 \le t < t_0$.*

Proof. The proof is an application of the implicit function theorem as in [8]. □

Finally, we recall from [3, 4, 2] that a holomorphic k-pair $(A, \vec{\varphi}) \in \mathcal{H}$ is called τ-stable if for all holomorphic subbundles $0 \neq F \subset E$, $\mu(F) < \tau$; and for all proper holomorphic subbundles $E_\varphi \subset E$ containing each φ_i, $\mu(E/E_\varphi) > \tau$. Here, μ denotes the Shatz slope $\mu = \deg/\text{rank}$. The version of the theorem of Bradlow, Tiwari that we shall need is the following (see [2] for more details):

Proposition 2.4. *For generic values of the parameter τ, a holomorphic k-pair $(A, \vec{\varphi})$ is τ-stable if and only if there exists a pair $(\widetilde{A}, \vec{\tilde{\varphi}})$, related to $(A, \vec{\varphi})$ by an element of $\mathfrak{G}^{\mathbb{C}}$, satisfying the k-τ-vortex equations:*

$$\sqrt{-1}\Lambda F_{\widetilde{A}} + \frac{1}{2}\sum_{i=1}^{k} \tilde{\varphi}_i \otimes \tilde{\varphi}_i^* - \frac{\tau}{2}\mathbf{I} = 0 \ .$$

Moreover, such a solution is unique up to real gauge equivalence.

3. Technical Lemmas

In this section we collect several results needed for the proof of the Main Theorem. Throughout, E will denote a holomorphic bundle of rank r and degree d on the compact Riemann surface Σ.

Lemma 3.1. *Let E be as above with $0 \le d \le r$ and $h^0(E) = k$. If either* (i) *E is semistable, or* (ii) *E satisfies the k-τ-vortex equation for some $0 < \tau < 1$ and E does not contain the trivial bundle as a split factor; then $k \le r$ and if $d \neq 0$, $r \le d + (r - k)g$.*

Proof. We first show that $k \le r$. Suppose $k \ge r$. Thus, E has at least r linearly independent holomorphic sections. If the sections generate E at every point, then $E \simeq \mathcal{O}^{\oplus r}$; in which case $d = 0$ and $k = r$. Suppose the sections fail to generate, so that we can find a point $p \in \Sigma$ and a section of E vanishing at p. Thus E contains $\mathcal{O}(p)$ as a subsheaf, which is a contradiction to (ii) (see the definition of τ-stability above). If (i) is assumed, then E is strictly semistable with $d = r$, and the bound $k \le r$ follows from induction on the rank. Note that the second inequality is also satisfied in this case.

Assume $0 < d < r$. In both cases (i) and (ii) we obtain $0 \to \mathcal{O}^{\oplus k} \xrightarrow{\pi} E \to F \to 0$, where F is locally free. By dualizing and taking the resulting long exact sequence in cohomology, we find

$$0 \longrightarrow H^0(F^*) \longrightarrow H^0(E^*) \xrightarrow{\delta} H^0(\mathcal{O}^{\oplus k}) \longrightarrow H^1(F^*) \ .$$

We are going to show that $H^0(E^*) = 0$. The result then follows by the Riemann-Roch formula. For (i), $H^0(E^*) = 0$ by semistability. For (ii), note first that $\delta = 0$. For if not, there would be a section $s : \mathcal{O} \to E^*$ with $\pi^* \circ s = \sigma \neq 0$. But σ could not have any zeros, and so $\mathcal{O}$ would be a split factor in E^*; hence, also in E. Secondly, we show that $H^0(F^*) = 0$. Let $L \subset F^*$ be a subbundle. Then τ-stability immediately implies $c_1(L^*) > \tau > 0$. Thus, in particular, F^* cannot contain $\mathcal{O}$ as a subsheaf. This completes the proof. □

Lemma 3.2. *Let E_1, E_2 be holomorphic bundles of rank r_1, r_2 and degree d_1, d_2, satisfying $0 \leq \mu_1 = d_1/r_1 \leq d_2/r_2 = \mu_2 \leq 1$. Suppose $h^0(E_1) = k_1 \leq r_1$, $h^0(E_2) = k_2 \leq r_2$, and*

$$d_2 + (r_2 - k_2 - 1)g < r_2 \leq d_2 + (r_2 - k_2)g \ .$$

Furthermore,

- *If $d_1 \neq 0$ assume $r_1 \leq d_1 + (r_1 - k_1)g$, and $k_1 r_2 \neq k_2 r_1$.*
- *If $d_1 = 0$ and $k_1 = r_1$, assume $r_2 < d_2 + (r_2 - k_2)g$.*

Then there exists a nontrivial extension $0 \to E_1 \to E \to E_2 \to 0$ such that $h^0(E) = k_1 + k_2$.

Proof. If $k_2 = 0$, the result follows from Riemann-Roch. Suppose $k_2 \geq 1$. The condition that the k_2 sections of E_2 lift for some nontrivial extension is $k_2 h^1(E_1) < h^1(E_1 \otimes E_2^*)$. Notice that

$$\begin{aligned} h^1(E_1) &= h^0(E_1) - d_1 + r_1(g-1) = k_1 - d_1 + r_1(g-1) \\ h^1(E_1 \otimes E_2^*) &= h^0(E_1 \otimes E_2^*) + r_1 r_2(\mu_2 - \mu_1 + g - 1) \\ &\geq r_1 r_2(\mu_2 - \mu_1 + g - 1) \ , \end{aligned}$$

hence, it suffices to show that

$$k_2(k_1 - d_1 + r_1(g-1)) < r_1 r_2(\mu_2 - \mu_1 + g - 1) \ , \tag{3.1}$$

or equivalently, that

$$r_1(d_2 - r_2 + (r_2 - k_2)g) - r_2 d_1 + k_2 d_1 - k_1 k_2 + k_2 r_1 > 0 \ . \tag{3.2}$$

Now if $k_2 = r_2 = d_2$, then (3.2) is trivially satisfied by the hypotheses. Similarly for $d_1 = 0$. So assume $k_2 \leq r_2 - 1$, $d_1 \neq 0$. Write $d_2 = r_2 - (r_2 - k_2)g + p$, where $0 \leq p < g$ by assumption. On the other hand,

$$d_1 \leq r_1 \frac{d_2}{r_2} \leq (d_1 + (r_1 - k_1)g) \, \frac{r_2 - (r_2 - k_2)g + p}{r_2}$$

where if $p = 0$ then either the first or the second inequality is strict. This is equivalent to

$$-\frac{d_1 p}{g} + k_1 p + (r_1 - k_1)(r_2 - k_2)(g-1) \leq r_1 p - r_2 d_1 + k_2 d_1 - k_1 k_2 + k_2 r_1 \ ,$$

and $<$ if $p = 0$. Therefore, (3.2) will follow from

$$-\frac{d_1 p}{g} + k_1 p + (r_1 - k_1)(r_2 - k_2)(g-1) > 0 \qquad (\geq \text{ if } p = 0.) \tag{3.3}$$

Now if $p = 0$ then (3.3) is trivially satisfied. Assume that $1 \leq p \leq g-1$. Then

$$\begin{aligned} -\frac{d_1 p}{g} &+ k_1 p + (r_1 - k_1)(r_2 - k_2)(g-1) \\ &> -d_1 + r_1 p - (r_1 - k_1)p + (r_1 - k_1)(r_2 - k_2)(g-1) \\ &\geq (r_1 - d_1) + (r_1 - k_1)(r_2 - k_2 - 1)(g-1) \\ &\geq 0 \ , \end{aligned}$$

which proves (3.3), (3.2), and hence the Lemma. □

In order to get an upper bound on the infimum of the Yang-Mills-Higgs functional in the next section, we shall need the following construction and energy estimate:

Lemma 3.3. *Assume* $0 < d < r$, $k \geq 1$, *and* $r \leq d + (r-k)g$. *Let* F *be a holomorphic bundle of degree* d *and rank* $r-1$ *with* $h^0(F) = k-1$. *Then there exists a non-split extension* $0 \to \mathcal{O} \to E \to F \to 0$ *with* $h^0(E) = k$.

Proof. The condition for all of the sections of F to lift is

$$\begin{aligned} (k-1)h^1(\mathcal{O}) < h^1(F^*) \quad &\Longleftrightarrow \quad g(k-1) < d + (r-1)(g-1) \\ &\Longleftrightarrow \quad r < d + (r-k)g + 1 \ , \end{aligned}$$

and hence the result. □

Proposition 3.4 (cf. [6, Prop. 3.5])**.** *Let* E_1, E_2 *be hermitian bundles with slope* μ_1, μ_2. *Let* A_1, A_2 *be hermitian connections on* E_1, E_2, *and* $\vec{\varphi}^1, \vec{\varphi}^2$ *be* k_1 *and* k_2 *tuples of holomorphic sections. Set* $k = k_1 + k_2$. *Let* τ_1, τ_2 *and* τ *be real numbers satisfying* $\mu_1 \leq \tau_1 \leq \tau < \mu_2 \leq \tau_2$, *and assume that* $(A_1, \vec{\varphi}^1)$ *and* $(A_2, \vec{\varphi}^2)$ *satisfy the* τ_1 *and* τ_2 *vortex equations, respectively. Set* $E = E_1 \oplus E_2$, $\varphi_i = (\varphi_i^1, 0)$ *for* $i = 1, \ldots, k_1$, *and* $\varphi_{k_1+i} = (0, \varphi_i^2)$ *for* $i = 1, \ldots, k_2$. *Then there exist constants* $\varepsilon_1, \varepsilon_2, \eta > 0$ *such that for all*

$$\beta \in H^{0,1}\left(\Sigma, \mathrm{Hom}(E_2, E_1)\right) \ , \quad \vec{\psi} \in \Omega^0(E)^{\oplus k} \ ,$$

with $\|\beta\| = \varepsilon_1$, $\|\vec{\psi}\| \leq \varepsilon_2$, *and*

$$\left(A_\beta = \begin{pmatrix} A_1 & \beta \\ 0 & A_2 \end{pmatrix}, \vec{\varphi} + \vec{\psi} \right) \in \mathcal{H} \ ,$$

it follows that $f_\tau(A_\beta, \vec{\varphi} + \vec{\psi}) < f_\tau(A_1 \oplus A_2, \vec{\varphi}) - \eta$.

Proof. By assumption,

$$\sqrt{-1}\Lambda F_{A_\ell} + \frac{1}{2}\sum_{j=1}^{k_\ell} \varphi_j^\ell \otimes (\varphi_j^\ell)^* = \frac{\tau}{2}\mathbf{I}_\ell \ , \quad \ell = 1, 2 \ .$$

It follows that

$$\sqrt{-1}\Lambda F_{A_1\oplus A_2}+\frac{1}{2}\sum_{j=1}^{k_1}\varphi_j^1\otimes(\varphi_j^1)^*+\frac{1}{2}\sum_{j=1}^{k_2}\varphi_j^2\otimes(\varphi_j^2)^*-\frac{\tau}{2}\mathbf{I}=\begin{pmatrix}\frac{\tau_1-\tau}{2}\mathbf{I}_1 & 0\\ 0 & \frac{\tau_2-\tau}{2}\mathbf{I}_2\end{pmatrix}$$

The argument of [6, pp. 715-716] shows that there is a constant ε_1 such that for β and A_β as in the statement,

$$f_\tau\left(A_\beta,\varphi_1^1,\dots,\varphi_{k_1}^1,\varphi_1^2,\dots,\varphi_{k_2}^2\right)<f_\tau\left(A_1\oplus A_2,\varphi_1^1,\dots,\varphi_{k_1}^1,\varphi_1^2,\dots,\varphi_{k_2}^2\right)\ .$$

Now if we take ε_2 sufficiently small the Proposition follows (note that which norms we use is irrelevant, since β and $\vec{\varphi}+\vec{\psi}$ satisfy elliptic equations, and hence the L^2 norm is equivalent to any other). □

4. Proof of the Main Theorem

Necessity of the conditions follows from Lemma 3.1, and sufficiency for $d=0$ or $d=r$ is clear as well, simply by taking direct sums of trivial line bundles or effective divisors of degree 1, respectively. To prove sufficiency in general, we shall proceed by induction on the rank. The case $r=2$, $d=1$ is clear from a direct construction. Indeed, we may choose any non-trivial extension $0\to\mathcal{O}\to E\to L\to 0$ where $\deg L=1$, and E will be stable and have one non-trivial section. Assume the Main Theorem holds for bundles of rank $<r$. We show that it holds for r as well. Let $\mathcal{H}^*\subset\mathcal{H}$ denote the subspace of k-pairs $(A,\vec{\varphi}=(\varphi_1,\dots,\varphi_k))$ such that the sections $\varphi_1,\dots,\varphi_k$ are linearly independent. Fix τ as in Assumption 1 of [4], i.e. $\mu(E)<\tau=\mu(E)+\gamma<\mu_+$, where μ_+ is the smallest possible slope greater that $\mu=\mu(E)$ of a subbundle of E (note that $0<\tau<1$ and that we also normalize the volume of Σ to be 4π).

Lemma 4.1. *Let $m=\inf_{\mathcal{H}^*}f_\tau$. Then $0\le m<\pi/(r-1)$.*

Proof. Let F be a vector bundle of degree d and rank $r-1$. Then by the inductive hypothesis, Lemma 3.2, and Proposition 2.4, we may assume there exist hermitian connections A_1 and A_2 on $\mathcal{O}$ and F, respectively, and holomorphic sections $\varphi_1\neq 0$ in $H^0(\Sigma,\mathcal{O})$, and $\varphi_2,\dots,\varphi_k$ linearly independent sections in $H^0(\Sigma,F)$, such that (A_1,φ_1) and $(A_2,\varphi_2,\dots,\varphi_k)$ satisfy the τ_1 and τ_2 vortex equations, respectively, for $\tau_1=\tau$, $\tau_2=d/(r-1)+\gamma$. It follows from Lemma 3.3 and Proposition 3.4 that there is a nontrivial extension $\beta:0\to\mathcal{O}\to E\to F\to 0$, an $\eta>0$, and a (smooth) $\vec{\psi}$ such that $(A_\beta,\vec{\varphi}+\vec{\psi})\in\mathcal{H}^*$ and

$$\begin{aligned}f_\tau(A_\beta,\vec{\varphi}+\vec{\psi})&<f_\tau(A_1\oplus A_2,\varphi_1,\dots,\varphi_k)-\eta\\&=\left\|\frac{1}{2}\left(\frac{d}{r-1}-\frac{d}{r}\right)\mathbf{I_F}\right\|^2-\eta<\frac{\pi}{r-1}\ .\end{aligned}$$

□

Let $(A^i, \vec{\varphi}^i)$ be a minimizing sequence in $\mathcal{H}^*$. Thus, $f_\tau(A^i, \vec{\varphi}^i) \to m$. By weak compactness (more precisely, see the argument in [4, Lemma 5.2]) there is a subsequence converging to $(A^\infty, \vec{\varphi}^\infty)$ in the C^∞ topology. By the continuity of f_τ with respect to the C^∞ topology, Propositions 2.3 and 2.2, and equation (2.3), it follows that $(A^\infty, \vec{\varphi}^\infty)$ is a critical point of f_τ. If the holomorphic structure E^∞ defined by A^∞ is semistable, then by upper semicontinuity of the dimension of the space of sections we are finished. We therefore assume E^∞ is unstable and derive a contradiction. According to Proposition 2.1 (iii) we must consider the following cases:

$$\text{(I)} \qquad \vec{\varphi}^\infty = 0 \ , \quad E^\infty = E_1 \oplus \cdots \oplus E_\ell$$

$$\text{(II)} \qquad \vec{\varphi}^\infty \neq 0 \ , \quad E^\infty = E_\varphi \oplus E_1 \oplus \cdots \oplus E_\ell$$

Set $\mu_j = \mu(E_j)$, and assume $\mu_1 < \cdots < \mu_\ell$. If $\mu_\ell > 1$, then

$$f_\tau\left(A^\infty, \vec{\varphi}^\infty\right) \geq \pi(\mu_\ell - \tau)^2 r_\ell \geq \pi(\mu_\ell - 1)^2 r_\ell \geq \frac{\pi}{r_\ell} \geq \frac{\pi}{r-1} > m \ ,$$

contradicting Lemma 4.1. Similarly, if $\mu_1 < 0$, then

$$f_\tau\left(A^\infty, \vec{\varphi}^\infty\right) \geq \pi(\mu_1 - \tau)^2 r_1 \geq \pi(\mu_1)^2 r_1 \geq \frac{\pi}{r_1} \geq \frac{\pi}{r-1} > m \ ;$$

also a contradiction. We therefore rule out these possibilities. We will consider Cases I and II separately.

Case I. Let $k_i = h^0(E_i)$. By upper semicontinuity, $\sum_{i=1}^{\ell} k_i \geq k$. If $\mu_\ell = 1$, then we may replace E_ℓ by a Hermitian-Yang-Mills bundle $\widehat{E}_\ell$ with exactly $\hat{k}_\ell = r_\ell$ sections. Hence, we may assume that

$$d_\ell + (r_\ell - \hat{k}_\ell - 1)g < r_\ell \leq d_\ell + (r_\ell - \hat{k}_\ell)g \ .$$

For $1 < i < \ell$, the inductive hypothesis implies that we may replace E_i by a Hermitian-Yang-Mills bundle $\widehat{E}_i$ with

$$h^0(\widehat{E}_i) = \hat{k}_i = \left[\frac{d_i + r_i(g-1)}{g}\right] \ ,$$

the maximal number of sections allowed for d_i, r_i, and g. Note that

$$d_i + (r_i - \hat{k}_i - 1)g < r_i \leq d_i + (r_i - \hat{k}_i)g \ . \tag{4.1}$$

If $\mu_1 \neq 0$, then we can replace E_1 by $\widehat{E}_1$ as above. If $\mu_1 = 0$, we may replace E_1 with $\mathcal{O}^{\oplus r_1}$, with $\hat{k}_1 = r_1 \geq k_1$ sections. By our choices of $\hat{k}_i$, $\sum_{i=1}^{\ell} \hat{k}_i \geq \sum_{i=1}^{\ell} k_i \geq k$.

Let $0 \leq \mu_1 < \cdots < \mu_s \leq \mu < \mu_{s+1} < \cdots < \mu_\ell \leq 1$. Suppose first that $\mu_s \neq 0$. By Lemma 3.2 there is a nontrivial extension $0 \to \widehat{E}_s \to G \to \widehat{E}_{s+1} \to 0$, with $h^0(G) = \hat{k}_s + \hat{k}_{s+1}$. Thus,

$$h^0\left(\widehat{E}_1 \oplus \cdots \oplus \widehat{E}_{s-1} \oplus G \oplus \widehat{E}_{s+1} \oplus \cdots \oplus \widehat{E}_\ell\right) = \sum_{i=1}^{\ell} \hat{k}_i \geq k \ .$$

On the other hand, by Proposition 3.4 there is a hermitian connection on $\widehat{E}_1 \oplus \cdots \oplus \widehat{E}_{s-1} \oplus G \oplus \widehat{E}_{s+1} \oplus \cdots \oplus \widehat{E}_\ell$ and linearly independent sections

$\varphi_1, \ldots, \varphi_k$ such that $f_\tau(A, \vec{\varphi}) < f_\tau(A_\infty, 0) = m$, contradicting the minimality of $(A_\infty, 0)$.

Now suppose $\mu_s = \mu_1 = 0$, $\mu < \mu_i$ for $2 \leq i \leq \ell$. If for any $2 \leq i \leq \ell$ we have $r_i < d_i + (r_i - \hat{k}_i)g$, then by Lemma 3.2 there is a nontrivial extension $0 \to \widehat{E}_1 \to G \to \widehat{E}_i \to 0$, with $h^0(G) = \hat{k}_1 + \hat{k}_i$, and Proposition 3.4 yields a contradiction as before. Suppose that for all $2 \leq i \leq \ell$, $r_i = d_i + (r_i - \hat{k}_i)g$. We claim that $\sum_{i=1}^{\ell} \hat{k}_i > k$. For if $\sum_{i=1}^{\ell} \hat{k}_i = k$, then $\sum_{i=2}^{\ell}(r_i - \hat{k}_i) = r - k$, and hence

$$r > \sum_{i=2}^{\ell} r_i = \sum_{i=2}^{\ell} d_i + (r_i - \hat{k}_i)g = d + (r-k)g \ ;$$

a contradiction. Thus, we may replace $\widehat{E}_1$ by a bundle $\widehat{E}_1'$ having $\hat{k}_1' = \hat{k}_1 - 1$ sections. According to Lemma 3.2 there is a nontrivial extension $0 \to \widehat{E}_1' \to G \to \widehat{E}_2 \to 0$, with $h^0(G) = \hat{k}_1' + \hat{k}_2$, $\hat{k}_1' + \sum_{i=2}^{\ell} \hat{k}_i \geq k$, and Proposition 3.4 yields a contradiction as before.

Case II. First notice that by the invariance of the Yang-Mills-Higgs equations under the natural action by $\mathrm{U}(k)$, we may assume that $\varphi_1^\infty, \ldots, \varphi_k^\infty$ form an L^2-orthogonal set of sections. In particular, we may assume that there exists $s \leq k$ such that $\varphi_1^\infty, \ldots, \varphi_s^\infty$ are linearly independent and $\varphi_{s+1}^\infty, \ldots, \varphi_k^\infty \equiv 0$. Write $E_\varphi = E_\varphi' \oplus \mathcal{O}^{\oplus t}$, where E_φ' contains no split factor of $\mathcal{O}$. Set $k_i = h^0(E_i)$, $k_\varphi = h^0(E_\varphi)$, $k_\varphi' = h^0(E_\varphi') = k_\varphi - t$. By upper semicontinuity, $k_\varphi + \sum_{i=1}^{\ell} k_i \geq k$. As in Case I, we may replace each E_i by a Hermitian-Yang-Mills bundle $\widehat{E}_i$ such that $h^0(\widehat{E}_i) = \hat{k}_i \geq k_i$, and (4.1) is satisfied for $i = 1, \ldots, \ell$. On the other hand, since E_φ satisfies the k-τ-vortex equation for $\tau = \mu + \gamma$ as above, it follows that E_φ' is τ-stable. Therefore, $0 \neq \mu(E_\varphi') \leq \mu = \mu(E)$; and since $\tau < 1$, we obtain from Lemma 3.1 that $r_\varphi' \leq d_\varphi + (r_\varphi' - k_\varphi')g$. Finally, notice that since E^∞ is unstable, $\mu_\ell > \mu$. We may now apply Lemma 3.2 to E_φ' and $\widehat{E}_\ell$ to obtain a nontrivial extension $0 \to E_\varphi' \to G \to \widehat{E}_\ell \to 0$, with $h^0(G) = k_\varphi' + \hat{k}_\ell$. It follows that

$$h^0\left(G \oplus \mathcal{O}^{\oplus t} \oplus \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_{\ell-1}\right) = k_\varphi + \sum_{i=1}^{\ell} \hat{k}_i \geq k \ .$$

By Proposition 3.4 there is a hermitian connection A on $G \oplus \mathcal{O}^{\oplus t} \oplus \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_{\ell-1}$ and linearly independent sections $\varphi_1, \ldots, \varphi_k$ extending $\varphi_1^\infty, \ldots, \varphi_s^\infty$ such that $f_\tau(A, \vec{\varphi}) < f_\tau(A_\infty, \vec{\varphi}^\infty) = m$, again contradicting the minimality of m. This completes the proof of the Main Theorem.

5. Proof of the Corollary

Let $\mathfrak{B}_\tau$ denote the set of gauge equivalence classes of solutions to the k-τ-vortex equation for bundles of rank r and degree d, where τ is chosen as in the proof of the Main Theorem. Let $\mathfrak{B}_\tau^*$ denote the open subset of pairs $(E, \varphi_1, \ldots, \varphi_k)$ such that the φ_i are linearly independent as sections of E. By the Main Theorem and Lemma 3.2 it follows that $\mathfrak{B}^*_\tau$ is nonempty. One can therefore show as in [2, 4] that $\mathfrak{B}^*_\tau$ is a smooth complex manifold of dimension $r^2(g-1)+k(d-r(g-1))$ with a holomorphic map $\psi : \mathfrak{B}^*_\tau \to \mathfrak{M}(r,d)$, where $\mathfrak{M}(r,d)$ is the moduli space of semistable bundles of rank r and degree d and where the map ψ sends a pair $[E,\vec{\varphi}]$ to $[E]$. Let $\mathfrak{B}'_\tau \subset \mathfrak{B}^*_\tau$ denote the subset where the bundle E is stable.

Proposition 5.1. *Let $W^{k-1}_{r,d}$ denote the closure of $\psi(\mathfrak{B}'_\tau)$ in $\mathfrak{M}(r,d)$. If $\mathfrak{B}'_\tau \neq \emptyset$, then every irreducible component of $W^{k-1}_{r,d}$ has dimension $\rho^{k-1}_{r,d} = r^2(g-1)+1-k(k-d+r(g-1))$.*

Proof. Consider first a pair $[E,\vec{\varphi}] \in \mathfrak{B}'_\tau$ where $h^0(E)=k$. Notice that $\mathfrak{B}'_\tau$ is smooth. Moreover,

$$\dim_{[E,\vec{\varphi}]} \mathfrak{B}'_\tau = \dim_{\psi([E])} W^{k-1}_{r,d} + \dim \psi^{-1}([E])$$

and the dimension formula holds, since $\dim \psi^{-1}([E]) = k^2-1$. Since $h^0(E) = k$ is an open condition in $W^{k-1}_{r,d}$, the Proposition follows from

Lemma 5.2. *Suppose that E_0 is a semistable (resp. stable) bundle of rank r, degree d, $0<d\leq r$, and $h^0(E_0)=k\geq 1$. Then there exists a sequence of semistable (resp. stable) bundles E_j of the same rank and degree with $h^0(E)=k-1$ and $E_j \to E_0$ in $\mathfrak{M}(r,d)$.*

Proof. By Lemma 3.1, $k \leq r$. The case where $d=r$ and E_0 is strictly semistable is trivial. In the other cases, $k<r$, and we may write

$$\beta_0 : 0 \to \mathcal{O}^{\oplus k} \to E_0 \to F \to 0 \ ,$$

where by assumption the connecting homomorphism $\delta_0 : H^0(F) \to H^1(\mathcal{O}^{\oplus k})$ is injective. Consider $\{L_t : t \in D\}$ a smooth local family of line bundles parametrized by the open unit disk $D \subset \mathbb{C}$ and satisfying $L_0 = \mathcal{O}$ and $H^0(L_t)=0$, $t \neq 0$. Set $G_t = \mathcal{O}^{k-1} \oplus L_t$. The semistability of E_0 implies that $H^0(F^* \otimes G_t)=0$. Hence, $\{H^1(F^*\otimes G_t) : t \in D\}$ defines a smooth vector bundle V over D. Let $\beta = \{\beta(t) : t \in D\}$ be a nowhere vanishing section of V with $\beta(0)=\beta_0$. Then β defines a smooth family of nonsplit extensions $0 \to G_t \to E_t \to F \to 0$ and a smooth family of connecting homomorphisms

$$\delta_t : H^0(F) \longrightarrow H^1(G_t) \subset \Omega^{0,1}(U) \ ,$$

where U is the trivial rank k, C^∞ vector bundle on Σ. By assumption, δ_0 is injective; hence, δ_t is injective for small t. It follows that $h^0(E_t)=k-1$ for small t. Furthermore, since E_0 is semistable (resp. stable) then E_t is also semistable (resp. stable) for small t. □

Continuing with the proof of the Corollary, we first take care of a borderline situation:

Lemma 5.3. *If $r = d+(r-k)g$, $0<d<r$, then there exists a stable bundle of rank r and degree d with k linearly independent holomorphic sections.*

Proof. Note that $k < r$. Let F be a stable bundle of rank $r-k$ and degree d. Consider an extension

$$\beta : 0 \to \mathcal{O}^{\oplus k} \to E \to F \to 0 \ ,$$

obtained by choosing a basis for $H^1(F^*)$. We claim that any such E is stable. For suppose there is a proper semistable quotient $E \to Q \to 0$ with $\mu(Q) \leq \mu(E)$. By the stability of F we also have $\mu(Q) \geq 0$. Let ℓ be the rank of the image of the induced map $\mathcal{O}^{\oplus k} \to Q$. Note that by the choice of β we cannot have $Q \simeq \mathcal{O}^{\oplus \ell}$, and so from Lemma 3.1 we obtain a locally free quotient $Q' \simeq Q/\mathcal{O}^{\oplus \ell}$ of F. Again applying Lemma 3.1 we find

$$\mu(Q') = \frac{\deg Q}{\operatorname{rk} Q - \ell} \leq \frac{\mu(Q)}{1-\mu(Q)} g \leq \frac{\mu(E)}{1-\mu(E)} g = \mu(F) \ .$$

By the stability of F we must have $Q' \simeq F$, which contradicts the properness of Q. $\square$

The proof of the Corollary is completed by the following

Lemma 5.4. *Let d and r be as in the statement of the Corollary. We choose k to be the maximal integer such that $r \leq d + (r-k)g$. By Lemma 5.3, we may also assume $r < d + (r-k)g$. Let $\mathfrak{B}_\tau^* \subset \mathfrak{B}_\tau$ be as above. Then no irreducible component of the image of $\mathfrak{B}_\tau^*$ under the map ψ can be contained in $\mathfrak{M}(r_1, d_1) \times \mathfrak{M}(r_2, d_2) \subset \mathfrak{M}(r, d)$ for any choice of integers r_1, r_2, d_1, d_2 satisfying $r_1 + r_2 = r$, $d_1 + d_2 = d$, and $d_1/r_1 = d_2/r_2 = d/r$.*

Proof. Assume the image of ψ is contained in such a locus; we shall derive a contradiction. Let $[E, \vec{\varphi}] \in \mathfrak{B}_\tau^*$, and suppose $\psi(E, \vec{\varphi}) \simeq E_1 \oplus E_2$. By semicontinuity of cohomology, $k \leq k_1 + k_2$, where $k_i = h^0(E_i)$. On the other hand, notice that $d_1 - (r_1 - k_1 - 1)g < r_1$, and $d_2 - (r_2 - k_2 - 1)g < r_2$, since otherwise by Lemma 3.1, $r \leq d + (r-(k+1))g$, contradicting the maximality of k. Now as in the proof of Lemma 3.2 (see (3.1)) we obtain

$$k_2(k_1 - d_1 + r_1(g-1)) < r_1 r_2 (g-1) - 1 \, , \tag{5.1}$$

$$k_1(k_2 - d_2 + r_2(g-1)) < r_1 r_2 (g-1) - 1 \, . \tag{5.2}$$

On the other hand, we may assume that E_1 and E_2 are stable and non-isomorphic (otherwise the inequalities are even sharper), and by Proposition 5.1 we may assume $E_1 \oplus E_2$ lies in a subvariety $S \subset \mathfrak{M}(r_1, d_1) \times \mathfrak{M}(r_2, d_2)$ of dimension at most $\rho^{k_1 - 1}_{r_1, d_1} + \rho^{k_2 - 1}_{r_2, d_2}$. Finally, we have

$$\dim_{[E, \vec{\varphi}]} \mathfrak{B}_\tau^* \leq \dim_{[E_1 \oplus E_2]} S + \dim \psi^{-1}([E_1 \oplus E_2]) \, . \tag{5.3}$$

Since $\psi^{-1}([E_1 \oplus E_2])$ consists of extensions E of E_2 by E_1 such that the sections of E_2 lift, or vice versa, together with k sections of E, it follows that

$$\dim \psi^{-1}([E_1 \oplus E_2]) = \max \begin{cases} h^1(E_1^* \otimes E_2) - k_1 h^1(E_2) + k^2 - 1 \\ h^1(E_2^* \otimes E_1) - k_2 h^1(E_1) + k^2 - 1 \ . \end{cases} \tag{5.4}$$

By combining (5.3) and (5.4) we obtain either

$$k_1(k_2 - d_2 + r_2(g-1)) \geq r_1 r_2(g-1) - 1$$

or

$$k_2(k_1 - d_1 + r_1(g-1)) \geq r_1 r_2(g-1) - 1\,,$$

contradicting either (5.1) or (5.2). This completes the proof of the Lemma. □

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