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The Yang–Mills flow near the boundary of Teichmüller space

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Abstract. We study the behavior of the Yang-Mills flow for unitary connections on compact and non-compact oriented surfaces with varying metrics. The flow can be used to define a one dimensional foliation on the space of $SU(2)$ representations of a once punctured surface. This foliation universalizes over Teichmüller space and is equivariant with respect to the action of the mapping class group. It is shown how to extend the foliation as a singular foliation over the augmented boundary of Teichmüller space obtained by adding nodal Riemann surfaces. Continuity of this extension is the main result of the paper.

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1. Introduction

The Morse theory of the Yang-Mills functional on the space of gauge equivalence classes of unitary connections on a hermitian vector bundle over a Riemann surface was introduced in the seminal paper of Atiyah and Bott [AB]. Further properties of the gradient flow of the functional were obtained in [D] and [R]. The minimal critical set can be identified on the one hand with conjugacy classes of (projective) unitary representations of the fundamental group of the surface via the holonomy map, and on the other hand with the moduli space of semistable holomorphic vector bundles via the theorem of Narasimhan and Seshadri (cf. [NS, Do1]), and the analysis of [AB] shows that the Yang-Mills flow can be used effectively to study the topology of this space.

In [DW2] we studied the behavior of the moduli space of vector bundles as the conformal structure on the Riemann surface degenerates. A natural question

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to pose is whether the Yang-Mills flow itself behaves in a reasonable fashion under similar degenerations. In the case where the metric degeneration is to a *cone metric*, one may regard such a description as a non-linear version of the convergence of eigenvalues and eigenfunctions of the Laplace operator obtained in [JW1].

The specific problem we consider is the following: let X denote a compact surface of genus $g \geq 2$, with a prescribed point p, and set $X^* = X \setminus \{p\}$. We define

$$\mathcal{R}_\alpha = \mathrm{Hom}_\alpha\left(\pi_1(X^*), SU(2)\right)\big/ SU(2) \tag{1.1}$$

where the subscript α indicates that the holonomy of the representation around p is conjugate to the diagonal matrix with entries $e^{\pm 2\pi i\alpha}$. We take $0 \leq \alpha \leq 1/2$. Thus, $\mathcal{R}_0$ is identified with conjugacy classes of $SU(2)$ representations of the fundamental group of the closed surface X. Fixing a conformal structure $[\sigma^*]$ on $(X, \{p\})$ we can define a smooth, surjective map (the "Hecke Correspondence"; see [DDW] and below) $\pi_\alpha^{[\sigma^*]} : \mathcal{R}_\alpha \longrightarrow \mathcal{R}_0$ for $\alpha \neq 0, 1/2$, which is an S^2-fibration over the irreducibles. Roughly speaking, the map is defined as follows: for a flat $SU(2)$ connection A with holonomy α about p, we act on A by a singular *complex* gauge transformation g to bring the connection $A_0 = g(A)$ into the standard, trivial form d on a neighborhood of p. Thus, A_0 may be regarded as a connection on the closed surface X which is, however, no longer flat. The map is then defined by using the Yang-Mills flow to obtain from A_0 a flat connection $\pi_\alpha^{[\sigma^*]}[A]$. This definition clearly involves the choice of conformal structure in an important way, and understanding this behavior is the motivation for this paper.

The map $\pi_\alpha^{[\sigma^*]}$ can be generalized to a map $\pi_{\alpha\beta}^{[\sigma^*]} : \mathcal{R}_\alpha \to \mathcal{R}_\beta$ for any $0 \leq \beta \leq \alpha < 1/2$, and this will be a homeomorphism for $\beta \neq 0$ (see Sect. 2.3). We obtain from this a (real) 1-dimensional foliation $\mathcal{F}^{[\sigma^*]}$ of $\mathcal{R} = \bigcup_{0<\alpha<1/2} \mathcal{R}_\alpha$. A similar question to the one posed above is the dependence of this foliation on $[\sigma^*]$. For example, while it may be intuitively clear that $\mathcal{F}^{[\sigma^*]}$ varies continuously with $[\sigma^*]$, a differentiable structure is less obvious. Our first result is thus an explicit determination of variational formulas governing $\mathcal{F}^{[\sigma^*]}$. As a consequence, we prove that the foliation is C^1. The second result is a description of the behavior when the conformal structure degenerates. Here we show that for certain kinds of ideal boundary points on Teichmüller space, the foliations actually converge away from some singularities.

More precisely, let $\mathcal{T}_{aug.}(g, 1)$ denote the augmented Teichmüller space obtained by adding *nodal* Riemann surfaces (cf. [A]). These are obtained by collapsing a collection Φ of disjoint simple closed boundary incompressible curves on X^*. In the topology of $\mathcal{T}_{aug.}(g, 1)$, nodal surfaces may be approached by the "pinching" degeneration familiar from the Deligne-Mumford compactification of the moduli of curves. Thus, there is a family of conformal structures $\sigma^*(\ell)$

on X^* and a conformal structure $\sigma^*(0)$ on $X^* \setminus \Phi$ such that: (i) $\sigma^*(\ell) \to \sigma^*(0)$ uniformly on compact sets of $X^* \setminus \Phi$ as $\ell \to 0$; and (ii) for each $c \in \Phi$ there is a tubular neighborhood

$$C = \{(x, y) : -1 \le x \le 1,\ 0 \le y \le 2\pi\} \big/ \{(x, 0) \sim (x, 2\pi)\}\ ,$$

where c is given by $x = 0$, and $\sigma^*(\ell)$ is given by the conformal metric $ds_\ell^2 = dx^2 + \left(\ell + (1-\ell)x^2\right)\kappa^2 dy^2$ for some fixed $0 < \kappa \le 1$. The collapsed curve c, corresponding to the origin, is called the *node*, and we shall refer to the union of the model tubular neighborhoods for $c \in \Phi$ as the *pinching region* (see Sect. 4.2 for more details). To see the effect of pinching on the representation variety, we first make the following:

Definition 1.1. *Given a system Φ of disjoint simple closed boundary incompressible curves on X^* and an irreducible representation $\rho : \pi_1(X^*) \to SU(2)$, we say that ρ is* accidentally reducible *with respect to Φ if the restriction of ρ to the fundamental group of any component of $X^* \setminus \Phi$ is reducible. We denote by $\mathcal{R}^\Phi \subset \mathcal{R}$ the closed subspace of conjugacy classes of accidentally reducible representations.*

In Sect. 5 we prove the:

Main Theorem. *For each equivalence class $[\sigma^*(0)]$ of nodal conformal structures on $X^* \setminus \Phi$ there exists a smooth 1-dimensional foliation $\mathcal{F}^{[\sigma^*(0)]} \subset \mathcal{R} \setminus \mathcal{R}^\Phi$ such that for all paths $[\sigma^*(\ell)] \to [\sigma^*(0)]$ in $\mathcal{T}_{aug.}(g, 1)$ as above, $\mathcal{F}^{[\sigma^*(\ell)]} \longrightarrow \mathcal{F}^{[\sigma^*(0)]}$ uniformly on compact sets of $\mathcal{R} \setminus \mathcal{R}^\Phi$ in the Hausdorff sense.*

This result is in contrast to the algebraic situation: for each conformal structure $[\sigma^*]$ one can identify the space $\mathcal{R}_\alpha$ with the moduli space of parabolic stable bundles $\mathcal{M}_\alpha^{[\sigma^*]}$ (cf. [MS]). This universalizes over $\mathcal{T}(g, 1)$ to define a holomorphic fibration $\widetilde{\mathcal{M}_\alpha}$. Furthermore, the Mehta-Seshadri Theorem defines an identification of $\widetilde{\mathcal{M}_\alpha}$ with the trivial fibration $\widetilde{\mathcal{R}_\alpha} = \mathcal{T}(g, 1) \times \mathcal{R}_\alpha$. In this setting, the map $\pi_\alpha^{[\sigma^*]} : \mathcal{M}_\alpha^{[\sigma^*]} \longrightarrow \mathcal{M}$ is the elementary transformation at p; for $0 < \beta < \alpha < 1/2$, $\pi_{\alpha\beta}^{[\sigma^*]}$ is simply the identity. Using algebro-geometric methods, it is possible to compactify $\widetilde{\mathcal{M}_\alpha}$ and $\widetilde{\mathcal{M}}$ over the Deligne-Mumford compactification of $\mathcal{M}(g, 1)$ by adding the appropriate moduli space of torsion-free sheaves on nodal curves. Furthermore, one can show that the maps $\pi_\alpha^{[\sigma^*]}$ extend holomorphically over the compactification. This is similar in spirit to the degeneration used in [DW2]. But the map defined in the Main Theorem differs significantly from this algebraic compactification in the sense that there does not appear to be a version of the Narasimhan-Seshadri theorem relating the moduli space of torsion-free sheaves on a nodal curve to the subspace of $\mathcal{R}_\alpha$ mentioned above.

The definition of accidental reducibles, Def. 1.1, may be motivated as follows: in the case where the Yang-Mills flow starting from a connection $[A]$ converges to an irreducible flat connection $[B]$, one can find a complex gauge transformation g such that $[B] = g[A]$. This is not true, however, if $[B]$ is reducible. As $[B]$ moves closer to the reducibles one expects to lose control of the C^0 bound on g. Reducibility of $[B]$ can be detected via the existence of a kernel for the associated Laplace operator Δ_B acting on the traceless, skew-hermitian endomorphisms of E. As $[B]$ approaches the reducibles, the first eigenvalue of this operator goes to zero, and one might therefore look for an explicit C^0 bound on g which depends on $[B]$ only through the first eigenvalue of Δ_B. Such an estimate exists and will be used in Sect. 5. As the surface degenerates to a cone metric we have convergence of eigenvalues. The C^0 bound on g mentioned in the previous paragraph will persist, provided that the limiting eigenvalues are non-zero.

This paper is organized as follows: in Sect. 2 we review Donaldson's approach to the Yang-Mills flow and Råde's version of L. Simon's estimate for the behavior of the curvature along the flow (see Prop. 2.1). This takes the form

$$\|D_A^* F_A\| \geq c\|F_A\| \tag{1.2}$$

where A is a connection along the flow, F_A is its curvature, and c is a constant. The norms are taken with respect to the metrics on X^* and the $SU(2)$ bundle. We also discuss Simpson's flow for singular metrics and show that the flow at infinity of a singular connection preserves the conjugacy class of the holonomy about the singularity (see Cor. 2.4). An interpretation of this construction via branched covers is provided. All this permits a definition of the foliation $\mathcal{F}^{[\sigma^*]}$ and of the extended maps $\pi_{\alpha\beta}^{[\sigma^*(0)]}$. Sect. 3 contains the proof of the first variational formula Thm. 3.1 for the action of the complex gauge group. In particular, we show that the first variation for a path $[g_\varepsilon A_\varepsilon]$ of gauge equivalence classes of flat connections is independent of the derivative $\dot{g}$. This may be regarded as a kind of analogue of Ahlfors' result for the first variation of the hyperbolic area element for quasi-conformal maps. In Sect. 4 we prove estimates for the eigenvalue problems for sections of vector bundles and degenerations to cone metrics, i.e. families of the type ds_ℓ^2 above. Finally, the Main Theorem is proved in Sect. 5.

Let us here briefly outline the proof of the Main Theorem (see Sect. 5.1 for more details). Let $[\sigma^*(\ell)] \to [\sigma^*(0)]$ be a degeneration of conformal structures associated to a collection Φ of simple closed curves. It suffices to show that for $[A] \in \mathcal{R}_\alpha$ away from the accidental reducibles, and for β sufficiently close to α, the family $\pi_{\alpha\beta}^{[\sigma^*(\ell)]}[A]$ converges to $\pi_{\alpha\beta}^{[\sigma^*(0)]}[A]$. Convergence of a conjugacy class of flat connections amounts to convergence of the associated holonomy maps on $\pi_1(X^*)$. The problem naturally divides into two parts: convergence of the holonomy about closed loops supported in $X^* \setminus \Phi$, and convergence of the "gluing parameters," which essentially measure the holonomy across arcs transverse to the pinching curves $c \in \Phi$. These two considerations are dealt with in Thm.'s 5.1 and 5.2, respectively.

To show convergence of the holonomy in $X^* \setminus \Phi$, the key idea is to show that the estimate (1.2) holds with respect to a uniform constant c independent of the degeneration parameter ℓ (see Prop. 5.9). This is achieved by choosing a conic degeneration of representative metrics (i.e. $\sigma^*(\ell)$ conformal metrics of the form ds_ℓ^2 in the pinching region) and using the results of Sect. 4. This basic estimate can then be used to obtain exponential decay of the curvature along the flow (Prop. 5.8), which in turn gives control over the connection, and hence the holonomy, away from the pinching region.

For the second part, we use the fact that the connections before and after the flow are flat in the pinching region. By the formula for the change of curvature under a complex gauge transformation (5.2) this means that the bundle metric relating the two connections is harmonic. A Bochner formula then gives us à priori control of the metric, and therefore also of the gluing parameters, within the pinching region. An important ingredient in all of these arguments is the C^0 bound on the gauge transformation g mentioned above.

A word concerning notation: if σ is a Riemannian metric on a surface X, then integrals over X, unless otherwise specified, will be assumed to be taken with respect to σ. If σ is a conformal metric and z a conformal coordinate, we sometimes write $\sigma(z)|dz|^2$ for the area form. If f is a function on X, or more generally a section of a vector bundle V with a fiber metric H, and $\nu \geq 1$, we set

$$\|f\|_\nu = \left\{ \int_X |f|^\nu \right\}^{1/\nu}$$

In the above, if f is a section, then $|f| = |f|_H$ involves the fiber metric H. When we want to emphasize the choice of metrics, we will write $\|f\|_{\nu;\sigma}$ or $\|f\|_{\nu;\sigma,H}$ or perhaps even $\|f\|_{\nu;H}$ when there is no risk of confusion of the two metrics. Other Sobolev norms will be denoted with an explicit subscript, e.g. $\|f\|_{L_1^2(\sigma)}$, with $\|f\|_\infty$ (resp. $\|f\|_{\infty,H}$) for the L^∞ norm of $|f|$ (resp. $|f|_H$). Finally, we also use the following abbreviation for 2×2 diagonal matrices:

$$\operatorname{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} .$$

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2. Gauge theory

2.1. Review of the Yang–Mills flow

Let E be a rank 2 hermitian vector bundle on a closed, compact surface X. The Yang-Mills flow is given by the equation:

$$\frac{dA(t)}{dt} + D^*_{A(t)} F_{A(t)} = 0 \ , \tag{2.1}$$

where $A(t)$ is a time dependent connection, F_A is the curvature, and $D^*_A = - * D_A *$ is the L^2 formal adjoint of covariant differentiation D_A with respect to the connection A. Eq. (2.1) depends on a choice of a Riemannian metric on X through the Hodge star, and this dependence will be the main theme of the paper.

Let $\mathcal{A}$ denote the space of unitary connections on E and $\mathfrak{G}$ the group of unitary gauge transformations. Donaldson proved in [Do2] that the Yang-Mills flow equations (2.1) with initial condition:

$$A(0) = A_0 \ , \tag{2.2}$$

have a unique solution for all time in $A/\mathfrak{G}$. Subsequently, Råde was able to prove that the initial value problem (2.1)-(2.2) has a unique solution for all time in $\mathcal{A}$ *before* we mod out by the gauge group. Råde's estimates on the asymptotics of the flow as $t \to \infty$ will be important for our arguments.

Eq. (2.1) is the L^2-gradient flow for the Yang-Mills functional $YM(A) = \|F_A\|_2^2$, and for a solution $A(t)$ of (2.1),

$$\frac{d}{dt} YM(A(t)) = -\nabla YM(A(t)) = -\|D^*_{A(t)} F_{A(t)}\|_2^2 \ . \tag{2.3}$$

The critical points of $YM(A)$ are solutions of the Yang-Mills equations $D^*_A F_A = 0$, and the minima are given by the Hermitian-Yang-Mills (or projectively flat) equation $*F_A = \mu \, \mathbb{I}$ for a constant μ.

Donaldson's approach to solving eq.'s (2.1)-(2.2) up to gauge is to solve instead the non-linear heat equation:

$$H^{-1}(t) \frac{dH(t)}{dt} = -\sqrt{-1} * F_{\bar\partial_E, H(t)} \ , \tag{2.4}$$

for a family of metrics $H(t)$ with initial conditions:

$$H(0) = H_0 \ . \tag{2.5}$$

In the above, $F_{\bar\partial_E, H(t)}$ denotes the curvature of the unique connection compatible with holomorphic structure $\bar\partial_E$ on E and unitary with respect to the metric $H(t)$.

The two systems yield the same solution in $\mathcal{A}/\mathfrak{G}$, for if $A(t) = g(t)A_0$ is a solution of (2.1), then $H(t) = g(t)g(t)^* H_0$ is a solution to (2.4); conversely, if $H(t) = h(t)H_0$ is a solution to (2.4), then $A(t) = h^{1/2}(t)A_0$ is real gauge equivalent to a solution of (2.1). See [Do2] for more details. Also notice that it is easy to factor out the trace part of the connection and gauge transformations in eq.'s (2.1)-(2.2) and (2.4)-(2.5). Therefore, in the following we shall assume that the solutions to the above equations all preserve determinants.

We now review the analogue of L. Simon's result, due to Råde, concerning the asymptotics of the solutions to (2.1)-(2.2):

Proposition 2.1 ([R, Prop. 7.2]). *Let A_∞ be an* irreducible *Yang-Mills connection. There exist constants $\varepsilon_1, c > 0$ such that for any A satisfying $\|A - A_\infty\|_{L^4} \leq \varepsilon_1$, we have:*

$$\|D_A^* F_A\|_2 \geq c\,|YM(A) - YM(A_\infty)|^{1/2} \ . \tag{2.6}$$

Let $\mathrm{ad}\,E$ denote the bundle of skew-hermitian endomorphisms of E, and let $(\mathrm{ad}\,E)_0$ denote the subbundle of traceless ones. Then the constants ε_1, c depend only on the first eigenvalue of the Laplacian associated to A_∞ acting on sections of $(\mathrm{ad}\,E)_0$ and on the constant governing the inclusion $L_1^2 \hookrightarrow L^4$ (notice that by Kato's inequality this is essentially the Sobolev constant $\mathfrak{s}_1$ for functions – see (4.2) below).

The initial value problem (2.4)-(2.5) has solutions over non-compact surfaces as well. More precisely, let X be a compact Riemann surface as before, choose "punctures" $p_1, \ldots, p_k \in X$, and let $X' = X \setminus \{p_1, \ldots, p_k\}$. Fix a metric on X' whose expression in terms of a conformal coordinate z on X centered at any one of the p_i's has the form $ds^2 = \sigma(z)|dz|^2$, with:

$$\int_X |\sigma(z)|^p |dz|^2 < \infty \ , \tag{2.7}$$

for some $p > 1$ (cf. [Si1, Prop. 2.4] and note that the finiteness of (2.7) is independent of the choice of coordinate, though the actual value of the integral may vary). Let E be a holomorphic bundle on X. Given a hermitian metric H_0 on E with $\| * F_{\bar\partial_E, H_0}\|_\infty < \infty$, we define:

$$\deg(E, H_0) = \sqrt{-1}\int_X \mathrm{Tr}\left(*F_{\bar\partial_E, H_0}\right) \ .$$

Then E is said to be H_0-*stable* (resp. *semistable*) if for any proper holomorphic subbundle F of E we have:

$$\frac{\deg(F, H_0)}{\mathrm{rk}\,F} < \frac{\deg(E, H_0)}{\mathrm{rk}\,E} \qquad (\text{resp. } \leq) \ .$$

Simpson proved the following:

Theorem 2.2 (cf. [Si1], Prop. 6.6 and the proof of Thm. 1). (i) *Given a holomorphic vector bundle E on X' with hermitian metric H_0 such that $\|*F_{\bar\partial_E,H_0}\|_{\infty,H_0} < \infty$, there exists a unique solution $H(t) = h(t)H_0$ to (2.4)-(2.5) with constant determinant and having the property that for any finite $T > 0$,*

$$\sup_{T\geq t\geq 0} \|h(t)\|_{\infty;H_0} < \infty \ . \tag{2.8}$$

(ii) *If, in addition, E is H_0-stable, then (2.8) holds uniformly for all T. Furthermore, $H(t) = h(t)H_0$ converges weakly in $L^p_{2,loc.}$ to a solution $H(\infty) = h(\infty)H_0$ of the Hermitian-Yang-Mills equation with $\|h(\infty)\|_{\infty;H_0} < \infty$.*

This result has important consequences which we will need in the proof of our main theorem: given a holomorphic bundle $E \to X^*$ as above and a puncture $p \in X \setminus X^*$, choose a local holomorphic coordinate $z : U \to \Delta$ centered at p and defined on a neighborhood U, and a local holomorphic frame $\{f_1, f_2\}$ over U such that H_0 is in the standard form $\mathrm{diag}(|z|^{2\alpha}, |z|^{-2\alpha})$. The identification is chosen so that a unitary frame $\{e_1, e_2\}$ is given by $e_1 = |z|^{-\alpha} f_1$, $e_2 = |z|^{\alpha} f_2$, and the hermitian connection D_0 with respect to this frame is in the form $D_0 = d + \mathrm{diag}(i\alpha, -i\alpha)d\theta$.

Proposition 2.3. *The gauge transformation $h(\infty)$ from Thm. 2.2, Part (ii), is independent of the choice of conformal metric satisfying (2.7). Furthermore, if $\alpha < 1/2$ then $h(\infty)$ extends continuously at p, and $h(\infty)(p)$ is diagonal with respect to the frame $\{e_1, e_2\}$.*

Proof. For the first statement, note that the action of the complex gauge group $\mathfrak{G}^{\mathbb{C}}$ on $\mathcal{A}$ is independent of the conformal factor. Therefore, the C^0 bound from Thm. 2.2, (ii), and the argument in [Do1] prove uniqueness. To show that $h(\infty)$ extends if $\alpha < 1/2$, denote by D_0 the hermitian connection on E associated to H_0. Let g_1 be a singular gauge transformation of the form $g_1 = \mathrm{diag}(|z|^{-\alpha}, |z|^{\alpha})$ near p. Then $g_1(D_0) = d$ near p. Furthermore, there exists $0 \leq \beta \leq 1/2$ and a real gauge transformation ℓ with $\det \ell = 1$ such that if $g_2 = \mathrm{diag}(|z|^{-\beta}, |z|^{\beta})$ near p, then $g_2\ell h^{1/2}(\infty)(D_0) = d$ (cf. [DW1, Lemma 2.7]). It follows that $g_2\ell h^{1/2}(\infty)g_1^{-1}(d) = d$, hence $g_2\ell h^{1/2}(\infty)g_1^{-1}$ is holomorphic on the punctured disk Δ^*. On the other hand, since $|\alpha + \beta| < 1$, this matrix cannot have a pole at p, and it therefore extends continuously. Thus, we may write:

$$g_2\ell h^{1/2}(\infty)g_1^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

where a, b, c, d are holomorphic in Δ. Then:

$$\ell h^{1/2}(\infty) = g_2^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_1 = \begin{pmatrix} |z|^{\beta-\alpha}a & |z|^{\alpha+\beta}b \\ |z|^{-(\alpha+\beta)}c & |z|^{\alpha-\beta}d \end{pmatrix} .$$

Suppose that $\alpha < \beta$. Since the entries of the last matrix are bounded, we must have $c(0) = 0$ and $d(0) = 0$, which contradicts $\det h^{1/2}(\infty) = 1$ (as mentioned above we always fix the determinant by projecting away the trace part of the connection; see [Si1]). A similar argument holds for $\alpha > \beta$. Hence, $\alpha = \beta$. Finally, since b and c are holomorphic, $c(0) = 0$, and $\ell h^{1/2}(\infty)$ is diagonal at p. In particular, $(\ell h^{1/2}(\infty))^*(\ell h^{1/2}(\infty))(p) = h(\infty)(p)$ is diagonal at p. □

Corollary 2.4. *If E is a holomorphic bundle on X^* which is H_0-stable, then the Yang-Mills flow at infinity preserves the conjugacy class of the holonomy around the punctures.*

Proof. The case $0 < \alpha < 1/2$ follows from the proof of Prop. 2.3 above. For $\alpha = 0$, the metric H_0 is smooth at p and the flow (2.4)-(2.5) extends smoothly on X. It follows that the holonomy of the limit remains trivial. The case $\alpha = 1/2$ also follows from the proof of Prop. 2.3, for if $\beta < 1/2$ then we again have $|\alpha + \beta| < 1$, and the same argument gives a contradiction. □

Remark 2.5. It follows by the arguments in [Si2] that if H_0 is of the form $\mathrm{diag}(|z|^{2\alpha}, |z|^{-2\alpha})$ with respect to a holomorphic frame $\{f_1, f_2\}$ near p, then H_0-stability coincides with Seshadri's parabolic stability with respect to the weights $\{\alpha, -\alpha\}$. For more details we refer to [DW1,DW2].

2.2. Representation varieties and branched covers

Let X be a compact surface, $p \in X$, and $X^* = X \setminus \{p\}$. We denote by $\mathcal{R}(X^*)$ the space of conjugacy classes of $SU(2)$ representations of the free group $\pi_1(X^*)$. We may also identify $\mathcal{R}(X^*)$ with the space of gauge equivalence classes of flat connections on a trivial rank two hermitian bundle E on X^*. Given a real number $0 \le \alpha \le 1/2$, we denote by $\mathcal{R}_\alpha$ the subspace of flat connections on E with holonomy matrix conjugate to $\mathrm{diag}(e^{2\pi i\alpha}, e^{-2\pi i\alpha})$ around the puncture p. Then $\mathcal{R}_0$ is naturally identified with the space $\mathcal{R}(X)$ of equivalence classes of flat connections on the trivial bundle over the compact surface X. Notice that for $\alpha \neq 0$, $\mathcal{R}_\alpha$ consists entirely of irreducible representations. For future reference, we let $\mathcal{R}(X)_{irr} \subset \mathcal{R}(X)$ denote the open set of irreducible representations of $\pi_1(X)$.

In this section, we will sketch how to reduce certain analytical questions for the Yang-Mills flow on E over X^* to the flow on a bundle over a branched cover of the compact surface X. Let $\beta = k/n$ where k, n are positive coprime integers. Consider a regular, cyclic, n-fold, holomorphic branched cover $\widehat{X}$ of X with p in the ramification divisor B. Let $q : \widehat{X} \to X$ be the covering map, $\hat{p} = q^{-1}(p)$, $\widehat{B} = q^{-1}(B)$, $\widehat{X}^* = \widehat{X} \setminus \{\hat{p}\}$, $\widehat{U} = q^{-1}(U)$, and $\widehat{U}^* = \widehat{U} \cap \widehat{X}^*$. Let $E = X^* \times \mathbb{C}^2$ be the trivial rank 2 vector bundle on X^*. We construct a bundle $\widehat{E}$ over $\widehat{X}$ by gluing $q^*(E)$ on $\widehat{X} \setminus \widehat{U}$ with $\widehat{U} \times \mathbb{C}^2$ via the gauge transformation:

$$\hat{s} : \widehat{U}^* \longrightarrow SU(2) \quad : \quad \hat{s}(w) = \hat{s}(\hat{r}, \hat{\theta}) = \begin{pmatrix} e^{-ik\hat{\theta}} & 0 \\ 0 & e^{-ik\hat{\theta}} \end{pmatrix} .$$

Since $\det \hat{s} = 1$, it follows that $\deg \widehat{E} = 0$; hence, $\widehat{E}$ is isomorphic to the trivial bundle. Although there is no natural global trivialization, there is a trivialization of $\widehat{E}\big|_{\widehat{X}^*} = q^* E\big|_{\widehat{X}^*}$ induced by the one on E.

On E we fix the trivial hermitian metric H_0. Also, since $\hat{s}$ is unitary, the trivial metrics on q^*E and $\widehat{U} \times \mathbb{C}^2$ glue together to define a hermitian metric $\widehat{H}_0$ on $\widehat{E}$. Let $\mathcal{A}_\beta$ denote the space of unitary connections of E on X^*, flat in a neighborhood of p with holonomy conjugate to $\mathrm{diag}(e^{2\pi i\beta}, e^{-2\pi i\beta})$. Given $A \in \mathcal{A}_\beta$, choose a real gauge transformation g such that:

$$g(A)\big|_U = d + \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} d\theta \ .$$

It follows that:

$$q^* g(A)\big|_{\widehat{U}^*} = d + \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix} d\hat{\theta} \ .$$

By gluing $q^*g(A)$ with the trivial connection $d_{\widehat{U}}$ via $\hat{s}$, we obtain a unitary connection $\widehat{A}$ on $\widehat{E}$. Let $\widehat{\mathcal{A}}$ denote the space of unitary connections on $\widehat{E}$. Define:

$$\hat{q} : \mathcal{A}_\beta \longrightarrow \widehat{\mathcal{A}} \quad , \quad \hat{q}(A) = q^*(g^{-1})\widehat{A} \ ,$$

where in the above, $q^*(g) = g \circ q$. It is easily checked that $\hat{q}$ is well-defined and real gauge equivariant. In particular, it induces a map $\hat{q} : \mathcal{R}_\beta \to \widehat{\mathcal{R}} = \mathcal{R}(\widehat{X}) = \widehat{\mathcal{A}}_{flat}/\widehat{\mathfrak{G}}$, where $\widehat{\mathcal{A}}_{flat}$ are the flat connections on $\widehat{E}$ and $\widehat{\mathfrak{G}}$ is the real gauge group. We note the following:

Proposition 2.6. *If $\beta = k/n$ with n odd, then $\hat{q}(\mathcal{R}_\beta) \subset \widehat{R}_{irr}$. Furthermore, given a collection Φ of simple closed boundary incompressible curves in X^*, mutually disjoint and disjoint from B, $\widehat{\Phi} = q^{-1}(\Phi)$, and $[A] \in \mathcal{R}_\beta$ which is not accidentally reducible with respect to Φ (see Def. 1.1), then $\hat{q}([A])$ is not accidentally reducible with respect to $\widehat{\Phi}$.*

Proof. We first prove:

Lemma 2.7. *Consider an exact sequence of groups $1 \to H \to G \to Q \to 1$, and suppose that Q is abelian with no index 2 subgroup. Then the restriction to H of any irreducible $SU(2)$ representation of G is also irreducible.*

Proof. Suppose, to the contrary, that there is an irreducible $\rho : G \to SU(2)$ which is reducible on H. Then there is a maximal torus $\mathbf{T}$ in $SU(2)$ such that the image of H lies in $\mathbf{T}$. Since Q is abelian and ρ is irreducible, the image of H cannot be contained entirely in the center of $SU(2)$. This implies that the image of G lies in $N(\mathbf{T})$, the normalizer of $\mathbf{T}$. Now we have the exact sequence for the Weyl group $1 \to \mathbf{T} \to N(\mathbf{T}) \to \mathbb{Z}/2\mathbb{Z} \to 1$. Since the image of G is not contained in $\mathbf{T}$, we must have a surjection $Q \to \mathbb{Z}/2\mathbb{Z} \to 1$; contradiction. □

Continuing with the proof of Prop. 2.6: as remarked above, a point $[A] \in \mathcal{R}_\beta$ gives a conjugacy class of irreducible representations of $\pi_1(X^*)$ which, in turn, induce irreducible representations of $\pi_1(X \setminus B)$. Now $\hat{q}[A]$ is irreducible as a representation of $\pi_1(\widehat{X})$ if and only if it induces an irreducible representation of $\pi_1(\widehat{X} \setminus \widehat{B})$. The first statement then follows from Lemma 2.7 by setting $H = \pi_1(\widehat{X} \setminus \widehat{B})$, $G = \pi_1(X \setminus B)$, and $Q = \mathbb{Z}/n\mathbb{Z}$. For the second statement, consider a connected component Y of $X \setminus \Phi$. If $Y \cap B \neq \emptyset$, then $\widehat{Y} = q^{-1}(Y)$ is a connected component of $\widehat{X} \setminus \widehat{\Phi}$, and the argument is as above. If $Y \cap B = \emptyset$, then for each connected component $\widehat{Y}_i$ of $q^{-1}(Y)$, $\pi_1(\widehat{Y}_i) \simeq \pi_1(Y)$, so $\hat{q}[A]$ is irreducible there as well. □

Now consider the effect of eq.'s (2.4)-(2.5) under $\hat{q}$. First we choose a conformal metric σ on $X \setminus B$ with the following property: for every $b \in B$, express the map q locally as $z = w^n$ for coordinates z on centered at b and w centered at $\hat{b}$. Then we assume that $\sigma(z) = 1/n^2|z|^{2(1-1/n)}$. Notice that such a metric satisfies condition (2.7). This is an example of a cone metric (see (4.1)). Let $\hat{\sigma}$ be the pull-back metric on $\widehat{X} \setminus \widehat{B}$. Then the condition on σ implies that $\hat{\sigma}$ extends to a smooth conformal metric on $\widehat{X}$.

Next, fix a connection A_0 on E over X^* and let $\widehat{A}_0 = \hat{q}(A_0)$. Let $H(t) = h(t)H_0$ be a solution of (2.4)-(2.5) on E over $X \setminus B$, where the holomorphic structure on E is defined by $A_0^{0,1}$. Then $\widehat{H}(t) = q^*H(t) = \hat{h}(t)\widehat{H}_0$ is a solution of the same equations on $\widehat{X} \setminus \widehat{B}$ with respect to the holomorphic structure $q^*A_0^{0,1}$. Since q is smooth and $h(t)$ satisfies the estimate (2.8), the same is true for $\hat{h}(t)$. Hence, by the uniqueness properties of the flow, we obtain

Lemma 2.8. *The restriction of the flow $\widehat{H}(t)$ to $\widehat{X} \setminus \widehat{B}$ coincides with $q^*H(t)$.*

This allows us to reduce estimates of $h(t)$ over X^* to estimates of $\hat{h}(t)$ over $\widehat{X}$. For example:

Lemma 2.9. *Let Ω be an open set with compact closure in $X \setminus B$. Then there exists a constant $C = C(\Omega)$ such that:*

$$\|h(t)\|_{L^p_k(\Omega)} \leq C\, \|\hat{h}(t)\|_{L^p_k(\widehat{\Omega})} .$$

In the above, $h(t)$ is the solution to (2.4)-(2.5) on X^*, and $\hat{h}(t)$ is the solution over $\widehat{X}$.

2.3. Definition of the foliation

Let X, X^* be as above. Given $0 \leq \beta < \alpha < 1/2$, or $0 < \alpha < \beta \leq 1/2$, and $[\sigma^*] \in \mathcal{T}(g, 1)$, we will now give a rigorous definition of the twist maps

$\pi^{[\sigma^*]}_{\alpha\beta} : \mathcal{R}_\alpha \to \mathcal{R}_\beta$. The conformal structure on X^* extends to X. Let σ denote a choice of smooth metric on X compatible with this conformal structure. Let $z : U \to \Delta$ be a holomorphic coordinate centered at p, and let φ_1 be a smooth cut-off function supported in $\Delta_{1/3}$ and identically equal to 1 on $\Delta_{1/6}$. We define a singular complex gauge transformation of E by setting:

(2.9)
$$g_{\alpha\beta}(\xi) = \begin{pmatrix} \exp\{\varphi_1(z(\xi))(\alpha-\beta)\log|z(\xi)|\} & 0 \\ 0 & \exp\{\varphi_1(z(\xi))(\beta-\alpha)\log|z(\xi)|\} \end{pmatrix},$$

for $\xi \in U$ and extending by the identity elsewhere. Given $[A] \in \mathcal{R}_\alpha$, choose a representative A in the standard form $\mathrm{diag}(i\alpha, -i\alpha)d\theta$ when pulled back to Δ via z. Then $\widetilde{A} = g_{\alpha\beta}(A)$ will be of the form $\mathrm{diag}(i\beta, -i\beta)d\theta$ over $\Delta_{1/6}$. Furthermore, it is easy to check (cf. [MS,DW1]) that the holomorphic structure induced by $\widetilde{A}$ is parabolic stable, and hence by Remark 2.5 we can flow the connection $\widetilde{A}$ with respect to the metric σ at infinite time. Thm. 2.2 guarantees that $\widetilde{A}$ flows to a flat connection $\widetilde{A}(\infty)$ which, by Cor. 2.4 has holonomy conjugate to $\mathrm{diag}(e^{2\pi i\beta}, e^{-2\pi i\beta})$. We set:

$$\pi^{[\sigma^*]}_{\alpha\beta}[A] = [\widetilde{A}(\infty)]\ , \tag{2.10}$$

the class of $\widetilde{A}(\infty)$ in $\mathcal{R}_\beta$.

It is straightforward to show that the definition of $\pi^{[\sigma^*]}_{\alpha\beta}$ is independent of the choices made, i.e. the coordinate z, the cutoff function φ_1, the lift of the conformal structure, and the lift of $[A]$. In other words, the only dependence of $\pi^{[\sigma^*]}_{\alpha\beta}$ is through the class $[\sigma^*] \in T(g,1)$. Set $\mathcal{R} = \bigcup_{0<\alpha<1/2} \mathcal{R}_\alpha \subset \mathcal{R}(X^*)$. The next two lemmas are left to the reader:

Lemma 2.10. *For* $0 \le \gamma < \beta < \alpha < 1/2$ *or* $0 < \alpha < \beta < \gamma \le 1/2$ *and* $[\sigma^*] \in \mathcal{T}(g,1)$, $\pi^{[\sigma^*]}_{\alpha\gamma} = \pi^{[\sigma^*]}_{\beta\gamma} \circ \pi^{[\sigma^*]}_{\alpha\beta}$.

Lemma 2.11. *Given* $[\sigma^*] \in \mathcal{T}(g,1)$, $\alpha \in (0,1/2)$, *and* $[A] \in \mathcal{R}_\alpha$, *the map* $[0,1/2] \to \mathcal{R} : \beta \mapsto \pi^{[\sigma^*]}_{\alpha\beta}[A]$ *is continuous. Furthermore, its restriction to* $(0,1/2)$ *is smooth.*

For convenience, for $\beta = \alpha$ we define $\pi^{[\sigma^*]}_{\alpha\beta}$ to be the identity.

Definition 2.12. *Fix* $[\sigma^*] \in T(g,1)$, $[A] \in \mathcal{R}_\alpha$, $0 < \alpha < 1/2$, *and let:*

$$\mathcal{F}^{[\sigma^*]}_{[A]} = \bigcup_{0<\beta<1/2} \pi^{[\sigma^*]}_{\alpha\beta}[A]\quad ,\quad \mathcal{F}^{[\sigma^*]} = \bigcup_{[A]\in\mathcal{R}^*} \mathcal{F}^{[\sigma^*]}_{[A]}\ .$$

It follows from the above discussion that $\mathcal{F}^{[\sigma^*]}$ is a smooth 1-dimensional foliation of $\mathcal{R}^* = \mathcal{R} \setminus \mathrm{hol}^{-1}\{0, 1/2\}$.

We now turn to the definition of the limiting foliation $\mathcal{F}^{[\sigma^*(0)]}$ discussed in the Introduction. Let Φ denote a collection of disjoint simple closed curves on X^*, and let $\sigma^*(0)$ be a conformal structure on the pinched surface $X_0^* = X^* \setminus \Phi$. Let us elaborate on this (see Sect. 4.2 for more details). By definition, the conformal structure on the pinched surface X_0^* arises from a conformal structure σ^* on X^* as follows: for each $c \in \Phi$ there is a tubular, cylindrical neighborhood C of c which is conformally equivalent, with respect to σ^*, to the intersection in $\mathbb{C}^2$ of a neighborhood of the origin with the annulus $zw = \varepsilon$ for some non-zero complex number ε. In these coordinates, c is the set where $|z| = |w| = |\varepsilon|^{1/2}$. The conformal structure $\sigma^*(0)$ then replaces C with a neighborhood of the origin of the pinched annulus $zw = 0$.

Let $\mathcal{R}^{\Phi} \subset \mathcal{R}$ be the set of Φ-accidental reducibles. Given $[A] \in \mathcal{R}_\alpha \setminus \mathcal{R}^{\Phi}$, let A be a lift of $[A]$ to a connection which has the standard form $d+\mathrm{diag}(i\gamma, -i\gamma)d\theta$ in a coordinate neighborhood z of each $c \in \Phi$ as described above. Of course, the holonomy γ depends on the component c. By assumption, the restriction of A to any component of X_0^* is irreducible. Hence, for $\varepsilon([A]) > 0$ sufficiently small and $|\beta - \alpha| < \varepsilon([A])$, the holomorphic structure associated to the twists $g_{\alpha\beta}(A)$ on X_0^* is parabolic stable for the choice of weights β and $\{\gamma_c\}_{c\in\Phi}$.

Set $A(0, \infty)$ to be the flow of $g_{\alpha\beta}(A)$ (at infinite time) with respect to a metric on X_0^* compatible with the conformal structure and satisfying the condition (2.7). As before, the real gauge equivalence class $[A(0, \infty)]$ of $A(0, \infty)$ is independent of all the choices made.

Theorem 2.13. *There is a well-defined lift of* $[A(0, \infty)]$ *to an element in* $\mathcal{R}_\beta$.

Proof. Let $U_\pm$ be coordinate neighborhoods z and w corresponding to a curve $c \in \Phi$ as before, and suppose the holonomy is γ. Thus we have:

$$A\big|_{U_\pm} = d \pm \begin{pmatrix} i\gamma & 0 \\ 0 & -i\gamma \end{pmatrix} d\theta_\pm \ ,$$

with respect to unitary frames $e_1^\pm, e_2^\pm$ of $E|_{U_\pm}$. By Cor. 2.4 there are real gauge transformations $g_\pm$ of $E|_{U_\pm}$ such that:

$$g_\pm\left(A(0,\infty)\right)\big|_{U_\pm} = d \pm \begin{pmatrix} i\gamma & 0 \\ 0 & -i\gamma \end{pmatrix} d\theta_\pm \ . \tag{2.11}$$

The $g_\pm$ may be extended to a global real gauge transformation g of $E|_{X_0^*}$ which is the identity away from a small neighborhood of $U_\pm$. By using the identification of the frame $\{e_1^+, e_2^+\}$ with $\{e_1^-, e_2^-\}$, (2.11) implies that the pull-back of $g(A(0, \infty))$ to $X^* \setminus c$ extends smoothly over c. By repeating the above for every curve $c \in \Phi$ we obtain a flat connection on X^* with the correct holonomy. The resulting conjugacy class is unique, because a conjugacy class of connections on $E|_{X_0^*}$ together with gluing data determine a unique conjugacy class of connections on $E|_{X^*}$. □

The construction above may be summarized as follows: the connection A on X^* determines a flat connection on $X^* \setminus \Phi$, along with "gluing parameters" across the curves $c \in \Phi$. The twisted connection $g_{\alpha\beta}(A)$ has a small amount of curvature in the component Y of $X^* \setminus \Phi$ containing $\{p\}$, and it agrees with A on all the other components. The flow $A(0,\infty)$ then runs the Simpson flow on Y, applied to $g_{\alpha\beta}(A)$, and leaves the connection on the other components fixed. Since the holonomies around the curves $c \in \Phi$ bounding Y remain unchanged for the flow at infinite time (Cor. 2.4), we may use the original gluing parameters to reconstruct a flat connection on X^*.

For each $[A] \in \mathcal{R}_\alpha$ and β satisfying $|\alpha - \beta| < \varepsilon([A])$ as above, define $\pi_{\alpha\beta}^{[\sigma^*(0)]}[A]$ to be the lift of $[A(0,\infty)]$ described in Thm. 2.13. Furthermore, we set:

$$\mathcal{F}_{[A]}^{[\sigma^*(0)]} = \bigcup_{\alpha-\varepsilon([A])<\beta<\alpha+\varepsilon([A])} \pi_{\alpha\beta}^{\Phi}[A]\ .$$

It is straightforward to check a composition rule as in Lemma 2.10. We may therefore define:

$$\mathcal{F}^{[\sigma^*(0)]} = \bigcup_{[A]\in\mathcal{R}} \mathcal{F}_{[A]}^{[\sigma^*(0)]}\ .$$

Then $\mathcal{F}^{[\sigma^*(0)]}$ is a smooth foliation of $\mathcal{R} \setminus \mathcal{R}^{\Phi}$.

3. Differentiability of the foliation

3.1. First order variational formula

In describing the behavior of the Yang-Mills flow applied to a connection A as the conformal structure σ on a closed surface varies, there are two considerations: first, the complex gauge transformation g describing the flow, i.e. such that $g(A)$ is flat, will depend in a complicated way on σ. Second, while the real gauge group acts on the space of unitary connections in a manner independent of the conformal structure, the complex gauge group does not.

Assume that we have fixed a conformal structure on X. Let μ_ε be a differentiable family of Beltrami differentials on X with $\mu_0 = 0$, $\dot{\mu}_\varepsilon\big|_{\varepsilon=0} = \nu$. Let g_ε be a differentiable family of complex gauge transformations on E with $g_0 = g$, $\dot{g}_\varepsilon\big|_{\varepsilon=0} = \dot{g}$. Finally, let A_ε be differentiable family of unitary connections on E with $A_0 = A$, $\dot{A}_\varepsilon\big|_{\varepsilon=0} = \dot{A}$. Set $\gamma_\varepsilon = g_\varepsilon A_\varepsilon$, and $\dot{\gamma} = \dot{\gamma}_\varepsilon\big|_{\varepsilon=0}$. We emphasize that the action of the complex gauge group is with respect to the μ_ε-deformed conformal structure. Also, we regard Beltrami differentials ν as endomorphisms $\Omega^{1,0}(X) \to \Omega^{0,1}(X)$, which extend to endomorphisms on forms with values in E.

Theorem 3.1. *Let μ_ε, g_ε, A_ε, and γ_ε be as above. Then:*

$$\dot{\gamma}^{0,1} = \bar{\partial}_{g(A)}\left(g^{-1}\dot{g}\right) - \nu\left[\left(\partial_A g^*\right)(g^*)^{-1} + (g^*)^{-1}\partial_A g^*\right] + g^{-1}\dot{A}^{0,1}g \ ,$$

where the $(0,1)$ *part is taken with respect to the fixed conformal structure on X, and g^* denotes the fiberwise adjoint of g with respect to the background metric H_0.*

Remark 3.2. If γ_ε is a path of flat connections, then in $\mathcal{A}/\mathfrak{G}$ a representative for the tangent vector $[\dot{\gamma}^{0,1}]$ may be taken to be harmonic. Projecting to the harmonics, we see that the first term in the expression above vanishes. Hence, we conclude that the first variation of the Yang-Mills flow is independent of $\dot{g}$.

Proof of Thm. 3.1. We first note how the Cauchy-Riemann operators deform: fix a unitary connection A on E and a Beltrami differential μ.

Lemma 3.3. *Let $\bar{\partial}_{A,\mu} : \Omega^0(E) \to \Omega^{0,1}(X_\mu, E) \subset \Omega^1(E)$ be the $\bar{\partial}$-operator on X_μ associated to A. Then $\bar{\partial}_{A,\mu}$ is given by*

$$\bar{\partial}_{A,\mu}s = \frac{1}{1-|\mu|^2}\left(\bar{\partial}_A s - \mu\partial_A s + \bar{\mu}\bar{\partial}_A s - |\mu|^2\partial_A s\right) \ ,$$

for smooth sections $s \in \Omega^0(E)$.

Proof. By choosing local frames, the lemma follows from the corresponding statement for $\bar{\partial}$ acting on functions. Thus, let f be a function, z a local conformal coordinate on X, and $w = w_\varepsilon$ solutions to the Beltrami equation $w_{\bar{z}} = \mu_{\bar{z}}^{z} w_z$. Here we have expressed $\mu = \mu_{\bar{z}}^{z} d\bar{z} \otimes (\partial/\partial z)$. Write:

$$df = f_z dz + f_{\bar{z}} d\bar{z} = f_w dw + f_{\bar{w}} d\bar{w} \ ,$$

and use $dw = w_z(dz + \mu_{\bar{z}}^{z} d\bar{z})$ to obtain:

$$f_z = f_z w_z + f_{\bar{w}}\bar{w}_z\bar{\mu}_{z}^{\bar{z}} \ , \ f_{\bar{z}} = f_w w_z \mu_{\bar{z}}^{z} + f_{\bar{w}}\bar{w}_z \ .$$

Multiplying the first equation by μ and subtracting the second, we have:

$$f_{\bar{w}}\bar{w}_z(1-|\mu|^2) = f_{\bar{z}} - f_z\mu_{\bar{z}}^{z} \ .$$

Now multiply through by $d\bar{z} + \bar{\mu}_{\bar{z}}^{z} dz$ to obtain:

$$\bar{\partial}_\mu f = f_{\bar{w}} d\bar{w} = \frac{1}{1-|\mu|^2}\left(f_{\bar{z}} d\bar{z} - \mu_{\bar{z}}^{z} d\bar{z} f_z + \bar{\mu}_{\bar{z}}^{z} dz f_{\bar{z}} - |\mu|^2 f_z dz\right) \ .$$

The result follows by observing that:

$$\mu_{\bar{z}}^{z} d\bar{z} f_z = \mu_{\bar{z}}^{z} d\bar{z} \otimes (\partial/\partial z)(f_z dz) = \mu\partial f \ ,$$
$$\bar{\mu}_{\bar{z}}^{z} dz f_{\bar{z}} = \bar{\mu}_{z}^{\bar{z}} dz \otimes (\partial/\partial z)(f_{\bar{z}} d\bar{z}) = \bar{\mu}\bar{\partial} f \ .$$

□

Continuing with the proof of Thm. 3.1: the action of the complex gauge group is given by:

$$g(D_A) = d + g^{-1}A^{0,1}g + g^*A^{1,0}(g^*)^{-1} + g^{-1}\bar{\partial}_\mu g - (\partial_\mu g^*)(g^*)^{-1}\ , \tag{3.1}$$

where

$$\begin{aligned} A^{0,1} &= A_{\bar{w}}d\bar{w} = A_{\bar{z}}d\bar{z} - \varepsilon\nu(A_z dz) + \varepsilon\bar{\nu}(A_{\bar{z}}d\bar{z}) + O(\varepsilon^2)\ , \\ A^{1,0} &= A_w dw = A_z dz - \varepsilon\bar{\nu}(A_{\bar{z}}d\bar{z}) + \varepsilon\nu(A_z dz) + O(\varepsilon^2)\ . \end{aligned} \tag{3.2}$$

From Lemma 3.3 we have:

$$\begin{aligned} \bar{\partial}_\mu g &= \bar{\partial}g + \varepsilon\left(\bar{\partial}\dot{g} - \nu(\partial g) + \bar{\nu}(\bar{\partial}g)\right) + O(\varepsilon^2)\ , \\ \partial_\mu g^* &= \partial g^* + \varepsilon\left(\partial\dot{g}^* - \bar{\nu}(\bar{\partial}g^*) + \nu(\partial g^*)\right) + O(\varepsilon^2)\ . \end{aligned}$$

Applying this and (3.2) to (3.1) yields the result. □

3.2. Differentiability of the twisted connection

The aim of this section is to prove the following:

Proposition 3.4. *Let σ_ε^* be a continuously differentiable family of the metrics representing a path $[\sigma_\varepsilon^*] \subset \mathcal{T}(g,1)$. Then there is a family of singular gauge transformations $g_{\alpha\beta}^\varepsilon$ such that $g_{\alpha\beta}^\varepsilon(A)$ is a continuously differentiable path of connections.*

Proof. We differentiate at $\varepsilon = 0$. Let μ_ε be the differentiable path of Beltrami differentials on X with $\mu_0 = 0$ and $\dot{\mu}_\varepsilon\big|_{\varepsilon=0} = \nu$, associated to σ_ε^*. Let $z : U \to \Delta$ be a local coordinate on a neighborhood U of p, conformal with respect to the conformal structure determined by σ_0^*. Fix φ_0 a smooth cut-off function supported in Δ and identically 1 on $\Delta_{2/3}$. We obtain Beltrami differentials $\tilde{\mu}$ on $\mathbb{C}$ by extending $\varphi_0\mu$ by zero. For $\varepsilon \geq 0$ small, we consider the solution w_ε to the Beltrami equation on $\mathbb{C}$:

$$w_{\bar{z}} = \tilde{\mu}_{\bar{z}}^{z} w_z\ , \tag{3.3}$$

normalized such that:

$$w_\varepsilon(0) = 0\ , \quad w_\varepsilon(1) = 1\ , \quad w_\varepsilon(\infty) = \infty\ . \tag{3.4}$$

Let $\dot{w}$ denote $\partial w_\varepsilon/\partial\varepsilon$ at $\varepsilon = 0$. We also set $\tilde{\nu} = \dot{\tilde{\mu}}$.

To compute the derivative, we must take into account the change of frame. The problem is local, so suppose we have a fixed (trivial) hermitian rank 2 vector bundle over the complex plane with global unitary frame $e_\pm$ and singular connection A_α such that:

$$D_A e_\pm = \pm i\alpha d\theta \otimes e_\pm\ . \tag{3.5}$$

Let θ_ε denote the theta coordinate of w_ε.

Lemma 3.5. *Define a frame* $e_\varepsilon^\pm = \exp\left\{\pm i u_\alpha^\varepsilon\right\} e^\pm$ *such that* $D_A e_\varepsilon^\pm = \pm i\alpha d\theta_\varepsilon \otimes e_\pm$*. Normalize by setting* $u_\alpha^\varepsilon(1) = 0$, $u_\alpha^0(z) \equiv 0$*, and set* $\dot{u}_\alpha = \dot{u}_\alpha^\varepsilon\big|_{\varepsilon=0}$*. Then*

$$i\dot{u}_\alpha(z) = \frac{\alpha}{2}\left(\frac{\dot{w}}{z} - \overline{\frac{\dot{w}}{z}}\right) .$$

Proof. Differentiate to obtain:

$$D_A e_\varepsilon^+ = i du_\alpha^\varepsilon \otimes e_\varepsilon^+ + i\alpha d\theta \otimes e_\varepsilon^+ \implies du_\alpha^\varepsilon = \alpha d\theta_\varepsilon - \alpha d\theta .$$

Hence, $d\dot{u}_\alpha = \alpha d\dot{\theta}$. Now:

$$d\theta_\varepsilon = \frac{1}{2i}\left(\frac{dw}{w} - \overline{\frac{dw}{w}}\right) \implies d\dot{\theta} = \frac{1}{2i}\left(d\left(\frac{\dot{w}}{z}\right) - d\left(\overline{\frac{\dot{w}}{z}}\right)\right) .$$

Since $\dot{w}(1) = 0$ by the normalization (3.4), the result follows upon integration. □

We now choose one more smooth cut-off function in addition to φ_0 and φ_1: let φ_2 be smooth, supported on Δ and is identically 1 on $\Delta_{1/3}$. Set $u_{\alpha\beta}^\varepsilon = u_\alpha^\varepsilon - u_\beta^\varepsilon$, and define the complex gauge transformation:

$$g_{\alpha\beta}^\varepsilon = \begin{pmatrix} \exp\left\{i\varphi_2 u_{\alpha\beta}^\varepsilon + \varphi_1(\alpha-\beta)\log|w_\varepsilon|\right\} & 0 \\ 0 & \exp\left\{-i\varphi_2 u_{\alpha\beta}^\varepsilon - \varphi_1(\alpha-\beta)\log|w_\varepsilon|\right\} \end{pmatrix} .$$

Now consider the family of singular connections $\gamma_\varepsilon = g_\varepsilon A_\alpha$, where the action of the complex gauge transformation is with respect to the complex structure w_ε. The derivative in the e^+ direction, for example, is given by (see Thm. 3.1):

$$\begin{aligned}\left(\dot{\gamma}^{0,1}\right)_{e^+} = \frac{\partial}{\partial\bar{z}}\left[i\varphi_2\dot{u}_{\alpha\beta} + \varphi_1\frac{\alpha-\beta}{2}\left(\frac{\dot{w}}{z} + \overline{\frac{\dot{w}}{z}}\right)\right] \\ -2(\alpha-\beta)\tilde{\nu}\frac{\partial}{\partial z}\left(\varphi_1 \log|z|^2\right) .\end{aligned}$$

By Lemma 3.5 this is:

$$\begin{aligned}\left(\dot{\gamma}^{0,1}\right)_{e^+} = \frac{\partial}{\partial\bar{z}}\left[\frac{\varphi_2(\alpha-\beta)}{2}\left\{\left(\frac{\dot{w}}{z} - \overline{\frac{\dot{w}}{z}}\right)\right.\right. \\ \left.\left. + \ \varphi_1\left(\frac{\dot{w}}{z} + \overline{\frac{\dot{w}}{z}}\right)\right\}\right] - 2(\alpha-\beta)\tilde{\nu}\frac{\partial}{\partial z}\left(\varphi_1 \log|z|^2\right) .\end{aligned}$$

By the choice of cut-off functions it is easily verified that the support of $\dot{\gamma}^{0,1}$ lies in the annulus $1/6 \leq |z| \leq 2/3$. The continuity follows from this expression. □

3.3. Differentiability

We combine Thm. 3.1 and Prop. 3.4 to prove:

Theorem 3.6. *Let σ_ε^* be a continuously differentiable family of metrics representing a path $[\sigma_\varepsilon^*] \in \mathcal{T}(g,1)$. Then for each $[A] \in \mathcal{R}_\alpha$, $\pi_{\alpha\beta}^{[\sigma_\varepsilon^*]}[A]$ is a continuously differentiable path in $\mathcal{R}_\beta$.*

As an immediate consequence:

Corollary 3.7. *Given $0 < \beta < \alpha < 1/2$, the universal Hecke correspondence $\tilde{\pi}_{\alpha\beta} : \widetilde{\mathcal{R}}_\alpha \to \widetilde{\mathcal{R}}_\beta$ is continuously differentiable. The same is true for $\beta = 0$ on the preimage of the irreducible representations.*

Proof of Thm. 3.6. First, notice that by choosing a rational number $\beta < k/n < \alpha$ and using Lemma 2.10 it suffices to prove the result for rational holonomies. Second, by the definition (2.10) of the foliation and Prop. 3.4 it suffices to show that given a continuously differentiable path of connections $A_\varepsilon \in \mathcal{A}_\beta$, the path $[A_\varepsilon(\infty)] \in \mathcal{R}_\beta$ is also continuously differentiable. Finally, by passing to a branched cover as in Sect. 2.2, it suffices to prove the result for closed surfaces. We continue with the notation as in Sect. 2.2.

Let $\widehat{A}_\varepsilon \in \widehat{\mathcal{A}}$ be a continuously differentiable path. Note that by Prop. 2.6 and Lemma 2.8 we may assume the $\widehat{A}_\varepsilon$ are stable. Since $\widehat{A}_\varepsilon(\infty) = g_\varepsilon \widehat{A}_\varepsilon$, where g_ε is a complex gauge transformation, it suffices by the first variational formula (Thm. 3.1) to show that h_ε is a smooth family of complex gauge transformations. This can be achieved by the implicit function theorem as follows: consider the map

$$f : \Omega^0(\sqrt{-1}\,\mathrm{ad}_0\,\widehat{E}) \times \widehat{\mathcal{A}}_{stable} \times \mathrm{Met}_{-1} \longrightarrow \Omega^0(\sqrt{-1}\,\mathrm{ad}_0\,\widehat{E})$$
$$f(u, \widehat{A}, \hat{\sigma}) = \sqrt{-1} *_{\hat{\sigma}} \bar{\partial}^{\hat{\sigma}}_{\widehat{A}} \left(e^{-u} \partial^{\hat{\sigma}}_{\widehat{A}} e^u \right) .$$

It is easily verified that the map f is smooth, and:

$$(\delta_u f)_{(u,\widehat{A},\hat{\sigma})}(\delta u) = \Delta^{\hat{\sigma}}_{e^u \widehat{A}}(\delta u) .$$

By the implicit function theorem, the solution $u = u(\widehat{A}, \hat{\sigma})$ depends smoothly on $(\widehat{A}, \hat{\sigma})$, and this completes the proof.

4. Eigenvalue and eigenfunction estimates

4.1. Estimates

We shall need eigenfunction and eigenvalue estimates on sections of a vector bundle V equipped with a Riemannian metric, a metric connection A, and Laplace operator Δ_A. In the following, we shall assume X is a compact surface, possibly with boundary, with a smooth metric of area $\mathfrak{a}$. We define (cf. [Li, eq. (0.4)]) the Sobolev constant $\mathfrak{s}_1$ to be the supremum over all constants s satisfying: $s \inf_{a\in\mathbb{R}} \|f-a\|_2^2 \leq \|df\|_1^2$, for all smooth functions f. In case $\partial X \neq \emptyset$, we define another constant associated to the Dirichlet problem: let $\mathfrak{s}_2$ be the supremum over all constants s satisfying: $s\,\|f\|_2^2 \leq \|df\|_1^2$, for all smooth compactly supported functions f. In this section we prove the following:

Theorem 4.1. *There is a universal constant C with the following property: let φ be an eigensection of V with eigenvalue λ. Then:*

(1) If $\partial X = \emptyset$ or if $\partial X \neq \emptyset$ and φ satisfies Neumann boundary conditions, then:

$$\|\varphi\|_\infty^2 \leq C\left(\mathfrak{a}^{-1} + \left(\frac{4\lambda}{\mathfrak{s}_1}\right)^2 \mathfrak{a}\right)\|\varphi\|_2^2 \,.$$

(2) If $\partial X \neq \emptyset$ and φ satisfies Dirichlet boundary conditions, then:

$$\|\varphi\|_\infty^2 \leq C\left(\frac{4\lambda}{\mathfrak{s}_2}\right)^2 \mathfrak{a}\|\varphi\|_2^2 \,.$$

We also need a lower bound on the growth of eigenvalues. While heat kernel estimates as in [CL] might perhaps give more precise estimates, we shall only need the following:

Theorem 4.2. *There is a universal constant C with the following property:*

(1) If λ_k denotes the k-th eigenvalue for sections of V for the closed or Neumann boundary problem, then:

$$k \leq C \operatorname{rk} V \left(1 + \frac{4\lambda_k \mathfrak{a}}{\mathfrak{s}_1}\right)^3 \,.$$

(2) If μ_k denotes the k-th eigenvalue for sections of V for the Dirichlet boundary problem, then:

$$k \leq C \operatorname{rk} V \left(\frac{4\mu_k \mathfrak{a}}{\mathfrak{s}_2}\right)^3 \,.$$

Thm.'s 4.1 and 4.2 stated above are generalizations of the results of P. Li [Li] (see also [CL]) on eigenvalues and eigenfunctions of the Laplacian on forms. For the sake of completeness, however, we shall sketch the important steps involved in the proofs of the results for sections. We point out that the main difference in the estimates is the extra term $\mathfrak{a}^{-1}$ in Thm. 4.1, Part (1), and the 1 in Thm. 4.2, Part (1). These are basically due to the lack of an à priori lower bound on the eigenvalues in terms of Sobolev constants. The key estimate is the following (cf. [Li, eq. (2.7) and (3.1)]):

Lemma 4.3. *Let φ be a smooth section of V and A a smooth metric connection on V. If $\partial X \neq \emptyset$, we assume that φ satisfies either Dirichlet or Neumann boundary conditions. Then for any $\nu > 1$,*

$$\int_X |\varphi|^{2\nu-2}\langle\varphi, \Delta_A\varphi\rangle \geq \frac{2\nu-1}{\nu^2}\int_X |d|\varphi|^\nu|^2 \ .$$

Proof. Using $2\langle\varphi, D_A\varphi\rangle = d|\varphi|^2$, we have for $\nu > 1$:

$$\left\langle D_A\left(|\varphi|^{2\nu-2}\varphi\right), D_A\varphi\right\rangle = \frac{\nu-1}{2}|\varphi|^{2\nu-4}\langle d|\varphi|^2, d|\varphi|^2\rangle + |\varphi|^{2\nu-2}|D_A\varphi|^2 \ .$$

By Kato's inequality, the right hand side above is

$$\geq \frac{\nu-1}{2}|\varphi|^{2\nu-4}\langle d|\varphi|^2, d|\varphi|^2\rangle + |\varphi|^{2\nu-2}\,|d|\varphi||^2 = \frac{2\nu-1}{\nu^2}\,|d|\varphi|^\nu|^2 \ .$$

The lemma then follows from integration by parts. □

Proof of Thm. 4.1. We shall only prove Part (1), the proofs of the other statements being similar. Assume $\Delta_A\varphi = \lambda\varphi$. Applying (4.2) to $f = |\varphi|^\nu$ and using Lemma 4.3, we obtain:

$$\lambda\int_X |\varphi|^{2\nu} \geq \frac{2\nu-1}{\nu^2}\frac{\mathfrak{s}_1}{4}\left(\mathfrak{a}^{-1/2}\left(\int_X|\varphi|^{4\nu}\right)^{1/2} - \mathfrak{a}^{-1}\int_X|\varphi|^{2\nu}\right) \ ,$$

or,

$$\left(\mathfrak{a}^{-1/2} + \frac{\nu^2}{2\nu-1}\frac{4\lambda\mathfrak{a}^{1/2}}{\mathfrak{s}_1}\right)^{1/2\nu}\|\varphi\|_{2\nu} \geq \|\varphi\|_{4\nu} \ .$$

Set $\nu = 2^k$ to obtain, by iteration,

$$\|\varphi\|_2\prod_{j=0}^{k}\left(\mathfrak{a}^{-1/2} + \frac{2^{2j}}{2^{j+1}-1}\frac{4\lambda\mathfrak{a}^{1/2}}{\mathfrak{s}_1}\right)^{1/2^{j+1}} \geq \|\varphi\|_{2^{k+2}} \ .$$

Letting $k \to \infty$ yields $\|\varphi\|_\infty^2 \leq \|\varphi\|_2^2\mathfrak{a}^{-1}\prod_{j=0}^{\infty}\left(1 + \frac{2^{2j}}{2^{j+1}-1}\frac{4\lambda\mathfrak{a}}{\mathfrak{s}_1}\right)^{1/2^j}$. The result then follows from the following simple:

Lemma 4.4. *Fix $\gamma > 0$. Then for $\beta > 1$ there is a constant $C(\beta)$ independent of γ such that:*

$$\prod_{j=0}^{\infty}\left(1 + \frac{\gamma\beta^{2j}}{2\beta^j-1}\right)^{1/\beta^j} \leq C(\beta)\left(1+\gamma^{\beta/\beta-1}\right) \ .$$

Proof of Thm. 4.2. Again, we shall concentrate on Part (1). Let the first k eigenvalues and eigensections be denoted $0 \leq \lambda_1 \leq \dots \leq \lambda_k$, and $\varphi_1, \dots, \varphi_k$, respectively.

Lemma 4.5. *For any $\psi \in span\{\varphi_1, \dots, \varphi_k\}$ and any $l \geq 1$,*

$$\frac{\|\psi\|_{2^{l+2}}}{\mathfrak{a}^{1/2^{l+2}}} \leq \left(1 + \frac{4\lambda_k \mathfrak{a}}{\mathfrak{s}_1} \frac{2^{2l}}{2^{l+1}-1}\right)^{1/2^{l+1}-1} \frac{\|\psi\|_{2^{l+1}}}{\mathfrak{a}^{1/2^{l+1}}} .$$

Proof. By Lemma 4.3 for $\nu = 2^l$ and (4.2) we have:

$$\begin{aligned}\frac{(2^{l+1}-1)}{2^{2l}} \frac{\mathfrak{s}_1}{4} \left\{ \mathfrak{a}^{-1/2} \|\psi\|_{2^{l+2}}^{2^{l+1}} - \mathfrak{a}^{-1} \|\psi\|_{2^{l+1}}^{2^{l+1}} \right\} &\leq \int_X |\psi|^{2^{l+1}-2} \langle \psi, \Delta_A \psi \rangle \\ &\leq \left\{ \int_X |\psi|^{2^{l+1}} \right\}^{(2^{l+1}-1)/2^{l+1}} \left\{ \int_X |\Delta_A \psi|^{2^{l+1}} \right\}^{1/2^{l+1}} .\end{aligned}$$

Write $\psi = \sum_{j=1}^k c_j \varphi_j$. Now it follows as in [Li, Lemma 17] that there is a subset $J \subset \{1, \dots, k\}$ such that:

$$\begin{aligned}\left\{ \int_X |\Delta_A \psi|^{2^{l+1}} \right\}^{1/2^{l+1}} &\leq \left\{ \int_X |\Delta_A \psi|^{2^{l+2}} \right\}^{1/2^{l+2}} \mathfrak{a}^{1/2^{l+2}} \\ &\leq \left\{ \int_X \Big|\sum_{j=1}^k \lambda_j c_j \varphi_j\Big|^{2^{l+2}} \right\}^{1/2^{l+2}} \mathfrak{a}^{1/2^{l+2}} \leq \lambda_k \left\{ \int_X \Big|\sum_{j\in J} c_j \varphi_j\Big|^{2^{l+2}} \right\}^{1/2^{l+2}} \mathfrak{a}^{1/2^{l+2}} \\ &\leq \lambda_k \|\psi\|_{2^{l+2}} \mathfrak{a}^{1/2^{l+2}} .\end{aligned}$$

Therefore,

$$\mathfrak{a}^{-1/2} \left(\frac{\|\psi\|_{2^{l+2}}}{\|\psi\|_{2^{l+1}}} \right)^{2^{l+1}-1} \leq \frac{4\lambda_k}{\mathfrak{s}_1} \frac{2^{2l}}{2^{l+1}-1} \mathfrak{a}^{1/2^{l+2}} + \mathfrak{a}^{-1} \frac{\|\psi\|_{2^{l+1}}}{\|\psi\|_{2^{l+2}}} .$$

Using the fact that $\|\psi\|_{2^{l+1}} \leq \mathfrak{a}^{1/2^{l+2}} \|\psi\|_{2^{l+2}}$, we have:

$$\left(\frac{\|\psi\|_{2^{l+2}}}{\|\psi\|_{2^{l+1}}} \right)^{2^{l+1}-1} \leq \left(\mathfrak{a}^{-1/2} + \frac{4\lambda_k}{\mathfrak{s}_1} \frac{2^{2l}}{2^{l+1}-1} \mathfrak{a}^{1/2} \right) \mathfrak{a}^{1/2^{l+2}} ,$$

from which the lemma follows. □

For each $l \geq 1$, choose $\psi_l \in \text{span}\{\varphi_1, \dots, \varphi_k\}$ so that $\|\varphi\|_{2^{l+2}} \|\psi_l\|_2 \leq \|\psi_l\|_{2^{l+2}} \|\varphi\|_2$ for all $\varphi \in \text{span}\{\varphi_1, \dots, \varphi_k\}$. Then repeated application of Lemma 4.5 gives:

$$\frac{\|\varphi\|_\infty}{\|\varphi\|_2} \leq \frac{\|\psi_0\|_4}{\|\psi_0\|_2} \frac{1}{\mathfrak{a}^{1/4}} \prod_{l=1}^{\infty} \left(1 + \frac{4\lambda_k \mathfrak{a}}{\mathfrak{s}_1} \frac{2^{2l}}{(2^{l+1}-1)}\right)^{1/2^{l+1}-1} .$$

From Lemma 4.3 and (4.2) we also have:

$$\frac{\|\psi_0\|_4^2}{\|\psi_0\|_2^2} \leq \mathfrak{a}^{-1/2}\left(1+\frac{4\lambda_k\mathfrak{a}}{\mathfrak{s}_1}\right) .$$

Using Lemma 4.4 and the fact that $1/(2^{l+1}-1) \leq 1/2^l$ we conclude that for all $\varphi \in \text{span}\,\{\varphi_1, \ldots, \varphi_k\}$

$$\frac{\|\varphi\|_\infty^2}{\|\varphi\|_2^2} \leq C\mathfrak{a}^{-1}\left(1+\frac{4\lambda_k\mathfrak{a}}{\mathfrak{s}_1}\right)^3 .$$

Part (1) of Thm. 4.2 now follows from the following:

Lemma 4.6 ([Li, Lemma 11]). *Let W be a finite dimensional subspace of $L^2(V) \cap C^0$. Then there is $\varphi \in W$ such that* $\dim W\,\|\varphi\|_2^2 \leq (\text{rk}\,V)\,\mathfrak{a}\|\varphi\|_\infty^2$.

4.2. Conic degeneration

Recall the notion of a cone metric on a manifold $C(Y) = (0,1) \times Y$ (cf. [Ch1], [Ch2], [JW1]): this is a metric of the form $ds^2 = dr^2 + r^2\tilde{\sigma}$, where $\tilde{\sigma}$ is a (smooth) metric on Y. An n-dimensional manifold X with metric σ on $X \setminus \{p\}$ is said to have a cone metric if for some choice of $(Y, \tilde{\sigma})$ and some neighborhood U of p in X, $U \setminus \{p\}$ is isometric to $C(Y)$; in this case we call p a cone point. We generalize this notion to that of a cone double point, by which we mean locally the union of two copies of $C(Y)$ with the singularity identified. For example, for surfaces it is natural to view such a singularity as arising from the following family of degenerating metrics: on the cylinder $C = \{(x,y) : -1 \leq x \leq 1,\ 0 \leq y \leq 2\pi\}\big/\{(x,0) \sim (x,2\pi)\}$, define, for $0 \leq \ell \leq 1$, the metrics $ds_\ell^2 = dx^2 + \left(\ell + (1-\ell)x^2\right)\kappa^2 dy^2$. The parameter $0 < \kappa \leq 1$, introduced here for convenience, is a fixed cone angle. The metric for $\ell = 0$ has a cone double point as described above. We shall refer to a family of metrics $\sigma(\ell)$ on a compact, connected surface X as a *conic degenerating family* if X contains a finite collection of disjoint cylinders $C_1, \ldots, C_k$, all of which are isometric to $C(\ell) = (C, ds_\ell^2)$, and $\sigma(\ell)$ converges smoothly to a Riemannian metric on $X \setminus C_1 \cup \cdots \cup C_k$.

Eigenvalue problems are still well-behaved on compact manifolds with cone metrics. We shall be considering the following situation: let A be a connection on a (real) vector bundle $V \to X$ with Riemannian metric. If X has cone points, then we allow A to be singular at those points. It will suffice to assume further that with respect to some choice of orthonormal frame and conformal coordinates (r,θ) near the cone point, A has the form $b \otimes d\theta$, where b is a constant diagonal matrix. Consider the operator $\Delta_A = D_A^* D_A$ acting on sections φ of V satisfying $\varphi,\ D_A\varphi,\ \Delta_A\varphi$ in L^2. Then by a generalization of the result in [Ch1], Stokes theorem holds for sections of V and the cone metric. Hence, the L^2 extension of Δ_A is self-adjoint. Moreover, the spectrum of Δ_A is discrete with finite multiplicity, and the eigenfunctions are smooth away from the singularities (cf. [Ch2]).

We will be using three important properties of conic degenerating families. The first follows from a direct computation:

Proposition 4.7. *Let $(X, \sigma(\ell))$ be a conic degenerating family of surfaces. Then the eigenvalues of the Ricci curvature tensor are uniformly bounded by a constant times $1/\ell$.*

Proof. We briefly sketch the computation: it clearly suffices to compute the Ricci tensor ${R_k}^i$ for $C(\ell) = (C, ds_\ell^2)$. We use coordinates $x_1 = x$, $x_2 = y$, so that:

$$\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \left(\ell + (1-\ell)x^2\right)\kappa^2 \end{pmatrix} .$$

Then one finds for the Christoffel symbols: $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0$, and

$$\Gamma_{22}^1 = -\frac{1}{2}\frac{\partial \sigma_{22}}{\partial x} = -(1-\ell)x\kappa^2 \quad , \quad \Gamma_{12}^2 = \frac{1}{2}\sigma^{22}\frac{\partial \sigma_{22}}{\partial x} = \frac{(1-\ell)x}{\ell + (1-\ell)x^2} .$$

The operator

$${R_k}^i = \sigma^{lj}\left(\frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{jl}^\mu \Gamma_{\mu k}^i - \Gamma_{jk}^\mu \Gamma_{\mu l}^i\right) .$$

From this, one verifies ${R_1}^2 = {R_2}^1 = 0$, and

$${R_1}^1 = \sigma^{22}\left(\frac{\partial \Gamma_{22}^1}{\partial x} - \Gamma_{12}^2 \Gamma_{22}^1\right) = \frac{-\ell(1-\ell)}{(\ell + (1-\ell)x^2)^2} ,$$

$${R_2}^2 = -\frac{\partial \Gamma_{12}^2}{\partial x} - \Gamma_{12}^2 \Gamma_{12}^2 = \frac{-\ell(1-\ell)}{(\ell + (1-\ell)x^2)^2} .$$

The result follows. □

The second fact is a comparison between conic degeneration and the well-known "plumbing" construction used to study holomorphic degenerating families of Riemann surfaces (cf. [DW2]). Consider the annuli given by: $\Upsilon_\varepsilon = \left\{(z, w) \in \mathbb{C}^2 : |z|, |w| \leq 1,\ zw = \varepsilon\right\}$. This is a holomorphic family for ε in a punctured disk, but we will take ε to be real. The central fiber $\varepsilon = 0$ is a "pinched" annulus – two disks with coordinates z and w identified at $z = w = 0$. We shall be interested in metrics on Υ_ε which are conformal with respect to the complex structure induced from $\mathbb{C}^2$ and which degenerate to cone metrics ds_0^2 on Υ_0 proportional to metrics of the following form: for some $0 < \kappa \leq 1$ and in the coordinate z,

$$ds_0^2 = \kappa^2 |z|^{2(\kappa-1)} |dz|^2 \tag{4.1}$$

(cf. [JW1, Lemma 6.1]). These can always be constructed:

Proposition 4.8. *Fix κ, $0 < \kappa \leq 1$. Then there is a function $\varepsilon = \varepsilon(\ell)$ depending on κ and a conformal conic degenerating family of metrics on $\Upsilon_{\varepsilon(\ell)}$ converging to the metric (4.1). Furthermore, the parameters ε and ℓ are related by the following bound:*

$$\frac{\ell^{2/\kappa}}{4^{1/\kappa}} \leq \varepsilon(\ell) \leq \ell^{1/\kappa} ,$$

for $\ell \leq 3/4$.

Proof. We solve for conformal coordinates $z = re^{i\theta}$ on the portion of $\Upsilon_{\varepsilon(\ell)}$ where $0 \leq x \leq 1$. Take $\theta = \theta(y) = y$, and $r = f(x, \ell)$ for f increasing, $f(1, \ell) = 1$. For such a solution, we may write $ds_\ell^2 = \sigma(z, \ell)|dz|^2$, or:

$$\frac{dr^2}{(f')^2(x, \ell)} + (\ell + (1 - \ell)x^2)\kappa^2 d\theta^2 = \sigma(z, \ell)(dr^2 + r^2 d\theta^2) .$$

where $f'(x, \ell)$ is the partial derivative with respect to x. This implies:

$$\frac{\partial}{\partial x} \log f(x, \ell) = \frac{1}{\kappa(\ell + (1 - \ell)x^2)^{1/2}} \quad , \quad \sigma(z, \ell) = \frac{1}{(f')^2(x, \ell)} .$$

Applying the initial condition, we find the solution

$$f(x, \ell) = \left[\frac{(1 - \ell)^{1/2} x}{1 + (1 - \ell)^{1/2}} + \frac{(1 - \ell)^{1/2}}{1 + (1 - \ell)^{1/2}} \sqrt{x^2 + \frac{\ell}{1 - \ell}} \right]^{1/\kappa(1-\ell)^{1/2}}$$

$$\varepsilon(\ell) = f^2(0, \ell) = \left[\frac{\ell}{(1 + (1 - \ell)^{1/2})^2} \right]^{1/\kappa(1-\ell)^{1/2}}$$

Notice that the relationship between ε and f in the second line follows from the fact that the middle of the cylinder corresponds to $|z| = |w| = \sqrt{\varepsilon}$. The convergence and bounds on $\varepsilon(\ell)$ easily follow. □

Lastly, we comment on the behavior of the Sobolev constants under conic degeneration. We quote the following result:

Proposition 4.9 ([JW1, Thm. 2.4 and Prop. 2.6]). *For the conic degenerating cylinder $C(\ell)$ in Sect. 4.2, there is a constant c such that $\mathfrak{s}_2(C(\ell)) \geq c > 0$ for all $\ell > 0$.*

This result bears on the inclusion $L_1^2 \hookrightarrow L^4$. Recall that for a smooth function f the definition of the Sobolev constant implies (see [Li, Lemma 4]):

$$\|df\|_2^2 \geq \frac{\mathfrak{s}_1}{4}\left(\mathfrak{a}^{-1/2}\|f\|_4^2 - \mathfrak{a}^{-1}\|f\|_2^2\right)\ , \tag{4.2}$$

where $\mathfrak{a}$ denotes the area of X. For a conic degenerating family, $\mathfrak{s}_1 \to 0$ if and only if there is a separating pinching cylinder (see [JW1, Cor. 2.9]). However, by the proposition above, the Sobolev constants $\mathfrak{s}_2$ of the component regions, i.e. the degenerating cylinders and their complements, all remain bounded away from zero. Thus, applying a cut-off function to (4.2), one easily proves

Proposition 4.10. *For a conic degenerating family $(X, \sigma(\ell))$ there is a constant $c > 0$ independent of ℓ such that for all smooth functions f on X and all $\ell > 0$,*

$$\|f\|_{4;\sigma(\ell)}^2 \leq c\left(\|df\|_{2;\sigma(\ell)}^2 + \|f\|_{2;\sigma(\ell)}^2\right)\ .$$

4.3. Uniform bounds

If $(X, \sigma(\ell))$ is a conic degeneration, then a vector bundle $V \to X$ with connection A such that A has the standard form $b\otimes d\theta$, b constant diagonal, on the cylinder C, naturally defines a family of operators Δ_A on $(X, \sigma(\ell))$. By the comments at the end of Sect. 4.2, it is reasonable to ask whether the eigenvalues and eigenfunctions of Δ_A on $(X, \sigma(\ell))$ converge as $\ell \to 0$ to those of the limiting cone metric. This is what we call *spectral convergence*. The estimates from Thm.'s 4.1 and 4.2 are the key elements needed to prove spectral convergence for this degenerating family (cf. [JW1,JW2]). The first step is uniform C^0 bounds on eigenfunctions and uniform growth of eigenvalues:

Corollary 4.11. *If $(X, \sigma(\ell))$ is a conic degenerating family with vector bundle V and connection A, then there are constants C_1, C_2 independent of ℓ such that if $\varphi(\ell)$ is a normalized eigensection of V with eigenvalue $\lambda(\ell)$ then $\|\varphi(\ell)\|_\infty \leq C_1 + C_2\lambda^5(\ell)$ for all ℓ.*

Corollary 4.12. *If $(X, \sigma(\ell))$ is a conic degenerating family, then there is a constant C and an integer N, both independent of ℓ, such that if $\lambda_k(\ell)$ denotes the k-th eigenvalue for the closed problem for sections of V, then $\lambda_k(\ell) \geq Ck^{1/3}$ for all ℓ and all $k \geq N$.*

The arguments in [JW1,JW2] then apply to give:

Theorem 4.13. *For a conic degenerating family we have spectral convergence for sections of V.*

Applying this to the particular case of a unitary connection A on E and the eigenvalue λ_1 of Δ_A acting on sections of $(\mathrm{ad}\, E)_0$, we have:

Corollary 4.14. *Let A be an irreducible connection on X. Then for a conic degenerating family $(X, \sigma(\ell))$, $\lambda_1(\ell) \to 0$ if and only if A is accidentally reducible.*

Proof of Cor. 4.11. In the non-separating case, it follows from [JW1, Cor. 2.9 and Thm. 2.4] that the Sobolev constant $\mathfrak{s}_1$ is bounded away from zero. The result then follows from Part (1) of Thm. 4.1. In the separating case, following [JW2] we divide the degenerating surface into three regions $X^\pm$ and C with the induced metrics from $\sigma(\ell)$ (denoted also by $\sigma(\ell)$). Let $\widehat{C} \subset C$ be a fixed subcylinder and $\widehat{X}^\pm$ the corresponding complementary regions. By definition, the degenerating family $(\widehat{C}, \sigma(\ell))$ is isometric to the standard degenerating cylinder; therefore, by Prop. 4.9 it follows that the Sobolev constant $\mathfrak{s}_2$ for $(\widehat{C}, \sigma(\ell))$ is bounded away from zero. The same is true for $(\widehat{X}^\pm, \sigma(\ell))$, since these form a smooth family of metrics on the surfaces with boundary for $\ell \geq 0$.

Choose a smooth cut-off function η on X such that $\eta \equiv 1$ on $\widehat{C}$ and $\eta \equiv 0$ on $X^\pm$. Suppose $\varphi(\ell)$ is a normalized eigensection on $(X, \sigma(\ell))$ with eigenvalue $\lambda(\ell)$. We denote by $((1-\eta)\varphi(\ell))^\pm$ the restriction of $((1-\eta)\varphi(\ell))$ to $\widehat{X}^\pm$. It clearly suffices to find bounds on $((1-\eta)\varphi(\ell))^\pm$ and $\eta\varphi$ separately. Notice that $\Delta_A^m((1-\eta)\varphi(\ell))^\pm$ and $\Delta_A^m \eta\varphi$ have L^2 bounds depending on m and $\lambda(\ell)$, but otherwise independent of ℓ. More precisely, the bound may be taken of the form $C_1 + C_2\lambda^m(\ell)$. This is because $d\eta$ and $d(1-\eta)$ are supported in $\widehat{X}^\pm$ where the higher derivatives of φ may be uniformly bounded by an application of the elliptic estimate.

Let $\psi_k^\pm(\ell)$ and $\psi_k^C(\ell)$ denote normalized eigensections with eigenvalues $\mu_k^\pm(\ell)$ and $\mu_k^C(\ell)$ for the Dirichlet problems on $\widehat{X}^\pm$ and $\widehat{C}$, respectively. Consider the Fourier expansions:

$$((1-\eta)\varphi(\ell))^\pm = \sum_{k=1}^{\infty} a_k^\pm(\ell)\psi_k^\pm(\ell) \quad , \quad \eta\varphi(\ell) = \sum_{k=1}^{\infty} b_k(\ell)\psi_k^C(\ell) \ .$$

Since the Sobolev constants for the individual pieces are bounded away from zero, it follows from Thm. 4.1 that we have uniform bounds:

$$\|\psi_k^\pm(\ell)\|_\infty \leq C\mu_k^\pm(\ell) \quad , \qquad \|\psi_k^C(\ell)\|_\infty \leq C\mu_k^C(\ell) \ .$$

Here the constant C may be chosen independent of k and ℓ. Hence, it suffices to show that there is some B of the required form satisfying:

$$\sum_{k=1}^{\infty} \left|a_k^\pm(\ell)\right| \mu_k^\pm(\ell) \leq B \ , \tag{4.3}$$

$$\sum_{k=1}^{\infty} |b_k(\ell)| \, \mu_k^C(\ell) \leq B \ , \tag{4.4}$$

for all ℓ. Consider (4.4). By definition, we have:

$$b_k(\ell) = \int_C \langle \eta\varphi(\ell), \psi_k^C(\ell)\rangle d\sigma(\ell) = \frac{1}{(\mu_k^C(\ell))^m}\int_C \langle \eta\varphi(\ell), \Delta_A^m \psi_k^C(\ell)\rangle d\sigma(\ell)$$
$$= \frac{1}{(\mu_k^C(\ell))^m}\int_C \langle \Delta_A^m\left(\eta\varphi(\ell)\right), \psi_k^C(\ell)\rangle d\sigma(\ell)\ .$$

Since we assume uniform L^2 bounds on $\Delta_A^m\left(\eta\varphi(\ell)\right)$, we obtain a uniform bound

$$\mu_k^C(\ell)\,|b_k(\ell)| \leq \left(C_1 + C_2\lambda^m(\ell)\right)(\mu_k^C(\ell))^{1-m}\ ,$$

where the constants C_1, C_2 may be chosen independent of ℓ. By Part (2) of Thm. 4.2 above we have $\mu_k^C(\ell) \geq C' k^{1/3}$ for some constant C' independent of k and ℓ. Hence,

$$\mu_k^C(\ell)\,|b_k(\ell)| \leq \left(C_1' + C_2'\lambda^m(\ell)\right) k^{(1-m)/3}\ ,$$

where the constants C_1', C_2' may be chosen independent of k and ℓ. Since the sum over k of the terms on the right hand side converges for $m \geq 5$, the desired bound in (4.4) is obtained. The proof for (4.3) is similar. □

Proof of Cor. 4.12. In the non-separating case, it follows from [JW1, Cor. 2.9 and Thm. 2.4] that the Sobolev constant $\mathfrak{s}_1$ is bounded away from zero. The result then follows from Part (1) of Thm. 4.2. In the separating case, consider the three regions $X^\pm$, C as in the proof of Cor. 4.12 above. By domain monotonicity, it suffices to prove the result for the Neumann spectra for $(X^\pm, \sigma(\ell))$ and $(C, \sigma(\ell))$. Since $(X^\pm, \sigma(\ell))$ form a smooth family of metrics on the surfaces with boundary, it follows that the Sobolev constants $\mathfrak{s}_1$ for these regions are uniformly bounded away from zero; hence, by Part (1) of Thm. 4.2 the Neumann spectra of these surfaces has uniform growth as in the statement of the corollary. By Prop. 4.9, the Sobolev constant $\mathfrak{s}_2$ for $(C, \sigma(\ell))$ is also uniformly bounded away from zero. Hence, by Part (2) of Thm. 4.2 the Dirichlet spectra of these surfaces has uniform growth as in the statement of the corollary. On the other hand, because the connection is rotationally symmetric on C, the Neumann spectrum may be bounded below by the Dirichlet spectrum after shifting the index by two as in [W]. This completes the proof. □

4.4. Heat kernel estimates

It is well-known that eigenvalue and eigenfunction estimates produce estimates on solutions to the linear heat equation. This will also be useful in the non-linear Yang-Mills flow. The result we need is the following:

Theorem 4.15. *Let $(X, \sigma(\ell))$ is a conic degenerating family and fix $T > 0$. There is a constant C depending on T but independent of ℓ such that if $v(t,x)$ is the solution to the heat equation with initial conditions $v(0,x) = v_0$, then* $\sup_x |v(t,x)| \leq C\|v_0\|_{2;\sigma(\ell)}$ *for all $t \geq T$.*

Proof. The solution $v(t, x)$ may be written:

$$v(t,x)=\sum_{i=0}^{\infty} e^{-t\lambda_i} a_i \varphi_i(x)\ ,$$

where $|a_i| \le \|v_0\|_{2;\sigma(\ell)}$. By Cor. 4.11 for the ordinary Laplacian on functions, there are constants C_1 and C_2 independent of ℓ such that:

$$\sup_x |v(t,x)| \le \|v_0\|_{2;\sigma(\ell)} \sum_{i=0}^{\infty} e^{-t\lambda_i(\ell)} \left(C_1 + C_2\lambda_i^5(\ell)\right)\ .$$

Thus, we must show that for fixed $T_0 > 0$ there is a constant C depending only on T_0, C_1, and C_2 such that for all $t \ge T_0$ and all $\ell > 0$,

$$\sum_{i=0}^{\infty} e^{-t\lambda_i(\ell)} \left(C_1 + C_2\lambda_i^5(\ell)\right) \le \widetilde{C}\ .$$

First choose Λ_0 such that for all $\Lambda \ge \Lambda_0$ we have $\Lambda^6(C_1+C_2\Lambda_i^5)e^{-\Lambda T} \le 1$. Now by Cor. 4.12 we can find a constant C_3 independent of ℓ such that $\lambda_i(\ell) \ge C_3 i^{1/3}$, for i sufficiently large, say $i \ge N$, and all ℓ. We further prescribe N such that for $i \ge N$, $\lambda_i(\ell) \ge \Lambda_0$. Finally, let:

$$C_4 = \sup_{\Lambda>0}(C_1 + C_2\Lambda_i^5)e^{-\Lambda T}\ .$$

Then for $t \ge T$ and all ℓ,

$$\sum_{i=0}^{\infty} e^{-t\lambda_i(\ell)}(C_1 + C_2\Lambda_i^5) = \sum_{i=0}^{N-1} e^{-t\lambda_i(\ell)}(C_1 + C_2\Lambda_i^5) + \sum_{i=N}^{\infty} e^{-t\lambda_i(\ell)}(C_1 + C_2\Lambda_i^5)$$
$$\le NC_4 + \frac{1}{C_3^6}\sum_{i=N}^{\infty}\frac{1}{i^2}\ .$$

This proves the result. □

5. Proof of the main theorem

5.1. Outline of the proof

Let $[\sigma^*(\ell)]$ be a degeneration in $\mathcal{T}_{aug.}(g,1)$ to a nodal Riemann surface with conformal structure $[\sigma^*(0)]$ associated to a collection Φ of simple closed curves. Recall that $\mathcal{R}^{\Phi} \subset \mathcal{R}$ denotes the Φ-accidentally reducible representations. In this section we are going to show that for a given $[A] \in \mathcal{R}_\alpha \setminus \mathcal{R}^{\Phi}$, and $|\beta - \alpha| < \varepsilon_0$, where ε_0 is sufficiently small as in Sect. 2.3,

$$\lim_{\ell \to 0} \pi_{\alpha\beta}^{[\sigma^*(\ell)]}[A] = \pi_{\alpha\beta}^{[\sigma^*(0)]}[A] \ . \tag{5.1}$$

By standard compactness arguments, this will suffice to prove the Main Theorem. As representatives for the degenerating conformal structure we may choose lifts $\sigma^*(\ell) \to \sigma^*(0)$ to be a conic degeneration as in Sect. 4.2 with cone angle $0 < \kappa < 1$. The fact that κ may be chosen strictly less than 1 will be important (see (5.12)). Given $[A] \in \mathcal{R}_\alpha \setminus \mathcal{R}^{\Phi}$, let $g_{\alpha\beta}(A)$ denote the twists of A in the standard form around the point p. Let $A(\ell, t)$, $A(0, t)$ denote the Yang-Mills flow of $g_{\alpha\beta}(A) = A_0$ with respect to $\sigma^*(\ell)$, $\sigma^*(0)$, respectively. Since a conjugacy class of flat connections is completely determined by its holonomy around all homotopy classes of closed curves on X^*, our strategy of proof will be to first show that the holonomies of $A(\ell, \infty)$ converge for curves supported away from the pinching cylinders. This does not suffice, however. Indeed, this statement, combined with the results on Simpson's flow from Sect. 2, show only that the limiting holonomies of $A(\ell, \infty)$ around the pinching cylinders return to the initial holonomies of A_0 as $\ell \to 0$. This would still allow for the possibility of a change of framing, or gluing parameters, across the cylinder (cf. the discussion in the second to last paragraph of Sect. 2.3). So the second part of the proof is to show that the holonomies *across* the pinching cylinders, as measured with respect to the framing coming from A_0, are very nearly trivial. We present these two results as Thm.'s 5.1 and 5.2 below:

Theorem 5.1. *For any set $\{\Xi_j\}_{j=1}^N$ of closed curves supported in $\pi_1(X_0^*)$ we have*

$$\left\{\mathrm{hol}_{\Xi_j}\, A(\ell, \infty)\right\}_{j=1}^N \longrightarrow \left\{\mathrm{hol}_{\Xi_j}\, A(0, \infty)\right\}_{j=1}^N \ ,$$

modulo overall conjugation by $SU(2)$.

Next, recall from Thm. 2.13 that the manner by which a connection $[A(0, \infty)]$ produces a point in $\mathcal{R}_\beta$ is to use the initial framings. Consider a cylinder (C, ds_ℓ^2) in X^* on which the twisted initial connection A_0 is flat, and recall the coordinates

$$C = \{(x, y) : -1 \le x \le 1,\ 0 \le y \le 2\pi\} \big/ \{(x, 0) \sim (x, 2\pi)\}$$

from Sect. 4.2. We may choose a unitary frame $\{e_1, e_2\}$ such that A_0 has the form $d_{A_0} = d + \mathrm{diag}(i\gamma, -i\gamma)dy$. We fix this frame once and for all throughout the degeneration. Now for a small transverse arc $\Gamma_\varepsilon = \{(x, y) \in C : -\varepsilon \le x \le \varepsilon,\ y = y_0\}$ and flat connection $A(\ell, \infty)$, we measure the holonomy $\mathrm{hol}_{\Gamma_\varepsilon}(A(\ell, \infty))$ by parallel translating the frame $\{e_1, e_2\}$ along Γ_ε. For example, notice that by our choice of lift $[A(0, \infty)]$ in Thm. 2.13, $\mathrm{hol}_{\Gamma_\varepsilon}(A(0, \infty)) = \mathbb{I}$ for any choice of ε.

Since any closed curve on X^* may be written, up to homotopy, as a concatenation of curves of the form Ξ in Thm. 5.1 and transverse arcs Γ_ε^i, one for each component $c_i \in \Phi$, we see that (5.1) will follow from Thm. 5.1 and the following:

Theorem 5.2. *For any $\delta > 0$ there is $\varepsilon_0 > 0$ and ℓ_0 such that for all $\Gamma^i_{\varepsilon_0}$ and all $\ell \geq \ell_0$,*

$$\left|\mathrm{hol}_{\Gamma^i_{\varepsilon_0}}\left(A(\ell,\infty)\right) - \mathbb{I}\right| < \delta \ .$$

The proof of Thm. 5.1 will occupy the next two subsections, and the proof of Thm. 5.2 will be given in subsection 5.4. One of the key ingredients in the proofs is the C^0 estimate for the metric and the curvature found in Cor. 5.10 below.

5.2. Proof of Theorem 5.1

Throughout this section, Ω is an open set with compact closure in $X^* \setminus \Phi$. As before, we fix $[A] \in \mathcal{R}_\alpha$. Choose a representative A, and let $A_0 = g_{\alpha\beta}(A)$ be the twist of A at p. We assume that $\beta = k/n$, where k, n are positive coprime integers, and n is odd.

We denote by $A(\ell,t)$ the solution of the Yang-Mills flow (2.1)-(2.2), by $h(\ell,t)$ the solution to the non-linear heat equation (2.4)-(2.5) on $(X^*, \sigma^*(\ell))$, and by $h(0,t)$ the solution to the same equations on $X_0^* = X^* \setminus \Phi$ with the degenerate metric $\sigma^*(0)$, the holomorphic structure on the bundle being determined by A_0.

As a first step, we show that as we degenerate the metric on X^*, $h(\ell,t)$ converges to $h(0,t)$ uniformly on compact sets. In the following, p is any number strictly greater than 1. Also, $*_\ell$ and $*_0$ will denote the Hodge stars on X^* and $X^* \setminus \Phi$ with respect to the metrics $\sigma^*(\ell)$ and $\sigma^*(0)$, respectively.

Proposition 5.3. *Given $T > 0$, $\log h(\ell,T) \to \log h(0,T)$ weakly in $L^p_{2,loc.}$. In particular, the convergence is strong in $C^1(\Omega)$.*

Proof. In Sect. 5.3, we will obtain C^0 bounds for $h(\ell,t)$ and $*_\ell F_{A(\ell,t)}$ independent of ℓ (see Cor. 5.10). Assuming these results, since

$$h^{-1/2}(\ell,t) F_{A(\ell,t)} h^{1/2}(\ell,t) = F_{A_0} + \bar{\partial}_{A_0}\left(h^{-1}(\ell,t)\partial_{A_0} h(\ell,t)\right) \ , \tag{5.2}$$

standard elliptic estimates imply $L^p_{2,loc.}$ estimates for $h(\ell,t)$ uniform in ℓ and $0 \leq t \leq T$. By eq. (2.4), this implies an $L^p_{2,1,loc.}$ estimate on $X^* \times [0,T]$, uniform in ℓ, where the 1 refers to the time derivative. It follows that $h(\ell,t)$ converges to some $\tilde{h}(0,t)$ weakly in $L^p_{2,loc.}$. The uniform C^0 bounds imply that $\tilde{h}(0,t)$ and $*_0 F_{\bar{\partial}_{A_0},\tilde{h}(0,t)H_0}$ are also bounded uniformly for $t \in [0,T]$. The uniqueness part of Thm. 2.2 shows that $\tilde{h}(0,T) = h(0,T)$ as desired. □

Corollary 5.4. *Given $\varepsilon > 0$ there exists $T > 0$ and $\ell_0 = \ell_0(\varepsilon,T) > 0$ such that for $\ell \geq \ell_0$,*

$$\|\log h(\ell,T) - \log h(0,\infty)\|_{C^1(\Omega)} < \varepsilon \ ,$$

and similarly,

$$\|h^{1/2}(\ell,T)A_0 - h^{1/2}(0,\infty)A_0\|_{C^0(\Omega)} < \varepsilon \ .$$

Proof. By Thm. 2.2, $h(0,t) \to h(0,\infty)$ uniformly in $C^1(\Omega)$, so we can choose T_1 such that for all $t \geq T_1$,

$$\| \log h(0,t) - \log h(0,\infty) \|_{C^1(\Omega)} < \varepsilon/2 \ .$$

Take $T \geq T_1$. By Prop. 5.3, $\exists\, \ell_0 = \ell_0(\varepsilon, T)$ such that for all $\ell \geq \ell_0$,

$$\| \log h(\ell,t) - \log h(0,T) \|_{C^1(\Omega)} < \varepsilon/2 \ .$$

The result follows. □

Now let $A(\ell,t)$ be the solution of the Yang-Mills flow on X^* as before. Write

$$A(\ell,\infty) = \tilde{g}(\ell,t) A(\ell,t) \quad , \quad \tilde{h}(\ell,t) = \tilde{g}^*(\ell,t)\tilde{g}(\ell,t) \ . \tag{5.3}$$

Proposition 5.5. *Given $\varepsilon > 0$, there is a $T_0 > 0$ independent of ℓ such that for all $t \geq T_0$, $|\log \tilde{h}(\ell,t)|_{C^1(\Omega)} < \varepsilon$.*

We shall also prove this result in the following subsection. Here we show how Prop. 5.5 implies Thm. 5.1. Let $T = \max(T_0, T_1)$, where T_0 and T_1 are as in Prop. 5.5 and Prop. 5.3, respectively. Let ℓ_0 be chosen as in Cor. 5.4, and choose $\ell \geq \ell_0$. Notice that both $h^{1/2}(\ell,T)A_0$ and $\tilde{h}^{1/2}(\ell,T)A(\ell,\infty)$ are real gauge equivalent to $A(\ell,T)$. We therefore can write:

$$\tilde{h}(\ell,T)A(\ell,\infty) = k(\ell,T)h^{1/2}(\ell,T)A_0 \ ,$$

where $k(\ell,T)$ is a real gauge transformation. By Cor. 5.4:

$$\left\| \tilde{h}^{1/2}(\ell,T)A(\ell,\infty) - k(\ell,T)h^{1/2}(0,\infty)A_0 \right\|_{C^0(\Omega)} \leq c\,\varepsilon \ ,$$

for c depending only on the Sobolev embedding $L^p_1(\Omega) \hookrightarrow L^\infty(\Omega)$, and may be taken independent of ℓ. On the other hand, by Prop. 5.5:

$$\left\| \tilde{h}^{1/2}(\ell,T)A(\ell,\infty) - A(\ell,\infty) \right\|_{L^\infty(\Omega)} \leq \ varepsilon \ .$$

It follows that:

$$\left\| A(\ell,\infty) - k(\ell,T)h^{1/2}(0,\infty)A_0 \right\|_{L^\infty(\Omega)} \leq (c+1)\varepsilon \ .$$

Since $A(\ell,\infty)$ and $k(\ell,T)h^{1/2}(0,\infty)A_0$ are C^0-close, their holonomies around the Ξ_j are also close. Finally, since $k(\ell,T)h^{1/2}(0,\infty)A_0$ and $A(0,\infty)$ are real gauge equivalent, the theorem follows.

5.3. Proof of Proposition 5.5

We begin with some preliminary results:

Proposition 5.6. *Let* $[A] \in \mathcal{R}_\alpha$ *be as before. Then given* $\varepsilon_0 > 0$ *there is* $\delta > 0$ *such that for* $|\beta - \alpha| < \delta$ *there exists a twist* $A_0 = g_{\alpha\beta}(A)$ *and a flat connection* A_∞ *with the standard form of holonomy* β *around* p*, and such that* $\|A_0 - A_\infty\|_{4;\sigma(\ell)} < \varepsilon_0$.

Proof. Let C denote a component of the pinching region and A_0 the twist of A at $p \notin C$. Let h denote the flow at ∞ for eq.'s (2.4)-(2.5) associated to $\bar\partial_{A_0}$, the initial hermitian structure H_0, and the degenerate metric $\sigma(0)$ on X_0^*. Let $\widetilde{A}_\infty$ denote the hermitian connection associated to $\bar\partial_{A_0}$ and hH_0. We may assume that A_0 is in the standard form $d + \mathrm{diag}(i\gamma, -i\gamma)d\theta$ on a slightly larger cylinder $C_1 \supset \overline{C}$. Let C_2 be an open cylinder such that $\overline{C} \subset C_2 \subset \overline{C}_2 \subset C_1$. Since the initial curvature $\| * F_{A_0}\|_\infty$ can be made arbitrarily small for $|\beta - \alpha|$ small, and since A_0 is stable on X_0^*, it follows (cf. [Si1]) that $\|\log h\|_{C^1(V)}$, and hence also $\|A_0 - \widetilde{A}_\infty\|_{C^0(V)}$, can be made arbitrarily small for a relatively compact set $V \subseteq X_0^*$. By [DW1, Lemma 2.7] there is a real gauge transformation g such that $g(\widetilde{A}_\infty) = A_0$ on C_2. Moreover, it is clear from the proof of that lemma that by taking V so that $X \setminus V \subset C$ and using the fact that $\|A_0 - \widetilde{A}_\infty\|_{C^0(V)}$ is small, we may conclude that $\|\log g\|_{C^0(C_1 \setminus C_2)}$ is small. By bootstrapping we find that $\|\log g\|_{L_1^4(C_1 \setminus C_2)}$ is small; hence, we can extend g to a real gauge transformation of E over X^* with $g \equiv \mathbb{I}$ on $X \setminus C_1$ and $\|g - \mathbb{I}\|_{L_1^4(X \setminus C_2)}$ small. Set $A_\infty = g(\widetilde{A}_\infty)$. Then A_∞ extends to a connection over X^*, and the desired estimate holds. □

In order to get our estimate for the metric in X^*, we need again to pass to branched covers. Assume that $p \in X^* \setminus X$ is outside the pinching region, and let $q : \widehat{X} \to X$ be a regular cyclic branched cover of degree n, chosen so that all branch points lie outside the pinching region. We choose metrics $\hat\sigma(\ell)$ on $\widehat{X}$ so that $\hat\sigma(\ell) \to \hat\sigma(0)$ is a conic degeneration, and with respect to the induced conformal structures from $\hat\sigma(\ell)$ and $\sigma(\ell)$, the map q is holomorphic. Recall the map $\hat q : \mathcal{A}_{k/n} \to \widehat{\mathcal{A}} : A \mapsto \widehat{A}$.

Lemma 5.7. *Let* $A \in \mathcal{A}_{k/n}$ *be a flat connection which is lies outside the* Φ*-accidental reducibles. Then there is a* λ *such that* $\lambda_1\big(\Delta_{\widehat{A}}^{\hat\sigma(\ell)}\big) \geq \lambda > 0$ *for all* ℓ.

Proof. According to Cor. 4.14, $\lambda_1\big(\Delta_{\widehat{A}}^{\hat\sigma(\ell)}\big) \to 0$ if and only if $\widehat{A}$ is accidentally reducible on $\widehat{X}_0$. But this is ruled out by the assumptions and Prop. 2.6. □

The next step is to reduce to the case of a closed surface. First, notice that it suffices to get C^0 estimates for $\hat h(\ell, t)$ and $*_\ell F_{\widehat{A}(\ell,t)}$. Also, by Lemma 2.9, and shrinking Ω slightly to avoid branch points, it suffices to show:

(5.4) $$\left\|\log \widetilde{\widehat{h}}(\ell,t)\right\|_{C^1(\widehat{\Omega})} < \varepsilon\ .$$

Note also that by Prop. 5.6 we have a flat connection $\widehat{A}_\infty$ such that $\|\widehat{A}_\infty - \widehat{A}_0\|_4 < \varepsilon_0$, where ε_0 can be taken arbitrarily small. Furthermore, by Lemma 5.7 we may assume that $\lambda_1\big(\Delta_{\widehat{A}}^{\hat\sigma(\ell)}\big) \geq \lambda > 0$ uniformly in ℓ. It follows by the fundamental estimate [R, Prop. 7.2 – second method] that if ε_0 is less than a universal constant ε_1, depending only on λ, then

(5.5) $$\left\| D_{\widehat{A}_0}^{*_\ell} F_{\widehat{A}_0} \right\|_{2,\hat\sigma(\ell)} \geq c \left\| *_\ell F_{\widehat{A}_0} \right\|_{2,\hat\sigma(\ell)}\ .$$

Since from now on we will work on a closed surface $\widehat{X}$, we henceforth omit the hats from the notation. Again, $\sigma(\ell) \to \sigma(0)$ is a conic degeneration of a closed surface X. We begin with the following:

Proposition 5.8. *Let $A(\ell,t)$ be a solution of the Yang-Mills flow with initial condition* A_0,
$\| *_\ell F_{A_0}\|_\infty < B$, *and such that the fundamental estimate*

$$\left\| D_{A(\ell,t)}^{*_\ell} F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} \geq c\ \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)}\ ,$$

holds for a constant c independent of ℓ and for $0 \leq t < \widehat{T} \leq \infty$. Then there are constants c_1, c_2 depending on c and B, but independent of ℓ and $\widehat{T}$, such that for all $0 \leq t \leq \widehat{T}$:

(1) $\left\| *_\ell F_{A(\ell,t)} \right\|_\infty \leq c_1 e^{-ct/2}$*; and,*
(2) $\|h(\ell,t)\|_\infty \leq c_2$.

Proof. For (1), first note that from (2.3):

(5.6) $$\begin{aligned} \frac{d}{dt} \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} &= -\frac{1}{2} \left\| D_{A(\ell,t)}^{*_\ell} F_{A(\ell,t)} \right\|_{2;\sigma(\ell)}^2 \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)}^{-1} \\ &\leq -\frac{c}{2} \left\| D_{A(\ell,t)}^{*_\ell} F_{A(\ell,t)} \right\|_{2;\sigma(\ell)}\ , \end{aligned}$$

from which we obtain $\left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} \leq \tilde c_1 e^{-ct/2}$, where $\tilde c_1$ depends only on B and an upper bound for the total area of $\sigma(\ell)$. Let $u(t,x) = \left| *_\ell F_{A(\ell,t)} \right|(x)$ be the pointwise norm. By [Do2, Prop. 16] it follows that $\dot u + \Delta_\sigma u \leq 0$, where Δ_σ is the ordinary Laplacian with respect to the metric σ. By the maximum principle, $\sup_x u(t,x) \leq \sup_x u(0,x) \leq B$, and therefore there is a constant c_1 as in the statement of the proposition such that $u(t,x) \leq c_1 e^{-ct/2}$ for $0 \leq t \leq 1$. For $t > 1$, we apply Thm. 4.15 to find C independent of ℓ such that

$$\sup_x u(t,x) \leq C\|u(t-1,x)\|_{2;\sigma(\ell)} \leq C\tilde c_1 e^{-c(t-1)/2} = \left(C\tilde c_1 e^{c/2}\right) e^{-ct/2}$$

By adjusting c_1 if necessary, (1) follows.

For (2) consider the metric flow (2.4):

$$h^{-1}\frac{dh}{dt} = -\sqrt{-1} *_\ell F_{hH_0} \quad , \quad h(0) = \mathbb{I} \ .$$

Multiplying the equation through by h and taking traces, we find:

$$\frac{d}{dt} \log \operatorname{Tr} h \le \widetilde{C} \left\| *_\ell F_{h(t)H_0} \right\|_\infty \ ,$$

for some numerical constant $\widetilde{C}$. By (1), the right hand side is uniformly integrable on $0 \le t \le \widehat{T}$. Since $\det h(t) = 1$, the result follows. □

Proposition 5.9. *Let A_0 be as before and $\| *_\ell F_{A_0}\|_\infty < \varepsilon_2$. Then there exists a constant c independent of ℓ such that:*

$$\left\| D^{*_\ell}_{A(\ell,t)} F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} \ge c \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} \ , \tag{5.7}$$

for all $t \ge 0$, where $A(\ell, t)$ is the solution of the Yang-Mills flow (2.1) with initial condition $A(\ell, 0) = A_0$, and with respect to the metric $\sigma(\ell)$.

Proof. By (5.5), there is $c > 0$ such that:

$$\left\| D^{*_\ell}_{A(\ell,0)} F_{A(\ell,0)} \right\|_{2;\sigma(\ell)} \ge c \left\| *_\ell F_{A(\ell,0)} \right\|_{2;\sigma(\ell)} \ ,$$

uniformly in ℓ. Let $\mathcal{J} = \{t \in [0, \infty) :$ the estimate (5.7) holds on $[0, t]\}$. Then $\mathcal{J} \ne \emptyset$. Let $T = \sup \mathcal{J}$. We assume $T < \infty$, and derive a contradiction.

Claim 1. For the constant c appearing in (5.7), $\|A(\ell, t) - A(\ell, 0)\|_{2;\sigma(\ell)} < c^{-1}\varepsilon_2$.

By (5.6),

$$\frac{d}{dt} \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} \le -c \left\| D^{*_\ell}_{A(\ell,t)} F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} = -c \left\| \frac{\partial A}{\partial t}(\ell, t) \right\|_{2;\sigma(\ell)} \ .$$

Hence, by integrating,

$$\begin{aligned} - \left\| *_\ell F_{A(\ell,0)} \right\|_{2;\sigma(\ell)} &\le \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} - \left\| *_\ell F_{A(\ell,0)} \right\|_{2;\sigma(\ell)} \\ &\le -c \int_0^t \left\| \frac{\partial A}{\partial t}(\ell, s) \right\|_{2;\sigma(\ell)} ds \le -c \left\| \int_0^t \frac{\partial A}{\partial t}(\ell, s) ds \right\|_{2;\sigma(\ell)} \\ &= -c \left\| A(\ell, t) - A(\ell, 0) \right\|_{2;\sigma(\ell)} \ . \end{aligned}$$

It follows that

$$\|A(\ell, t) - A(\ell, 0)\|_{2;\sigma(\ell)} \le c^{-1} \left\| *_\ell F_{A(\ell,0)} \right\|_{2;\sigma(\ell)} < c^{-1}\varepsilon_2 \ ,$$

and hence the claim.

Claim 2. Write $A(\ell, T) = g(\ell)A_0$, $h(\ell) = g(\ell)g^*(\ell)$. Then $\|g(\ell)\|_\infty$ and $\|g^{-1}(\ell)\|_\infty$ are uniformly bounded independent of ℓ.

This is immediate from Prop. 5.8.

Claim 3. There exists a real gauge transformation $u(\ell)$ such that

$$\|u(\ell)A(\ell, T) - A_0\|_{L^2_1(A_0);\sigma(\ell)} < c\,\varepsilon_2\ ,$$

where c is independent of ℓ.

Write $A(\ell, T) = g(\ell)A_0$ as before. By Claim 1, $\|g^{-1}(\ell)\bar\partial_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq c\,\varepsilon_2$, and by Claim 2, $\|\bar\partial_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq c\varepsilon_2$ (here and for the remainder of the proof, c will denote a generic constant independent of ℓ but possibly depending upon T; since $T = \sup\mathcal{J}$ is fixed throughout the argument, we omit the explicit dependence of the constants on T). By elliptic regularity and the fact that $\lambda_1\left(\Delta^{\sigma(\ell)}_{A_0}\right) \geq \lambda > 0$ (see Prop. 2.6) it follows that

$$\begin{aligned}\|\partial_{A_0}g(\ell)\|^2_{2;\sigma(\ell)} &\leq \|d_{A_0}g(\ell)\|^2_{2;\sigma(\ell)} \leq \|g(\ell)^\perp\|^2_{L^2_1(A_0);\sigma(\ell)}\\ &\leq \lambda^{-1}\|\bar\partial_{A_0}g(\ell)^\perp\|^2_{2;\sigma(\ell)} = \lambda^{-1}\|\bar\partial_{A_0}g(\ell)\|^2_{2;\sigma(\ell)} \leq c\,\varepsilon_2\ ,\end{aligned}$$

where $g(\ell)^\perp$ is the L^2-orthogonal projection to the perp space of $\ker d_{A_0}$. This result, combined with Claim 2, implies that $\|h^{-1}(\ell)\partial_{A_0}h(\ell)\|_{2;\sigma(\ell)} \leq c\,\varepsilon_2$. On the other hand,

$$g^{-1}(\ell)F_{A(\ell,t)}g(\ell) = F_{A_0} + \bar\partial_{A_0}\left(h^{-1}(\ell)\partial_{A_0}h(\ell)\right)\ .$$

Hence, by again applying elliptic regularity, $\|h^{-1}(\ell)\bar\partial_{A_0}h(\ell)\|_{L^2_1(A_0);\sigma(\ell)} \leq c\,\varepsilon_2$. Since the connection defined from $\bar\partial_{A_0}$ and $h(\ell)H_0$ is real gauge equivalent to $A(\ell, T)$, the proof of Claim 3 is complete.

For notational simplicity, set $\underset{\sim}{A}(\ell, 0) = u(\ell)A(\ell, T)$. Then:

$$\left\|\underset{\sim}{A}(\ell, 0) - A_0\right\|_{L^2_1(A_0);\sigma(\ell)} < c\,\varepsilon_2\ . \tag{5.8}$$

Claim 4. Let $\underset{\sim}{A}(\ell, t)$ denote the Yang-Mills flow with initial condition $\underset{\sim}{A}(\ell, 0)$ and with respect to the metric $\sigma(\ell)$. Then there is $\delta > 0$ and a real gauge transformation $v(\ell, t)$ such that:

$$\left\|v(\ell, t)\underset{\sim}{A}(\ell, t) - A_0\right\|_{4;\sigma(\ell)} < \varepsilon_1/2\ ,$$

for $0 < t < \delta$ and all ℓ. Here, ε_1 is the universal constant so that Råde's estimate (2.6) holds.

The proof of Claim 4 will be accomplished in three stages:
(i) Given $\eta > 0$, $\exists\, \delta_1 > 0$ such that for $0 \le t \le \delta_1$ and all ℓ, $\|\log \underset{\sim}{h}(\ell,t)\|_\infty < \eta$, where $\underset{\sim}{h}(\ell,t)$ is the solution of (2.4)-(2.5) with respect to the holomorphic structure defined by $\bar\partial_{\underset{\sim}{A}(\ell,0)}$. This follows by Prop. 5.8, as in Claim 2.
(ii) Given $\eta > 0$, $\exists\, \delta_2 > 0$ such that for $0 \le t \le \delta_2$ and all ℓ, $\|\underset{\sim}{A}(\ell,t) - \underset{\sim}{A}(\ell,0)\|_{2;\sigma(\ell)} < \eta$. This obtains from the following inequalities:

$$\begin{aligned}
\|\underset{\sim}{A}(\ell,t) - \underset{\sim}{A}(\ell,0)\|_{2;\sigma(\ell)} &= \left\| \int_0^t \frac{d}{dt}\, \underset{\sim}{A}(\ell,t)dt \right\|_{2;\sigma(\ell)} \\
&\le \int_0^t \left\| \frac{d}{dt}\, \underset{\sim}{A}(\ell,t) \right\| dt = \int_0^t \left\| D^{*\ell}_{\underset{\sim}{A}(\ell,t)} F_{\underset{\sim}{A}(\ell,t)} \right\|_{2;\sigma(\ell)} dt \\
&\le t^{1/2} \left(\int_0^t \left\| D^{*\ell}_{\underset{\sim}{A}(\ell,t)} F_{\underset{\sim}{A}(\ell,t)} \right\|^2_{2;\sigma(\ell)} dt \right)^{1/2} = t^{1/2} \\
&\quad \left(\left\| *_\ell F_{\underset{\sim}{A}(\ell,0)} \right\|^2_{2;\sigma(\ell)} - \left\| *_\ell F_{\underset{\sim}{A}(\ell,t)} \right\|^2_{2;\sigma(\ell)} \right) \le c\, t^{1/2} \varepsilon\ .
\end{aligned}$$

(iii) Let $\underset{\sim}{A}(\ell,t) = \underset{\sim}{g}(\ell,t)\underset{\sim}{A}(\ell,0)$, $\underset{\sim}{h}(\ell,t) = \underset{\sim}{g}^*(\ell,t)\underset{\sim}{g}(\ell,t)$, and $\delta = \min(\delta_1,\delta_2)$. As in Claim 3, we first obtain $\|\bar\partial_{\underset{\sim}{A}(\ell,0)}\underset{\sim}{g}(\ell,t)\|_{2;\sigma(\ell)} \le c\,\eta$ for $0 \le t \le \delta$. On the other hand, by choosing ε_2 sufficiently small, it follows by (5.8) that we may assume that $\lambda_1\big(\Delta^{\sigma(\ell)}_{\underset{\sim}{A}(\ell,0)}\big) \ge \lambda/2 > 0$. As in Claim 3, we obtain for $0 \le t \le \delta$,

$$\|\underset{\sim}{h}^{-1}(\ell,t)\bar\partial_{\underset{\sim}{A}(\ell,0)}\underset{\sim}{h}(\ell,t)\|_{L^2_1\left(\underset{\sim}{A}(\ell,0)\right);\sigma(\ell)} \le c(\varepsilon_2 + \eta)\ .$$

By taking ε_2 and η sufficiently small with respect to ε_1, and applying this result together with Kato's inequality and the uniform embedding $L^2_1 \hookrightarrow L^4$ for functions (see Prop. 4.10), we obtain Claim 4.

Now we are ready to complete the proof of the proposition. By Claim 4, and by taking ε_0 in Prop. 5.6 sufficiently small, it follows that $\|v(\ell,t)\underset{\sim}{A}(\ell,t) - A_\infty\|_{4;\sigma(\ell)} < \varepsilon_1$; hence,

$$\begin{aligned}
\left\| D^{*\ell}_{\underset{\sim}{A}(\ell,t)} F_{\underset{\sim}{A}(\ell,t)} \right\|_{2;\sigma(\ell)} &= \left\| D^{*\ell}_{v(\ell,t)\underset{\sim}{A}(\ell,t)} F_{v(\ell,t)\underset{\sim}{A}(\ell,t)} \right\|_{2;\sigma(\ell)} \ge c \left\| *_\ell F_{v(\ell,t)\underset{\sim}{A}(\ell,t)} \right\|_{2;\sigma(\ell)} \\
&= c \left\| *_\ell F_{\underset{\sim}{A}(\ell,t)} \right\|_{2;\sigma(\ell)} \qquad (0 \le t \le \delta)
\end{aligned}$$

where c is the same constant as before (cf. [R, Proof of Prop. 7.2, second method]) depending on $\lambda_1(\Delta_{A_\infty})$ and Sobolev constants coming from the embedding

$L_1^2 \hookrightarrow L^4$ (see Prop. 4.10). It follows again by the real gauge equivalence of the heat flow and the invariance of the L^2 norm that

$$\left\| D^{*_\ell}_{A(\ell,t)} F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} \geq c \left\| *_\ell F_{A(\ell,t)} \right\|_{2;\sigma(\ell)} ,$$

for $0 \leq t \leq T + \delta$, contradicting the assumption that $T = \sup \mathcal{J}$. □

The following corollary is an immediate consequence of Prop.'s 5.8 and 5.9, and completes the proof of the C^0 estimate:

Corollary 5.10. *Let A_0 be as in Prop. 5.9. Then there are constants c_1, c_2, and c_3 independent of ℓ and t such that*

(1) $\left\| *_\ell F_{A(\ell,t)} \right\|_\infty \leq c_1 e^{-c_2 t}$ *for* $0 \leq t \leq \widehat{T}$;
(2) $\|h(\ell, t)\|_\infty \leq c_3$.

We now proceed with the proof of (5.4):

Corollary 5.11. *Let $\Omega \subset X$ be as before. Write $A(\ell, t) = \tilde{g}(\ell, t) A(\ell, \infty)$, $\tilde{h}(\ell, t) = \tilde{g}^*(\ell, t)\tilde{g}(\ell, t)$. Given $\varepsilon > 0$, $\exists\, T_0 > 0$ independent of ℓ such that for all $t \geq T_0$, $\| \log \tilde{h}(\ell, t)\|_{C^1(\Omega)} < \varepsilon$.*

Proof. By Prop.'s 5.9 and 5.8, we have $\left\| *_\ell F_{A(\ell,t)} \right\|_\infty \to 0$ as $t \to \infty$ uniformly in ℓ, and:

$$\int_0^\infty \left\| *_\ell F_{A(\ell,t)} \right\|_\infty dt < \infty ,$$

uniformly in ℓ. From this we deduce as in Claims 1 and 2 that:

$$\lim_{t\to\infty} \|A(\ell, t) - A(\ell, \infty)\|_{2;\sigma(\ell)} = 0 ,$$

and that $\|\tilde{g}(\ell, t)\|_{2;\sigma(\ell)}$ and $\|\tilde{g}^{-1}(\ell, t)\|_{2;\sigma(\ell)}$ are bounded, all uniformly in ℓ. Hence,

$$\|\bar{\partial}_{A(\ell,\infty)} \tilde{g}(\ell, t)\|_{2;\sigma(\ell)} ,$$

can be made arbitrarily small independent of ℓ. Since $\lambda_1\big(\Delta^{\sigma(\ell)}_{A(\ell,\infty)}\big) \geq \lambda > 0$ uniformly in ℓ, it follows that $\|g(\ell, t)\|_{L^2_{1;\sigma(\ell)}}$ can be made arbitrarily small uniformly in ℓ. Finally, by using the curvature estimate:

$$\tilde{g}^{-1}(\ell, t) F_{A(\ell,t)} \tilde{g}(\ell, t) = \bar{\partial}_{A(\ell,\infty)} \left(\tilde{h}^{-1}(\ell, t) \partial_{A(\ell,\infty)} \tilde{h}(\ell, t) \right) ,$$

we obtain the corollary by bootstrapping in Ω. □

This proves (5.4), and thus completes the proof of Prop. 5.5.

5.4. Proof of Theorem 5.2

Let $h(\ell, \infty)$ denote the limit at infinite time of the solution to eq.'s (2.4)-(2.5). Then up to real gauge, $A(\ell, \infty)$ is $h^{1/2}(\ell, \infty) \cdot A_0$. It suffices to obtain an estimate of the form:

$$\int_{\sqrt{\varepsilon}(\ell)}^{\varepsilon_0} \left| \iota_{\partial/\partial r} \left\{ h^{-1/2}(\ell, \infty) \partial_{A_0} h^{1/2}(\ell, \infty) - c.c. \right\} \right| dr < \delta , \tag{5.9}$$

for sufficiently small ε_0 and ℓ. In the above, the coordinate r is $|z|$ for the conformal coordinates on (C, ds_ℓ^2) from Prop. 4.8, and $\iota_{\partial/\partial r}$ denotes contraction of the form in the r direction. The notation $c.c.$ means hermitian conjugation of the previous term. Also, note that we have taken the square root $\sqrt{\varepsilon}$, since with respect to the conformal coordinates in Sect. 4.2 this corresponds to a point in the middle of the cylinder, i.e. $x = 0$. The estimate for the whole arc $\Gamma_{\varepsilon_0}^i$ follows from the estimates in the z and w coordinates separately.

The square root of the gauge transformation h is difficult to work with directly, so we will eliminate it by using a C^0 bound. Namely, to prove (5.9), is suffices to prove:

(1) There is B independent of ℓ such that $\|h(\ell, \infty)\| \leq B$ for all ℓ;
(2) $\displaystyle\int_{\sqrt{\varepsilon}(\ell)}^{\varepsilon_0} \left| \iota_{\partial/\partial r} \left\{ h^{-1} \partial_{A_0} h - c.c. \right\} \right| dr < \delta$ for ε_0 sufficiently small and $\sqrt{\varepsilon}(\ell) \leq \varepsilon_0$.

We further reduce this with the following:

Proposition 5.12. *Suppose that there is a constant B such that (1) holds. Suppose in addition that $\|h^{-1}\partial_{A_0} h - c.c.\|_{2;\sigma(\ell)} \leq B$ for all ℓ. Then (2) also holds.*

Proof. Set $u = h^{-1}\partial_{A_0} h - c.c.$. The first step of the proof is to show that the hypotheses imply an estimate of the form:

$$\|u\|_{4;\sigma(\ell)}^4 \leq B/\ell^2 , \tag{5.10}$$

where B is independent of ℓ. Consider the function $|u|$. By Prop. 4.10, it suffices to estimate $d|u|$. By Kato's inequality, $|d|u||^2 \leq \left|\nabla_{A_0} u\right|^2$. So it suffices to estimate the right hand side. We now apply the Weitzenböck formula for a 1-form u with values in the self-adjoint bundle:

$$\Delta_{A_0} u = -\nabla_{A_0}^* \nabla_{A_0} u + \{R, u\} + \left\{F_{A_0}, u\right\} .$$

Now F_{A_0} is uniformly bounded in any norm, say C^1, and we have $\Delta_{A_0} h^{-1}\partial_{A_0} h = -\bar{\partial}_{A_0}^* F_{A_0}$; so $\Delta_{A_0} u$ is bounded as well. Therefore, the desired estimate is obtained, provided we can estimate the term $\{R, u\}$. The explicit formula can found in [Wu], and it involves the operator ${R_k}^i = \sigma^{lj} {R_{klj}}^i$ (see [Wu, p. 953]). Then (5.10) follows from Prop. 4.7.

Now we consider the problem on a cylinder (C, ds^2_ℓ). Choose a local holomorphic frame $\{f_i\}$ for A_0 adapted to $\{e_i\}$ as in Sect. 2.1. In conformal coordinates, we may write

$$h^{-1}\partial_{A_0} h = \sum_{n\in\mathbb{Z}} c_n^{ij} z^n f_i \otimes f_j^* \otimes dz ,$$

since $\bar\partial_{A_0}\left(h^{-1}\partial_{A_0}h\right) = 0$ (see (5.2)). The coefficients c_n^{ij} depend on ℓ, and we wish to estimate them. For convenience, we set $|c_{-1}|^2 = |c_{-1}^{11}|^2 + |c_{-1}^{22}|^2$. We assume the frame has been chosen such that $|f_i \otimes f_j^*| = |z|^{\gamma_i-\gamma_j}$, where $\gamma_1 = \gamma$, $\gamma_2 = -\gamma$. We will assume that $\gamma \neq 0$, the argument in the case $\gamma = 0$ being similar. Also, since we assume that $\sigma(\ell)$ converges to a cone metric of the type (4.1), the integrals below will be carried out with respect to this metric. A simple computation shows that these estimates are valid. We have:

$$\left|h^{-1}\partial_{A_0}h\right|^2 = \sum_{m,n\in\mathbb{Z}} c_m^{ij}\bar c_n^{ij} z^m \bar z^n |z|^{\gamma_i-\gamma_j}|dz|^2 .$$

The L^2 bound on w implies one on $|h^{-1}\partial_{A_0}h|$, which in turn implies that there is a constant B independent of ℓ such that:

$$|c_m^{ij}|^2 \int_{\varepsilon(\ell)\le|z|\le 1} |z|^{2m+\gamma_i-\gamma_j}|dz|^2 \le B .$$

Therefore, there is another constant independent of ℓ, which we also denote by B, such that:

(i) For $m \ge 0$, $|c_m^{ij}| \le B$. Also, $|c_{-1}^{12}| \le B$;
(ii) For $m \le -2$, or $(m, i, j) = (-1, 2, 1)$, $|c_m^{ij}| \le B\varepsilon(t)^{-m-\frac{1}{2}(\gamma_i-\gamma_j)-1}$;
(iii) $|c_{-1}|^2 \log(1/\varepsilon(\ell)) \le B$.

We now apply this to:

$$\int_{\varepsilon^{1/2}}^{\varepsilon_0} \left|\iota_{\partial/\partial r}\left\{h^{-1}\partial_{A_0}h\right\}\right| dr \le \sum_{(m,i,j)\neq(-1,i,i)} |c_m^{ij}| \int_{\varepsilon^{1/2}}^{\varepsilon_0} r^{m+\frac{1}{2}(\gamma_i-\gamma_j)} dr + 2|c_{-1}|\log(1/\varepsilon) .$$

By the estimates (i) and (ii) above, the first term on the right hand side may be made arbitarily small for small ε_0, independent of ℓ. To estimate the second term, we use the L^4 bound (5.10). Because of the log term, it suffices to show that $|c_{-1}|$ vanishes as some power of ε. The estimate (iii) is not sufficient.

To use the L^4 bound, we first isolate the $m = -1$ terms. Again, (5.10) implies a similar bound on the L^4 norm of $|h^{-1}\partial_{A_0}h|$. Write:

$$|h^{-1}\partial_{A_0}h|^2 = \frac{|c_{-1}|^2 + |c_{-1}^{12}||z|^{2\gamma} + |c_{-1}^{21}||z|^{-2\gamma}}{|z|^2} + g ,$$

Integrating over a subcylinder C_a, we have:

(5.11)

$$\int_{C_a} \frac{|c_{-1}|^4}{|z|^4} ds_0^2 \le B \int_{C_a} \left(|h^{-1}\partial_{A_0} h|^4 + g^2 + \frac{|c_{-1}^{12}|^4 |z|^{4\gamma}}{|z|^4} + \frac{|c_{-1}^{21}|^4 |z|^{-4\gamma}}{|z|^4} \right) ds_0^2 \; .$$

The subcylinder is given by $C_a = \{z \in C : \varepsilon^a \le |z| \le \varepsilon_0\}$, where a will be chosen as follows:

Claim. For:

$$\frac{1+\kappa}{3-\kappa} > a > \frac{\kappa}{2-\kappa} \; , \tag{5.12}$$

there is a constant B independent of ℓ so that:

$$\int_{C_a} g^2 ds_0^2 \le \frac{B}{\varepsilon^{2\kappa}} \; .$$

Assuming this claim, we complete the proof. By (5.10) and Prop. 4.8 applied to (5.11), we have:

$$\int_{C_a} \frac{|c_{-1}|^4}{|z|^4} ds_0^2 \le \frac{B}{\varepsilon^{2\kappa}} + \int_{C_a} \left(\frac{|c_{-1}^{12}|^4 |z|^{4\gamma}}{|z|^4} + \frac{|c_{-1}^{21}|^4 |z|^{-4\gamma}}{|z|^4} \right) ds_0^2 \; .$$

Carrying out the integrals on both sides, this implies $|c_{-1}|^4 \le B \left(\varepsilon^{(4-2\kappa)a-2\kappa} + \varepsilon^{4\gamma(1-a)}\right)$. By the choice of a in (5.12), $(4-2\kappa)a - 2\kappa > 0$, so we are finished.

It remains to prove the claim. First, by the expression for $|h^{-1}\partial_{A_0} h|^2$, note that the terms in the series for g involving the coefficients c_m and c_n are bounded uniformly by $\varepsilon_0^{(|m|+|n|)b}$ for some $b > 0$ and $|m|$, $|n|$ large, depending upon a. Therefore, to bound the integral of g^2, it suffices to bound the squares of the individual terms. Of the terms which appear, there are three types which need to be estimated:

I. $|c_m^{ij}|^2 |c_n^{ij}|^2 |z|^{2(m+n)+2(\gamma_i-\gamma_j)}$, where $m, n \le -2$;
II. $|c_{-1}^{21}|^2 |c_n^{21}|^2 |z|^{2(n-1)-2\gamma}$ and $|c_{-1}^{12}|^2 |c_n^{12}|^2 |z|^{2(n-1)+2\gamma}$, where $n \le -2$;
III. $|c_{-1}|^2 |c_n^{ii}|^2 |z|^{2(n-1)}$, where $n \le -2$.

Terms of type I give integrals of the form (using (i) above):

$$\begin{aligned} \int_{\varepsilon^a}^{\varepsilon_0} |c_m^{ij}|^2 |c_n^{ij}|^2 r^{2(m+n)+2(\gamma_i-\gamma_j)-1+2\kappa} dr &\le B |c_m^{ij}|^2 |c_n^{ij}|^2 \varepsilon^{(2(m+n)+2(\gamma_i-\gamma_j)+2\kappa)a} \\ &\le B\varepsilon^{-2(m+n)(1-a)-2(\gamma_i-\gamma_j)(1-a)-2(2-\kappa a)} \; . \end{aligned}$$

But:

$$
\begin{aligned}
&-2(m+n)(1-a)-2(\gamma_i-\gamma_j)(1-a)-2(2-\kappa a)\\
&\geq -2(m+n)(1-a)-2(1-a)-2(2-\kappa a)\\
&\geq -2(m+n+3)(1-a)\ ,
\end{aligned}
$$

so these terms are, in fact, bounded. For type III, we apply the same estimate to get:

$$
\int_{\varepsilon^a}^{\varepsilon_0} |c_{-1}|^2|c_n^{ii}|^2 r^{2(n-1)-1+2\kappa}dr \leq B|c_{-1}|^2\varepsilon^{-2(n-1)(1-a)-2(2-\kappa a)}\ .
$$

Since $n\leq -2$, the exponent of ε is $\geq -2\kappa$ by the assumption on a in (5.12). Type II is similar to these two computations. □

Since the hypotheses of Prop. 5.12 are satisfied by Cor. 5.10 (2), and the uniform L^2 bound on $|h^{-1/2}\partial_{A_0}h^{1/2}|$, this completes the proof of (5.1) and also of the Main Theorem.

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