Commun. Math. Phys. 282, 339–355 (2008)
Digital Object Identifier (DOI) 10.1007/s00220-008-0560-z

Communications in **Mathematical Physics**

Precise Constants in Bosonization Formulas on Riemann Surfaces. I

Richard A. Wentworth⋆

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA.
E-mail: wentworth@jhu.edu

Received: 2 March 2007 / Accepted: 13 March 2008
Published online: 3 July 2008 –

Abstract: A computation of the constant appearing in the spin-1 bosonization formula is given. This constant relates Faltings' delta invariant to the zeta-regularized determinant of the Laplace operator with respect to the Arakelov metric.

1. Introduction

The bosonization formulas on Riemann surfaces relate zeta-regularized determinants of Laplace operators acting on sections of line bundles and on scalars [AMV, VV, ABMNV, Sn, DS, F2]. They play an important role in conformal field theory and perturbative string theory (for a survey of the subject, see [DP3]). Proofs of these identities in the mathematical literature generally proceed by computing either first or second variations of certain combinations of Green's and theta functions with respect to the Riemann moduli. As a consequence, all current formulations leave undetermined constants of integration, depending only on the genus and spin, which must be evaluated by other means. There has been renewed interest recently in the precise values of these constants (cf. [DGP]).

In [W1], using ideas of Belavin-Knizhnik [BK] and D'Hoker-Phong [DP1], along with the results in [W2] on the behavior of the Arakelov metric on degenerating surfaces, values for the constants c_g associated to genus $= g$ and spin $= 1$ were obtained. The argument was heuristic, however, since it involved techniques from functional integration where normalizations can be somewhat arbitrary. The goal of this note is to give a rigorous proof of the result in [W1].

For the purposes of this paper, we may define c_g in terms of the relationship between Faltings' delta invariant $\delta(M)$ of a compact Riemann surface M of genus $g \geq 1$ and the determinant of the Laplace operator on functions with respect to the Arakelov metric (cf. [Fl]),

$$\delta(M) = c_g - 6 \log \frac{\det' \Delta_{(M,\text{Arak.})}}{\text{area}(M, \text{Arak.})}. \tag{1.1}$$

⋆ Research supported in part by NSF grant DMS-0505512.

The spin-1 bosonization formula states that

$$\left(\frac{\det' \Delta_{(M,\text{Arak.})}}{\text{area}(M, \text{Arak.}) \det \operatorname{Im} \Omega}\right)^{3/4} = e^{c_g/8} \frac{\|\vartheta\|(p_1 + \cdots + p_g - z - \Delta) \prod_{i<j} G(p_i, p_j)}{\|\det \omega_i(p_j)\| \prod_{i=1}^{g} G(p_i, z)}, \tag{1.2}$$

where ϑ is the theta function associated to M, Δ is the Riemann divisor, ω_i is a basis of abelian differentials, Ω is the period matrix, $G(z, w)$ is the Arakelov-Green's function, and $\{p_i, z\} \subset M$ are generic points.

The main result is

Theorem 1.3. *The value of* c_g *in* (1.1) *is*

$$\begin{aligned} c_g &= (1-g)c_0 + gc_1, \\ c_1 &= -8\log(2\pi), \\ c_0 &= -24\zeta'(-1) + 1 - 6\log(2\pi) - 2\log(2), \end{aligned}$$

where $\zeta(s)$ *is the Riemann zeta function.*

Remark 1.4. In [W1] one of the terms in the computation of the conformal anomaly was mistakenly neglected (see [W3] and the proof of Lemma 5.2 below). With this correction, the expression in Theorem 1.3 agrees with the result in [W1]. It is noteworthy that the path integral approach to factorization computes this constant exactly.

Remark 1.5. The constant:

$$\begin{aligned} a_g &= (1-g)a_0, \\ a_0 &= c_0 - c_1 = -24\zeta'(-1) + 1 + 2\log(2\pi) - 2\log(2), \end{aligned}$$

is called "Deligne's constant" and is the normalization in another formulation of the bosonization formulas (see [D]). We note that this value of c_g implies that formally: $\delta(\mathbb{P}^1) = 0$ (see [J2, §7]).

Remark 1.6. In addition to [W1], expressions for c_g have previously appeared in the mathematical literature (cf. [GSo, So, J2]). These disagree slightly with the result obtained in Theorem 1.3.

The proof of Theorem 1.3 is obtained via degeneration. Let M_t denote a family of genus g Riemann surfaces degenerating as $t \to 0$ to a semistable nodal curve M_0 with irreducible components M^+ and M^- of genus g^+ and g^-, respectively. We will use the following

Theorem 1.7 ([W2], Main Theorem; see also [J1]).

$$\lim_{t\to 0}\left(\delta(M_t) + \frac{4g^+g^-}{g}\log|t|\right) = \delta(M^+) + \delta(M^-).$$

To evaluate c_g we compare this result with the degeneration of the determinant of the Laplace operator for the Arakelov metric, and then we use (1.1). The asymptotic behavior of determinants of laplacians is a widely studied subject (cf. [DP2, Sa, K, V, GIJR, H, Wo1]). The new ingredient that is useful here is the result of [BFK]. Let γ denote a simple closed separating curve on M, with $R^+ \cup R^- = M - \gamma$ the connected components. Given a Riemannian metric h on M, let $R^\pm$ have the induced metrics.

Theorem 1.8 ([BFK], Theorem B*).

$$\frac{\det' \Delta_{(M,h)}}{\mathrm{area}(M,h)} = \det\Delta_{(R^+,h)}\det\Delta_{(R^-,h)}\frac{\det' \mathcal{N}_{(M,\gamma,h)}}{\ell_h(\gamma)}.$$

Here, the determinants on the right-hand-side are evaluated with respect to the spectra for the Dirichlet problems on the surfaces with boundary $R^{\pm}$, $\ell_h(\gamma)$ denotes the length of γ with respect to the metric h, and $\mathcal{N}_{(M,\gamma,h)}$ denotes the Neumann jump operator for functions on γ; in this case, it is the sum of the Dirichlet-to-Neumann operators for $R^{\pm}$.

In Sect. 2 we show how Theorem 1.8 can be used to factorize $\det' \Delta_M$ in terms of $\det' \Delta_{M^{\pm}}$ (see Proposition 2.5). This argument is closely parallel to the one in [W1], where the sewing property of path integrals was used in lieu of Theorem 1.8. In Sect. 3 we make a simple observation concerning the asymptotic behavior of the Neumann jump operator $\mathcal{N}_{(M,\gamma,h)}$ as the surface degenerates along γ. The key result, Proposition 3.5, is that $\mathcal{N}_{(M,\gamma,h)}$ is equal to a universal operator on L^2-functions on the circle, modulo trace-class operators with asymptotically vanishing norm (for more detailed treatments, see [GG,PW,L,MM]). In Sect. 4 we briefly review the results needed on the Arakelov metric, and in Sect. 5 we complete the proof of Theorem 1.3.

As a final remark, we emphasize that the method described above gives the asymptotic behavior of determinants up to the zero$^{\text{th}}$ order term for *any* family of conformal metrics, provided one has sufficient information on the degeneration (essentially, a C^0 estimate on the metric and its curvature, with bounded growth of derivatives; see Proposition 4.6). This paper illustrates the case of the Arakelov metric, but the technique applies equally well to the hyperbolic metric, for example. Indeed, using the expansion in [Wo2] one can quickly recover the asymptotic behavior for the case of pinching along a separating curve,

$$\log\det'\Delta_{(M_t,hyp.)} = (1/6)\log|t| + O(1)$$

that is a consequence of the (more precise) expression given in [Wo1, Theorem 5.3], without passing through the explicit evaluation of determinants in terms of Selberg zeta functions. This also explains why evaluating the next order term is more difficult in this case, since the procedure in [Wo2] relates the hyperbolic metric on M_t to the complete hyperbolic metrics on the punctured surfaces $M^{\pm} - \{p^{\pm}\}$, and not to the hyperbolic metrics on the closed surfaces $M^{\pm}$. The conformal factors relating these two are, of course, quite complicated (cf. [JL]).

2. Factorization of Determinants

Let M^+ and M^- be a pair of closed Riemann surfaces with genera $g^{\pm} \geq 1$. For a complex parameter t, $0 < |t| \leq 1$ we construct a degenerating family M_t of closed surfaces of genus $g = g^+ + g^-$ using the "plumbing construction": $z^+z^- = t$, for local coordinates $z^{\pm}$ on $M^{\pm}$ centered at points $p^{\pm}$ (for more details, see [W2]). Let γ_t denote the curve in M_t given by $|z^+| = |z^-| = |t|^{1/2}$. If we set $B_t = \{z \in \mathbb{C} : |z| \leq |t|^{1/2}\}$, identify z with $z^{\pm}$, and let $R_t^{\pm} = M^{\pm} - B_t$, then $M_t - \gamma_t$ is conformally equivalent to the disjoint union $R_t^+ \cup R_t^-$. Fix conformal metrics $h^{\pm}$ on $M^{\pm}$. For simplicity, we will assume in advance that $z^{\pm}$ are normalized in the sense that with respect to these coordinates $h^{\pm}(0) = 1$. Let h_t, $t \neq 0$ be a family of conformal metrics on M_t. Then on $R_t^{\pm}$ there are conformal factors

$$h_t\big|_{M_t-\gamma_t} = h^{\pm}e^{2\sigma_t^{\pm}}. \tag{2.1}$$

Set

$$\mathcal{Q}_{(M_t,h_t)} = \frac{\det' \Delta_{(M_t,h_t)}}{\mathrm{area}(M_t, h_t)}, \tag{2.2}$$

where $\det' \Delta$ denotes the zeta-regularized determinant of the (positive) Laplace-Beltrami operator over the nonzero spectrum $\{\lambda_j\}_{j=1}^{\infty}$:

$$\log \det{}' \Delta = -\zeta'_{\Delta}(0) , \qquad \zeta_{\Delta}(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} .$$

Similarly, we define

$$\mathcal{Q}_{(M^{\pm},h^{\pm})} = \frac{\det' \Delta_{(M^{\pm},h^{\pm})}}{\mathrm{area}(M^{\pm}, h^{\pm})}.$$

We wish to derive an expression for $\mathcal{Q}_{M_t}$ in terms of $\mathcal{Q}_{M^{\pm}}$. To do this, we pass through the relative determinants. Let

$$\mathcal{Q}_{(R_t^{\pm},h_t)} = \det \Delta_{(R_t^{\pm},h_t)} , \qquad \mathcal{Q}_{(R_t^{\pm},h^{\pm})} = \det \Delta_{(R_t^{\pm},h^{\pm})}$$

denote the determinants for the Laplacian on $R_t^{\pm}$ with respect to the restriction of the metric h_t on the one hand, and the metrics $h^{\pm}$ on the other. In this note we always choose Dirichlet boundary conditions on a manifold with boundary. Since h_t and $h^{\pm}$ are conformally related as in (2.1), we have the Polyakov-Alvarez formula (cf. [A]):

$$\mathcal{Q}_{(R_t^{\pm},h_t)} = \mathcal{Q}_{(R_t^{\pm},h^{\pm})} e^{S_{(R_t^{\pm},h^{\pm})}(\sigma_t^{\pm})}, \tag{2.3}$$

$$S_{(R,h)}(\sigma) = -\frac{1}{12\pi} \int_R dA_h \left\{2K_h\sigma + |\nabla\sigma|^2\right\} - \frac{1}{12\pi} \int_{\partial R} ds_h \left\{2k_h\sigma + 3\partial_n\sigma\right\} , \tag{2.4}$$

where K_h and k_h are the Gauss and geodesic curvatures, respectively, for the metric h. Let $\mathcal{N}_t$ denote the Neumann jump operator for (M_t, γ_t, h_t), and $\mathcal{N}_t^{\pm}$ the Neumann jump operators for the surfaces $(M^{\pm}, \gamma_t^{\pm}, h^{\pm})$, where $\gamma_t^{\pm} = \partial R_t^{\pm}$. Finally, we set

$$\mathcal{Q}_{B_t} = \det \Delta_{B_t} , \quad \mathcal{Q}_{(B_t,h)} = \det \Delta_{(B_t,h)},$$

where B_t has the Euclidean metric, or a general metric h, respectively. Then we have the following

Proposition 2.5.

$$\lim_{t\to 0} \left\{ \mathcal{Q}_{(M_t,h_t)} (\mathcal{Q}_{B_t})^2 \exp\left\{-S_{(R_t^{+},h^{+})}(\sigma_t^{+}) - S_{(R_t^{-},h^{-})}(\sigma_t^{-})\right\} \frac{\ell_{h_t}(\gamma_t) \det' \mathcal{N}_t^{+} \det' \mathcal{N}_t^{-}}{\ell_{h^{+}}(\gamma_t)\ell_{h^{-}}(\gamma_t) \det' \mathcal{N}_t} \right\}$$
$$= \mathcal{Q}_{(M^{+},h^{+})} \mathcal{Q}_{(M^{-},h^{-})}.$$

Proof. Applying Theorem 1.8 and (2.3) we have

$$\mathcal{Q}_{(M_t,h_t)} = \mathcal{Q}_{(R_t^{+},h^{+})} \mathcal{Q}_{(R_t^{-},h^{-})} \exp\left\{S_{(R_t^{+},h^{+})}(\sigma_t^{+}) + S_{(R_t^{-},h^{-})}(\sigma_t^{-})\right\} \frac{\det' \mathcal{N}_t}{\ell_{h_t}(\gamma_t)}.$$

Using Theorem 1.8 again for $(M^{\pm}, h^{\pm})$:

$$\mathcal{Q}_{(M^{\pm},h^{\pm})} = \mathcal{Q}_{(R_t^{\pm},h^{\pm})}\mathcal{Q}_{(B_t,h^{\pm})}\frac{\det' \mathcal{N}_t^{\pm}}{\ell_{h^{\pm}}(\gamma_t^{\pm})}.$$

Now $\mathcal{Q}_{(B_t,h^{\pm})}$ and $\mathcal{Q}_{B_t}$ are related by the Liouville action (2.4) for a smooth conformal factor. Hence, $\mathcal{Q}_{(B_t,h^{\pm})}/\mathcal{Q}_{B_t} \to 1$ as $t \to 0$ (recall the normalization $h^{\pm}(0) = 1$). The result follows. □

3. Asymptotics of the Neumann Jump Operator

Let M be a compact Riemann surface and $z = re^{i\theta}$ a local coordinate. For a real parameter ε, $0 < \varepsilon \le 1$, set $B(\varepsilon) = \{x \in M : |z(x)| < \varepsilon\}$, and $R(\varepsilon) = M - B(\varepsilon)$ (n.b. in the notation of the previous section, $B(\varepsilon) = B_t$ and $R^{\pm}(\varepsilon) = R_t^{\pm}$, where $|t| = \varepsilon^2$). Assume M is equipped with a conformal metric that is euclidean $|dz|^2$ in a neighborhood of $B(1)$. Let $\mathcal{P}_{R(\varepsilon)} : L^2(S^1) \to L^2(S^1)$ (resp. $\mathcal{P}_{B(\varepsilon)}$) denote the Dirichlet-to-Neumann operator for $R(\varepsilon)$ (resp. $B(\varepsilon)$). Normals are always taken to be outward pointing. Note that $\ker \mathcal{P}_{R(\varepsilon)}$ consists of the constants, and by Stokes' theorem, $\mathcal{P}_{R(\varepsilon)} : L_0^2(S^1) \to L_0^2(S^1)$, where $L_0^2(S^1)$ is the subspace of $L^2(S^1)$ orthogonal to the constants. Define the following auxiliary operators on $L^2(S^1)$:

$$\mathcal{T}_\varepsilon : \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \longmapsto \sum_{n\in\mathbb{Z}} \left(\frac{\varepsilon^n - \varepsilon^{-n}}{\varepsilon^n + \varepsilon^{-n}}\right) a_n e^{in\theta},$$

$$\mathcal{U}_\varepsilon^{\pm} : \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \longmapsto \frac{1}{2}\sum_{n\in\mathbb{Z}} (\varepsilon^n \pm \varepsilon^{-n}) a_n e^{in\theta},$$

$$\mathcal{V} : \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \longmapsto \sum_{n\in\mathbb{Z}} n a_n e^{in\theta},$$

$$|\mathcal{V}| : \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \longmapsto \sum_{n\in\mathbb{Z}} |n| a_n e^{in\theta}.$$

Remark 3.1. On $L_0^2(S^1)$, the operators $\mathcal{V}(\mathcal{U}_\varepsilon^-)^{-1}$ and $\left\{\mathcal{V}(\mathcal{T}_\varepsilon)^{-1} + |\mathcal{V}|\right\}$ are trace-class with norm tending to zero as $\varepsilon \to 0$.

Remark 3.2. By direct computation: $\varepsilon\mathcal{P}_{B(\varepsilon)} = |\mathcal{V}|$.

We also define the (unbounded) operator $\mathcal{E}_{R(\varepsilon)} : L^2(S^1) \to L^2(S^1)$ as follows. For $f \in L^2(S^1)$, let u be the harmonic function on $R(1)$ with boundary values f. Extend u to a harmonic function on $R(\varepsilon)$ (also denoted u). Then $\mathcal{E}_{R(\varepsilon)}(f) = u\big|_{\partial R(\varepsilon)}$. We have the following simple

Lemma 3.3. $\|\mathcal{E}_{R(\varepsilon)}(f)\| \ge \|f\|$ *for all* $0 < \varepsilon \le 1$ *and all* $f \in L^2(S^1)$. *In particular,* $\mathcal{E}_{R(\varepsilon)}^{-1}$ *is uniformly bounded as* $\varepsilon \to 0$.

Proof. It suffices to prove the estimate for f smooth. Let u be harmonic on $R(\varepsilon)$ with $u\big|_{\partial R(1)} = f$. Then

$$\begin{aligned}
\|f\|^2 - \|\mathcal{E}_{R(\varepsilon)}(f)\|^2 &= \int_0^{2\pi} d\theta \, \left\{u^2(1,\theta) - u^2(\varepsilon,\theta)\right\} \\
&= \int_\varepsilon^1 dr \, \frac{d}{dr} \int_0^{2\pi} d\theta \, u^2(r,\theta) \\
&= 2\int_\varepsilon^1 \frac{dr}{r} \int_0^{2\pi} r d\theta \, u(r,\theta) \frac{\partial u}{\partial r}(r,\theta) \\
&= -2\int_\varepsilon^1 \frac{dr}{r} \int_{\partial R(r)} ds \, u \partial_n u \\
&= -2\int_\varepsilon^1 \frac{dr}{r} \int_{R(r)} dA \, |\nabla u|^2 \\
&\le 0.
\end{aligned}$$

□

We also have

Lemma 3.4. *$\mathcal{E}_{R(\varepsilon)}$ preserves the orthogonal splitting $L^2(S^1) = \mathbb{C} \oplus L^2_0(S^1)$.*

Proof. Let u be a harmonic function on $R(\varepsilon)$ with $u\big|_{\partial R(1)} = f$, $u\big|_{\partial R(\varepsilon)} = g$. Then applying Green's theorem with the harmonic function $v = \log r$ on the annulus $B(1) - B(\varepsilon)$,

$$\begin{aligned}
0 &= \int_{\partial(R(\varepsilon)-R(1))} (u\partial_n v - v\partial_n u)\, ds \\
&= \int_{S^1} f d\theta - \int_{S^1} g d\theta - \int_{\partial R(\varepsilon)} \mathcal{P}_{R(\varepsilon)}(g) \log \varepsilon \, ds.
\end{aligned}$$

Since $\mathcal{P}_{R(\varepsilon)}(g) \in L^2_0(S^1)$, we have

$$\int_{S^1} f d\theta = \int_{S^1} g d\theta = \int_{S^1} \mathcal{E}_{R(\varepsilon)}(f) d\theta.$$

□

The following result shows that the operators $\varepsilon\mathcal{P}_{R(\varepsilon)}$ and $\varepsilon\mathcal{P}_{B(\varepsilon)}$ are asymptotically close as $\varepsilon \to 0$.

Proposition 3.5. *On $L^2_0(S^1)$, $\varepsilon\mathcal{P}_{R(\varepsilon)} = -\mathcal{V}(\mathcal{T}_\varepsilon)^{-1} + \mathcal{V}(\mathcal{U}^-_\varepsilon)^{-1}\mathcal{E}^{-1}_{R(\varepsilon)}$.*

Proof. First, note that by Lemma 3.4 the composition $(\mathcal{U}^-_\varepsilon)^{-1}\mathcal{E}^{-1}_{R(\varepsilon)}$ is well-defined on $L^2_0(S^1)$. With this understood, the result follows by a direct calculation. Namely, by expanding functions in Fourier modes, one computes

$$\begin{aligned}
\mathcal{E}_{R(\varepsilon)} &= \mathcal{U}^+_\varepsilon - \mathcal{V}^{-1}\mathcal{U}^-_\varepsilon\mathcal{P}_{R(1)}, \\
-\varepsilon\mathcal{P}_{R(\varepsilon)}\mathcal{E}_{R(\varepsilon)} &= \mathcal{V}\mathcal{U}^-_\varepsilon - \mathcal{U}^+_\varepsilon\mathcal{P}_{R(1)}.
\end{aligned}$$

The proposition is then a consequence of these two facts. □

Corollary 3.6. *Let* Λ_t *be any of* $\det' \mathcal{N}_t/\ell_{h_t}(\gamma_t)$, *or* $\det' \mathcal{N}_t^{\pm}/\ell_{h^{\pm}}(\gamma_t^{\pm})$ *from Sect. 2. Then*

$$\lim_{t\to 0} \Lambda_t = 1/2.$$

Proof. Consider $\mathcal{N}_t^+$, the argument being similar for the other two cases. By conformal invariance of $\det' \mathcal{N}_t^+/\ell_{h^+}(\gamma_t^+)$ (cf. [GG], the method in [EW], or note that this follows from (2.3) and Theorem 1.8) it suffices to consider the locally euclidean case. Also, notice that

$$\zeta_{\mathcal{N}_t^+}(0) = -1 \tag{3.7}$$

for all t. Indeed, if we scale the metric by c^2,

$$\ell_{h^+}(\gamma_t^+) \mapsto c\,\ell_{h^+}(\gamma_t^+)\,, \qquad \mathcal{N}_t^+ \mapsto c^{-1}\mathcal{N}_t^+.$$

Then by conformal invariance

$$\log \frac{\det \mathcal{N}_t^+}{\ell_{h^+}(\gamma_t^+)} = \log \frac{\det(c^{-1}\mathcal{N}_t^+)}{c\,\ell_{h^+}(\gamma_t^+)} = -(\zeta_{\mathcal{N}_t^+}(0)+1)\log c + \log \frac{\det \mathcal{N}_t^+}{\ell_{h^+}(\gamma_t^+)},$$

and since c is arbitrary, (3.7) holds. As a consequence,

$$\begin{aligned} \log\det(|t|^{1/2}\mathcal{N}_t^+) &= \zeta_{\mathcal{N}_t^+}(0)\log|t|^{1/2} + \log\det\mathcal{N}_t^+ \\ &= -\log|t|^{1/2} + \log\det\mathcal{N}_t^+, \\ \log\det(|t|^{1/2}\mathcal{N}_t^+) &= \log\frac{\det\mathcal{N}_t^+}{\ell_{h^+}(\gamma_t^+)} + \log(2\pi). \end{aligned} \tag{3.8}$$

On the other hand, by Proposition 3.5, Remarks 3.1 and 3.2, and Lemma 3.3, it follows that

$$|t|^{1/2}\mathcal{N}_t^+ = 2|\mathcal{V}| + \{\text{trace-class}\},$$

where the trace norm of the remainder tends to zero as $t \to 0$. Hence by (3.8) and [L, Lemma 4.1],

$$\lim_{t\to 0}\log\Lambda_t = \lim_{t\to 0}\Big(\log\det(|t|^{1/2}\mathcal{N}_t^+) - \log(2\pi)\Big) = \log\det(2|\mathcal{V}|) - \log(2\pi).$$

Now $\operatorname{spec}(2|\mathcal{V}|) = \{2n\}_{n=0}^{\infty}$, and each of the nonzero eigenvalues has multiplicity 2. Hence, $\zeta_{2|\mathcal{V}|}(s) = 2^{1-s}\zeta(s)$. The result then follows from the special values of the Riemann zeta function: $\zeta(0) = -1/2$, $\zeta'(0) = -(1/2)\log(2\pi)$. □

4. The Arakelov Metric

Recall the definition of the Arakelov metric (cf. [Ar,Fl,J1,W2,F2]). Given a compact Riemann surface M of genus $g \geq 1$, let $\{A_i, B_i\}_{i=1}^g$ be a symplectic set of generators of $H_1(M)$ and choose $\{\omega_i\}_{i=1}^g$ to be a basis of abelian differentials normalized such that $\int_{A_i} \omega_j = \delta_{ij}$. Let $\Omega_{ij} = \int_{B_i} \omega_j$ be the associated period matrix with theta function ϑ. Set

$$\mu = \frac{\sqrt{-1}}{2g} \sum_{i,j=1}^{g} (\mathfrak{Im}\,\Omega)^{-1}_{ij} \omega_i \wedge \overline{\omega}_j. \tag{4.1}$$

Then $\int_M \mu = 1$. The Arakelov-Green's function $G(z, w)$ is symmetric with a zero of order one along the diagonal satisfying

$$\partial_z \partial_{\bar{z}} \log G(z, w) = \pi \sqrt{-1} \mu_{z\bar{z}}\ , z \neq w\ ; \tag{4.2}$$

$$\int_M \mu(z) \log G(z, w) = 0, \tag{4.3}$$

and the Arakelov metric $h = h_{z\bar{z}} |dz|^2$ is defined by

$$\log h_{z\bar{z}} = 2 \lim_{w \to z} \{\log G(z, w) - \log |z - w|\}\,. \tag{4.4}$$

The Arakelov metric is "admissible" in the sense of [Fl]; hence,

$$\mathrm{Ric}(h) = 4\pi \sqrt{-1}(g - 1)\mu. \tag{4.5}$$

For more details we refer to the papers cited above.

Consider now the situation in Sect. 2, where h_t and $h^{\pm}$ denote the Arakelov metrics on M_t and $M^{\pm}$, respectively (see [Ar,Fl]). The purpose of this section is to prove

Proposition 4.6 (cf. [W2], Eq. (8.1)). *Let $h_t = e^{2\sigma_t^{\pm}} h^{\pm}$. Then for $z \in R_t^{\pm}$,*

$$\sigma_t^{\pm}(z) = \left(\frac{g^{\mp}}{g}\right)^2 \log |t| - 2 \left(\frac{g^{\mp}}{g}\right) \log G^{\pm}(z, p^{\pm}) + r^{\pm}(t, z), \tag{4.7}$$

where $\lim_{t \to 0} \sup_{z \in R_t^{\pm}} |r^{\pm}(t, z)| = 0$. Moreover, if $z^{\pm}$ are the plumbing coordinates then there is a constant $C > 0$ independent of t such that

$$\sup_{|z^{\pm}| = |t|^{1/2}} |\partial_{z^{\pm}} \sigma_t^{\pm}(z^{\pm})| \leq C |t|^{-1/2}. \tag{4.8}$$

In the statement above, $G^{\pm}(z, w)$ denote the Arakelov-Green's functions on $M^{\pm}$. The uniformity of the estimate is an improvement on the result of [W2] and is made possible by the explicit expression for the Arakelov metric in [F2] (Eq. (4.28) below). We require two preliminary technical results.

Lemma 4.9. *Let ω_t be a holomorphic family of abelian differentials on M_t such that (pointwise) $\omega_t(z) \to \omega_0(z)$ for $z \in M^+ - \{p^+\}$, and $\omega_t(z) \to 0$ for $z \in M^- - \{p^-\}$, where ω_0 is an abelian differential on M^+. Let $W_t^\pm$, W_0^+ be the local expressions for ω_t and ω_0 in the plumbing coordinates $z^\pm$. Then there are functions $f(t,x)$, $g(t,x)$, analytic in a neighborhood of $(0,0)$, and $h(t)$ analytic in a neighborhood of 0, such that*

$$W_t^+(z^+) = \frac{t}{z^+}h(t) + f(t, z^+ + t/z^+) + \frac{t^2}{(z^+)^2}g(t, z^+ + t/z^+), \tag{4.10}$$

$$W_t^-(z^-) = -\frac{t}{z^-}h(t) - \frac{t}{(z^-)^2}f(t, z^- + t/z^-) - tg(t, z^- + t/z^-). \tag{4.11}$$

In particular, if we assume $f(0,0) \neq 0$, then for $|z^\pm|$ sufficiently small,

$$\log|W_t^+(z^+)| = \log|W_0^+(z^+)| + r^+(t, z^+), \tag{4.12}$$

$$\log|W_t^-(z^-)| = \log|t| + \log|\dot{W}_0^-(z^-)| + r^-(t, z^-), \tag{4.13}$$

where

$$W_0^+(z^+) = f(0, z^+),$$

$$\dot{W}_0^-(z^-) = -\frac{h(0)}{z^-} - \frac{f(0,z^-)}{(z^-)^2} - g(0, z^-),$$

and $\lim_{t\to 0} \sup_{|z^\pm| \geq |t|^{1/2}} |r^\pm(t, z^\pm)| = 0$. *Moreover, there is a constant C independent of t such that*

$$\sup_{|z^\pm| = |t|^{1/2}} \left|\partial_{z^+} W_t^\pm(z^\pm)\right| \leq C|t|^{-1/2}. \tag{4.14}$$

Proof. By [F1, p. 40] we have an expansion

$$\omega_t = \sum_{m\geq 0} a_m(t)\mathcal{X}^m d\mathcal{X} + \sum_{n \geq 0} b_n(t)\mathcal{X}^n \frac{d\mathcal{X}}{\mathcal{Y}}, \tag{4.15}$$

where

$$\mathcal{X} = (1/2)(z^+ + z^-) = (1/2)(z^+ + t/z^+), \tag{4.16}$$

$$\mathcal{Y} = (1/2)(z^+ - z^-) = -(1/2)(z^- - t/z^-). \tag{4.17}$$

The $a_m(t)$, $b_n(t)$ are analytic near $t = 0$, and the series are convergent for $(\mathcal{X}, t)$ in a neighborhood of $(0,0)$. Moreover, using the assumption that $\omega_t(z) \to 0$ on $M^- - \{p^-\}$, we have $b_0(0) = 0$, and $b_{n+1}(0) = a_n(0)$ for all $m \geq 0$. Now, substituting the expressions for $\mathcal{X}$, $d\mathcal{X}$, and $\mathcal{Y}$, we find

$$\begin{aligned} W_t^+(z^+) &= \sum_{m\geq 0} \frac{a_m(t)}{2^{m+1}}\left(1 - \frac{t}{(z^+)^2}\right)\left(z^+ + \frac{t}{z^+}\right)^m + \sum_{n\geq 0}\frac{b_n(t)}{2^n}\frac{1}{z^+}\left(z^+ + \frac{t}{z^+}\right)^n \\ &= \frac{b_0(t)}{t}\frac{t}{z^+} + \sum_{m\geq 0}\frac{1}{2^{m+1}}(a_m(t) + b_{m+1}(t))\left(z^+ + \frac{t}{z^+}\right)^m \\ &\quad - \sum_{n\geq 0}\frac{1}{2^{n+1}}\frac{(a_n(t) - b_{n+1}(t))}{t}\frac{t^2}{(z^+)^2}\left(z^+ + \frac{t}{z^+}\right)^n. \end{aligned}$$

Since $b_0(t)/t$ and $(a_n(t) - b_{n+1}(t))/t$ are regular at $t = 0$, this gives the expression in (4.10) with

$$h(t) = \frac{b_0(t)}{t},$$
$$f(t, x) = \sum_{m\geq 0} \frac{1}{2^{m+1}}(a_m(t) + b_{m+1}(t))x^m,$$
$$g(t, x) = -\sum_{n\geq 0} \frac{1}{2^{n+1}} \frac{(a_n(t) - b_{n+1}(t))}{t} x^n.$$

Equation (4.11) follows from (4.10) and the fact that $dz^+ = -t dz^-/(z^-)^2$.

For fixed z^+ write

$$F(t) = f(t, z^+ + t/z^+) + \frac{t}{z^+}h(t) + \frac{t^2}{(z^+)^2}g(t, z^+ + t/z^+),$$
$$F'(t) = \partial_1 f + \frac{1}{z^+}\partial_2 f + \frac{1}{z^+}h + \frac{t}{z^+}h' + \frac{2t}{(z^+)^2}g + \frac{t^2}{(z^+)^2}\partial_1 g(t, z^+ + t/z^+) + \frac{t^2}{(z^+)^3}\partial_2 g(t, z^+ + t/z^+).$$

Hence, for $|z^+| \geq |t|^{1/2}$, we have $F(t) = f(0, 0) + O(|t|^{1/2})$, uniformly. This implies (4.12). For fixed z^- write

$$G(t) = f(t, z^- + t/z^-) + z^- h(t) + (z^-)^2 g(t, z^- + t/z^-),$$
$$G'(t) = \partial_1 f + \frac{1}{z^-}\partial_2 f + z^- h' + (z^-)^2 \partial_1 g(t, z^- + t/z^-) + z^- \partial_2 g(t, z^- + t/z^-).$$

It follows that $G(t) = G(0) + O(|t|^{1/2})$, uniformly for $|z^-| \geq |t|^{1/2}$. By assumption, $G(0) \neq 0$ for $|z^-|$ sufficiently small, and hence

$$W_t^-(z^-) = -\frac{t}{(z^-)^2}G(0)(1 + O(|t|^{1/2})),$$

so the estimate (4.13) follows. The estimate (4.14) follows immediately from (4.10) and (4.11). □

For the second result, let $E_t(z, w)$, $E^{\pm}(z, w)$ denote the Schottky prime forms on M_t, $M^{\pm}$ (cf. [F1,F2] and Eq. (4.22) below). When evaluated at points on the Riemann surface, we will assume that lifts to a fixed fundamental domain have been chosen. Also, when points are in the pinching region we will assume the expressions for the prime forms are given with respect to the coordinates $z^{\pm}$. The following asymptotics are well-known. The point again is the uniformity of the estimates.

Lemma 4.18. *For $z, w \in R_t^+$,*

$$\log|E_t(z, w)| = \log|E^+(z, w)| + r_1^+(z, w, t), \tag{4.19}$$

where for fixed w, $\lim_{t\to 0} \sup_{z\in R_t^+} |r_1^+(z, w, t)| = 0$ (and similarly for fixed z). A similar result holds for points in R_t^-. For $z \in R_t^+$ and $w \in R_t^-$,

$$\log|E_t(z, w)| = -\log|t|^{1/2} + \log|E^+(z, p^+)| + \log|E^-(w, p^-)| + r_2(z, w, t), \tag{4.20}$$

where for fixed w, $\lim_{t\to 0}\sup_{z\in R_t^+}|r_2(z,w,t)|=0$ (*and similarly for fixed* z). *Moreover, if* $z^\pm$ *are the plumbing coordinates and* w *is fixed, then there is a constant* $C>0$ *independent of* t *such that*

$$\sup_{|z^\pm|=|t|^{1/2}} |\partial_{z^\pm} E_t^\pm(z^\pm,w)| \le C|t|^{-1/2}. \tag{4.21}$$

Proof. Recall that the prime form may be defined as follows. Let $\vartheta(Z)$, $Z\in\mathbb{C}^g$, denote the theta function for the period matrix Ω. Choose a nonsingular $\delta\in\Theta$, where $\Theta\subset Jac(M)$ is the theta divisor in the Jacobian of M. Then by [F1, Corollary 2.3],

$$-E^2(z,w)=\frac{\vartheta(z-w+\delta)\vartheta(z-w-\delta)}{H_\delta(z)H_\delta(w)}, \tag{4.22}$$

where

$$H_\delta(z)=\sum_{i=1}^g \partial_{Z_i}\vartheta(\delta)\omega_i(z). \tag{4.23}$$

Now consider the degenerating family M_t. The collection $\{A_i,B_i\}_{i=1}^g$ may be chosen so that $\{A_i,B_i\}_{i=1}^{g^+}$ is a symplectic homology basis for M^+ and $\{A_i,B_i\}_{i=g^++1}^{g}$ is a symplectic homology basis for M^-. Let $\omega_{i,t}$, $\omega_i^\pm$ be abelian differentials on M_t and $M^\pm$, respectively, normalized with respect to $\{A_i\}$, and let Ω_t and $\Omega^\pm$ be the associated period matrices. Then as $t\to 0$,

$$\omega_{i,t}(z)\longrightarrow\begin{cases}\omega_i^+(z) & i\le g^+,\, z\in M^+-\{p^+\}\\ 0 & i\le g^+,\, z\in M^--\{p^-\}\\ \omega_i^-(z) & i>g^+,\, z\in M^--\{p^-\}\\ 0 & i>g^+,\, z\in M^+-\{p^+\}\end{cases} \tag{4.24}$$

and Ω_t becomes block diagonal (Ω^+,Ω^-) (cf. [F1, p. 38]). Choose δ_t so that

$$\lim_{t\to 0}\delta_t=(\delta^+,\delta^-)\in\Theta^+\times Jac(M^-).$$

If $z,w\in R_t^+$, we also have $\vartheta_t(z-w+\delta_t)\to\vartheta^+(z-w+\delta^+)\vartheta^-(\delta^-)$, and this is uniform in z,w (cf. [W2, Sect. 3 and Prop. A.1]). For simplicity and without loss of generality, assume $\vartheta^-(\delta^-)\neq 0$ and $H^+_{\delta^+}(p^+)\neq 0$. Then from (4.23),

$$H_{\delta_t}(z)\to H^+_{\delta^+}(z)\vartheta^-(\delta^-), \tag{4.25}$$

and by Lemma 4.9 this is uniform for $z\in R_t^+$. Hence, the expression (4.22) immediately implies (4.19). Now suppose $z,w\in R_t^-$. From [W2, Prop. 3.8] we have

$$\vartheta_t(z-w\pm\delta_t)=t\,\frac{\vartheta^-(z-p^-\pm\delta^-)\vartheta^-(w-p^-\mp\delta^-)E^-(z,w)}{\vartheta^-(\delta^-)E^-(z,p^-)E^-(w,p^-)}H^+_{\delta^+}(p^+)+O(|t|^2), \tag{4.26}$$

where the $O(|t|^2)$ term depends on two derivatives of $\vartheta(Z,\Omega)$ with respect to Ω_{ij} and is smooth in Z. Hence, again this is uniform. We then also have,

$$H_{\delta_t}(z) = t\,\frac{\vartheta^-(z-p^-+\delta^-)\vartheta^-(z-p^--\delta^-)}{\vartheta^-(\delta^-)(E^-(z,p^-))^2}H^+_{\delta^+}(p^+) + O(|t|^2). \tag{4.27}$$

Again, (4.19) follows. The uniformity for the remainder term in $\log|H_{\delta_t}(z)|$ comes from (4.13) as applied to $H_{\delta_t}(z)$ and $H_{\delta_t}(w)$, and as mentioned, the expansion (4.26) is also uniform. The expression (4.20) follows. For (4.21), note that from (4.22),

$$2\partial_z E_t(z,w) = E_t(z,w)\left\{\frac{\sum_{i=1}^g \partial_{Z_i}\vartheta_t(z-w+\delta_t)\omega_i(z)}{\vartheta_t(z-w+\delta_t)}\right.$$
$$\left.+\frac{\sum_{i=1}^g \partial_{Z_i}\vartheta_t(z-w-\delta_t)\omega_i(z)}{\vartheta_t(z-w-\delta_t)} - \frac{\partial_z H_{\delta_t}(z)}{H_{\delta_t}(z)}\right\}.$$

Now fix $w \in R_t^-$. By (4.19), $|E_t(z,w)|$ is uniformly bounded from above for $|z^+| = |t|^{1/2}$, and $|H_{\delta_t}(z^+)|$ is uniformly bounded from below (see Lemma 4.9). Similarly, $|\vartheta_t(z-w\pm\delta_t)| \sim |t|^{1/2}$ (see (4.26)). Hence,

$$\sup_{|z^\pm|=|t|^{1/2}} |\partial_{z^+}E_t(z^+,w)| \le C_1|t|^{-1/2} + C_2 \sup_{|z^\pm|=|t|^{1/2}} |\partial_{z^+}H_{\delta_t}(z^+)|$$

and the desired bound in (4.21) follows from (4.14) applied to $H_{\delta_t}(z)$. The case $w \in R_t^+$ is similar. Finally, in case $z \in R_t^+$, $w \in R_t^-$, use $\vartheta_t(z-w+\delta_t) \to \vartheta^+(z-p^++\delta^+)\vartheta^-(w-p^-+\delta^-)$, (4.25), and (4.27) to obtain (4.20). The uniformity and statement about derivatives follows as above. This completes the proof. □

Proof of Proposition 4.6. From [F2, Eq. (1.31)], the Arakelov metric may be expressed in local conformal coordinates as

$$h(z) = C(M)|s(z)|^{4/g}\exp\{(4/g(g-1))B[k^z,k^z]\}\,|dz|^2, \tag{4.28}$$

where $C(M)$ is an explicit constant depending on M but independent of z, k^z is the vector of Riemann constants associated to z,

$$k_j^z = \frac{1+\Omega_{jj}}{2} + \sum_{i\neq j}\int_{A_i}\omega_i(x)\int_x^z \omega_j,$$

and

$$B[k^z,k^z] = \pi\sum_{i=1}^g (\Im\mathfrak{m}\,\Omega)^{-1}_{ij}(\Im\mathfrak{m}\,k^z)_i(\Im\mathfrak{m}\,k^z)_j.$$

The function $s(z)$ is given in [F2, Prop. 1.2]. Important here is the relation

$$\frac{s(z)}{s(z_0)} = \frac{\vartheta(\sum_{i=1}^g p_i - z - \Delta)}{\vartheta(\sum_{i=1}^g p_i - z_0 - \Delta)}\prod_{i=1}^g \frac{E(p_i,z)}{E(p_i,z_0)}, \tag{4.29}$$

where $p_1,\ldots,p_g$ are generic points. Using Theorem 1.7 one can in fact recover the asymptotics in [W2, Eq. (8.1)] from (4.28). Relevant to the proof of Proposition 4.6, however, is the uniformity. From (4.28) and (4.29) this amounts to uniformity and bound on derivatives of the prime form, since the uniformity of the theta function and Riemann constants follows from the results in [W2]. Hence, Proposition 4.6 is a consequence of Lemma 4.18. □

5. Proof of the Main Theorem

We now apply the formulation of Sects. 2 and 3 to $\mathcal{Q}_{(M_t,h_t)}$ and $\mathcal{Q}_{(M^\pm,h^\pm)}$, where h_t and $h^\pm$ are the Arakelov metrics on M_t and $M^\pm$, respectively. The main result is

Proposition 5.1.

$$\lim_{t\to 0}\left(\log \mathcal{Q}_{(M_t,h_t)} - \frac{2g^+g^-}{3g}\log|t|\right) = \log(\mathcal{Q}_{(M^+,h^+)}\mathcal{Q}_{(M^-,h^-)}) - \frac{c_0}{6},$$

where c_0 is defined in Theorem 1.3.

For this we require

Lemma 5.2. *In the notation of Sect. 2,*

$$\lim_{t\to 0}\left\{S_{(R_t^+,h^+)}(\sigma_t^+) + S_{(R_t^-,h^-)}(\sigma_t^-) + \left(\frac{1}{3} - \frac{2g^+g^-}{3g}\right)\log|t| + 1\right\} = 0.$$

Proof. This is essentially [W1, Eq. (3.5)]. However, there is an error in the computation due to the omission of the last term in (2.4). See [W3] or (5.8) below for the correction. Moreover, the uniformity of the asymptotics of the Arakelov metric, established in the previous section, was assumed in that paper. Hence, for the sake of completeness, we include the full computation here. Let μ_t, $\mu^\pm$ be the forms corresponding to (4.1) on M_t and $M^\pm$. It follows from (4.24) that

$$\mu_t \longrightarrow (g^\pm/g)\mu^\pm, \tag{5.3}$$

and from Lemma 4.9, μ_t is uniformly bounded in the plumbing coordinates as $t \to 0$. By (4.5) we have $K_{h^\pm}dA_{h^\pm} = -4\pi(g^\pm - 1)\mu^\pm$. Using this and Proposition 4.6,

$$\begin{aligned} -\frac{1}{12\pi}\int_{R_t^\pm} dA_{h^\pm}2K_{h^\pm}\sigma_t^\pm &= \frac{2}{3}(g^\pm - 1)\int_{R_t^\pm}\mu^\pm\Big\{(g^\mp/g)^2\log|t| \\ &\qquad -2(g^\mp/g)\log G^\pm(z,p^\pm)\Big\} + o(1) \\ &= \frac{2}{3}(g^\pm - 1)(g^\mp/g)^2\log|t| + o(1), \end{aligned} \tag{5.4}$$

where we have also used the normalization (4.3). For the next term in (2.4), use the estimates in Proposition 4.6 and the fact, again from (4.5) and also (5.3), that

$$dA_{h^\pm}\Delta\sigma_t^\pm = 4\pi(g^\mp/g)\mu^\pm + o(1), \tag{5.5}$$

where $o(1)$ converges pointwise to zero and is uniformly bounded on $R_t^{\pm}$ with respect the euclidean measure in the plumbing coordinates. By (4.7), (4.8), and (5.5), we have

$$\begin{aligned}
&-\frac{1}{12\pi}\int_{R_t^{\pm}} dA_{h^{\pm}}|\nabla\sigma_t^{\pm}|^2\\
&=\frac{1}{12\pi}\int_{R_t^{\pm}} dA_{h^{\pm}}\sigma_t^{\pm}\Delta\sigma_t^{\pm}-\frac{1}{12\pi}\int_{\partial R_t^{\pm}} ds_{h^{\pm}}\sigma_t^{\pm}\partial_n\sigma_t^{\pm}\\
&=\frac{1}{12\pi}\int_{R_t^{\pm}} dA_{h^{\pm}}\sigma_t^{\pm}\Delta\sigma_t^{\pm}\\
&\quad+\frac{1}{12\pi}\int_{\partial R_t^{\pm}} ds_{h^{\pm}}\left\{(g^+g^-/g^2)\log|t|+o(1)\right\}\partial_n\sigma_t^{\pm}\\
&=\frac{1}{12\pi}\int_{R_t^{\pm}} dA_{h^{\pm}}\Delta\sigma_t^{\pm}\left\{(g^{\mp}/g)^2\log|t|-2(g^{\mp}/g)\log G^{\pm}(z,p^{\pm})\right.\\
&\quad\left.+\ (g^+g^-/g^2)\log|t|\right\}+o(1)\\
&=\frac{1}{3}\left(\frac{g^{\mp}}{g}\right)^2\log|t|+o(1).
\end{aligned}\tag{5.6}$$

Since the metrics $h^{\pm}$ are normalized at $p^{\pm}$,

$$-\frac{1}{6\pi}\int_{\partial R_t^{\pm}} ds_{h^{\pm}}k_{h^{\pm}}\sigma_t^{\pm}=-\frac{1}{3}(g^+g^-/g^2)\log|t|+o(1)\qquad(\text{by }(4.7)).\tag{5.7}$$

Finally,

$$-\frac{1}{4\pi}\int_{\partial R_t^{\pm}} ds_{h^{\pm}}\partial_n\sigma_t^{\pm}=-\frac{1}{4\pi}\int_{R_t^{\pm}} dA_{h^{\pm}}\Delta\sigma_t^{\pm}=-\frac{g^{\mp}}{g}+o(1)\qquad(\text{by }(5.5)).\tag{5.8}$$

Combining the results (5.4) and (5.6)-(5.8) for R_t^+ and R_t^-, we obtain the statement of the lemma. □

Proof of Proposition 5.1. It suffices to consider the factors in Proposition 2.5. By Corollary 3.6,

$$\frac{\ell_{h_t}(\gamma_t)\det\mathcal{N}_t^+\det\mathcal{N}_t^-}{\ell_{h^+}(\gamma_t)\ell_{h^-}(\gamma_t)\det\mathcal{N}_t}\longrightarrow\frac{1}{2}.$$

By [Wb, Eq. (28)],

$$(\mathcal{Q}_{B_t})^2=2^{2/3}(2\pi)^{-1}|t|^{-1/3}\exp(-4\zeta'(-1)-5/6).$$

The result now follows from Lemma 5.2. □

Proof of Theorem 1.3. Comparing Proposition 5.1 with Theorem 1.7 and the definition (1.1) of c_g, we find: $c_{g+1}-c_g=c_1-c_0$, where c_1 is the constant for $g=1$ surfaces and c_0 is as in Proposition 5.1. The $g=1$ constant c_1 has been evaluated (cf. [Fl,P]): $c_1=-8\log(2\pi)$. The expression for c_g now follows. □

Appendix

At the suggestion of the referee, in this appendix we clarify the relationship between the expansions for holomorphic abelian differentials on degenerating surfaces found in [F1] and [Y]. Let $\omega_t(z) \to \omega_0(z)$ be as in the statement of Lemma 4.9. We assume moreover that the periods $\int_{A_j} \omega_t$ are fixed independent of t.

From (4.10) and (4.11) we have

$$W_0^+(z^+) = f(0, z^+), \tag{5.9}$$

$$\dot{W}_0^+(z^+) = \frac{h(0)}{z^+} + \partial_1 f(0, z^+) + \frac{\partial_2 f(0, z^+)}{z^+}, \tag{5.10}$$

$$\ddot{W}_0^+(z^+) = 2\frac{h'(0)}{z^+} + \partial_1^2 f(0, z^+) + 2\frac{\partial_1\partial_2 f(0, z^+)}{z^+} + \frac{\partial_2^2 f(0, z^+)}{(z^+)^2} + \frac{2}{(z^+)^2} g(0, z^+), \tag{5.11}$$

$$\dot{W}_0^-(z^-) = -\frac{h(0)}{z^-} - \frac{f(0, z^-)}{(z^-)^2} - g(0, z^-), \tag{5.12}$$

where the dots indicate derivatives with respect to t. From (5.10), the restriction of

$$\dot{\omega}_0(z) = \lim_{t\to 0} \frac{\omega_t(z) - \omega_0(t)}{t}$$

to M^+ is an abelian differential with a pole of order at most one at p^+. Hence, it is holomorphic. On the other hand, by the normalization it must have zero A_j periods. It therefore vanishes identically. From (5.12), the restriction of $\dot{\omega}_0$ to M^- is an abelian differential with a pole of order two at p^-, and the coefficient of the $1/(z^-)^2$ term is $-f(0, 0) = -W_0^+(0)$, from (5.9). Hence, for $z \in M^-$ we have the t-expansion

$$\omega_t(z) = -t\, W_0^+(0)\omega^-(z, p^-) + O(t^2), \tag{5.13}$$

where $\omega^-(z, w)$ is the abelian differential of the second kind on M^-, normalized to have zero A_j-periods and an expansion

$$\omega^-(z, w) = \left(\frac{1}{(z - w)^2} + \frac{1}{6}S^-(p) + \cdots\right) dz dw$$

in local coordinates about a point p, where S^- is a holomorphic projective connection on M^- (see [F1, Cor. 2.6]). Examining the constant term in the t-expansion (5.12) we find

$$\frac{1}{2}\partial_2^2 f(0, 0) + g(0, 0) = \frac{1}{6} W_0^+(0) S^-(p^-).$$

From (5.11), the restriction of $\ddot{\omega}_0$ to M^+ is an abelian differential with a pole of order two at p^+, and the coefficient of the $1/(z^+)^2$ term is

$$\partial_2^2 f(0, 0) + 2g(0, 0) = \frac{1}{3} W_0^+(0) S^-(p^-).$$

Hence, for $z \in M^+$ we have the t-expansion

$$\omega_t(z) = \omega_0(z) + \frac{t^2}{6} W_0^+(0) S^-(p^-)\omega^-(z, p^-) + O(t^3). \tag{5.14}$$

The expressions (5.13) and (5.14) agree with [Y, Eq. (36)]. In [F1, Eq. 47], the t-expansion was carried out with respect to a local expression of the differentials in the "pinching coordinate" $\mathcal{X}$. Since the restrictions of $d\mathcal{X}$ to $M^{\pm} - \{p^{\pm}\}$ are differentials that themselves depend on t (see (4.16)), the $O(t)$ terms calculated there are incorrect.

Acknowledgement. The author is grateful to J. Jorgenson, D.H. Phong, and S. Zelditch for very useful discussions. Many thanks also to Alexey Kokotov for numerous suggestions during the revision of an earlier version of this paper.

References

[A] Alvarez, O.: Theory of strings with boundaries: fluctuations, topology, and quantum geometry. Nucl. Phys. B **216**, 125–184 (1983)

[AMV] Alvarez-Gaumé, L., Moore, G., Vafa, C.: Theta functions, modular invariance, and strings. Commun. Math. Phys. **106**(1), 1–40 (1986)

[Ar] Arakelov, S.: An intersection theory for divisors on an arithmetic surface. Math. USSR Izv. **8**, 1179–1192 (1974)

[BK] Belavin, A., Knizhnik, V.: Complex geometry and the theory of quantum strings. Soviet Phys. JETP **64** no. 2, 214–228 (1986), translated from Zh.Eksper. Teoret. Fiz. **91**(2), 364–390 (1986) (Russian)

[ABMNV] Alvarez-Gaumé, L., Bost, J.-B., Moore, G., Nelson, P., Vafa, C.: Bosonization on higher genus Riemann surfaces. Commun. Math. Phys. **112**(3), 503–552 (1987)

[BFK] Burghelea, D., Friedlander, L., Kappeler, T.: Meyer-Vietoris type formula for determinants of elliptic differential operators. J. Funct. Anal. **107**(1), 34–65 (1992)

[D] Deligne, P.: *Le déterminant de la cohomologie*. Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., **67**, Providence, RI: Amer. Math. Soc., 1987, pp. 93–177

[DP1] D'Hoker, E., Phong, D.H.: Length-twist parameters in string path integrals. Phys. Rev. Lett. **56**(9), 912–915 (1986)

[DP2] D'Hoker, E., Phong, D.H.: On determinants of laplacians on Riemann surfaces. Commun. Math. Phys. **104**(4), 537–545 (1986)

[DP3] D'Hoker, E., Phong, D.H.: The geometry of string perturbation theory. Rev. Mod. Phys. **60**(4), 917–1065 (1988)

[DGP] D'Hoker, E., Gutperle, M., Phong, D.H.: Two-loop superstrings and s-duality. Nucl. Phys. B **722**(1-2), 81–118 (2005)

[DS] Dugan, M., Sonoda, H.: Functional determinants on Riemann surfaces. Nucl. Phys. B **289**(1), 227–252 (1987)

[EW] Edward, J., Wu, S.: Determinant of the Neumann operator on smooth jordan curves. Proc. Amer. Math. Soc. **111**(2), 357–363 (1991)

[Fl] Faltings, G.: Calculus on arithmetic surfaces. Ann. of Math. (2) **119**(2), 387–424 (1984)

[F1] Fay, J.: *Theta functions on Riemann surfaces*. Lecture Notes in Mathematics, Vol. **352**. Berlin-New York:Springer-Verlag, 1973

[F2] Fay, J.: *Kernel functions, analytic torsion, and moduli spaces*. Mem. Amer. Math. Soc. **96**, no. 464 (1992)

[GIJR] Gava, E., Iengo, R., Jayaraman, T., Ramachandran, R.: Multiloop divergences in the closed bosonic string theory. Phys. Lett. B **168**(3), 207–211 (1986)

[GSo] Gillet, H., Soulé, C.: Analytic torsion and the arithmetic Todd genus. With an Appendix by D. Zagier. Topology **30**(1), 21–54 (1991)

[GG] Guillarmou, C., Guillopé, L.: *The determinant of the Dirichlet-to-Neumann map for surfaces with boundary*. http://arXiv.org/list/math/0701727, 2007

[H] Hejhal, D.: Regular b-groups, degenerating Riemann surfaces, and spectral theory. Mem. Amer. Math. Soc. **88**, no. 437 (1990)

[J1] Jorgenson, J.: Asymptotic behavior of Faltings's delta function. Duke Math. J. **61**(1), 221–254 (1990)

[J2] Jorgenson, J.: *Degenerating hyperbolic Riemann surfaces and an evaluation of the constant in Deligne's arithmetic Riemann-Roch theorem*. Preprint, 1991

[JL] Jorgenson, J., Lundelius, R.: Continuity of relative hyperbolic spectral theory through metric degeneration. Duke Math. J. **84**(1), 47–81 (1996)

[K] Kierlanczyk, M.: *Determinants of Laplacians*. MIT thesis, 1986
[L] Lee, Y.: Burghelea-Friedlander-Kappeler's gluing formula for the zeta-determinant and its applications to the adiabatic decompositions of the zeta-determinant and the analytic torsion. Trans. Amer. Math. Soc. **355**(10), 4093–4110 (2003)
[MM] Müller, J., Müller, W.: Regularized determinants of laplace-type operators, analytic surgery, and relative determinants. Duke Math. J. **133**(2), 259–312 (2006)
[P] Polchinski, J.: Evaluation of the one loop string path integral. Commun. Math. Phys. **104**(1), 37–47 (1986)
[PW] Park, J., Wojciechowski, K.: Adiabatic decomposition of the ζ-determinant and Dirichlet to Neumann operator. J. Geom. Phys. **55**(3), 241–266 (2005)
[Sa] Sarnak, P.: Determinants of laplacians. Commun. Math. Phys. **110**(1), 113–120 (1987)
[Sm] Smit, D.-J.: String theory and algebraic geometry of moduli spaces. Commun. Math. Phys. **114**, 645–685 (1988)
[Sn] Sonoda, H.: Calculation of a propagator on a Riemann surface. Phys. Lett. B **178**(4), 390–394 (1986)
[So] Soulé, C.: Géométrie d'Arakelov des surfaces arithmétiques. Séminaire Bourbaki, Vol. 1988/89. Astérisque No. 177-178, Exp. No. 713, 327–343 (1989)
[VV] Verlinde, E., Verlinde, H.: Chiral bosonization, determinants and the string partition function. Nucl. Phys. B **288**(2), 357–396 (1987)
[V] Voros, A.: Spectral functions, special functions and the selberg zeta function. Commun. Math. Phys. **110**(3), 439–465 (1987)
[Wb] Weisberger, W.: Conformal invariants for determinants of laplacians on Riemann surfaces. Commun. Math. Phys. **112**(4), 633–638 (1987)
[W1] Wentworth, R.: Asymptotics of determinants from functional integration. J. Math. Phys. **32**(7), 1767–1773 (1991)
[W2] Wentworth, R.: The asymptotics of the Arakelov-Green's function and Faltings' delta invariant. Commun. Math. Phys. **137**, 427–459 (1991)
[W3] Wentworth, R.: *Erratum to "Asymptotics of determinants from functional integration*". Available at:http://math.jhu.edu/~raw/papers/
[Wo1] Wolpert, S.: Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces. Commun. Math. Phys. **112**(2), 283–315 (1987)
[Wo2] Wolpert, S.: The hyperbolic metric and the geometry of the universal curve. J. Diff. Geom. **31**, 417–472 (1990)
[Y] Yamada, A.: Precise variational formulas for abelian differentials. Kodai Math. J. **3**(1), 114–143 (1980)

Communicated by L. Takhtajan