PRECISE CONSTANTS IN BOSONIZATION FORMULAS ON RIEMANN SURFACES II

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ABSTRACT. We give explicit formulas for the constants appearing in bosonization formulas on Riemann surfaces relating zeta regularized determinants of Laplace type operators associated to holomorphic line bundles of varying degree. We find that the constants $B_{g,d}$, which depend on the genus g of the surface and the degree d of the line bundle, satisfy the relation $B_{g,d} = (2\pi)^{2g-2-d} B_{g,2g-2}$. The value of $B_{g,2g-2}$ has been determined in an earlier work. One may interpret this formula in terms of the relationship between the Quillen and Faltings metrics on the determinant of cohomology.

1. INTRODUCTION

The bosonization formulas on Riemann surfaces relate zeta-regularized determinants of Laplace operators acting on sections of line bundles to determinants of scalar laplacians (see [2, 7, 36, 8, 6, 12, 33, 14, 17], and for their role in string theory [13]). They are tantamount to a relationship between the metrics defined by Quillen and Faltings on the determinant of cohomology [28, 15]. Following initial work on the scalar laplacian [39], the results presented in this paper give a precise expression for all of the previously undetermined constants appearing in these formulas.

Given a conformal metric ρ on a closed Riemann surface M of genus g, denote the associated scalar laplacian by Δ_M. If h is a hermitian metric on a holomorphic line bundle $L \to M$, let $\Box_L = 2\bar\partial_L^*\bar\partial_L$ be the Dolbeault laplacian. Determinants $\operatorname{Det}\Box_L$ are defined as the zeta regularized product of eigenvalues and are functions of ρ, h, and the moduli of M and L (see Section 2.4; in the case of a kernel the notation $\operatorname{Det}^*\Box_L$ is used to emphasize that the zeta function is defined using only nonzero eigenvalues). The bosonization formulas, which hold for general metrics, are most conveniently expressed in the case where ρ is the Arakelov metric on M and h is an admissible metric on L (see Section 4.3 for the definitions). Let $G(z,w)$ be the Arakelov-Green's function. A marking of M determines a Riemann divisor $\mathfrak{D}$. This satisfies $2\mathfrak{D} = [K]$, where K is the canonical bundle of M and $[K]$ is its associated divisor class of degree $2g-2$. The marking also gives a period matrix Ω and theta function $\vartheta(Z,\Omega)$. For the following statement of the bosonization formulas see the references cited above (and in particular [17, Theorems 4.9, 5.8 and 5.11]).

Theorem 1.1. *Consider M with the Arakelov metric ρ. For $d \geq g-1$, there are positive constants $B_{g,d}$ such that for any line bundle L of degree d with admissible metric h and satisfying $h^1(L) = 0$, the following hold.*

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- **Spin-1/2 bosonization.** *If $d = g-1$, then*

$$\operatorname{Det}\Box_L = B_{g,g-1}\left(\frac{\operatorname{Det}^*\Delta_M}{\operatorname{area}(M)\det\mathfrak{Im}\,\Omega}\right)^{-1/2}\|\vartheta\|^2([L]-\mathfrak{D},\Omega) \tag{1.2}$$

where $\operatorname{area}(M)$ *is the area of M with respect to the Arakelov metric.*

- **Higher spin bosonization.** *If $d \geq g$, then*

$$\frac{\operatorname{Det}^*\Box_L}{\det\langle\omega_i,\omega_j\rangle} = B_{g,d}\left(\frac{\operatorname{Det}^*\Delta_M}{\operatorname{area}(M)\det\mathfrak{Im}\,\Omega}\right)^{-1/2}\frac{\prod_{i\neq j}G(p_i,p_j)}{\|\det\omega_i(p_j)\|^2}\|\vartheta\|^2\big([L]-\sum_{i=1}^{m}p_i-\mathfrak{D},\Omega\big) \tag{1.3}$$

where $m = d-g+1$, $\{p_i\}_{i=1}^m$ are generic points of M, $\{\omega_i\}_{i=1}^m$ is any basis for $H^0(M,L)$, and the pointwise and L^2-metrics are taken with respect to h.

- **Scalar laplacian.** *If $g \geq 2$ and $B_{g,2g-2}$ is as above, the scalar determinant is given by*

$$\left(\frac{\operatorname{Det}^*\Delta_M}{\operatorname{area}(M)\det\mathfrak{Im}\,\Omega}\right)^{3/2} = \frac{B_{g,2g-2}}{4\pi^2}\frac{\prod_{i\neq j}G(p_i,p_j)}{\|\det\omega_i(p_j)\|^2\prod_{i=1}^{g}G^2(p_i,z)}\|\vartheta\|^2\Big(\sum_{i=1}^{g}p_i-z-\mathfrak{D},\Omega\Big) \tag{1.4}$$

where $\{p_i\}_{i=1}^g$ are generic points of M, $\{\omega_i\}_{i=1}^g$ is any basis for $H^0(M,K)$, and the pointwise metric is the one induced by ρ.

Let

$$\begin{aligned} c_g &= (1-g)c_0 + gc_1 \\ c_1 &= -8\log(2\pi) \\ c_0 &= -24\zeta'(-1)+1-6\log(2\pi)-2\log(2) \end{aligned} \tag{1.5}$$

where $\zeta(s)$ is the Riemann zeta function. Then we have previously shown the following

Theorem 1.6 ([39, Theorem 1.3])**.** $B_{g,2g-2} = 4\pi^2\exp(c_g/4)$.

The main result of this paper is a computation of the constants for arbitrary degree. We will prove the

Main Theorem. *For any $d \geq g-1$, $B_{g,d} = (2\pi)^{2g-2-d}B_{g,2g-2}$.*

The importance of the constants $B_{g,d}$ was emphasized by Fay, who made a systematic study of bosonization formulas. Our result provides a complete solution to the problem stated in the last paragraph of [17]. The following generalization of the genus 1 computation in [17, p. 117] is a consequence of Theorem 1.6 and the Main Theorem.

Corollary 1.7. *Fay's constants δ_g and $\varepsilon_{g,d}$ defined in* [17, Thms. 5.9 and 5.11] *have values:*

$$\begin{aligned} \delta_g &= (2\pi)^{g+1}\exp(c_g/6) \\ \varepsilon_{g,d} &= (2\pi)^{g-1-d} \end{aligned}$$

What follows is a brief outline of the paper and an explanation of the difference with the scalar case. Our approach to prove the Main Theorem is to study the behavior of both sides of eqs. (1.2) and (1.3) as the Riemann surface structure degenerates in moduli. Doing so gives a recursive formula for $B_{g,d}$ in the genus and consequently an expression in terms of the genus 1 constants which have been determined explicitly (cf. [17, p. 117]). The right hand sides of the bosonization formulas consist of holomorphic data and the determinant of the scalar laplacian, and the asymptotic behavior of these is known (cf. [38, 39]). Hence, it remains to study the asymptotics of the determinant of the Dolbeault laplacian.

For the *scalar* laplacian (1.4) this analysis depends on two key ingredients: (1) the Polyakov-Alvarez formula which expresses the dependency of the determinant on conformal changes in the metric on a surface, possibly with boundary (see [1]); and (2) the Burghelea-Friedlander-Kappeler factorization of the determinant in terms of Dirichlet determinants when the surface is cut along simple closed curves (see [10] and also [18]). Roughly speaking, the strategy is to use (2) to track the behavior of the determinant when the surface degenerates by pinching a simple closed curve, rescale the degenerating metrics to admissible metrics on the components of the nodal surface using (1), and then cap off the components of the cut surface by again applying (2).

It turns out that these steps are incompatible when one considers the Dolbeault operators in (1.2) and (1.3). The reason is that the proof of the Polyakov-Alvarez formula exploits the laplacian for the adjoint operator as well. Hence, in imposing boundary conditions, one requires ellipticity of the Dolbeault *complex*, rather than just the Dolbeault operator (this requirement is also, of course, natural from the point of view of the determinant of cohomology). On the other hand, it is well-known that such (local) boundary conditions fail to exist for the Dolbeault laplacian, except in the scalar case (cf. [20]).

In [1] Alvarez computed the conformal variation of the determinant of the laplacian acting on traceless symmetric tensors. In this case, the appropriate boundary condition was found to be of a mixed type arising from the natural splitting of the bundle induced along the boundary. One may view the bundle of traceless symmetric tensors as the real bundle underlying a power of the canonical (complex) line bundle. Motivated by this example, in Section 2.2, we introduce elliptic boundary conditions for Dolbeault operators on holomorphic bundles equipped with a *framing*, by which we mean a choice of trivialization near the boundary. These *Alvarez boundary conditions* are of mixed Dirichlet–Robin type, and come from the splitting of sections near the boundary into real and imaginary parts made possible by the framing. This is essentially the second order counterpart to the boundary conditions discussed in [40]. Because of the asymmetry, the boundary conditions are manifestly not complex linear. In particular, the Dolbeault laplacian must be regarded as a real operator (see Section 2.1). The advantage, however, is that the boundary value problem is compatible with a similar BVP on the adjoint bundle. This allows for an index theorem and a generalization of the Polyakov-Alvarez formula to arbitrary conformal changes of (ρ, h) (see Theorems 2.28 and 2.33).

The next step in our approach is to prove the corresponding Burghelea-Friedlander-Kappeler factorization theorem for Alvarez boundary conditions. The result is stated in Theorems 3.21, 3.26 and 3.38. A very special case of this factorization theorem is the well-known product formula

$$\mathrm{Det}^* \Delta_{S^2_R} = (\mathrm{Det}^*_{Neu.} \Delta_{H^2_R})(\mathrm{Det}_{Dir.} \Delta_{H^2_R}) \tag{1.8}$$

expressing the scalar determinant on the sphere S^2_R of radius R in terms of the determinants on the hemisphere H^2_R with Neumann and Dirichlet boundary conditions. Theorem 3.38 is therefore a kind of generalization of this result to arbitrary curves and to arbitrary line bundles. An interesting novelty is the fact that the Neumann jump operator for this problem is a pseudo-differential operator of order zero, rather than of order one as in the scalar case. Following Friedlander-Guillemin [19], we define its determinant by choosing a *regularizer* (see the definition (3.18)). This leads to an overall constant in the factorization which depends on the choice of regularizer. This constant can, however, be computed explicitly.

With these two ingredients, we apply the same degeneration techniques used in [39] to compute the asymptotics of the determinant of the Dolbeault laplacian. In Section 4.1, we study the asymptotics of the determinant of the Neumann jump operator. The analogous result was a key step in [39]. As in that case, we find that the Neumann jump operator takes a standard form when the surface degenerates. In particular, the aforementioned dependency on a choice of regularizer cancels out in the limit. In Sections 4.2 and 4.3, we describe the behavior of holomorphic sections and admissible metrics using the results of [38]. The Main Theorem is then proven in Section 4.4.

Finally, we should point out that there are many papers on gluing formulas for determinants of Laplace type operators with various boundary conditions (see, for example, [27] and the references therein), as well as results on degeneration of Quillen metrics [5, 22, 41]. We also note that expressions for the constant $B_{g,2g-2}$ have previously appeared in the mathematical literature (cf. [21, 34, 23]). These disagree slightly with the value found in [39, Theorem 1.3].

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2. THE MIXED BOUNDARY VALUE PROBLEM

2.1. **Real structures.** We begin with a construction that is completely elementary but will nevertheless serve to make precise the notions of a real operator and a real structure used in this paper. Let V be a complex Hilbert space with hermitian inner product $\langle\cdot,\cdot\rangle$ and dual space V^*. Let $\mathcal{R} : V^* \to V$ be the complex antilinear isomorphism given by the Riesz representation: $f(a) = \langle a, \mathcal{R}(f)\rangle$, for all $a \in V$, $f \in V^*$. Note that the complex antilinear involution

$$\imath : V \oplus V^* \longrightarrow V \oplus V^* : (a, f) \mapsto (\mathcal{R}(f), \mathcal{R}^{-1}(a))$$

satisfies $\langle \imath(a_1, f_1), \imath(a_2, f_2)\rangle = \overline{\langle (a_1, f_1), (a_2, f_2)\rangle}$ for the induced inner product on $V \oplus V^*$. Define

$$V_{\mathbb{R}} = \mathrm{Fix}(\imath) = \left\{(a, \mathcal{R}^{-1}(a)) : a \in V\right\} \tag{2.1}$$

The map $\jmath : V \to V_{\mathbb{R}} : a \mapsto A = (a, \mathcal{R}^{-1}(a))$ is then an $\mathbb{R}$-linear isomorphism. The real vector space $V_{\mathbb{R}}$ inherits a complete inner product $(\cdot,\cdot)$ from $V \oplus V^*$, and

$$(\jmath a_1, \jmath a_2) = 2\,\mathfrak{Re}\langle a_1, a_2\rangle \tag{2.2}$$

Let $T : V \to W$ be a (possibly unbounded) linear operator between Hilbert spaces. Then $\mathcal{R}^{-1}T\mathcal{R} : V^* \to W^*$ is also linear (with domain $\mathcal{R}^{-1}(\mathrm{Dom}\, T)$). The associated operator $(T, \mathcal{R}^{-1}T\mathcal{R}) : V \oplus V^* \longrightarrow W \oplus W^*$ commutes with the involution $\imath$ and hence induces a real linear map $P_T : V_{\mathbb{R}} \to W_{\mathbb{R}}$ that makes the following diagram commute.

$$\begin{tikzcd} V \arrow[r, "\jmath"] \arrow[d, "T"] & V_{\mathbb{R}} \arrow[d, "P_T"] \\ W \arrow[r, "\jmath"] & W_{\mathbb{R}} \end{tikzcd}$$

We call P_T the **real operator** associated to T. Note that in the case $W = V$, it follows that the spectrum of $P_T : V_{\mathbb{R}} \to V_{\mathbb{R}}$ coincides with the real spectrum of $T : V \to V$ with *twice* the multiplicity: if $a \in V$ is nonzero with $Ta = \lambda a$ and $\lambda \in \mathbb{R}$, then $\jmath a$ and $\jmath(ia)$ are independent eigenvectors of P_T, both with eigenvalue λ.

Finally, suppose that V has a **real structure**. By this we mean a complex antilinear involution $\sigma : V \to V$ satisfying

$$\langle \sigma a_1, \sigma a_2 \rangle = \overline{\langle a_1, a_2 \rangle} \tag{2.3}$$

Then $\sigma_{\mathbb{R}} = \jmath \circ \sigma \circ \jmath^{-1}$ gives an involution of $V_{\mathbb{R}}$ which, by (2.2) and (2.3), is an isometry. Let $V'_{\mathbb{R}}$, $V''_{\mathbb{R}}$ denote the $+1$, -1 eigenspaces of $\sigma_{\mathbb{R}}$, respectively. Then we have an orthogonal decomposition $V_{\mathbb{R}} = V'_{\mathbb{R}} \oplus V''_{\mathbb{R}}$. For $A \in V_{\mathbb{R}}$, $A = A' + A''$, where $A' = (1/2)(A + \sigma_{\mathbb{R}} A)$, $A'' = (1/2)(A - \sigma_{\mathbb{R}} A)$. We refer to A' and A'' as the *real* and *imaginary* parts of A. There is a natural almost complex structure J on $V_{\mathbb{R}}$ given by $JA = \jmath(i\jmath^{-1}(A))$. A calculation shows that $(JA_1, JA_2) = (A_1, A_2)$, and $J(V'_{\mathbb{R}}) \subset V''_{\mathbb{R}}$, $J(V''_{\mathbb{R}}) \subset V'_{\mathbb{R}}$. As a consequence, if we define a symplectic structure on $V_{\mathbb{R}}$ by the pairing (A_1, JA_2), then $V'_{\mathbb{R}}$ and $V''_{\mathbb{R}}$ are lagrangian subspaces (i.e. maximal isotropic).

2.2. **Alvarez boundary conditions.** We apply the construction of Section 2.1 to sections of hermitian holomorphic line bundles on M. Let M be a compact Riemann surface of genus g with a (non-empty) boundary ∂M and inclusion $\imath : \partial M \hookrightarrow M$. Without loss of generality, we may assume that M is obtained from a closed Riemann surface by deleting finitely many disjoint coordinate disks. Each component of ∂M has an open neighborhood in M biholomorphic to an annulus $\{r_1 \leq |z| < r_2\}$. We will refer to such a z as an *annular coordinate*.

Let $L \to M$ be a holomorphic line bundle. A holomorphic structure on L is equivalent to a Dolbeault operator $\bar{\partial}_L : \Omega^0(M, L) \to \Omega^{0,1}(M, L)$ satisfying the Leibniz rule. Equip M with a conformal metric ρ and L with a hermitian metric h. The holomorphic and hermitian structures on L give a unique unitary Chern connection $D = (\bar{\partial}_L, h)$, as well as an adjoint operator $\bar{\partial}_L^*$, and similarly on L^*. We will use the standard notation $h^0(L) = \dim_{\mathbb{C}} \ker \bar{\partial}_L$, $h^1(L) = \dim_{\mathbb{C}} \operatorname{coker} \bar{\partial}_L = \dim_{\mathbb{C}} \ker \bar{\partial}_L^*$.

There is a natural hermitian inner product on the space $\Omega^0(M, L)$ of smooth sections of L given by

$$\langle s_1, s_2 \rangle_M = \int_M dA_\rho \, \langle s_1, s_2 \rangle_h$$

where dA_ρ is the area form on M coming from the metric ρ. The dual space is given by integration on M: $\Omega^0(M, L)^* \simeq \Omega^{1,1}(M, L^*)$. Then

$$\Omega^0_{\mathbb{R}}(M, L) \subset \Omega^0(M, L) \oplus \Omega^{1,1}(M, L^*) \tag{2.4}$$

is the real vector space constructed as in (2.1). Strictly speaking, here we should work with the L^2 and Sobolev completions. These are defined using the Chern connection D. Since this is standard, for notational simplicity we omit this from the notation.

We can also carry out this construction on $(0,1)$-forms:

$$\Omega^{0,1}_{\mathbb{R}}(M,L) \subset \Omega^{0,1}(M,L) \oplus \Omega^{1,0}(M,L^*) \tag{2.5}$$

Denote the isomorphisms of real vector spaces

$$\begin{aligned} \jmath_0 &: \Omega^0(M,L) \longrightarrow \Omega^0_{\mathbb{R}}(M,L) : \varphi \mapsto \Phi \\ \jmath_1 &: \Omega^{0,1}(M,L) \longrightarrow \Omega^{0,1}_{\mathbb{R}}(M,L) : \psi \mapsto \Psi \end{aligned}$$

or simply by $\jmath$ when the meaning is clear.

As in Section 2.1, define a (real, unbounded) linear operator $P_L : \Omega^0_{\mathbb{R}}(M,L) \to \Omega^{0,1}_{\mathbb{R}}(M,L)$ making the following diagram commute:

$$\begin{tikzcd} \Omega^0(M,L) \arrow[r, "\jmath_0"] \arrow[d, "\bar\partial_L"] & \Omega^0_{\mathbb{R}}(M,L) \arrow[d, "P_L"] \\ \Omega^{0,1}(M,L) \arrow[r, "\jmath_1"] & \Omega^{0,1}_{\mathbb{R}}(M,L) \end{tikzcd}$$

In terms of the decompositions (2.4) and (2.5), it follows that

$$P_L = \begin{pmatrix} \bar\partial_L & 0 \\ 0 & (\bar\partial_{L^*})^* \end{pmatrix} \tag{2.6}$$

Now consider the boundary. There is an hermitian inner product on $\Omega^0(\partial M, \imath^* L)$ given by

$$\langle s_1, s_2\rangle_{\partial M} = \int_{\partial M} ds_\rho\, \langle s_1, s_2\rangle_h$$

where ds_ρ is the induced measure on ∂M. Note that ∂M inherits an orientation from M and the outward normal. Hence, integration gives an identification $\Omega^0(\partial M, \imath^*L)^*$ with $\Omega^1(\partial M, \imath^*(L^*))$. With this understood, let

$$\Omega^0_{\mathbb{R}}(\partial M, \imath^*L) \subset \Omega^0(\partial M, \imath^*L) \oplus \Omega^1(\partial M, \imath^*(L^*)) \tag{2.7}$$

be the real vector space constructed as in the previous section.

The trace map

$$\Omega^0(M,L) \longrightarrow \Omega^0(\partial M, \imath^*L) : \varphi \mapsto \varphi\big|_{\partial M} \tag{2.8}$$

is induced by restriction. Using the Hodge star on M to identify $\Omega^{1,1}(M,L^*) \simeq \Omega^0(M,L^*)$, and on ∂M to identify $\Omega^1(\partial M, \imath^*L^*) \simeq \Omega^0(\partial M, \imath^*L^*)$, there is a similar restriction map

$$\Omega^{1,1}(M,L^*) \simeq \Omega^0(M,L^*) \longrightarrow \Omega^0(\partial M, \imath^*L^*) \simeq \Omega^1(\partial M, \imath^*L^*)$$

The restriction maps combine to give a trace map $\Omega^0_{\mathbb{R}}(M,L) \to \Omega^0_{\mathbb{R}}(\partial M, \imath^*L)$. We carry out the same construction with $\Omega^{0,1}(M,L)$. Here, we define

$$\Omega^1_{\mathbb{R}}(\partial M, \imath^*L) \subset \Omega^1(\partial M, \imath^*L) \oplus \Omega^0(\partial M, \imath^*(L^*))$$

In this case, again using the Hodge star on ∂M the trace map $\Omega^{0,1}_{\mathbb{R}}(M, L) \to \Omega^1_{\mathbb{R}}(\partial M, \imath^* L)$ pulls-back the forms and restricts the section.

Definition 2.9. *We call*

$$\mathfrak{B}(\partial M, \imath^* L) = \Omega^0_{\mathbb{R}}(\partial M, \imath^* L) \oplus \Omega^1_{\mathbb{R}}(\partial M, \imath^* L)$$

the space of **Cauchy data**. *The trace map is the (real) linear map:*

$$\mathbf{b}_{\partial M} : \Omega^0_{\mathbb{R}}(M, L) \longrightarrow \mathfrak{B}(\partial M, \imath^* L) : \Phi \mapsto (\Phi, P_L \Phi)\big|_{\partial M}$$

defined as above.

In order to define elliptic boundary conditions we will need real structures. These come from a choice of trivialization of L near ∂M.

Definition 2.10. *A* **framing** *of a holomorphic line bundle $L \to M$ is a trivialization (i.e. a nowhere vanishing holomorphic section) τ_L of L near ∂M.*

An important example of a framing is the following

Example 2.11. *Let L be defined by a divisor D compactly supported in M. Then by construction L has a meromorphic section τ_L with zeros and poles exactly at D. In particular, τ_L gives a framing of L. While τ_L is only defined up to multiplication by a nonzero constant, we shall refer to any such choice as a canonical framing.*

Given a framing and a section φ of L defined in a neighborhood of ∂M, write $\varphi = (\varphi' + i\varphi'') \cdot \tau_L$, where φ', φ'' are real valued functions. Then let $\sigma(\varphi) = (\varphi' - i\varphi'') \cdot \tau_L$. This defines a real structure on $\Omega^0(\partial M, \imath^* L)$. As in Section 2.1, the boundary values of $\Phi \in \Omega^0_{\mathbb{R}}(M, L)$ therefore have real and imaginary parts Φ', Φ''. The framing also gives a real structure on boundary values of elements of $\Omega^{0,1}(M, L)$. Indeed, there is natural isomorphism $T^{0,1}M\big|_{\partial M} \simeq T(\partial M) \otimes \mathbb{C}$. Equivalently, the Hodge star gives a $\mathbb{C}$-linear isomorphism $* : \Omega^0(\partial M, \imath^* L) \simeq \Omega^1(\partial M, \imath^* L)$ with $*^2 = 1$. If σ_0 is the real structure on $\Omega^0(\partial M, \imath^* L)$, then $\sigma_1 = *\sigma_0 *$ is a real structure on $\Omega^1(\partial M, \imath^* L)$. We let $\mathfrak{B}'(\partial M, \imath^* L)$ (resp. $\mathfrak{B}''(\partial M, \imath^* L)$) be the subspaces of $\mathfrak{B}(\partial M, \imath^* L)$ consisting of elements (Φ', Ψ') (resp. (Φ'', Ψ'')).

- Note that there is a natural pairing of $\Omega^0_{\mathbb{R}}(\partial M, \imath^* L)$ and $\Omega^1_{\mathbb{R}}(\partial M, \imath^* L)$ defined as follows. If $\Phi = \jmath_0(\varphi)$, $\Psi = \jmath_1(\psi)$ then

$$(\Phi, \Psi)_{\partial M} = 2\,\mathfrak{Re} \int_{\partial M} \langle \varphi, \psi \rangle_h \tag{2.12}$$

- The real structure defines an almost complex structure on $\Omega^0_{\mathbb{R}}(\partial M, \imath^* L)$ and $\Omega^1_{\mathbb{R}}(\partial M, \imath^* L)$ as in Section 2.1. We extend this to an almost complex structure on the space of boundary values $\mathfrak{B}(\partial M, \imath^* L)$ by defining

$$J_{\partial M} = \begin{pmatrix} 0 & *J \\ *J & 0 \end{pmatrix}$$

(for simplicity, we will denote this operator simply by J as well). This almost complex structure and the pairing (2.12) give a symplectic structure on $\mathfrak{B}(\partial M, \imath^* L)$ defined by (f, Jg). As in Section 2.1, the subspaces $\mathfrak{B}'(\partial M, \imath^* L)$ and $\mathfrak{B}''(\partial M, \imath^* L)$ are then lagrangian.

Definition 2.13. *Let* $\mathbf{b}'_{\partial M}$ *and* $\mathbf{b}''_{\partial M}$ *be the projections to the real and imaginary parts of* $\mathbf{b}_{\partial M}$. *We call the equation* $\mathbf{b}'_{\partial M}(\Phi) = 0$ *(resp.* $\mathbf{b}''_{\partial M}(\Phi) = 0$*) the* real *(resp.* imaginary*)* Alvarez boundary conditions.

Note that $\mathbf{b}'_{\partial M}$ and $\mathbf{b}''_{\partial M}$ take values in lagrangian subspaces of $\mathfrak{B}(\partial M, \imath^* L)$. We will use the same notation for the boundary map on $\Omega^{0,1}_{\mathbb{R}}(M, L)$; namely,

$$\mathbf{b}_{\partial M} : \Omega^{0,1}_{\mathbb{R}}(M, L) \longrightarrow \mathfrak{B}(\partial M, \imath^* L) : \Psi \mapsto (P_L^\dagger \Psi, \Psi)\big|_{\partial M}$$

where $P_L^\dagger$ is the formal adjoint of P_L. Then $\mathbf{b}'_{\partial M}$ and $\mathbf{b}''_{\partial M}$ are defined similarly.

Since we here assume that $\partial M \neq \emptyset$, by a theorem of Grauert L admits a global holomorphic trivialization $\mathbb{1}_L$ on M. Then $\tau_L/\mathbb{1}_L$ is a nowhere vanishing holomorphic function in a neighborhood of ∂M. We define the **degree** $\deg(\tau_L)$ of a framed line bundle to be the winding number of $\tau_L/\mathbb{1}_L$ (with the outward normal, summed over all components of ∂M). Clearly, the definition of degree is independent of the choice of trivialization $\mathbb{1}_L$. Note the following two important examples.

Example 2.14.
(1) *Let s be a meromorphic section of L satisfying imaginary Alvarez boundary conditions and with divisor (s) compactly supported in the interior of M. Then $\deg(\tau_L) = \deg(s)$.*
(2) *Let $L = K^q$, where the framing is given by $\tau_L = (-idz/z)^q$ in local annular coordinates near ∂M. Then $\deg(\tau_L) = -\chi(M)$. One can check that the real structure is independent of the choice of annular coordinate.*

The Alvarez boundary conditions are of mixed Dirichlet-Robin type. Indeed, fix a framing τ_L of L, and let $h = \|\tau_L\|^2$. Then on ∂M, define

$$\nu_{L,h} = -\tfrac{1}{2}\partial_n \log h \tag{2.15}$$

where n is the *outward* normal. Also, let $\Pi_\pm = \frac{1}{2}(I \pm \sigma_{\mathbb{R}})$ be the projections to the real and imaginary parts. Then it is easy to see that $\mathbf{b}''_{\partial M}(\Phi) = 0$ is equivalent to the conditions

$$\begin{aligned} \Pi_- \Phi\big|_{\partial M} &= 0 \\ (\nabla_n + S)\Pi_+ \Phi\big|_{\partial M} &= 0 \end{aligned} \tag{2.16}$$

where n is the outward normal, ∇ is the induced connection on the bundle of real sections, and $S = \nu_{L,h}$. Indeed, write $\varphi = (\varphi' + i\varphi'')\tau_L$. The Alvarez boundary conditions are $\varphi'' = 0$ and $\partial_n \varphi' = 0$ on ∂M. A local unitary frame is given by $\mathbf{e}_L = h^{-1/2}\tau_L$. Since the connection form in the frame $\mathbf{e}_L$ is purely imaginary, $\mathbf{e}_L$ is parallel with respect to ∇, and the result follows from the expression $\Pi_+\Phi = (\varphi' h^{1/2})\mathbf{e}_L$.

2.3. **Heat kernels and an index theorem.** A straightforward calculation gives the following important integration by parts formula: for smooth sections $\Phi \in \Omega^0_{\mathbb{R}}(M, L)$ and $\Psi \in \Omega^{0,1}_{\mathbb{R}}(M, L)$,

$$(P_L\Phi, \Psi)_M - (\Phi, P_L^\dagger \Psi)_M = \tfrac{1}{2}(\Phi, J\Psi)_{\partial M} \tag{2.17}$$

where the pairing (2.12) appears on the right hand side. Define the laplacian $D_L = 2P_L^\dagger P_L$ on smooth sections $\Omega^0(M, L)$. Then from (2.17) we have

$$(D_L\Phi_1, \Phi_2)_M - (\Phi_1, D_L\Phi_2)_M = (\mathbf{b}_{\partial M}(\Phi_1), J\, \mathbf{b}_{\partial M}(\Phi_2)) \tag{2.18}$$

$$(P_L\Phi_1, P_L\Phi_2) - (\Phi_1, D_L\Phi_2) = \tfrac{1}{2}\left[(\Phi_1'', J(P_L\Phi_2)') - ((P_L\Phi_2)'', J\Phi_1')\right] \tag{2.19}$$

The right hand sides of (2.18) and (2.19) vanish identically for Alvarez boundary conditions. For the following result, see for example [20, Lemma 1.11.1].

Proposition 2.20. *Assuming either real or imaginary Alvarez boundary conditions, the formal adjoint $P_L^\dagger$ extends to an unbounded operator on $\Omega^{0,1}_{\mathbb{R}}(M,L)$ as the the L^2-adjoint of P_L on $\Omega^0_{\mathbb{R}}(M,L)$. Moreover, D_L extends to an unbounded self-adjoint non-negative elliptic operator D_L^A on sections $\Omega^0_{\mathbb{R}}(M,L)$ satisfying real (resp. imaginary) Alvarez boundary conditions. A similar statement holds for the laplacian $2P_LP_L^\dagger$ on $\Omega^{0,1}_{\mathbb{R}}(M,L)$.*

We now make a choice: henceforth, unless otherwise indicated, by *Alvarez boundary conditions* we will mean the condition $\mathbf{b}''_{\partial M}(\Phi) = 0$. We write D_L^A when we wish to emphasize that the laplacian D_L is acting on the space of sections satisfying Alvarez boundary conditions.

Remark 2.21. *By* (2.19), $\ker D_L^A \subset \ker P_L$. *Hence,* $\ker D_L^A$ *is real isomorphic to the space of holomorphic sections φ of L with local expression $\varphi = \varphi(z)\tau_L$ near ∂M, satisfying $\mathfrak{Im}(\varphi(z))\big|_{\partial M} = 0$.*

Remark 2.22. *If Φ is an eigensection of D_L satisfying Alvarez boundary conditions with eigenvalue $\lambda \neq 0$, then $P_L\Phi$ is an eigensection of $2P_LP_L^\dagger$ with the same eigenvalue λ, also satisfying Alvarez boundary conditions.*

This simple observation is the *raison d'être* of the mixed boundary conditions we have chosen. By contrast, if φ is an eigensection of $\Box_L$ satisfying Dirichlet conditions, then $\bar\partial_L\varphi$ is a formal eigensection of $\bar\partial_L\bar\partial_L^*$, but does not necessarily satisfy an elliptic boundary condition.

We also note the following

Proposition 2.23 (Serre duality)**.** *Fix a framing τ_L on $L \to M$. Then with respect to the duality*

$$\Omega^{0,1}(M,L) \simeq (\Omega^0(K \otimes L^*))^*$$

the framing on $K \otimes L^*$ is induced by that on L and $-idz/z$, where z is an annular coordinate near ∂M. In particular, with these Alvarez boundary conditions, $\operatorname{coker} P_L \simeq \ker P_L^\dagger \simeq (\ker P_{K\otimes L^*})^\dagger$.

Proof. The usual proof of Serre duality applies, modulo the boundary conditions. To understand these, choose a local annular coordinate z near ∂M. Then with respect to the trivialization τ_L, a smooth section $\psi d\bar z \in \Omega^{0,1}(M,L)$ with $\Psi \in \ker P_L^\dagger$ satisfies Alvarez boundary conditions if $\mathfrak{Im}(i\psi e^{-i\theta}) = 0$. The corresponding section of $\Omega^0(K\otimes L^*)$ is $\bar\psi h^{-1}dz = iz\bar\psi h^{-1}(-idz/z)$. On the boundary, $\mathfrak{Im}(i\psi e^{-i\theta}) = 0$ is equivalent to $\mathfrak{Im}(iz\bar\psi h^{-1}) = 0$. This proves the claim. □

We will need the following

Proposition 2.24. *Let $L \to M$ be the holomorphic bundle associated to an effective divisor D of degree d compactly supported in the interior of M, and give L a canonical framing. If $2d \geq 1 - \chi(M)$, then for generic choices of D,* $\operatorname{coker} P_L = \{0\}$.

Proof. By Proposition 2.23, it suffices to show $\ker P_{K\otimes L^*} = \{0\}$ for generic D. Suppose to the contrary that $0 \neq \Phi \in \ker P_{K\otimes L^*}$. Let N be the Schottky double of M. Then N has genus $G = 1 - \chi(M) \leq 2d$, and by Remark 2.21 and the reflection principle, φ extends to a holomorphic section of a a bundle on N associated to a symmetric divisor of degree $-2\chi(M) - 2d \leq -1 - \chi(M) \leq G - 2$ (see Example 2.14 (2)). On the other hand, by the discussion in [16, Ch. VI], every symmetric divisor of degree G is linearly equivalent to a positive one, whereas the space of positive symmetric divisors of degree $G - 1$ has codimension 1. The result follows from this fact. □

In order to state a result for the small time expansion of the trace of the heat kernel, we will need the following quantities. Let $\Omega_{L,h}$ denote the Hermitian-Einstein tensor (cf. [24, IV.1.2]). In a local holomorphic frame we have

$$\Omega_{L,h} = i * F_{(\bar\partial_L,h)} = -\tfrac{1}{2}\Delta_\rho \log h \tag{2.25}$$

where $F_{(\bar\partial_L,h)}$ is the curvature of the Chern connection. Note the following special case.

Lemma 2.26. *Let R_ρ and κ_ρ denote the scalar and geodesic curvatures of M and ∂M. With the hermitian metric on K induced from the metric on M, $\Omega_{K,\rho^{-1}} = -(1/2)R_\rho$. For the framing $-idz/z$, $\nu_{K,\rho^{-1}} = \kappa_\rho$.*

For the short time expansion of heat kernels, we refer to [9] and [20]. In particular, we use the result in [35, Sec. 5.3] and the expression for S in (2.16).

Proposition 2.27. *Let $L \to M$ be a holomorphic line bundle on M with framing τ_L. Let ρ and h be hermitian metrics on M and L, respectively. Then for any function f, the trace with the heat kernel for the operator D_L^A with Alvarez boundary conditions defined by τ_L has the following short time expansion:*

$$\operatorname{Tr}(fe^{-\varepsilon D_L^A}) = \frac{1}{2\pi\varepsilon}\int_M dA\, f + \frac{1}{12\pi}\int_M dA\, f(6\Omega_{L,h} + R_\rho) + \frac{1}{6\pi}\int_{\partial M} ds\, f(\kappa_\rho - 3\nu_{L,h}) + O(\varepsilon^{1/2})$$

Theorem 2.28 (Index theorem)**.** *Let $L \to M$ be a holomorphic line bundle on M with framing τ_L. Then for Alvarez boundary conditions,*

$$\operatorname{index} P_L = \dim_{\mathbb{R}} \ker P_L - \dim_{\mathbb{R}} \operatorname{coker} P_L = 2\deg(\tau_L) + \chi(M) \tag{2.29}$$

Proof. From Proposition 2.27, Lemma 2.26, Remark 2.22, and Proposition 2.23

$$\begin{aligned}
\operatorname{index} P_L &= \lim_{\varepsilon\to 0}\left\{\operatorname{Tr}(e^{-2\varepsilon P_L^\dagger P_L}) - \operatorname{Tr}(e^{-2\varepsilon P_L P_L^\dagger})\right\} \\
&= \frac{1}{2\pi}\int_M dA\,(\Omega_{L,h} - \Omega_{KL^*,(\rho h)^{-1}}) - \frac{1}{2\pi}\int_{\partial M} ds\,(\nu_{L,h} - \nu_{KL^*,(\rho h)^{-1}}) \\
&= \frac{1}{2\pi}\int_M dA\, 2\Omega_{L,h} - \frac{1}{2\pi}\int_{\partial M} ds\, 2\nu_{L,h} + \frac{1}{4\pi}\int_M dA\, R_\rho + \frac{1}{2\pi}\int_{\partial M} ds\, \kappa_\rho
\end{aligned}$$

By the Gauss-Bonnet Theorem, the last two terms give the Euler characteristic $\chi(M)$. Write $\tau_L = f\mathbb{1}_L$, and let $h_0 = \|\mathbb{1}_L\|^2$. Then near ∂M, $h = |f|^2 h_0$, and

$$\deg(\tau_L) = \frac{1}{2\pi}\int_{\partial M} ds\, \partial_n \log|f|$$

On the other hand,

$$\begin{aligned}
\frac{1}{2\pi}\int_M dA\, \Omega_{L,h} - \frac{1}{2\pi}\int_{\partial M} \nu_{L,h} &= -\frac{1}{4\pi}\int_M dA\, \Delta \log h_0 + \frac{1}{4\pi}\int_{\partial M} ds\, \partial_n \log h \\
&= \frac{1}{4\pi}\int_{\partial M} ds\, (-\partial_n \log h_0 + \partial_n \log h) \\
&= \frac{1}{4\pi}\int_{\partial M} ds\, \partial_n \log|f|^2 = \deg(\tau_L)
\end{aligned}$$

The result follows. □

Remark 2.30. *By Example 2.14, if K^q on M is given the framing $(-idz/z)^q$ for annular coordinates at each component of ∂M, then* $\deg(\tau_{K^q}) = -q\chi(M)$. *Hence, by Theorem 2.28,* $\operatorname{index} P_{K^q} = (1-2q)\chi(M)$. *This agrees with* [1, eq. (4.32)].

2.4. **Determinants of laplacians.** Following [29], we define determinants as follows. Suppose M is closed with conformal metric ρ and a hermitian holomorphic line bundle $L \to M$. Let $\{\lambda_j\}_{j=1}^{\infty}$ be the spectrum of $\square_L$ and form the zeta function $\zeta_{\square_L}(s) = \sum_{\lambda_j>0} \lambda_j^{-s}$. Then $\zeta_{\square_L}(s)$ converges for $\mathfrak{Re}(s)$ sufficiently large, and by a theorem of Seeley [30] it is known that $\zeta_{\square_L}(s)$ is regular at $s = 0$. Then $\log \operatorname{Det}^* \square_L := -\zeta'_{\square_L}(0)$. A similar definition applies to $\operatorname{Det}^* D_L$ on M, and to $\operatorname{Det}^* D_L^A$ when M has boundary, L has a framing, and we use Alvarez boundary conditions. When it is understood that the spectrum is strictly positive, we will omit the asterisk and write $\operatorname{Det} \square_L$, etc.

When M is closed, D_L acting on $\Omega^0_{\mathbb{R}}(M, L)$ is the same as $\square_L$ acting on $\Omega(M, L)$, regarded as a real operator (see Section 2.1), and hence it has the same spectrum but with twice the multiplicity. Taking into account also the factor of 2 in the definition of the real inner product (see (2.2)), we have the following

Lemma 2.31. *If M is a* **closed** *Riemann surface with line bundle $L \to M$. Then for all $\lambda > 0$,*

$$\operatorname{Det}(D_L + \lambda) = [\operatorname{Det}(\square_L + \lambda)]^2$$

Similarly,

$$\frac{\operatorname{Det}^* D_L}{\det(\Phi_i, \Phi_j)} = \left(2^{-h^0(L)} \frac{\operatorname{Det}^* \square_L}{\det\langle \omega_i, \omega_j\rangle}\right)^2$$

where $\{\omega_i\}_{i=1}^{h^0(L)}$ is a basis (over $\mathbb{C}$) for $H^0(M, L)$ and $\{\Phi_i\}_{i=1}^{2h^0(L)}$ is the associated basis (over $\mathbb{R}$) of $\ker D_L$ *given by*

$$\Phi_{2i} = \jmath(i\omega_i)\ ,\ \Phi_{2i-1} = \jmath(\omega_i) \tag{2.32}$$

for $i = 1, \ldots, h^0(L)$.

The main result of this section is the following

Theorem 2.33 (Polyakov-Alvarez formula)**.** *Let $\{\Phi_i\}_{i=1}^m$, $\{\Psi_j\}_{j=1}^n$ be bases for the kernel and cokernel of P_L on M with Alvarez boundary conditions. Suppose the following relation for hermitian metrics: $\rho = e^{2\sigma}\hat{\rho}$, $h = e^{2f}\hat{h}$. Then*

$$\left[\frac{\operatorname{Det}^* D_L^A}{\det(\Phi_i,\Phi_j)\det(\Psi_i,\Psi_j)}\right]_{(\rho,h)} = \left[\frac{\operatorname{Det}^* D_L^A}{\det(\Phi_i,\Phi_j)\det(\Psi_i,\Psi_j)}\right]_{(\hat{\rho},\hat{h})} \exp(S(\sigma,f))$$

where

$$\begin{aligned} S(\sigma,f) = &-\frac{1}{6\pi}\int_M dA_{\hat{\rho}}\left\{6\nabla f\cdot\nabla(\sigma+f)+|\nabla\sigma|^2\right\} \\ &-\frac{1}{6\pi}\int_M dA_{\hat{\rho}}\left\{6\Omega_{L,\hat{h}}(\sigma+2f)+R_{\hat{\rho}}(\sigma+3f)\right\} \\ &+\frac{1}{3\pi}\int_{\partial M} ds_{\hat{\rho}}\left\{3\nu_{L,\hat{h}}(\sigma+2f)-\kappa_{\hat{\rho}}(\sigma+3f)\right\} \end{aligned} \tag{2.34}$$

Proof. The argument exactly follows [1]; here we only sketch the important differences. Temporarily define $D_L^+ = 2P_L^\dagger P_L$, and $D_L^- = 2P_L P_L^\dagger$. Let $\{\Phi_j\}$ be an orthonormal basis of eigensections of D_L^+ with eigenvalues λ_j. Then by Remark 2.22, if $\Psi_j = (1/\sqrt{\lambda_j})P_L\Phi_j$, then $\{\Psi_j\}$ is an orthonormal basis of the subspace of eigensections of D_L^- with nonzero eigenvalues. Let $\sigma=\sigma(t)$, $f=f(t)$ be one parameter families of conformal deformations; $\dot{\sigma}$ and $\dot{f}$, their derivatives. One computes the variation of eigenvalues.

$$\dot{\lambda}_j = -2\lambda_j((\dot{\sigma}+\dot{f})\Phi_j,\Phi_j)+2\lambda_j(\dot{f}\Psi_j,\Psi_j)$$

Then as in [1, pp. 148-9], the corresponding variation of the determinant is given by

$$\begin{aligned} \frac{d}{dt}\log\operatorname{Det}^* D_L^+ &= f.p.\int_\varepsilon^\infty dt\sum_{\lambda_j\neq 0}\dot{\lambda}_j e^{-t\lambda_j} \\ &= f.p.\int_\varepsilon^\infty dt\sum_{\lambda_j\neq 0}\left\{-2\lambda_j((\dot{\sigma}+\dot{f})\Phi_j,\Phi_j)+2\lambda_j(\dot{f}\Psi_j,\Psi_j)\right\}e^{-t\lambda_j} \\ &= -f.p.\int_\varepsilon^\infty dt\frac{d}{dt}\left\{-2\operatorname{Tr}((\dot{\sigma}+\dot{f})e^{-tD_L^+})+2\operatorname{Tr}(\dot{f}e^{-tD_L^-})\right\} \end{aligned}$$

Applying Proposition 2.27 to the heat kernel expansion for the laplacian on L and KL^*,

$$\begin{aligned} \frac{d}{dt}\log\operatorname{Det}^* D_L^+ = &-\frac{1}{6\pi}\int_M dA_\rho\,(6\Omega_{L,h}+R_\rho)(\dot{\sigma}+\dot{f})+\frac{1}{6\pi}\int_M dA_\rho\,(6\Omega_{KL^*,(\rho h)^{-1}}+R_\rho)\dot{f} \\ &-\frac{1}{3\pi}\int_{\partial M} ds_\rho\,(\kappa_\rho-3\nu_{L,h})(\dot{\sigma}+\dot{f})+\frac{1}{3\pi}\int_{\partial M} ds_\rho\,(\kappa_\rho-3\nu_{KL^*,(\rho h)^{-1}})\dot{f} \end{aligned}$$

From Lemma 2.26 it follows that $\Omega_{KL^*,(\rho h)^{-1}} = -(1/2)R_\rho - \Omega_{L,h}$, and $\nu_{KL^*,(\rho h)^{-1}} = \kappa_\rho - \nu_{L,h}$. Hence,

$$\begin{aligned} \frac{d}{dt}\log\operatorname{Det}^* D_L^+ = &-\frac{1}{6\pi}\int_M dA_\rho\,\left\{6\Omega_{L,h}(\dot{\sigma}+2\dot{f})+R_\rho(\dot{\sigma}+3\dot{f})\right\} \\ &-\frac{1}{3\pi}\int_{\partial M} ds_\rho\,\left\{\kappa_\rho(\dot{\sigma}+3\dot{f})-3\nu_{L,h}(\dot{\sigma}+2\dot{f})\right\} \end{aligned} \tag{2.35}$$

We have the following variations with respect to conformal changes.

$$\begin{aligned} R_\rho &= e^{-2\sigma}(R_{\hat{\rho}}-2\Delta_{\hat{\rho}}\sigma) & \Omega_{L,h} &= e^{-2\sigma}(\Omega_{L,\hat{h}}-\Delta_{\hat{\rho}}f) \\ \kappa_\rho &= e^{-\sigma}(\kappa_{\hat{\rho}}+\partial_{\hat{n}}\sigma) & \nu_{L,h} &= e^{-\sigma}(\nu_{L,\hat{h}}-\partial_{\hat{n}}f) \end{aligned}$$

Plugging these into the above, the first term on the right hand side of (2.35) becomes

$$
\begin{aligned}
&-\frac{1}{6\pi}\int_M dA_{\hat\rho}\,\Big\{6\Omega_{L,\hat h}(\dot\sigma+2\dot f)+R_{\hat\rho}(\dot\sigma+3\dot f)\Big\}\\
&\qquad-\frac{1}{6\pi}\int_M dA_{\hat\rho}\,\Big\{6\nabla f\cdot\nabla\dot\sigma+12\nabla f\cdot\nabla\dot f+2\nabla\sigma\cdot\nabla\dot\sigma+6\nabla\sigma\cdot\nabla\dot f\Big\}\\
&\qquad+\frac{1}{6\pi}\int_{\partial M} ds_{\hat\rho}\Big\{12(\partial_{\hat n}f)\dot f+2(\partial_{\hat n}\sigma)\dot\sigma+6((\partial_{\hat n}f)\dot\sigma+(\partial_{\hat n}\sigma)\dot f)\Big\}
\end{aligned}
\tag{2.36}
$$

whereas the second term on the right hand side of (2.35) becomes

$$
\begin{aligned}
&-\frac{1}{3\pi}\int_{\partial M} ds_{\hat\rho}\,\Big\{\kappa_{\hat\rho}(\dot\sigma+3\dot f)-3\nu_{L,\hat h}(\dot\sigma+2\dot f)\Big\}\\
&\qquad-\frac{1}{3\pi}\int_{\partial M} ds_{\hat\rho}\,\Big\{(\partial_{\hat n}\sigma)(\dot\sigma+3\dot f)+3(\partial_{\hat n}f)(\dot\sigma+2\dot f)\Big\}
\end{aligned}
\tag{2.37}
$$

The last terms on the right hand sides of (2.36) and (2.37) cancel. The remaining terms can be integrated as in [1], giving the desired result. □

Remark 2.38. *Consider the following special cases:*

(1) $\partial M=\emptyset$. *Then the formula in* (2.34) *coincides with the result in* [17, Prop. 3.8]. *Note that there is an overall factor of* 2*, coming from the fact that the determinant* $\mathrm{Det}^* D_L$*, regarded as a real operator, is the square of the complex laplacian (see Lemma 2.31).*

(2) *If* $L=K^q$*,* h *the induced metric from* M*, and* $f=-q\sigma$*, then* (2.34) *coincides with the result in* [1, eq. (4.29)] *(see Lemma 2.26).*

(3) *If* σ *and* f *are constant, then*

$$S(\sigma,f)=-(2\deg(\tau_L)+\tfrac{2}{3}\chi(M))\sigma-(4\deg(\tau_L)+2\chi(M))f$$

gives the scaling law for determinants.

We will also need the following example. For the trivial bundle over the disk, the Alvarez boundary conditions on a complex valued function reduce to Dirichlet conditions on the imaginary part and Neumann conditions on the real part. Hence, the determinant is a product of the determinants for these boundary conditions, which have been evaluated in [37]: i.e. for the flat disk of radius R and trivial, flat L with parallel framing,

$$\mathrm{Det}^* D_L^A=[\mathrm{Det}^*_{Neu.}\Delta][\mathrm{Det}_{Dir.}\Delta]=2^{-1/3}R^{4/3}\exp(-4\zeta'(-1)+1/6)\tag{2.39}$$

3. Factorization of determinants

3.1. **The generalized Dirichlet-to-Neumann operator.** In this section we assume M has non-empty boundary. Let $L\to M$ be a hermitian holomorphic bundle with framing τ_L. The following is clear.

Lemma 3.1. *The real and imaginary Alvarez boundary conditions are complimentary in the sense of* [10, Def. 2.12].

Definition 3.2. *The Poisson operator is characterized by the condition*

$$\mathcal{P}_M(\lambda) : \mathfrak{B}''(\partial M, \imath^* L) \to \Omega^0_{\mathbb{R}}(M, L) : (f, g) \mapsto \mathcal{P}_M(\lambda)(f, g) = \Phi$$

where Φ *satisfies* $(D_L + \lambda)\Phi = 0$, *and* $\mathbf{b}''_{\partial M}(\Phi) = (f, g)$. *The boundary operator is defined by*

$$\mathcal{A}_M(\lambda) : \mathfrak{B}''(\partial M, \imath^* L) \to \mathfrak{B}''(\partial M, \imath^* L) : \mathcal{A}_M(\lambda) = J\, \mathbf{b}'_{\partial M}\, \mathcal{P}_M(\lambda)$$

Hence, $\mathcal{A}_M(\lambda)$ is the analog of the Dirichlet-to-Neumann operator. Like the DN operator, $\mathcal{A}_M(\lambda)$ is elliptic and, by (2.18) it is self-adjoint. In this case, however, it is a *zero-th* order pseudo-differential operator instead of first order.

In case $\lambda = 0$, the Poisson, and hence also boundary operators are not necessarily everywhere defined nor are they a priori well-defined. This can be seen from the integration by parts formula (2.18). The Poisson operator is defined at (f, g) only if (f, g) is orthogonal to the image by J of boundary values of sections $\Phi \in \ker D_L$ satisfying imaginary Alvarez boundary conditions. Similarly, given any such (f, g), the extension by the Poisson operator is only well-defined up to addition of such Φ. With this in mind, set

$$\mathbb{A}^{\mathrm{alv}}_M = \left\{ J\, \mathbf{b}'_{\partial M}(\Phi) : \Phi \in \ker D_L\ ,\ \mathbf{b}''_{\partial M}(\Phi) = 0 \right\} \tag{3.3}$$

Proposition 3.4. *On the orthogonal complement of* $\mathbb{A}^{\mathrm{alv}}_M$, *the family* $\mathcal{A}_M(\lambda)$ *extends continously as* $\lambda \to 0$ *to a pseudo-differential operator* $\mathcal{A}_M(0) = \mathcal{A}_M$.

Proof. Let $\{\Phi^A_i\}_{i=1}^\infty$ be a complete set of eigensections for D^A_L with eigenvalues $\{\lambda_i\}_{i=1}^\infty$, and $\lambda_i = 0$ if and only if $i \leq n$. Choose a smooth extension map $E : \mathfrak{B}''(\partial M, \imath^* L) \to L^2(M)$ satisfying $\mathbf{b}''_{\partial M}\, E = I$, $\mathbf{b}'_{\partial M}\, E = 0$. To compute $\mathcal{P}_M(\lambda)(f, g)$ we need to solve the boundary value problem

$$(D_L + \lambda)\Phi = 0\ ,\ \mathbf{b}''_{\partial M}(\Phi) = (f, g)$$

on M. From the definition of the extension, it suffices to solve

$$(D_L + \lambda)\widetilde{\Phi} = -(D_L + \lambda)E(f, g)\ ,\ \mathbf{b}''_{\partial M}(\widetilde{\Phi}) = 0$$

for then $\Phi = E(f, g) + \widetilde{\Phi}$. Moreover, by the assumption on E, $J\, \mathbf{b}'_{\partial M}(\widetilde{\Phi}) = \mathcal{A}_M(\lambda)(f, g)$. Now

$$\begin{aligned}
\widetilde{\Phi} &= -\sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda}((D_L + \lambda)E(f, g), \Phi^A_j)_M \Phi^A_j \\
&= -\sum_{j=1}^n \left\{ \frac{1}{\lambda}(D_L E(f, g), \Phi^A_j)_M + (E, \Phi_j)_M \right\} \Phi^A_j \\
&\qquad - \sum_{j=n+1}^\infty \frac{1}{\lambda_j + \lambda} \left((D_L + \lambda)E(f, g), \Phi^A_j\right)_M \Phi^A_j
\end{aligned}$$

By (2.18), the first sum on the right hand side reduces to (since $\mathbf{b}''_{\partial M}(\Phi_j^A)=0$)

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}(\mathbf{b}_{\partial M}(E(f,g)), J\,\mathbf{b}_{\partial M}(\Phi_j^A)) + (E(f,g),\Phi_j^A)_M\right\}\Phi_j^A \\
&= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}(\mathbf{b}''_{\partial M}(E(f,g)), J\,\mathbf{b}'_{\partial M}(\Phi_j^A)) + (E(f,g),\Phi_j^A)_M\right\}\Phi_j^A \\
&= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}((f,g), J\,\mathbf{b}'_{\partial M}(\Phi_j^A)) + (E(f,g),\Phi_j^A)_M\right\}\Phi_j^A
\end{aligned}
$$

Hence, if $(f,g)\in(\mathbb{A}_M^{\mathrm{alv}})^\perp$,

$$
\mathcal{A}_M(\lambda)(f,g) = -\sum_{j=1}^{n}(E(f,g),\Phi_j^A)_M J\,\mathbf{b}'_{\partial M}\,\Phi_j^A - \sum_{j=n+1}^{\infty}\frac{1}{\lambda_j+\lambda}\left((D_L+\lambda)E(f,g),\Phi_j^A\right)_M J\,\mathbf{b}'_{\partial M}\,\Phi_j^A
$$

This clearly extends continuously as $\lambda\to 0$, the second term giving the orthogonal projection to $(\mathbb{A}_M^{\mathrm{alv}})^\perp$. □

Example 3.5. *Consider the disk B_ε of radius ε with the euclidean metric and trivial line bundle, metric, and framing. Then $\mathbb{A}_{B_\varepsilon}^{\mathrm{alv}}=\{0\}\oplus\mathbb{R}$. By direct computation one shows that*

$$
\mathcal{A}_{B_\varepsilon}(f,g)(\theta) = \sum_{n\neq 0}\begin{pmatrix}0 & i\sigma(n)\\ -i\sigma(n) & \varepsilon/|n|\end{pmatrix}\begin{pmatrix}\hat f(n)\\ \hat g(n)\end{pmatrix}e^{in\theta} \tag{3.6}
$$

where

$$
f(\theta)=\sum_{n\in\mathbb{Z}}\hat f(n)e^{i\theta}\ ,\ g(\theta)=\sum_{n\neq 0}\hat g(n)e^{i\theta} \tag{3.7}
$$

and $\sigma(n)$ is the sign of n.

3.2. **The generalized Neumann jump operator.** Now suppose M is *closed*. Let $\Gamma\subset M$ be a union of simple closed disjoint curves in M, and define M_Γ to be the surface with boundary obtained from $M\setminus\Gamma$ by adjoining a double cover of Γ. We denote the connected components of M_Γ by $R^{(i)}$, and by g_i we mean the genus of $R^{(i)}$. Note that a conformal metric ρ on M induces one on M_Γ, and a holomorphic hermitian line bundle L on determines one on M_Γ. In both cases, we use the same notation for the objects on M and M_Γ.

Suppose that τ_L is a framing of $L\to M_\Gamma$. We will *always* assume such framings arise from local trivializations of L in a neighborhood of $\Gamma\subset M$. We have the following

Lemma 3.8. *Let d_i denote the degree of $L\to R^{(i)}$ defined by framing τ_L, and let d be the degree of $L\to M$. Then $d=\sum_i d_i$.*

Proof. Let s be a meromorphic section of L with no zeros or poles on Γ, and let s_i denote the induced meromorphic sections of $L\to R^{(i)}$. Clearly, $d=\deg(s)=\sum_i\deg(s_i)$. Write $\tau_L=fs$ for a nowhere vanishing function f defined in a neighborhood of Γ. Then the local winding number of τ_L is the sum of local winding numbers of f and s. On the other hand, for each component of Γ, the local winding numbers of f on the two copies in M_Γ cancel, since they are defined in terms of outward normals. Hence,

$$\sum_{i=1} \deg(\tau_L)_{R^{(i)}} = \sum_{i=1} \deg(s_i) = d$$

$\square$

The additivity of the Euler characteristic and Theorem 2.28 imply

Corollary 3.9. *Let M be a closed surface and $\Gamma \subset M$ a union of simple closed curves dividing M into surfaces $R^{(i)}$, $i = 1, \ldots, \ell$, with boundary. Let P_L be the real operator associated to $\bar{\partial}_L$ on $\Omega^0(M, L)$, and on $\Omega^0_{\mathbb{R}}(R^{(i)}, L)$ with Alvarez boundary conditions defined by a framing τ_L. Then*

$$\mathrm{index}(P_L) = \sum_{i=1}^{\ell} \mathrm{index}(P_L)_{R^{(i)}}$$

Choose an orientation for Γ. We define maps

$$\mathbf{b}_\Gamma : \Omega^0_{\mathbb{R}}(M, L) \to \mathfrak{B}(\Gamma, \imath^* L) := \Omega^0_{\mathbb{R}}(\Gamma, \imath^* L) \oplus \Omega^1_{\mathbb{R}}(\Gamma, \imath^* L)$$

(and $\mathbf{b}'_\Gamma$, $\mathbf{b}''_\Gamma$) by restriction. The double cover $\partial M_\Gamma \to \Gamma$ gives a *diagonal* and *difference* map

$$\begin{aligned} \imath_\Delta &: \mathfrak{B}(\Gamma, \imath^* L) \longrightarrow \mathfrak{B}(\partial M_\Gamma, \imath^* L) \\ \delta_\Gamma &: \mathfrak{B}(\partial M_\Gamma, \imath^* L) \longrightarrow \mathfrak{B}(\Gamma, \imath^* L) \end{aligned} \tag{3.10}$$

The maps $\imath_\Delta$ and δ_Γ depend on the choice of orientation of Γ. We assume that such an orientation has been fixed once and for all.

We now come the following crucial

Definition 3.11. *The Neumann jump operator $\mathcal{N}_\Gamma(\lambda) : \mathfrak{B}''(\Gamma, \imath^* L) \longrightarrow \mathfrak{B}''(\Gamma, \imath^* L)$ is defined by the composition: $\mathcal{N}_\Gamma(\lambda)(f, g) = \delta_\Gamma \mathcal{A}_{M_\Gamma}(\lambda)(\imath_\Delta(f, g))$.*

Then $\mathcal{N}_\Gamma(\lambda)$ is a self-adjoint elliptic pseudo-differential operator of order zero. A calculation similar to the one in [10, Prop. 4.4] leads to the following

Proposition 3.12. *Choose coordinates with $\rho \equiv 1$ on ∂M. Then the symbol of $\mathcal{N}_\Gamma(\lambda)$ is given by*

$$\sigma_{\mathcal{N}_\Gamma(\lambda)}(x, \xi) = 2(I + r_\lambda(x, \xi)) a_\lambda(\xi)$$

where $a_\lambda(\xi)$ is block diagonal with respect to the components of Γ, with blocks equal to

$$\frac{1}{(\xi^2 + \lambda)^{1/2}} \begin{pmatrix} \lambda/2 & i\xi \\ -i\xi & -2 \end{pmatrix}$$

and $r_\lambda(x, \xi)$ is a matrix symbol with parameter satisfying

$$\|\partial_x^m \partial_\xi^n r_\lambda(x, \xi)\| \leq C_{m,n}(1 + |\xi| + |\lambda|^{1/2})^{-2-n}$$

for all $m, n \geq 0$.

Let

$$\star : \mathfrak{B}''(\Gamma, \imath^* L) \to \mathfrak{B}''(\Gamma, \imath^* L) : (f, g) \mapsto (*g, *f) \tag{3.13}$$

Corollary 3.14. *For $\lambda > 0$ we have $\mathcal{N}_\Gamma(\lambda) = 2(I + R(\lambda))A(\lambda)$, where*

(1) *$A(\lambda)$ is an invertible elliptic pseudo-differential operator of order zero satisfying*

$$\star A(\lambda) = -A(\lambda)^{-1}\star$$

(2) *$R(\lambda)$ is a pseudo-differential operator with parameter of order -2 with uniform bound $O(\lambda^{-1})$.*

Proof. Let $A(\lambda)$ be the operator with total symbol $a_\lambda(\xi)$, and $R(\lambda)$ the operator with total symbol $r_\lambda(x,\xi)$. Then the expression for $\mathcal{N}_\Gamma(\lambda)$ follows from Proposition 3.12, (1) is clear from the definition, and (2) follows from [32, Cor. 9.1]. □

As with the boundary operator, the jump operator is not everywhere defined for $\lambda = 0$. In order to rectify this, let $\mathbb{A}_\Gamma = \mathbb{A}_\Gamma^{\mathrm{ker}} \oplus \mathbb{A}_\Gamma^{\mathrm{alv}}$, where

$$\begin{aligned}
\mathbb{A}_\Gamma^{\mathrm{ker}} &= \left\{\mathbf{b}''_\Gamma(\Phi) : \Phi \in \ker D_L \subset \Omega^0_{\mathbb{R}}(M,L)\right\} \\
\mathbb{A}_\Gamma^{\mathrm{alv}} &= \left\{\delta_\Gamma J(\mathbf{b}'_{\partial M_\Gamma}(\Phi)) : \Phi \in \ker D_L \subset \Omega^0_{\mathbb{R}}(M_\Gamma,L)\ ,\ \mathbf{b}''_{\partial M_\Gamma}(\Phi) = 0\right\}
\end{aligned}$$

Notice that $\mathbb{A}_\Gamma^{\mathrm{ker}} \subset \Omega^0_{\mathbb{R}}(\Gamma,\imath^*L) \oplus \{0\}$, $\mathbb{A}_\Gamma^{\mathrm{alv}} \subset \{0\} \oplus \Omega^1_{\mathbb{R}}(\Gamma,\imath^*L)$. In particular, $\mathbb{A}_\Gamma^{\mathrm{ker}} \perp \mathbb{A}_\Gamma^{\mathrm{alv}}$. Now Propositions 3.4 and 3.12 imply

Proposition 3.15. *On the orthogonal complement of $\mathbb{A}_\Gamma$, the family $\mathcal{N}_\Gamma(\lambda)$ extends continously as $\lambda \to 0$ to a zero-th order pseudo-differential operator $\mathcal{N}_\Gamma(0) = \mathcal{N}_\Gamma$.*

We also record the following

Lemma 3.16. *Assume $\operatorname{coker} P_L = \{0\}$ on M and on M_Γ. Then $\dim_{\mathbb{R}} \mathbb{A}_\Gamma^{\mathrm{ker}} = \dim_{\mathbb{R}} \mathbb{A}_\Gamma^{\mathrm{alv}}$.*

Proof. Let $V = \{\mathbf{b}'_\Gamma(\Phi) : \Phi \in \ker D_L\ ,\ \mathbf{b}''_\Gamma(\Phi) = 0\}$. Then since any holomorphic section vanishing on Γ must vanish identically, we have by the assumption on cokernels Corollary 3.9,

$$\dim_{\mathbb{R}} \mathbb{A}_\Gamma^{\mathrm{ker}} = 2h^0(L) - \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \ker D_L^{A} - \dim_{\mathbb{R}} V$$

On the other hand, consider the surjective map $\ker D_L^{A} \to \mathbb{A}_\Gamma^{\mathrm{alv}}$. Any element in the kernel corresponds to a global holomorphic section satisfying the extra condition $\mathbf{b}''_\Gamma(\Phi) = 0$. Hence,

$$\dim_{\mathbb{R}} \ker D_L^{A} - \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \mathbb{A}_\Gamma^{\mathrm{alv}}$$

and the result follows. □

3.3. **Determinants of zero-th order operators.** Let T be a positive elliptic self-adjoint pseudo-differential operator of order zero on the real Hilbert space $L^2(S^1) \oplus L^2(S^1)$ (where the L^2 functions are real valued). The usual zeta regularization procedure does not apply to T. In order to define its determinant, we need to choose a regularizer. By this we mean an elliptic pseudo-differential operator Q of order 1 on $L^2(S^1)$. Given Q, we extend it diagonally on $L^2(S^1) \oplus L^2(S^1)$ and denote this extended operator also by Q.

Next, define $\operatorname{Log} T$ as follows. Let $\gamma \subset \mathbb{C} \setminus \{\mathfrak{Re}\, z \leq 0\}$ be a closed curve containing the spectrum of T. Then define

$$\operatorname{Log} T = \frac{1}{2\pi i} \int_\gamma dz (\log z)(z - T)^{-1} \tag{3.17}$$

where $\log$ is the branch of the logarithm on $\mathbb{C} \setminus \{\mathfrak{Re}\, z \leq 0\}$ with $-\pi < \arg \log z < \pi$. Then following [19], we set

$$\log \operatorname{Det}_Q T = f.p.\ \operatorname{Tr} \left(Q^{-s} \operatorname{Log} T \right)_{s=0} \tag{3.18}$$

While this definition of the determinant depends on Q, it is nevertheless very suitable for our purposes. The main properties we will need are summarized below.

Proposition 3.19. (1) *Let B be a bounded operator satisfying $BT = T^{-1}B$. Then*

$$B(\operatorname{Log} T) = -(\operatorname{Log} T)B$$

(2) *Suppose in addition that B is an involution that commutes with Q. Then* $\operatorname{Det}_Q T = 1$.
(3) *If R is trace-class with norm $\leq 1/2$, then* $|\log \operatorname{Det}_Q(I + R)| \leq 2|\operatorname{Tr} R|$.
(4) *Suppose $T(\varepsilon)$ is a differentiable family of positive elliptic self-adjoint pseudo-differential operators of order zero. If $dT(\varepsilon)/d\varepsilon$ is trace-class, then*

$$\frac{d}{d\varepsilon} \log \operatorname{Det}_Q T(\varepsilon) = \operatorname{Tr}\Big(T(\varepsilon)^{-1} \frac{dT(\varepsilon)}{d\varepsilon} \Big)$$

Proof. These follow from directly from the definitions. For example, for (1) note that

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} (\log z)(z - T)^{-1} = (\operatorname{Log} T)T^{-1} \tag{3.20}$$

Indeed, from $z^{-1}(z-T)^{-1} = (z-T)^{-1}T^{-1} - z^{-1}T^{-1}$ we have

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} (\log z)(z - T)^{-1} = \frac{1}{2\pi i} \int_\gamma dz (\log z)(z - T)^{-1}T^{-1} - \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} (\log z)T^{-1}$$

Because of the choice of contour, the second term vanishes. Now

$$B(z - T) = (z - T^{-1})B \implies (z - T^{-1})^{-1}B = B(z - T)^{-1}$$

Hence,

$$\begin{aligned} B(\operatorname{Log} T) &= \frac{1}{2\pi i} \int_\gamma dz (\log z)(z - T^{-1})^{-1}B = \frac{1}{2\pi i} \int_\gamma dz (\log z)(Tz - I)^{-1}TB \\ &= -\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} (\log z)(z^{-1} - T)^{-1}TB \end{aligned}$$

Next make a change of variables $w = z^{-1}$. Without loss of generality, we may assume γ is invariant under this change. Then by (3.20).

$$B(\operatorname{Log} T) = -\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} (\log z)(z^{-1} - T)^{-1}TB = -\frac{1}{2\pi i} \int_\gamma \frac{dw}{w} (\log w)(w - T)^{-1}TB = -(\operatorname{Log} T)B$$

For (2), it follows from (1) that

$$\begin{aligned} f.p.\ \mathrm{Tr}\left(Q^{-s}\,\mathrm{Log}\,T\right)_{s=0} &= f.p.\ \mathrm{Tr}\left(BQ^{-s}(\mathrm{Log}\,T)B\right)_{s=0} = f.p.\ \mathrm{Tr}\left(Q^{-s}B(\mathrm{Log}\,T)B\right)_{s=0} \\ &= -f.p.\ \mathrm{Tr}\left(Q^{-s}\,\mathrm{Log}\,T\right)_{s=0} \end{aligned}$$

Parts (3) and (4) follow similarly. □

3.4. **The Burghelea-Friedlander-Kappeler formula.** We apply the definition of determinant in the previous section to the Neumann jump operator. The self-adjoint operator $\mathcal{N}_\Gamma(\lambda)$ has non-zero real eigenvalues for $\lambda \neq 0$, but is not positive. Hence, we *define* the logarithm and determinant by

$$\begin{aligned} \mathrm{Log}\,\mathcal{N}_\Gamma(\lambda) &= \tfrac{1}{2}\,\mathrm{Log}\,\mathcal{N}_\Gamma^2(\lambda) \\ \log \mathrm{Det}_Q\,\mathcal{N}_\Gamma(\lambda) &= \tfrac{1}{2}\log \mathrm{Det}_Q\,\mathcal{N}_\Gamma^2(\lambda) \end{aligned}$$

Note that we choose the regulator Q to be diagonal on each component of Γ. In what follows, let $\zeta_Q(s) = \mathrm{Tr}\,Q^{-s}$, and recall that $s = 0$ is a regular value of (the analytic continuation of) $\zeta_Q(s)$.

With this understood, we state the key factorization theorem (cf. [10, Thm. A]).

Theorem 3.21 (BFK formula)**.** *For all* $\lambda > 0$*,*

$$[\mathrm{Det}(D_L+\lambda)]_M = c_Q\ [\mathrm{Det}(D_L^A+\lambda)]_{M_\Gamma}\,\mathrm{Det}_Q\mathcal{N}_\Gamma(\lambda)$$

where $c_Q = 2^{-\zeta_Q(0)}$.

First, note the following

Lemma 3.22. *For all* $\lambda > 0$*,* $\mathcal{N}_\Gamma^{-1}(\lambda)d\mathcal{N}_\Gamma(\lambda)/d\lambda$ *is trace-class.*

Proof. In terms of the expression from Corollary 3.14,

$$\mathcal{N}_\Gamma^{-1}(\lambda)\dot{\mathcal{N}}_\Gamma(\lambda) = A^{-1}(\lambda)(I+R(\lambda))^{-1}\dot{R}(\lambda)A(\lambda) + A^{-1}(\lambda)\dot{A}(\lambda)$$

where the dot indicates the derivative with respect to the spectral parameter. Since $R(\lambda)$ has order -2 on the circle, the first term on the right hand side is trace class. It therefore suffices to prove that $A^{-1}(\lambda)\dot{A}(\lambda)$ is trace class. Its symbol is block diagonal with respect to the components of Γ, with blocks given by

$$\sigma_{A^{-1}(\lambda)\dot{A}(\lambda)}(x,\xi) = \frac{1}{(\xi^2+\lambda)}\begin{pmatrix} 1/2 & 0 \\ i\xi/2 & -1/2 \end{pmatrix}$$

Since the associated operator with this symbol is clearly trace class, the assertion follows. □

Lemma 3.23. $\log \mathrm{Det}_Q\,\mathcal{N}_\Gamma(\lambda) = \zeta_Q(0)\log 2 + \log \mathrm{Det}_Q\,A(\lambda) + O(\lambda^{-1/2})$.

Proof. From Corollary 3.14, $\log \mathrm{Det}_Q\,\mathcal{N}_\Gamma(\lambda) = \zeta_Q(0)\log 2 + \log \mathrm{Det}_Q\left[(I+R(\lambda))A(\lambda)\right]$. On the other hand, as in the proof of the previous lemma, $R(\lambda)A(\lambda)$ is trace-class. Applying Proposition 3.19 (4) to the family

$$T(\varepsilon) = [(I+\varepsilon R(\lambda))A(\lambda)]^2$$

and integrating ε, we have

$$\log \mathrm{Det}_Q\left[(I+R(\lambda))A(\lambda)\right] = \log \mathrm{Det}_Q\,A(\lambda) + \int_0^1 d\varepsilon\,\mathrm{Tr}\left[R(\lambda)(I+\varepsilon R(\lambda))^{-1}\right]$$

By the estimate on the symbol of $R(\lambda)$, its trace is bounded by $O(\lambda^{-1/2})$, say. The result follows. □

Proof of Theorem 3.21. Using Lemma 3.22, the proof of the existence of such a constant c_Q follows exactly as in [10, pp. 46-47]. By [10, Thm. 3.12 (2)] it therefore suffices to prove the estimate

$$\log \mathrm{Det}_Q \mathcal{N}_\Gamma(\lambda) = \zeta_Q(0) \log 2 + O(\lambda^{-1/2}) \tag{3.24}$$

By Lemma 3.23, this immediately follows if we can show $\log \mathrm{Det}_Q A(\lambda) = 0$. Using Corollary 3.14 and Proposition 3.19 (1), $\star \,\mathrm{Log}\, A(\lambda) = -(\mathrm{Log}\, A(\lambda))\star$, where $\star$ is given by (3.13). Since Q is a diagonal operator, $\star Q = Q\star$, and the claim follows from Proposition 3.19 (2). □

3.5. **The case of zero modes.** The goal of this section is to extend the formula in Theorem 3.21 as $\lambda \to 0$. We will need a preliminary

Definition 3.25. *A framing τ_L near Γ is* **generic** *if $\mathbf{b}''_\Gamma$ is injective on* $\ker D_L \subset \Omega^0_{\mathbb{R}}(M, L)$.

Note that an equivalent condition to the one above is that the difference map $\delta_\Gamma\, \mathbf{b}'_{\partial M_\Gamma}$ be injective on $\ker D_L^A$ on M_Γ. Indeed, if Φ is a global section in $\ker D_L$, then regarded as a section on M_Γ, we automatically have $\delta_\Gamma\, \mathbf{b}'_{\partial M_\Gamma}(\Phi) = 0$. If in addition, $\mathbf{b}''_\Gamma(\Phi) = 0$, then $\Phi \in \ker D_L^A$. Conversely, if $\Phi^A \in \ker D_L^A$ and $\delta_\Gamma\, \mathbf{b}'_{\partial M_\Gamma}(\Phi^A) = 0$, then since $\delta_\Gamma\, \mathbf{b}''_{\partial M_\Gamma}(\Phi^A) = 0$ automatically, it extends to a global section on M.

Theorem 3.26. *For a given framing τ_L near Γ, let $\{\Phi_i\}$ (resp. $\{\Phi_i^A\}$) be a basis for $\ker D_L$ on M (resp. for $\ker D_L^A$ on M_Γ). Assume the framing is generic in the sense of Definition 3.25. Then*

$$\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i, \Phi_j)}\right]_M = c_Q \left[\frac{\mathrm{Det}^* D_L^A}{\det(\Phi_i^A, \Phi_j^A)}\right]_{M_\Gamma} \frac{\det(\delta_\Gamma\, \mathbf{b}'_{\partial M_\Gamma}(\Phi_i^A), \delta_\Gamma\, \mathbf{b}'_{\partial M_\Gamma}(\Phi_j^A))}{\det(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))} \mathrm{Det}^*_Q \mathcal{N}_\Gamma$$

where $\mathcal{N}_\Gamma = \mathcal{N}_\Gamma(0)$ is the operator defined on the orthogonal complement of $\mathbb{A}_\Gamma$ in Proposition 3.15.

Proof. We apply Theorem 3.21 as $\lambda \downarrow 0$. By the definition of zeta regularization,

$$\log \mathrm{Det}(D_L + \lambda) = (\log \lambda) \dim_{\mathbb{R}} \ker D_L + \log \mathrm{Det}^* D_L + o(1)$$
$$\log \mathrm{Det}(D_L^A + \lambda) = (\log \lambda) \dim_{\mathbb{R}} \ker D_L^A + \log \mathrm{Det}^* D_L^A + o(1)$$

on M and M_Γ with Alvarez boundary conditions. Let $m = \dim_{\mathbb{R}} \ker D_L$ on M, and $n = \dim_{\mathbb{R}} \ker D_L^A$ on M_Γ. Hence, it suffices to compute $\lim_{\lambda \to 0} \{\log \mathrm{Det}_Q \mathcal{N}_\Gamma(\lambda) + (n - m) \log \lambda\}$. The key point is that there are small eigenvalues of $\mathcal{N}_\Gamma(\lambda)$, $\mu_j(\lambda) \to 0$, $j = 1, \ldots, m$, corresponding to global holomorphic sections of L, and large eigenvalues $\nu_j(\lambda) \to +\infty$, $j = 1, \ldots, n$, corresponding to $\ker D_L^A$. Moreover, it follows easily from the definition that

$$\log \mathrm{Det}\, \mathcal{N}_\Gamma(\lambda) = \log(\mu_1(\lambda) \cdots \mu_m(\lambda)) + \log(\nu_1(\lambda) \cdots \nu_n(\lambda)) + \log \mathrm{Det}^*_Q \mathcal{N}_\Gamma + o(1) \tag{3.27}$$

We need therefore to compute the contribution from both the $\{\mu_i\}$ and $\{\nu_i\}$.

Let $\mu_1(\lambda), \ldots, \mu_m(\lambda)$ be the small eigenvalues of $\mathcal{N}_\Gamma(\lambda)$, and let $\{\beta_j(\lambda)\}_{j=1}^m$ be orthonormal with eigenvalues $\mu_j(\lambda)$. Let $\{\Phi_j\}_{i=1}^\infty$ be a complete set of eigensections for D_L on M with eigenvalues $\{\lambda_j\}_{j=1}^\infty$,

$\lambda_j = 0$ if and only if $j \leq m$. Let $\pi : \mathfrak{B}''(\Gamma, \imath^* L) \to \mathfrak{B}''(\Gamma, \imath^* L)$ orthogonal projection to $\mathbb{A}_\Gamma^{\ker}$. Then we compute

$$\mathcal{N}_\Gamma^{-1}(\lambda) = \begin{pmatrix} \frac{1}{\lambda} A_1 + \pi B_1(\lambda)\pi & \pi B_1(\lambda)\pi^\perp \\ \pi^\perp B_1(\lambda)\pi & \pi^\perp B_1(\lambda)\pi^\perp \end{pmatrix}$$

where $A_1, B_1(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$ are given by

$$A_1(F, G) = \sum_{j=1}^m \big((F,G), \mathbf{b}''_\Gamma(\Phi_j)\big)_\Gamma \, \mathbf{b}''_\Gamma(\Phi_j) \tag{3.28}$$

$$B_1(\lambda)(F,G) = \sum_{j=m+1}^\infty \frac{1}{\lambda_j + \lambda} \big((F,G), \mathbf{b}''_\Gamma(\Phi_j)\big)\, \mathbf{b}''_\Gamma(\Phi_j)$$

To see this, let Φ be a section of $L \to M_\Gamma$, $(D_L + \lambda)\Phi = 0$, with $(F,G) = \delta_\Gamma J\, \mathbf{b}'_{\partial M_\Gamma}(\Phi)$, and $\delta_\Gamma\, \mathbf{b}''_{\partial M_\Gamma}(\Phi) = 0$. Then by (2.18),

$$\begin{aligned}
\Phi = \sum_{j=1}^\infty (\Phi, \Phi_j)_{M_\Gamma} \Phi_j &= \sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda} (\Phi, (D_L + \lambda)\Phi_j)_{M_\Gamma} \Phi_j \\
&= -\sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda} (\mathbf{b}_{\partial M_\Gamma}\Phi, J\, \mathbf{b}_{\partial M_\Gamma}\Phi_j)\Phi_j \\
&= -\sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda} (\delta_\Gamma\, \mathbf{b}_\Gamma\, \Phi, J\, \mathbf{b}_\Gamma\, \Phi_j)\Phi_j \\
&= -\sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda} (\delta_\Gamma\, \mathbf{b}'_\Gamma\, \Phi, J\, \mathbf{b}''_\Gamma\, \Phi_j)\Phi_j \\
&= \sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda} (\delta_\Gamma J\, \mathbf{b}'_\Gamma\, \Phi, \mathbf{b}''_\Gamma\, \Phi_j)\Phi_j \\
&= \sum_{j=1}^\infty \frac{1}{\lambda_j + \lambda} ((F,G), \mathbf{b}''_\Gamma\, \Phi_j)\Phi_j
\end{aligned}$$

and computing $\mathbf{b}''_\Gamma(\Phi)$ gives the result. We wish to relate the eigenvalues of A_1 to the $\mu_j(\lambda)$. Since

$$\mathcal{N}_\Gamma^{-1}(\lambda)\beta_j(\lambda) = \mu_j^{-1}(\lambda)\beta_j(\lambda)$$

we have

$$\begin{aligned}
\frac{1}{\lambda} A_1 \beta_j(\lambda) + \pi B_1(\lambda)\beta_j(\lambda) &= \mu_j^{-1}(\lambda)\pi\beta_j(\lambda) \\
\pi^\perp B_1(\lambda)\beta_j(\lambda) &= \mu_j^{-1}(\lambda)\pi^\perp\beta_j(\lambda)
\end{aligned} \tag{3.29}$$

Since $B_1(\lambda)$ is uniformly bounded as $\lambda \downarrow 0$, it follows that $\|\pi^\perp \beta_j(\lambda)\|_{L^2(\Gamma)} \leq C\mu_j(\lambda)$, for C independent of λ. In particular, $\|\pi\beta_j(\lambda)\|_{L^2(\Gamma)} \to 1$ as $\lambda \downarrow 0$, and so (after passing to a sequence $\lambda_k \downarrow 0$) there exist limits $\{\beta_j(0)\}$ which give a basis for $\mathbb{A}_\Gamma^{\ker}$. If we let v_j be an orthonormal basis for $\mathbb{A}_\Gamma^{\ker}$ such that $A_1 v_j = \sigma_j v_j$, and write

$$\pi\beta_j(\lambda) = \sum_{k=1}^m C_{jk}(\lambda) v_k$$

then the (subsequential) limit $C_{jk}(0)$ exists and is nonsingular. From (3.29) we have

$$\left\| A_1\pi\beta_j(\lambda) - \frac{\lambda}{\mu_j(\lambda)}\pi\beta_j(\lambda)\right\|_{L^2(\Gamma)} \leq C\lambda \tag{3.30}$$

In terms of the basis $\{v_j\}$,

$$A_1\pi\beta_j(\lambda) - \frac{\lambda}{\mu_j(\lambda)}\pi\beta_j(\lambda) = \sum_{k=1}^m C_{jk}(\lambda)\left(\sigma_k - \frac{\lambda}{\mu_j(\lambda)}\right)v_k$$

so by (3.30), $C_{jk}(\lambda)(\sigma_k - (\lambda/\mu_j(\lambda))) \to 0$, for all j,k. Since (C_{jk}) is non-singular, for each j, $C_{jk}(0)\neq 0$ for some k. Hence, $\sigma_k^{-1} = \lim_{\lambda\downarrow 0}\mu_j(\lambda)/\lambda = \hat{\mu}_j$ exists for each j, with $C_{jk}\sigma_k = \hat{\mu}_j^{-1}C_{jk}$. Again using the fact that (C_{jk}) is non-singular, we have

$$\log\det A_1 + \log(\prod \mu_j(\lambda)) = m\log\lambda + o(1)$$

Finally, note that by choosing $\mathbf{b}''_\Gamma(\Phi_j)$, $j=1,\ldots,m$, as a basis in (3.28), we have

$$\det A_1 = \det(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))$$

$$\log(\prod \mu_j(\lambda)) = m\log\lambda - \log\det(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j)) + o(1) \tag{3.31}$$

Let $\nu_1(\lambda),\ldots,\nu_n(\lambda)$ be the divergent eigenvalues of $\mathcal{N}_\Gamma(\lambda)$, and let $\{\beta_j^A(\lambda)\}_{j=1}^n$ be orthonormal with eigenvalues $\nu_j(\lambda)$. Let $\{\Phi_i^A\}_{i=1}^\infty$ be a complete set of eigensections for D_L^A on M_Γ with eigenvalues $\{\lambda_i\}_{i=1}^\infty$, and $\lambda_i = 0$ if and only if $i\leq n$. Let $\pi : \mathfrak{B}''(\Gamma, \imath^*L)\to \mathfrak{B}''(\Gamma,\imath^*L)$ be orthogonal projection to $\mathbb{A}_\Gamma^{\mathrm{alv}}$. We also choose a smooth extension map $E : \mathfrak{B}''(\Gamma,\imath^*L)\to L^2(M_\Gamma)$ satisfying $\mathbf{b}''_\Gamma E = I$, $\mathbf{b}'_\Gamma E = 0$. Then as above we compute

$$\mathcal{N}_\Gamma(\lambda) = \begin{pmatrix} \frac{1}{\lambda}A_2(\lambda) + \pi B_2(\lambda)\pi & \pi B_2(\lambda)\pi^\perp \\ \pi^\perp B_2(\lambda)\pi & \pi^\perp B_2(\lambda)\pi^\perp \end{pmatrix}$$

where $A_2(\lambda), B_2(\lambda) : L^2(\Gamma)\to L^2(\Gamma)$ are given by

$$A_2(\lambda)(f,g) = -\sum_{j=1}^n \left\{ \left((f,g), \delta_\Gamma J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi_j^A)\right) + \lambda\left(E(f,g),\Phi_j^A\right)_M \right\}\delta_\Gamma J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi_i^A) \tag{3.32}$$

$$B_2(\lambda)(f,g) = -\sum_{j=n+1}^\infty \frac{1}{\lambda_j+\lambda}\left((D_L+\lambda)E(f,g),\Phi_j^A\right)_M \delta_\Gamma J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi_j^A)$$

To see this, note that to compute $\mathcal{N}_\Gamma(\lambda)(f,g)$ we need to solve the boundary value problem

$$(D_L+\lambda)\Phi = 0\ ,\ \mathbf{b}''_{\partial M_\Gamma}(\Phi) = \imath_\Delta(f,g)$$

on M_Γ. From the definition of the extension, it suffices to solve

$$(D_L+\lambda)\widetilde{\Phi} = -(D_L+\lambda)E(f,g)\ ,\ \mathbf{b}''_{\partial M_\Gamma}(\widetilde{\Phi}) = 0$$

for then $\Phi = E(f,g) + \widetilde{\Phi}$, and by the assumption on E the jump in $\mathbf{b}'_{\partial M_\Gamma}(\widetilde{\Phi})$ gives $\mathcal{N}_\Gamma(\lambda)(f,g)$. Now

$$\begin{aligned}\widetilde{\Phi} &= -\sum_{j=1}^{\infty} \frac{1}{\lambda_j+\lambda}((D_L+\lambda)E(f,g),\Phi_j^A)_{M_\Gamma}\Phi_j^A \\ &= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}(D_L E(f,g),\Phi_j^A)_{M_\Gamma} + (E,\Phi_j)_{M_\Gamma}\right\}\Phi_j^A \\ &\qquad - \sum_{j=n+1}^{\infty}\frac{1}{\lambda_j+\lambda}\left((D_L+\lambda)E(f,g),\Phi_j^A\right)_{M_\Gamma}\Phi_j^A\end{aligned}$$

By (2.18), the first term on the right hand side is (since $\mathbf{b}''_{\partial M_\Gamma}(\Phi_j^A)=0$)

$$\begin{aligned}&= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}(\mathbf{b}_{\partial M_\Gamma}(E(f,g)), J\,\mathbf{b}_{\partial M_\Gamma}(\Phi_j^A)) + (E(f,g),\Phi_j)_{M_\Gamma}\right\}\Phi_j^A \\ &= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}(\mathbf{b}''_{\partial M_\Gamma}(E(f,g)), J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi_j^A)) + (E(f,g),\Phi_j^A)_{M_\Gamma}\right\}\Phi_j^A \\ &= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}(\mathbf{b}''_{\Gamma}(E(f,g)), \delta_\Gamma J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi_j^A)) + (E(f,g),\Phi_j^A)_{M_\Gamma}\right\}\Phi_j^A \\ &= -\sum_{j=1}^{n}\left\{\frac{1}{\lambda}((f,g), \delta_\Gamma J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi_j^A)) + (E(f,g),\Phi_j^A)_{M_\Gamma}\right\}\Phi_j^A\end{aligned}$$

We again relate the eigenvalues of $A_2(0)$ to the $\nu_j(\lambda)$. Since $\mathcal{N}_\Gamma(\lambda)\beta_j^A(\lambda) = \nu_j(\lambda)\beta_j^A(\lambda)$, we have

$$\begin{aligned}\frac{1}{\lambda}A_2(\lambda)\beta_j^A(\lambda) + \pi B_2(\lambda)\beta_j^A(\lambda) &= \nu_j(\lambda)\pi\beta_j^A(\lambda) \\ \pi^\perp B_2(\lambda)\beta_j^A(\lambda) &= \nu_j(\lambda)\pi^\perp\beta_j^A(\lambda)\end{aligned} \tag{3.33}$$

Since $B_2(\lambda)$ is uniformly bounded as $\lambda\downarrow 0$, it follows that $\|\pi^\perp\beta_j^A(\lambda)\|_{L^2(\Gamma)} \le C\nu_j^{-1}(\lambda)$, for C independent of λ. In particular, $\|\pi\beta_j^A(\lambda)\|_{L^2(\Gamma)}\to 1$ as $\lambda\downarrow 0$, and so the (sequential) limits $\{\beta_j^A(0)\}$ give a basis for $\mathbb{A}_\Gamma^{\mathrm{alv}}$. If we let v_j be an orthonormal basis for $\mathbb{A}_\Gamma^{\mathrm{alv}}$ such that $A_2(0)v_j=\sigma_j v_j$, and write

$$\pi\beta_j^A(\lambda) = \sum_{k=1}^{n} C_{jk}(\lambda)v_k\ ,$$

then $C_{jk}(0)$ exists and is nonsingular. From (3.33) we have

$$\left\|A_2(0)\pi\beta_j^A(\lambda) - \lambda\nu_j(\lambda)\pi\beta_j^A(\lambda)\right\|_{L^2(\Gamma)} \le C\lambda \tag{3.34}$$

In terms of the basis $\{v_j\}$,

$$A_2(0)\pi\beta_j^A(\lambda) - \lambda\nu_j(\lambda)\pi\beta_j^A(\lambda) = \sum_{k=1}^{n} C_{jk}(\lambda)\left(\sigma_k - \lambda\nu_j(\lambda)\right)v_k$$

so by (3.34), $C_{jk}(\lambda)(\sigma_k-\lambda\nu_j(\lambda))\to 0$ for all j,k. As before, $\lim_{\lambda\downarrow 0}\lambda\nu_j(\lambda)=\hat\nu_j$ exists for each j, and $C_{jk}\sigma_k = \hat\nu_j C_{jk}$ for all j,k. Hence, $\log\det A_2(0) = \log(\prod\nu_j(\lambda)) + m\log\lambda + o(1)$. Finally, note that by choosing $\delta_\Gamma J\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j)$ as a basis in (3.32), we have

$$\det A_2(0) = \det(\delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j))$$

$$\log(\prod \nu_j(\lambda)) = -m\log\lambda + \log\det(\delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j)) + o(1) \tag{3.35}$$

Putting together (3.27), (3.31) and (3.35) gives the result. □

In calculating the bosonization constants we will use the following special case of Theorem 3.26: let Γ be a simple closed connected curve separating M into components $R^{(1)}$ and $R^{(2)}$. Then for any choice of bases $\{\Phi_i\}_{i=1}^m$ for $\ker D_L$ on M, and $\{\Phi_i^{A,(1)}\}_{i=1}^{m_1}$, and $\{\Phi_i^{A,(2)}\}_{i=1}^{m_2}$ for $\ker D_L^A$ on $R^{(1)}$ and $R^{(2)}$, we have

$$\begin{aligned}\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i,\Phi_j)}\right]_M = c_Q &\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i^{A,(1)},\Phi_j^{A,(1)})}\right]_{R^{(1)}} \left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i^{A,(2)},\Phi_j^{A,(2)})}\right]_{R^{(2)}} \\ &\times \frac{\det(\delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j))}{\det(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))}\,\mathrm{Det}^*_Q \mathcal{N}_\Gamma\end{aligned} \tag{3.36}$$

where

$$\Phi_i^A = \begin{cases}\Phi_i^{A,(1)} & 1\le i\le m_1\\ \Phi_{i-m_1}^{A,(2)} & m_1 < i \le m_1+m_2\end{cases}$$

Actually, for the purpose of degeneration it will be useful to also have a slightly modified version of (3.36) in the case where the trivialization τ_L is in fact the restriction of a global holomorphic section. This is not a generic situation in the sense of Definition 3.25, since the global section τ_L also satisfies Alvarez boundary conditions, and hence $\det(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))_\Gamma = 0$ for any basis. Similarly, since the jump of τ_L is trivial, $\det(\delta_\Gamma\,\mathbf{b}'_\Gamma(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_\Gamma(\Phi^A_j))$ also vanishes. This motivates the following

Definition 3.37. *Let τ_L be a global holomorphic section of $L\to M$, nowhere vanishing near Γ. We call the framing τ_L* **good** *if the kernel of $\mathbf{b}''_\Gamma$ on $\ker D_L\subset\Omega^0_{\mathbb{R}}(M,L)$ is precisely the $\mathbb{R}$-span of τ_L. We say that bases $\{\Phi_i\}_{i=1}^m$, $\{\Phi_i^{A,(1)}\}_{i=1}^{m_1}$, and $\{\Phi_i^{A,(2)}\}_{i=1}^{m_2}$ for $\ker D_L$ on M and for $\ker D_L^A$ on $R^{(1)}$ and $R^{(2)}$, are* **adapted** *to τ_L if $\Phi_1=\tau_L$, $\Phi_1^{A,(1)} = \tau_L\big|_{R^{(1)}}$, $\Phi_1^{A,(2)} = \tau_L\big|_{R^{(2)}}$.*

For adapted bases, the notation $\widehat{\det}(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))$ will by definition denote the determinant of the (11)-minor of $(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))$. Similarly, $\widehat{\det}(\delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j))$ will denote the determinant of the (11)-minor of $(\delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j))$. Then after some linear algebra we have

Theorem 3.38. *Let τ_L be a global holomorphic section giving a framing of L near Γ, and let $\{\Phi_i\}$ (resp. $\{\Phi_i^{A,(1)},\Phi_j^{A,(2)}\}$) be an adapted basis for $\ker D_L$ on M (resp. on $R^{(1,2)}$ with Alvarez boundary conditions). Assume the framing is good in the sense of Definition 3.37. Then*

$$\begin{aligned}\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i,\Phi_j)}\right]_M = c_Q &\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i^{A,(1)},\Phi_j^{A,(1)})}\right]_{R^{(1)}} \left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i^{A,(2)},\Phi_j^{A,(2)})}\right]_{R^{(2)}} \\ &\times \frac{\widehat{\det}(\delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_i), \delta_\Gamma\,\mathbf{b}'_{\partial M_\Gamma}(\Phi^A_j))}{\widehat{\det}(\mathbf{b}''_\Gamma(\Phi_i), \mathbf{b}''_\Gamma(\Phi_j))}\,\mathrm{Det}^*_Q \mathcal{N}_\Gamma\end{aligned}$$

4. Asymptotics of Determinants

4.1. **Asymptotics of the generalized Neumann jump operator.** The goal of this section is to prove the following. Let M be a closed Riemann surface of genus g, and choose a coordinate neighborhood B with coordinate z centered at $p \in M$. Let $B_\varepsilon = \{|z| < \varepsilon\}$, and set $R_\varepsilon = M \setminus B_\varepsilon$. Let $L \to M$ be a hermitian holomorphic line bundle of degree d with a global holomorphic section τ_L that is nowhere vanishing on B. Also, assume coker $P_L = \{0\}$ on M and on R_ε, and that $\rho \equiv 1$ and $\|\tau_L\| = 1$ on B.

Proposition 4.1. *If $\mathcal{N}_{\Gamma_\varepsilon}$ denotes the Neumann jump operator with respect to Alvarez boundary conditions defined by a* global *section τ_L. Then as $\varepsilon \to 0$,*

$$\log \mathrm{Det}^*_Q \mathcal{N}_{\Gamma_\varepsilon} \longrightarrow (\zeta_Q(0) - 4h^0(L) + 2)\log 2$$

By direct computation, as in [39] one proves

Lemma 4.2. *For $1/2 \geq \varepsilon > 0$, $\mathcal{A}_{R_\varepsilon} = S_\varepsilon + \varepsilon U_\varepsilon \mathcal{A}_{R_1}(I + T_\varepsilon \mathcal{A}_{R_1})^{-1} U_\varepsilon$, where*

$$S_\varepsilon(f,g)(\theta) = \sum_{n\neq 0} \left(\frac{\varepsilon^n - \varepsilon^{-n}}{\varepsilon^n + \varepsilon^{-n}}\right) \begin{pmatrix} 0 & -i \\ i & -\varepsilon/n \end{pmatrix} \begin{pmatrix} \hat f(n) \\ \hat g(n) \end{pmatrix} e^{i\theta}$$

$$U_\varepsilon(f,g)(\theta) = \sum_{n\neq 0} \frac{2}{\varepsilon(\varepsilon^n + \varepsilon^{-n})} \begin{pmatrix} \hat f(n) \\ \varepsilon \hat g(n) \end{pmatrix} e^{i\theta}$$

$$T_\varepsilon(f,g)(\theta) = \sum_{n\neq 0} \left(\frac{\varepsilon^n - \varepsilon^{-n}}{\varepsilon^n + \varepsilon^{-n}}\right) \begin{pmatrix} 1/n & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \hat f(n) \\ \hat g(n) \end{pmatrix} e^{i\theta}$$

for functions f, g in (3.7).

We also note the following estimates.

Lemma 4.3. *Assume $1/2 \geq \varepsilon > 0$.*

(1) *$(A_\varepsilon - S_\varepsilon)$ is trace-class with norm bounded by $8\varepsilon^2$.*
(2) *U_ε is trace-class with uniformly bounded norm.*
(3) *If T_0 is defined by*

$$T_0(f,g)(\theta) = \sum_{n\neq 0} \begin{pmatrix} -1/|n| & i\sigma(n) \\ -i\sigma(n) & 0 \end{pmatrix} \begin{pmatrix} \hat f(n) \\ \hat g(n) \end{pmatrix} e^{i\theta}$$

then $(T_\varepsilon - T_0)$ is trace-class with norm bounded by $8\varepsilon^2$.

Lemma 4.4. *For $\varepsilon > 0$ sufficiently small , $I + T_\varepsilon \mathcal{A}_{R_1}$ is uniformly invertible on the orthogonal complement of $\mathbb{A}_\Gamma$.*

Proof. It suffices to show that $I + T_0 \mathcal{A}_{R_1}$ has no kernel on $\mathbb{A}_\Gamma^\perp$. But by a direct computation, if $(f,g) = -T_0\mathcal{A}_{R_1}(f,g)$, then $\mathcal{P}_{R_1}(f,g)$ extends to a global section in $\ker D_L$. $\square$

Proof of Proposition 4.1. By Lemma 4.2 we have on the orthogonal complement of $\mathbb{A}_{\Gamma_\varepsilon}$:

$$\begin{aligned}\operatorname{Log}\mathcal{N}_{\Gamma_\varepsilon} &= \log 2 + \tfrac{1}{2}\operatorname{Log}\left(\tfrac{1}{2}\mathcal{N}_{\Gamma_\varepsilon}\right)^2 = \log 2 + \tfrac{1}{2}\operatorname{Log}\left(I + C(\varepsilon)\right)\\ \log\operatorname{Det}^*_Q\mathcal{N}_{\Gamma_\varepsilon} &= (\zeta_Q(0) - \dim_{\mathbb{R}}\mathbb{A}_{\Gamma_\varepsilon})\log 2 + \tfrac{1}{2}\log\operatorname{Det}^*_Q(I + C(\varepsilon))\end{aligned}$$

More precisely, assume the orientation of Γ is chosen to agree with ∂R_ε. Then for f, g as in (3.7), and using (3.6),

$$\begin{aligned}\mathcal{N}_{\Gamma_\varepsilon} &= \mathcal{A}_{R_\varepsilon} + \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathcal{A}_{B_\varepsilon}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ \mathcal{N}_{\Gamma_\varepsilon}\begin{pmatrix}f\\ g\end{pmatrix} &= \sum_{n\neq 0}\left[\begin{pmatrix}0 & i\sigma(n)\\ -i\sigma(n) & \varepsilon/|n|\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & i\sigma(n)\\ -i\sigma(n) & \varepsilon/|n|\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\begin{pmatrix}\hat{f}(n)\\ \hat{g}(n)\end{pmatrix}e^{in\theta}\\ &\qquad + \{\text{trace class}\}\\ &= \sum_{n\neq 0} 2\begin{pmatrix}i\sigma(n)\hat{g}(n)\\ -i\sigma(n)\hat{f}(n)\end{pmatrix}e^{in\theta} + \{\text{trace class}\}\\ \mathcal{N}^2_{\Gamma_\varepsilon} &= 4I + \{\text{trace class}\}\end{aligned}$$

Now by Lemma 3.16,

$$\dim_{\mathbb{R}}\mathbb{A}_{\Gamma_\varepsilon} = \dim_{\mathbb{R}}\ker D_L - 1 + \dim_{\mathbb{R}}\ker D^{\mathcal{A}}_L - 1 = 2\dim_{\mathbb{R}}\ker D_L - 2 = 4h^0(L) - 2$$

Since $C(\varepsilon) \to 0$ in trace, the result follows from Proposition 3.19 (3). $\square$

4.2. **Degeneration of Riemann surfaces and holomorphic sections.** Suppose $g \geq 1$. Let M_1 and M_2 be a pair of closed Riemann surfaces with genera $g-1$ and 1, respectively. For a complex parameter t, $0 < |t| \leq 1$ we construct a degenerating family M_t of closed surfaces of genus g using the "plumbing construction": $z_1 z_2 = t$, for local coordinates on M_1, M_2 centered at points q_1, q_2 (for more details, see [38]). Let Γ_t denote the curve in M_t given by $|z_1| = |z_2| = |t|^{1/2}$. If we set $B^{(i)}_t = \{|z_i| \leq |t|^{1/2}\}$, and let $R^{(1)}_t = M_1 \setminus B^{(1)}_t$, $R^{(2)}_t = M_2 \setminus B^{(2)}_t$, then $M_t \setminus \Gamma_t$ is conformally equivalent to the disjoint union $R^{(1)}_t \cup R^{(2)}_t$.

We now consider holomorphic line bundles of degree $d \geq g-1 \geq 1$, and set $m = d-g+1$. We will need two different degeneration schemes: (I) for $d \geq g$, and another (II) for $d = g-1$. In scheme I, the bundle will be associated to an effective divisor $\sum_{i=1}^{d-1} x_i + y$, with $x_i \in M_1 \setminus \{q_1\}$, $y \in M_2 \setminus \{q_2\}$. By keeping the points fixed, this defines a holomorphic family of line bundles $L_t \to M_t$ degenerating to line bundles $L_1 \to M_1$ of degree $d-1$ and $L_2 \to M_2$ of degree 1, given by divisors $\sum_{i=1}^{d-1} x_i$ and y, respectively. In scheme II, the divisor is $\sum_{i=1}^{g-1} x_i - x_0 + y$, with $x_i \in M_1 \setminus \{q_1\}$, $y \in M_2 \setminus \{q_2\}$. In both cases, we assume the points have been chosen generically so that $h^1(L_1)$ (note that $h^1(L_2) = 0$ automatically). In scheme I, by the same argument as in the proof of Proposition 2.24 we may also assume $\operatorname{coker} P_L$ vanishes on $R^{(1)}_t$.

A divisor also determines a framing τ_{L_t} of L_t, unique up to scale, which we fix once and for all t. Note that τ_{L_t} is globally holomorphic in scheme I, whereas in scheme II it is only meromorphic with pole $y \in M_1$. In scheme I we assume, again by the genericity of the divisor, that the framing is good. The section τ_{L_t} clearly limits to sections τ_{L_1} and τ_{L_2} of L_1 and L_2.

More generally, for the following result we refer to [31] (see also [11] and the discussion in [39]).

Proposition 4.5. *Suppose $d \geq g$. Choose a basis $\{\tau_{L_1}, \omega_2^{(1)}, \dots, \omega_m^{(1)}\}$ for $H^0(M_1, L_1)$ with $\omega_i^{(1)}(q_1) = 0$. Then there is a family of bases $\{\omega_1(t), \dots, \omega_m(t)\}$ for $H^0(M_t, L_t)$ such that $\omega_1(t) = \tau_{L_t}$, and for $2 \leq i \leq m$,*

$$\omega_i(z,t) = \begin{cases} \omega_i^{(1)}(z) + o(1) \ , & z \in M_1 \setminus \{q_1\} \\ o(1) \ , & z \in M_2 \setminus \{q_2\} \end{cases}$$

Moreover, the $o(1)$ vanish uniformly on $R_t^{(1)}$ and $R_t^{(2)}$ as $t \to 0$.

4.3. **Admissible metrics and asymptotics of $S(\sigma, f)$.** Recall the definition of the Arakelov metric (cf. [3, 15, 38, 17]). Given a compact Riemann surface M of genus $g \geq 1$, let $\{A_i, B_i\}_{i=1}^g$ be a symplectic set of generators of $H_1(M)$ and choose $\{\omega_i\}_{i=1}^g$ to be a basis of abelian differentials normalized such that $\int_{A_i} \omega_j = \delta_{ij}$. Let $\Omega_{ij} = \int_{B_i} \omega_j$ be the associated period matrix with theta function ϑ. Set

$$\mu = \frac{i}{2g} \sum_{i,j=1}^{g} (\mathfrak{Im}\, \Omega)_{ij}^{-1} \omega_i \wedge \overline{\omega}_j$$

Then $\int_M \mu = 1$. The **Arakelov-Green's function** $G(z,w)$ is symmetric with a zero of order one along the diagonal satisfying $\partial\bar\partial \log G(z,w) = (\pi i)\mu$, for $z \neq w$, normalized by $\int_M \mu(z) \log G(z,w) = 0$. The **Arakelov metric** $\rho = \rho(z)|dz|^2$ is defined by

$$\log \rho(z) = 2 \lim_{w \to z} \{\log G(z,w) - \log |z-w|\} \tag{4.6}$$

A hermitian metric h on a line bundle $L \to M$ of degree is d is **admissible** in the sense of [15] if

$$\mathrm{Ric}(h) = -(2\pi i d)\mu \tag{4.7}$$

The Arakelov metric on M, considered as a hermitian metric on the anti-canonical bundle K^*, is admissible:

$$\mathrm{Ric}(\rho) = 4\pi i(g-1)\mu \tag{4.8}$$

In terms of the Hermitian-Einstein tensor and the scalar curvature, (4.7) and (4.8) become

$$dA\, \Omega_{L,h} = (2\pi d)\mu \quad , \quad dA\, R_\rho = -8\pi(g-1)\mu \tag{4.9}$$

For more details we refer to the papers cited above.

We now consider the degenerating families $L_t \to M_t$ discussed in the previous section. The choice of framing τ_{L_t} allows us to define admissible metrics. Let $G_t(z,w)$ denote the Arakelov-Green's function on M_t.

In scheme I, we define the metric on L_t, L_1, and L_2 by

$$\|\tau_{L_t}\|^2(z) = h_t(z) = G_t^2(z,q) \prod_{i=1}^{d-1} G_t^2(z,x_i) \tag{4.10}$$

$$\|\tau_{L_1}\|^2(z) = h_1(z) = \prod_{i=1}^{d-1} G_1^2(z,x_i) \left[\frac{G_2(q,p)}{\prod_{i=1}^{d-1} G_1(p,x_i)} \right]^{2/g} \tag{4.11}$$

$$\|\tau_{L_2}\|^2(z) = h_2(z) = G_2^2(z,q) \left[\frac{\prod_{i=1}^{d-1} G_1(p,x_i)}{G_2(q,p)} \right]^{2(g-1)/g} \tag{4.12}$$

Similarly, in scheme II, we define the metrics

$$\|\tau_{L_t}\|^2(z) = h_t(z) = G_t^{-2}(z,y)G_t^2(z,q)\prod_{i=1}^{g-1} G_t^2(z,x_i) \tag{4.13}$$

$$\|\tau_{L_1}\|^2(z) = h_1(z) = G_1^{-2}(z,y)\prod_{i=1}^{g-1} G_1^2(z,x_i)\left[\frac{G_1(y,p)G_2(q,p)}{\prod_{i=1}^{g-1} G_1(p,x_i)}\right]^{2/g} \tag{4.14}$$

$$\|\tau_{L_2}\|^2(z) = h_2(z) = G_2^2(z,q)\left[\frac{\prod_{i=1}^{d-1} G_1(p,x_i)}{G_1(y,p)G_2(q,p)}\right]^{2(g-1)/g} \tag{4.15}$$

For $z \in M_t$, define the conformal factor $f_t(z)$ by

$$h_t(z) = \begin{cases} e^{2f_t(z)}h_1(z) & z \in M_1 \setminus \{p\} \\ e^{2f_t(z)}h_2(z) & z \in M_2 \setminus \{p\} \end{cases}$$

Similarly, let ρ_1, ρ_2, and ρ_t denote the Arakelov metrics on M_1, M_2, and M_t, respectively. Define σ_t by

$$\rho_t(z) = \begin{cases} e^{2\sigma_t(z)}\rho_1(z) & z \in M_1 \setminus \{p\} \\ e^{2\sigma_t(z)}\rho_2(z) & z \in M_2 \setminus \{p\} \end{cases}$$

Applying [38, Thm. 6.10] to (4.10), (4.11), and (4.12) for scheme I, and to (4.13), (4.14), and (4.15) for scheme II, we have the following estimates:

$$G_t(z,w) = \begin{cases} |t|^{1/g^2}G_1(z,w)(G_1(z,p)G_1(w,p))^{-1/g} + o(1)\ , & z,w \in M_1 \\ \\ |t|^{(g-1)^2/g^2}G_2(z,w)(G_2(z,p)G_2(w,p))^{-(g-1)/g} + o(1)\ , & z,w \in M_2 \\ \\ |t|^{-(g-1)/g^2}G_1(z,p)^{(g-1)/g}G_2(w,p)^{1/g} + o(1)\ , & z \in M_1\ ,\ w \in M_2 \end{cases} \tag{4.16}$$

$$f_t(z) = \begin{cases} \dfrac{(d-g)}{g^2}\log|t| - \dfrac{(d-g)}{g}\log G_1(z,p) + o(1)\ , & z \in M_1 \setminus \{p\} \\ \\ \dfrac{(g-1)(g-d)}{g^2}\log|t| + \dfrac{(d-g)}{g}\log G_2(z,p) + o(1)\ , & z \in M_2 \setminus \{p\} \end{cases} \tag{4.17}$$

$$\sigma_t(z) = \begin{cases} \dfrac{1}{g^2}\log|t| - \dfrac{2}{g}\log G_1(z,p) + o(1)\ , & z \in M_1 \setminus \{p\} \\ \\ \left(\dfrac{g-1}{g}\right)^2\log|t| - \dfrac{2(g-1)}{g}\log G_2(z,p) + o(1)\ , & z \in M_2 \setminus \{p\} \end{cases} \tag{4.18}$$

The $o(1)$ terms converge to zero uniformly as $|t| \to 0$. By direct computation of the expression (2.34) using (4.16), (4.17), and (4.18), we have

Lemma 4.19. *Assume $d \geq g-1$ and let $m = d-g+1$. Then in both degeneration schemes,*

$$S_{R_t^{(1)}}(\sigma(t), f(t)) + S_{R_t^{(2)}}(\sigma(t), f(t)) = -\left(\frac{2}{3} + \frac{2(g-1)}{3g} + \frac{2m(m-1)}{g^2}\right)\log|t| + o(1)$$

4.4. **Proof of the Main Theorem.** Let $\widehat{B}_{g,d} = B_{g,d}\exp(-c_g/12)$. Then the Main Theorem is equivalent to

Proposition 4.20. *For $d \geq g-1$, $\widehat{B}_{g,d} = (2\pi)^{2g-d}\exp(c_g/6)$.*

Proof. Analyze the asymptotics of both sides of the bosonization formula under the degeneration described in Section 4.2. First, consider the case $d \geq g$, i.e. the degeneration scheme I, and again set $m = d-g+1$. Choose sections $\omega_i(z,t)$, $i = 1,\dots,m$, as in Proposition 4.5. In terms of $\widehat{B}_{g,d}$ the right hand side of the bosonization formula (1.3) is

$$RHS = \widehat{B}_{g,d} e^{\delta(M)/12} (\det \mathfrak{Im}\,\Omega)^{1/2} \frac{\prod_{i<j} G^2(p_i,p_j)}{\left|\det \omega_i(p_j)\right|^2 \prod_{i=1}^m h(p_i)} \|\vartheta\|^2\Big([L] - \sum_{i=1}^m p_i - \mathfrak{D}, \Omega\Big)$$

where $\delta(M)$ is Faltings' delta invariant (cf. [39, Theorem 1.3]). Apply this to M_t. Choose generic points $\{p_i\}_{i=1}^m \subset M_1 \setminus \{q_1\}$. The Jacobian degenerates to the product of Jacobians for M_1 and M_2. With respect to this decomposition,

$$([L_t] - \sum_{i=1}^m p_i - \mathfrak{D}_t) \longrightarrow ([L_1] - \sum_{i=1}^m p_i - \mathfrak{D}_1, [L_2] - q_2 - \mathfrak{D}_2)$$

as $t \to 0$, and the period matrix and theta function factorize

$$\det \mathfrak{Im}\,\Omega \longrightarrow \det \mathfrak{Im}\,\Omega_1 \det \mathfrak{Im}\,\Omega_2$$

$$\|\vartheta\|^2\big(L_t - \sum_{i=1}^m p_i - \mathfrak{D}_t, \Omega_t\big) \longrightarrow \|\vartheta\|^2\big(L_1 - \sum_{i=1}^m p_i - \mathfrak{D}_1, \Omega_1\big)\|\vartheta\|^2\left(L_2 - q_2 - \mathfrak{D}_2, \Omega_2\right) \tag{4.21}$$

(cf. [16] and [38, Lemma 3.7]). From (4.16),

$$\prod_{i<j} G_t^2(p_i,p_j) \sim |t|^{m(m-1)/g^2} \prod_{i<j} G_1^2(p_i,p_j) \prod_{i=1}^m G_1(p_i,p)^{-2(d-g)/g} \tag{4.22}$$

From (4.17)

$$\prod_{i=1}^m h_t(p_i) \sim |t|^{2m(d-g)/g^2} \prod_{i=1}^m G_1(p_i,p)^{-2(d-g)/g} \prod_{i=1}^m h_1(p_i) \tag{4.23}$$

and by Proposition 4.5, $|\det \omega_i(p_j,t)|^2 \to |\det \omega_i^{(1)}(p_j)|^2$. Finally, from [38, Main Theorem] we have

$$e^{\delta(M_t)/12} \sim |t|^{-(g-1)/3g} e^{\delta(M_1)/12} e^{\delta(M_2)/12} \tag{4.24}$$

Putting this altogether,

$$(RHS)_t \sim |t|^{-(g-1)/3g - m(m-1)/g^2} \widehat{B}_{g,d} \frac{RHS_1}{\widehat{B}_{g-1,d-1}} \frac{RHS_2}{\widehat{B}_{1,1}} h_2(q_2) \tag{4.25}$$

For the left hand side of (1.3), apply Theorem 3.38 to the pairs (M_t,Γ_t), (M_1,Γ_t) and (M_2,Γ_t). Let $\{\Phi_i^{(1)}(z)\}_{i=1}^{2m}$ be the basis associated to $\{\omega_i^{(1)}\}_{i=1}^m$, and let $\{\Phi_i(z,t)\}_{i=1}^{2m}$ be the basis associated to $\{\omega_i(z,t)\}_{i=1}^{2m}$ (see (2.32)). Let $\{\Phi_i^{(2)}(z)\}_{i=1}^2$ be the basis associated to τ_{L_2}. Also, choose a basis $\{\Phi_i^A(z,t)\}_{i=1}^{2m-1}$ of $\ker D_L^A$

on $R_t^{(1)}$, with $\Phi_1^A = \tau_{L_t}\big|_{R_t^{(1)}}$. Note that the $\mathbb{R}$-span of $\tau_{L_t}\big|_{R_t^{(2)}}$ is the entire one dimensional space $\ker D_L^A$ on $R_t^{(2)}$. Then we have

$$\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i, \Phi_j)}\right]_{(M_t,\rho_t,h_t)} = c_Q \left[\frac{\mathrm{Det}^* D_L^A}{\det(\Phi_i^A, \Phi_j^A)}\right]_{(R_t^{(1)},\rho_t,h_t))} \left[\frac{\mathrm{Det}^* D_L^A}{\|\tau_{L_t}\|^2}\right]_{(R_t^{(2)},\rho_t,h_t)} \times \frac{\det((\Phi_i^A)', (\Phi_j^A)')_{(\Gamma_t,\rho_t,h_t)}}{\widehat{\det}((\Phi_i)'', (\Phi_j)'')_{(\Gamma_t,\rho_t,h_t)}} \mathrm{Det}_Q^* \mathcal{N}_{(M_t,\Gamma_t)} \tag{4.26}$$

$$\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i^{(1)}, \Phi_j^{(1)})}\right]_{(M_1,\rho_1,h_1)} = c_Q \left[\frac{\mathrm{Det}^* D_L^A}{\det(\Phi_i^A, \Phi_j^A)}\right]_{(R_t^{(1)},\rho_1,h_1)} \left[\frac{\mathrm{Det}^* D_L^A}{\|\tau_{L_1}\|^2}\right]_{(B_t^{(1)},\rho_1,h_1)} \times \frac{\det((\Phi_i^A)', (\Phi_j^A)')_{(\Gamma_t,\rho_1,h_2)}}{\widehat{\det}((\Phi_i^{(1)})'', (\Phi_j^{(1)})'')_{(\Gamma_t,\rho_1,h_2)}} \mathrm{Det}_Q^* \mathcal{N}_{(M_1,\Gamma_t)} \tag{4.27}$$

$$\left[\frac{\mathrm{Det}^* D_L}{\det(\Phi_i^{(2)}, \Phi_j^{(2)})}\right]_{(M_2,\rho_2,h_2)} = c_Q \left[\frac{\mathrm{Det}^* D_L^A}{\|\tau_{L_2}\|^2}\right]_{(R_t^{(2)},\rho_2,h_2)} \left[\frac{\mathrm{Det}^* D_L^A}{\|\tau_{L_2}\|^2}\right]_{(B_t^{(2)},\rho_2,h_t)} \mathrm{Det}_Q^* \mathcal{N}_{(M_2,\Gamma_t)} \tag{4.28}$$

Use the expressions on the right hand side to compute the ratio of (4.26) by (4.27) $\times$ (4.28). The factor

$$\frac{\det((\Phi_i^A)', (\Phi_j^A)')_{(\Gamma_t,\rho_t,h_t)}}{\widehat{\det}((\Phi_i)'', (\Phi_j)'')_{(\Gamma_t,\rho_t,h_t)}} \mathrm{Det}_Q^* \mathcal{N}_{(M_t,\Gamma_t)}$$

is invariant under conformal rescalings of (ρ, h) (see Remark 2.38); hence, we may suppose the metrics are locally euclidean. Using the same argument as in the proof of Proposition 4.1, we have

$$c_Q \mathrm{Det}_Q^* \mathcal{N}_{M_t,\Gamma_t} \sim 2^{-4h^0(L)+2}$$

By the same argument,

$$c_Q \mathrm{Det}_Q^* \mathcal{N}_{M_1,\Gamma_t} \sim 2^{-4h^0(L_1)+2}\ ,\ c_Q \mathrm{Det}_Q^* \mathcal{N}_{M_2,\Gamma_t} \sim 2^{-4h^0(L_2)+2}$$

After the conformal rescaling, the factors involving $(\Phi_i^A)'$ in (4.26) and (4.27) cancel, while by Proposition 4.5, the same is true for the factors involving $(\Phi_i)''$ and $(\Phi_i^{(1)})''$. Applying Theorem 2.33 to $R_t^{(1)}$ and $R_t^{(2)}$, we find

$$\left[\frac{\det^* D_L}{\det(\Phi_i, \Phi_j)}\right]_{M_t} \sim \left[\frac{\det^* D_L^A}{\det(\Phi_i, \Phi_j)}\right]_{M_1} \left[\frac{\det^* D_L^A}{\det(\Phi_i, \Phi_j)}\right]_{M_2} \left[\frac{\mathrm{Det}^* D_L^A}{\|\tau_{L_1}\|^2}\right]^{-1}_{B_t^{(1)}} \left[\frac{\mathrm{Det}^* D_L^A}{\|\tau_{L_2}\|^2}\right]^{-1}_{B_t^{(2)}} \times \exp(S(\sigma_t, f_t)) \frac{2^{-4h^0(L)+2}}{(2^{-4h^0(L_1)+2})(2^{-4h^0(L_2)+2})}$$

Next, we scale the hermitian metrics on the line bundles on the disks by $h_1(q_1) = h_2(q_2)$ in order to have $\|\tau_{L_i}\|(q_i) = 1$ (see (4.11) and (4.12)). Use Remark 2.38 (3) to compute the scale, and then (2.39) to compute the determinant. Combining this with Lemma 4.19, we have

$$\left[\frac{\det^* D_L}{\det(\Phi_i,\Phi_j)}\right]_{M_t} \sim \left[\frac{\det^* D_L}{\det(\Phi_i,\Phi_j)}\right]_{M_1}\left[\frac{\det^* D_L}{\det(\Phi_i,\Phi_j)}\right]_{M_2}\left[\frac{1}{2\pi|t|}\,2^{-1/3}|t|^{2/3}\exp(-4\zeta'(-1)+1/6)\right]^{-2} \times 4(h_2(q_2))^2|t|^{-\{2/3+2(g-1)/3g+2m(m-1)/g^2\}}$$

Now use Lemma 2.31 to compute

$$\left[\frac{\det^* \Box_L}{\det\langle\omega_i,\omega_j\rangle}\right]_{M_t} \sim \left[\frac{\det^* \Box_L}{\det\langle\omega_i,\omega_j\rangle}\right]_{M_1}\left[\frac{\det^* \Box_L}{\det\langle\omega_i,\omega_j\rangle}\right]_{M_2}\left[(2\pi)\,2^{1/3}\exp(4\zeta'(-1)-1/6)\right] \times h_2(q_2)|t|^{-(g-1)/3g-m(m-1)/g^2}$$

Comparing with (4.25) and using definition (1.5), we arrive at the recursive formula for $g\geq 2$:

$$\widehat{B}_{g,d}=\widehat{B}_{g-1,d-1}\widehat{B}_{1,1}\exp(-c_0/6) \tag{4.29}$$

This implies by recursion on g that

$$\widehat{B}_{g,d}=(\widehat{B}_{1,1})^{g-1}\exp\{(1-g)c_0/6\}\,\widehat{B}_{1,d-g+1} \tag{4.30}$$

By [17, p. 117], we have $\widehat{B}_{1,d}=(2\pi)^{-d}\exp(-c_1/12)$. Plugging into (4.30) proves Proposition 4.20 for $d\geq g$.

For $d=g-1$, we use the degeneration scheme II. The RHS of (1.2) is calculated from (4.21) and (4.24). The assumptions imply that $\mathbb{A}_\Gamma^{\mathrm{ker}}=\{0\}$ and $\mathbb{A}_\Gamma^{\mathrm{alv}}=\{0\}\oplus\mathbb{R}$. Hence, for M_t and M_1 we use (3.36), and (4.26), (4.27) become in this case

$$[\mathrm{Det}\,D_L]_{(M_t,\rho_t,h_t)}=c_Q\,[\mathrm{Det}\,D_L^A]_{(R_t^{(1)},\rho_t,h_t))}\left[\frac{\mathrm{Det}^*D_L^A}{\|\tau_{L_t}\|^2}\right]_{(R_t^{(2)},\rho_t,h_t)}\|\tau_{L_t}\|^2_{\Gamma_t}\,\mathrm{Det}^*_Q\mathcal{N}_{(M_t,\Gamma_t)} \tag{4.31}$$

$$[\mathrm{Det}\,D_L]_{(M_1,\rho_1,h_1)}=c_Q\,[\mathrm{Det}\,D_L^A]_{(R_t^{(1)},\rho_1,h_1)}\left[\frac{\mathrm{Det}^*D_L^A}{\|\tau_{L_1}\|^2}\right]_{(B_t^{(1)},\rho_1,h_1)}\|\tau_{L_1}\|^2_{\Gamma_t}\,\mathrm{Det}^*_Q\mathcal{N}_{(M_1,\Gamma_t)} \tag{4.32}$$

The only difference from the argument above is in the degeneration of the Neumann jump operators: since there are no global sections, we may not restrict the boundary operators $\mathcal{A}_{R_t^{(1,2)}}$ orthogonal to the constants. On the other hand, the section τ_L does define an element of $\ker D_L^A$ on $R^{(2)}$ (resp. the disk). To analyze this further, given $f_0\in\mathbb{R}$, let F_0 to be the constant term in the Fourier expansion of $\mathcal{A}_{R_1^{(1)}}(f_0,0)$. Notice that since the section τ_{L_2} extends holomorphically on on $R^{(2)}$, $\mathcal{A}_{R_t^{(2)}}(f_0,0)=0$ for all t. Let $\Phi_0=\mathcal{P}_{R_1^{(1)}}(f_0,0)$. By the proof of Proposition 2.24 we may assume $P_L\Phi_0\neq 0$. On the other hand, $\Phi_0\in\ker P_L^\dagger P_L$ and $(P_L\Phi_0)''=0$ on $\partial R_1^{(1)}$. Applying (2.17) to Φ_0, we have

$$(P_L\Phi_0,P_L\Phi_0)_{R_1^{(1)}}=\frac{1}{2}(\Phi_0,JP_L\Phi_0)_{\partial R_1^{(1)}}=\frac{1}{2}(\Phi_0'',J(P_L\Phi_0)')_{\partial R_1^{(1)}}=2\pi f_0F_0$$

The left hand side is nonzero; hence, we have a lower bound $|F_0|\geq c|f_0|$ for a constant $c>0$. Next, compute the harmonic extension Φ_ε of Φ_0 to R_ε as in Lemma 4.2. One finds that the corresponding constant terms in the Fourier expansions of the first factors f^ε and F^ε in $\mathbf{b}''_{\partial M_\varepsilon}\Phi_\varepsilon$ and $J\mathbf{b}'_{\partial M_\varepsilon}(\Phi_\varepsilon)$, respectively, are related by $F_0^\varepsilon=(1/\varepsilon\log\varepsilon)(f_0^\varepsilon-f_0)$, and $f_0^\varepsilon=f_0-F_0\log\varepsilon$. Taken together, this implies

$$F_0^\varepsilon=\frac{1}{\varepsilon\log\varepsilon}(1+O(1/\log(1/\varepsilon)))f_0^\varepsilon$$

The computation for the remaining Fourier modes is unchanged as in Lemma 4.2. We deduce the following modification of Proposition 4.1.

$$\Big[\log \mathrm{Det}^*_Q \mathcal{N}_{(M_t,\Gamma_t)} - \log(|t|^{1/2}\log|t|^{1/2})\Big] \longrightarrow (\zeta_Q(0)-2)\log 2$$

and this applies as well to M_1. Hence, in the ratio of (4.31) by (4.32), the extra singular term cancels. Using the the same argument given above for scheme I, we obtain the recursion (4.29) for $d = g-1$ as well. This completes the proof of Proposition 4.20 . □

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