COMPACTNESS FOR Ω-YANG-MILLS CONNECTIONS

XUEMIAO CHEN AND RICHARD A. WENTWORTH

Abstract. On a Riemannian manifold of dimension n we extend the known analytic results on Yang-Mills connections to the class of connections called Ω-Yang-Mills connections, where Ω is a smooth, not necessarily closed, $(n-4)$-form on M. Special cases include Ω-anti-self-dual connections and Hermitian-Yang-Mills connections over general complex manifolds. By a key observation, a weak compactness result is obtained for moduli space of smooth Ω-Yang-Mills connections with uniformly L^2 bounded curvature, and it can be improved in the case of Hermitian-Yang-Mills connections over general complex manifolds. A removable singularity theorem for singular Ω-Yang-Mills connections on a trivial bundle with small energy concentration is also proven. As an application, it is shown how to compactify the moduli space of smooth Hermitian-Yang-Mills connections on unitary bundles over a class of balanced manifolds of Hodge-Riemann type. This class includes the metrics coming from multipolarizations, and in particular, the Kähler metrics. In the case of multipolarizations on a projective algebraic manifold, the compactification of smooth irreducible Hermitian-Yang-Mills connections with fixed determinant modulo gauge transformations inherits a complex structure from algebro-geometric considerations.

Contents

2020 *Mathematics Subject Classification.* Primary: 53C07, 58E15; Secondary: 14D20 .
Key words and phrases. Yang-Mills connections, Gauduchon metric, Uhlenbeck compactness, moduli space.
R.W.'s research is supported in part by NSF grant DMS-1906403.

1. Introduction

1.1. Ω-Yang-Mills equations. Let (M, g) be an oriented Riemannian manifold of dimension $n \geq 4$, Ω a smooth $(n-4)$-form on M, and $E \to M$ a vector bundle with a Riemannian metric[1]. The Ω-*Yang-Mills equations* for a metric connection A on E with curvature F_A are

$$d_A^* \left(F_A + *(F_A \wedge \Omega)\right) = 0 \ , \tag{1.1}$$

and a solution A to (1.1) will be called an Ω-*Yang-Mills connection* (or Ω-YM connection, for short). This equation is the Euler-Lagrange equation of the functional

$$\mathrm{YM}_\Omega(A) = \int_M |F_A|^2 \, dV - \int_M \mathrm{tr}(F_A \wedge F_A) \wedge \Omega \tag{1.2}$$

which may be viewed as a gauge invariant function on the infinite dimensional space of metric connections on E. The first term in (1.2) is the usual Yang-Mills functional $\mathrm{YM}(A)$. If we assume Ω is closed, then the second term in (1.2) is topological for compact M (or with respect to compactly supported variations), and so the critical points of YM_Ω are identical to those of YM, i.e. the Yang-Mills connections. Indeed, (1.1) reduces to $d_A^* F_A = 0$ in this case. The main goal of this paper is to extend the analysis of Yang-Mills connections to the more general solutions of (1.1) for the case where Ω is not closed and Ω-YM connections are not necessarily Yang-Mills.

To provide some motivation, let us note an interesting special case. We define the Ω-*ASD connections* to be the solutions to (1.1) of the form

$$* F_A + F_A \wedge \Omega = 0 \tag{1.3}$$

If $n = 4$, $\Omega = 1$, then connections satisfying (1.3) are the much studied anti-self-dual *instantons* (cf. [9, 6]). Higher dimensional instanton equations of the type (1.3) have been considered in a variety of contexts, and their formulation goes back to [4]. In the mathematics literature, we refer to [8, 22, 7], to list only a few of many recent papers. We again point out that an Ω-ASD connection is not necessarily Yang-Mills unless Ω is closed.

If we assume the comass $|\Omega| \leq 1$, then $\mathrm{YM}_\Omega(A) \geq 0$, and we say A is an absolute minimizer if $\mathrm{YM}_\Omega(A) = 0$. We have the following simple lemma.

Lemma 1.1. *Suppose $|\Omega| \leq 1$. Then a connection A is an absolute minimizer of YM_Ω if and only if it is an Ω-ASD connection.*

Now let us suppose that M is an m-dimensional hermitian manifold, $2m = n$, with Kähler form ω (not necessarily closed). If the connection A is integrable (i.e. F_A is of type $(1,1)$), then

$$\mathrm{YM}_\Omega(A) = \int_M |\Lambda F_A|^2 \, dV$$

[1]In this paper, if (M, g) is a hermitian complex manifold we assume bundles are also complex Hermitian; otherwise, E can be real or complex.

where $i\Lambda F_A$ is the Hermitian-Einstein tensor, and $\Omega = \omega^{m-2}/(m-2)!$. It follows that in this case the Ω-ASD connections are exactly the Hermitian-Yang-Mills (HYM) connections with $i\Lambda F_A = 0$. In case ω is a *Gauduchon* metric, then nontrivial solutions arise from *stable* holomorphic vector bundles on M (see [13])[2]. Even when M is a projective algebraic manifold, many interesting examples of solutions can be obtained from holomorphic bundles that are stable with respect to *multipolarizations* [16, 11]. For example, if $\omega_1, \dots, \omega_{m-1}$ are Kähler forms on M, then solutions to the equations

$$F_A \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0 \tag{1.4}$$

exist for holomorphic bundles that are stable with respect to $\omega_1, \dots, \omega_{m-1}$. On the other hand, $\omega_1 \wedge \cdots \wedge \omega_{m-1}$ determines a balanced hermitian metric ω, in general not Kähler, and solutions to (1.4) are Ω-ASD for $\Omega = \omega^{m-2}/(m-2)!$. Note once more that these are not, in general, Yang-Mills, even though the ω_i are Kähler forms. Multipolarizations are also considered in more detail in [3]. Another motivation is to hopefully give new nontrivial ways to deform the moduli space of Yang-Mills connections, which fits into the higher dimensional gauge theoretic picture described in [7, 8]. As indicated by the multipolarization case, the moduli space of HYM connections can be deformed nontrivially by moving the metric on the base complex manifold while at the same time giving a uniform L^2 bound on the curvature for all the connections. In general, we know the Kähler condition is often too rigid to deform nontrivially. In a sense, the results obtained here enrich the picture over complex manifolds by providing new structures to consider as well as examples arising from algebraic geometry.

1.2. **Main results.** In this paper, we always assume that (M, g) has bounded geometry in the sense that (M, g) can be isometrically embedded in a larger Riemannian manifold so that M has compact closure. In Section 2, we will prove a monotonicity formula and an ϵ-regularity result for Ω-YM connections. As a consequence, we obtain the following version of Uhlenbeck's weak compactness theorem (cf. [17, 24]).

Theorem 1.2. *Let $\{A_i\}$ be a sequence of smooth Ω-YM connections with $\|F_{A_i}\|_{L^2}$ uniformly bounded. Define the set Σ by*

$$\Sigma = \{x \in M : \lim_{r \to 0^+} \liminf_{i \to \infty} r^{4-n} \int_{B_r(x)} |F_{A_i}|^2 \geq \epsilon_0^2\}.$$

Then Σ is a closed subset of finite $(n-4)$-dimensional Hausdorff measure. There is a bundle $E_\infty \to M \setminus \Sigma$ with a metric that is locally isometric to E on $M \setminus \Sigma$. Moreover, there is and a smooth Ω-YM connection A_∞ on E_∞ so that after passing to a subsequence $\{j_i\}$, and modulo to gauge transformations, A_{j_i} converges (locally in the C^∞ topology) to an Ω-YM connection A_∞ outside Σ, i.e. for any compact subset $K \subset M \setminus \Sigma$, there exists a sequence of isometries $\Phi_K^{j_i} : E_\infty|_K \to E|_K$ so that $(\Phi_K^{j_i})^* A_{j_i}$ converges to A_∞ smoothly [3]. *Furthermore, at each point $x \in \Sigma$, by passing to a subsequence, up to gauge transformations, $\{\lambda_i^* A_{j_i}\}_i$ converges to a smooth nontrivial Ω_x-YM connection over $\mathbb{R}^n = T_x M$ endowed with the flat metric given by g_x. Here $\{\lambda_i\}_i$ denotes a sequence of blow-up rescalings centered at x.*

Remark 1.3.
- As pointed out in [17], we emphasize here that a priori we only know that E_∞ and $E|_{M \setminus \Sigma}$ are isometric on compact subsets away from Σ. This is due to the possible complexity of the topology of $M \setminus \Sigma$. But as we will see, a global isometry does

[2]HYM connections over hermitian manifolds are *not* Yang-Mills connections in general.

[3]Unless otherwise specified, convergence of connections is always taken in this sense.

exist in the case of Hermitian-Yang-Mills connections (see Corollary 7.4). This is due to the fact that we can show Σ is a subvariety in this case.
- A slightly more general statement about the bundle isometries can be obtained as [26]. We refer the interested reader there.

We will refer to Σ as the *bubbling set*. By passing to a subsequence, we can assume

$$\mu_i := |F_{A_i}|^2 \,\mathrm{dVol} \rightharpoonup \mu_\infty$$

as a sequence of Radon measures. So the limit of $\{A_i\}_i$ consists of a pair (A_∞, μ_∞). As we will see later (see Lemma 3.1), μ_∞ can recover Σ intrinsically. We will refer it as A_i sub-converges to (A_∞, μ_∞).

We also generalize Tian's results [22] for Yang-Mills connections to the case of Ω-YM connections.

Theorem 1.4. *Σ is $(n-4)$-rectifiable.*

Denote $\mathcal{A}_{\Omega,c}$ to be the space of smooth Ω-YM connections A on a fixed bundle E with $\|F_A\| \le c$. Now we consider the space $\overline{\mathcal{A}_{\Omega,c}}$ by adding limits (A_∞, μ_∞) of smooth Ω-YM connections $\{A_i\}$ with $\|F_{A_i}\|_{L^2(M)} \le c$ (see Section 4 for more details.) *Since the space of Radon measures $\{\mu_\infty\}$, which come from the limits of smooth ones, is compact, we get a natural control of the singularities of A_i.* In particular, the diagonal sequence argument gives the following (see Section 4 for details)

Theorem 1.5. *$\overline{\mathcal{A}_{\Omega,c}}$ is weakly sequentially compact in the sense that every sequence $\{(A_i, \mu_i)\}$ in $\overline{\mathcal{A}_{\Omega,c}}$ sub-converges to some $(A_\infty, \mu_\infty) \in \overline{\mathcal{A}_{\Omega,c}}$.*

Remark 1.6.
- Without assuming A_i coming from limits of smooth connections, even in the case of admissible YM connections, we do not know whether such a limit exists or not due to lack of control of $\mathrm{Sing}(A_i)$.
- Again, we emphasize here that the limiting bundle E_∞ is not known to be isometric to $E|_{M\setminus\Sigma}$ for different subsequences in general. That is why we cannot directly take the quotient of $\mathcal{A}_{\Omega,c}$ mod gauge here. Due to this, it does not make sense to put a topology on the moduli space at this point. Later in the case of HYM connections over general complex manifolds, the results can be improved.

Suppose A_i sub-converges to (A_∞, μ_∞) as above. In Section 5, it is straightforward by the argument in [22] to define a notion of *bubbling connections* associated to the sequence. Also the *tangent cones* associated to (A_∞, μ_∞) are shown to exist. Unlike [22] where the tangent cone is defined for stationary admissible Yang-Mills connections, the tangent cone here is defined for the pair (A_∞, μ_∞) rather than just for A_∞. This comes from the fact that a monotonicity formula still holds for the energy density of μ_∞ which suffices for our use.

By restricting to the case of Ω-ASD instantons, we can generalize Tian's results ([22]) without requiring Ω be closed.

Theorem 1.7. *Ω restricts to a volume form of $T_x\Sigma$ at $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$.*

In Section 6, using the argument in [20], we generalize the removable singularities theorem for Yang-Mills connections of Tao-Tian [21] to the case of Ω-YM connections.

Theorem 1.8. *The removable singularities theorem holds for Ω-YM connections on a trivial bundle with small energy concentration away from a closed Hausdorff codimension 4 set.*

In the last section, we restrict our discussion to the case of HYM connections over general complex manifolds. If we assume (A_∞, μ) is the limit of a sequence of Hermitian-Yang-Mills connections over a compact Hermitian manifold, then by using the argument in [22] for Hermitian-Yang-Mills connections over Kähler manifolds and the extension theorem in [1], we can show that (A_∞, μ) are all *holomorphic* and Σ is a complex subvariety of codimension at least 2. In particular, we can now take the quotient of $\overline{\mathcal{A}_{\Omega,c}}$ mod gauge to get $\overline{M_{HYM,c}}$. There exists a way to give it a topology that coincides with the four dimensional case (see [6]) so that

Theorem 1.9. *$\overline{M_{HYM,c}}$ is a first countable sequentially compact Hausdorff space.*

Assume now (X, ω) is balanced of Hodge-Riemann type (see Section 7.2 for definitions). It turns out there exists a natural L^2 bound for the HYM connections in this case. By choosing c large for $\overline{M_{HYM,c}}$, we get the analytic compactification of smooth HYM connections on a fixed unitary bundle, which we denote it as $\overline{M}_{HYM}$.

Theorem 1.10. *Over a compact balanced Hermitian manifold of Hodge-Riemann type, $\overline{M}_{HYM}$ is a first countable sequentially compact Hausdorff space.*

Remark 1.11. Here the Hodge-Riemann type condition on the metrics can give us a uniform bound on the curvature of all the Ω-YM connections considered. We also refer the interested readers to [7, Section 3.1 (Property B')] where a notion of *taming forms* has been introduced for almost Spin(7) manifold to achieve the L^2 bound of the curvature as well as a discussion reduced to dimension 6 (see [7, eqn. (28)]).

By the main results in [23], this gives the following

Corollary 1.12. *Over a complex Hermitian manifold (X, ω) so that $\omega^{m-1} = \omega_0 \wedge \cdots \omega_{m-2}$ where ω_i are positive $(1,1)$ forms with $d\omega^{m-1} = 0$ and $d(\omega_1 \wedge \cdots \omega_{m-2}) = 0$, $\overline{M}_{HYM}$ is a first countable sequentially compact Hausdorff space.*

Remark 1.13. We emphasize here that by [23], $\omega_0 \wedge \cdots \omega_{m-2}$ is always strictly positive and thus defines a positive $(1,1)$ form on X through $\omega^{m-1} = \omega_0 \wedge \cdots \omega_{m-2}$.

In particular, we have

Corollary 1.14. *Assume (X, ω) is a compact Kähler manifold, $\overline{M_{HYM}}$ is a first countable sequentially compact Hausdorff space.*

Remark 1.15.
- As mentioned in Theorem 1.5 above, the novelty here is that we do not need to consider a larger space as [22] (explained below). Rather, we use the crucial condition that the connections considered come from limits of smooth connections. The latter gives a natural control of the singularities of the singular connections on the boundary.
- In [22], in order to compactify the moduli space, a notion of ideal HYM connection is introduced that generalizes the situation in four dimension (see [6]); namely, those pairs (A, Σ) with certain natural curvature conditions but not necessarily coming from limits of smooth ones. In the case of four manifolds, the compactification works essentially due to the good control of the bubbling set, which consists of points, and Uhlenbeck's removable singularity theorem. In higher dimensions, essential difficulties arise if we insist on such a large space of ideal objects. One is the lack of control of Sing(A). Also, the removable singularity theorem does not automatically apply in this situation due to the fact that the limiting bundle E_∞, defined only away from the singular set, does not necessarily extend to all of M as a smooth bundle.

- In higher dimensions, and assuming (X, ω) is projective, it is shown in [10] that the space of ideal HYM connections modulo gauge is indeed compact. This is essentially due to a boundedness result from the algebraic geometric side which gives control of $\mathrm{Sing}(A)$, and a version of the removable singularity theorem for HYM connections by Bando and Siu ([1]). With this, one can take the closure of the space of smooth HYM connections mod gauge in such a space to get a compactification.
- It is an interesting question to find a characterization of the ideal HYM connections added on to the boundary of $\overline{M_{HYM}}$, i.e. determine whether a given ideal HYM connection be approximated by the smooth ones.

Following from the argument in [10], and using the results on compactification of semistable sheaves via multipolarizations in [11], we explain how to give a complex structure to the compactification $\overline{M^*_{HYM}}$, where M^*_{HYM} is the moduli space of smooth irreducible HYM connections with *fixed determinant*.

Finally, consider a finite energy HYM connection A_∞ over a complex Hermitian manifold, and denote by $\mathcal{E}_\infty$ the corresponding reflexive sheaf. Given the analytic results above the following follows directly from the argument in [2], to which we refer the interested reader for the concepts involved. Here the tangent cone can be directly defined for A_∞ (not necessarily coming from the limit of smooth ones).

Theorem 1.16. *The analytic tangent cone of A_∞ at a point x is uniquely determined by the optimal algebraic tangent cones of $\mathcal{E}_\infty$ at x.*

Acknowledgements. The authors are grateful for comments on this paper from Daniel Greb, Ben Sibley, Song Sun, Matei Toma, and Thomas Walpuski.

2. Sequential compactness of smooth Ω-Yang-Mills connections

2.1. **Monotonicity.** Following the argument used by Price for Yang-Mills connections [18], we will show that a monotonicity formula holds for Ω-YM connections. We also refer to [22, Thm. 2.1.1] for a slightly more general version of the following for Yang-Mills connections.

Theorem 2.1. *There exist positive constants a and r_0, depending only on the geometry of (M, g) and Ω, with the following significance. If A is a smooth solution to* (1.1) *and $0 < r_1 < r_2 \leq r_0$, then*

$$\int_{B_{r_2}(x)\setminus B_{r_1}(x)} r^{4-n} e^{ar} |\iota_{\partial_r} F_A|^2 \leq e^{ar_2} r_2^{4-n} \int_{B_{r_2}(x)} |F_A|^2 - e^{ar_1} r_1^{4-n} \int_{B_{r_1}(x)} |F_A|^2.$$

Remark 2.2. If we denote the scale invariant L^p norms by:

$$f_p(x, r) := \left\{ r^{2p-n} \int_{B_r(x)} |F_A|^p dV \right\}^{1/p} \tag{2.1}$$

then Theorem 2.1 implies, in particular, that $e^{ar} f_2(x, r)$ is increasing for sufficiently small r.

Proof of Theorem 2.1. Let $\pi : P \to M$ be the orthogonal (or unitary) frame bundle of E. Given any connection B on E, denote by $\widetilde{B}$ the associated connection 1-form on the principal bundle P. Given a vector field X on M with compact support, we denote by $\widetilde{X}$ the unique horizontal lift of X to P. Let $\widetilde{\Phi}_t$ (resp. Φ_t) be the family of diffeomorphisms generated by $\widetilde{X}$ (resp. X).

As in [18], we consider the family of connection 1-forms $\widetilde{A}_t = \widetilde{\Phi}_t^* \omega$, and we denote by A_t the corresponding family of connections on E. We have

$$\delta \widetilde{A}_t(0) = \iota_{\widetilde{X}} d\widetilde{A} = \pi^* \iota_X F_A$$

since $\widetilde{X}$ is the horizontal lift of X. In particular, $\delta A_t(0) = \iota_X F_A$. Indeed, choosing a local section σ of P, which gives a trivialization of E, then by definition: $A_t = \sigma^* \widetilde{A}_t$. This implies

$$\delta A_t(0) = \sigma^* \iota_{\widetilde{X}} d\widetilde{A} = \sigma^* \pi^* \iota_X F_A = (\pi\sigma)^* \iota_X F_A = \iota_X F_A$$

since $\pi\sigma = \mathrm{Id}$. Now we look at the variation of the Yang-Mills functional along A_t. As for this, there are two ways to calculate it. First, since A satisfies (1.1), we have

$$d_A^* F_A \pm *(F_A \wedge d\Omega) = 0 \ . \tag{2.2}$$

Then,

$$\begin{aligned} \frac{d}{dt} \int_M |F_{A_t}|^2 \Big|_{t=0} &= 2 \int_M \langle d_A \delta A_t(0), F_A \rangle = 2 \int_M \langle \iota_X F_A, d_A^* F_A \rangle \\ &= \mp 2 \int_M \langle \iota_X F_A, *(F_A \wedge d\Omega) \rangle. \end{aligned}$$

Alternatively, one may differentiate (1.2) at $t = 0$ and use the fact that A is critical for YM_Ω. In any case, this implies

$$\left| \frac{d}{dt} \int_M |F_{A_t}|^2 \Big|_{t=0} \right| \leq 2 \sup |d\Omega| \int_M |\iota_X F_A| |F_A|. \tag{2.3}$$

Now the second way to calculate the variation is as in [18]. We include the details here. By definition, we know

$$\int_M |F_{A_t}|^2 = \int_M |F_{A_t}(d\Phi_t \cdot, d\Phi_t \cdot)|^2 (\Phi_t \cdot) \, dV = \int_M |F_{A_t}(d\Phi_t(e_i), d\Phi_t(e_j))|^2 (x) J_{\phi_t^{-1}} \, dV$$

where $\{e_i\}$ is a local orthonormal frame near the point x. Taking derivatives and evaluating at $t = 0$ gives

$$\begin{aligned} \frac{d}{dt} \int_M |F_{A_t}|^2 |_{t=0} &= \int_M -|F_A|^2 \mathrm{div} X - 4 \langle F_{A_t}(L_X e_i, e_j), F_A(e_i, e_j) \rangle \\ &= \int_M -|F_A|^2 \mathrm{div} X + \sum_{i,j} 4 \int_M \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \rangle \ . \end{aligned}$$

Combined with (2.3), this implies

$$\left| \int_M -|F_A|^2 \mathrm{div} X + \sum_{i,j} 4 \int_M \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \rangle \right| \leq 2 \sup |d\Omega| \int_M |\iota_X F_A| |F_A|. \tag{2.4}$$

Near the point x we fix the normal coordinates and let $\{e_1 = \partial_r, e_2, \cdots, e_n\}$ be a normal frame. In particular, $\nabla_{\partial_r} \partial_r = 0$. Choose $X = \xi(r) r \partial_r$, where ξ is a compact supported function supported over $[0, 1 + \epsilon]$ with $\xi = 1$ on $[0, 1]$ and $\xi' \leq 0$. Then

- $\nabla_{\partial_r} X = (\xi' r + \xi) \frac{\partial}{\partial r}$
- for $i \geq 2$, $\nabla_{e_i} X = \xi r \nabla_{e_i} \frac{\partial}{\partial r} = \xi e_i + \xi O(r^2)$

which implies
(2.5)

$$\begin{aligned}
&\sum_{i,j} 4\int_M \langle F_A(\nabla_{e_i}X, e_j), F_A(e_i,e_j)\rangle \\
=&\sum_j 4\int_M \langle F_A(\nabla_{\partial_r}X, e_j), F_A(\partial_r, e_j)\rangle + \sum_{i\geq 2}\sum_j 4\int_M \langle F_A(\nabla_{e_i}X, e_j), F_A(e_i,e_j)\rangle \\
=&\int_M 4\xi' r|\iota_{\partial_r}F_A|^2 + \sum_j 4\int_M \xi|F_A(\partial_r, e_j)|^2 + \sum_{i\geq 2}\sum_j 4\int_M \xi|F_A(e_i,e_j)|^2 + \int_M O(r^2)\xi|F_A|^2 \\
=&\int_M 4\xi' r|\iota_{\partial_r}F_A|^2 + 4\int_M \xi|F_A|^2 + \int_M O(r^2)\xi|F_A|^2.
\end{aligned}$$

and

$$\mathrm{div}X = \xi' r + n\xi + \xi O(r^2).$$

Given this, we have

$$\begin{aligned}
\int_M |F_A|^2\mathrm{div}(X) - 2\sup|d\Omega|\int_M |X||F_A|^2 = &\int_M |F_A|^2(\xi' r + n\xi + O(r^2)) \\
&- 2\sup|d\Omega|\int_M |X||F_A|^2
\end{aligned} \tag{2.6}$$

Plugging eqns. (2.5) and (2.6) into (2.4), we have

$$\begin{aligned}
&\int_M |F_A|^2(\xi' r + (n-4)\xi + O(r^2)) - 2\sup|d\Omega|\int_M \xi r|F_A|^2 \\
&\qquad\leq \int_M 4\xi' r|\iota_{\partial_r}F_A|^2 + \int_M O(r^2)\xi|F_A|^2
\end{aligned} \tag{2.7}$$

Now by replacing ξ_τ with $\xi_\tau(r) = \xi(\tau^{-1}r)$ in (2.7), and using the fact that

$$\tau\frac{d\xi_\tau}{d\tau} = -r\xi'_\tau\ ,$$

we have

$$\begin{aligned}
&\int_M |F_A|^2(-\tau\frac{d\xi_\tau}{d\tau} + (n-4)\xi_\tau) - 2\sup|d\Omega|\int_M \xi_\tau r|F_A|^2 \\
\leq& -\int_M 4\tau\frac{d\xi_\tau}{d\tau}|\iota_{\partial_r}F_A|^2 + \int_M O(r^2)\xi_\tau|F_A|^2
\end{aligned}$$

i.e.

$$\begin{aligned}
&\int_M |F_A|^2(\tau\frac{d\xi_\tau}{d\tau} + (4-n)\xi_\tau) + 2\sup|d\Omega|\int_M \xi_\tau r|F_A|^2 \\
\geq&\int_M 4\tau\frac{d\xi_\tau}{d\tau}|\iota_{\partial_r}F_A|^2 + \int_M O(r^2)\xi_\tau|F_A|^2.
\end{aligned}$$

Multiply the above by $e^{a\tau}\tau^{3-n}$ where a is a constant to be determined later, and use the fact that $\xi_\tau r|F_A|^2 \leq \xi_\tau\tau|F_A|^2$, since ξ_τ is supported over $\{|x| \leq \tau\}$. We conclude

$$\begin{aligned}
&e^{a\tau}\frac{d}{d\tau}(\tau^{4-n}\int_M \xi_\tau|F_A|^2) + e^{a\tau}\tau^{4-n}2\sup|d\Omega|\int_M \xi_\tau|F_A|^2 \\
\geq& 4e^{a\tau}\tau^{4-n}\int_M \frac{d\xi_\tau}{d\tau}|\iota_{\partial_r}F_A|^2 + e^{a\tau}\tau^{3-n}\int_M O(r^2)\xi_\tau|F_A|^2.
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{d}{d\tau}(e^{a\tau}\tau^{4-n}\int_M \xi_\tau |F_A|^2)\\
\geq & 4e^{a\tau}\tau^{4-n}\int_M \frac{d\xi_\tau}{d\tau}|\iota_{\partial_r}F_A|^2 + e^{a\tau}\tau^{3-n}\int_M O(r^2)\xi_\tau|F_A|^2 + ae^{a\tau}\tau^{4-n}\int_M \xi_\tau|F_A|^2.\\
&- e^{a\tau}\tau^{4-n}2\sup|d\Omega|\int_M \xi_\tau|F_A|^2
\end{aligned}$$

Now choose a large so that $a \gg 2\max\{1, 2\sup|d\Omega|\}$. Since $\frac{d\xi_\tau}{d\tau} = -\frac{r}{\tau}\xi_\tau'$ is nonnegative,

$$\frac{d}{d\tau}(e^{a\tau}\tau^{4-n}\int_M \xi_\tau|F_A|^2) \geq 4e^{a\tau}\tau^{4-n}\int_M \frac{d\xi_\tau}{d\tau}|\iota_{\partial_r}F_A|^2 \geq 4\int_M e^{ar}r^{4-n}\frac{d\xi_\tau}{d\tau}|\iota_{\partial_r}F_A|^2$$

if $\tau < r_0$ for some r_0 so that $e^{a\tau}\tau^{4-n}$ is decreasing over $[0, r_0]$. By integrating the inequality above from r_1 to r_2 and letting $\epsilon \to 0$, Theorem 2.1 follows. □

2.2. **ϵ-Regularity.** The goal of this section is to prove the following ε-regularity result.

Theorem 2.3. *There exist positive constants ϵ_0, r_0, and C, depending only on the geometry of (M, g) and Ω, with the following property. If A is a smooth solution to the Ω-Yang-Mills equations* (1.1) *on M, and $x \in M$ is a point for which $f_2(x, r) \leq \epsilon_0$ for some $0 < r \leq r_0$, then*

$$\sup_{B_{r/4}(x)} r^2|F_A| \leq Cf_2(x, r)$$

There are two approaches to the regularity of Yang-Mills equations in higher dimensions, and both make use of the monotonicity formula. Nakajima [17] uses a Bochner-Weitzenböck formula for the curvature to directly get the bound in Theorem 2.3. This is similar to Schoen's approach for the harmonic map problem. Uhlenbeck [24] derives L^p estimates from L^2, and then uses a continuity method to reduce to the case of connections with L^p bounds. This has the advantage of applying to a larger class of connections satisfying curvature bounds rather than equations. Interestingly, both methods apply directly to the case of Ω-YM connections, and we find it useful to present each one here.

2.2.1. *Method I.* Suppose A is a smooth solution to (1.1). Then (2.2) implies

$$\Delta_A F_A = \mp d_A * (F_A \wedge d\Omega).$$

In particular, by the Weitzenböck formula, we have

$$\nabla_A^*\nabla_A F_A = \mp d_A * (F_A \wedge d\Omega) + \{F_A, R_g\} + \{F_A, F_A\}\ . \tag{2.8}$$

Proposition 2.4. *A solution to* (1.1) *satisfies*

$$\frac{1}{2}\Delta|F_A|^2 \geq -|F_A|^2 - c|R_g||F_A|^2 - \frac{c^2}{4}|d\Omega|^2|F_A|^2 - c|\nabla d\Omega||F_A|^2$$

for some constant c depending only on (M, g).

Proof. Indeed, from (2.8) we have

$$\begin{aligned}\frac{1}{2}\Delta|F_A|^2 &= -<\nabla_A^*\nabla_A F_A, F_A> + <\nabla_A F_A, \nabla_A F_A> \\ &\geq -|F_A|^3 - |R_g||F_A|^2 - |d_A * (F_A \wedge d\Omega)||F_A| + |\nabla_A F_A|^2 \\ &\geq -|F_A|^3 - |R_g||F_A|^2 - c(|d\Omega||\nabla_A F_A||F_A| + |\nabla d\Omega||F_A|^2) + |\nabla_A F_A|^2 \\ &\geq -|F_A|^3 - |R_g||F_A|^2 - \frac{c^2}{4}|d\Omega|^2|F_A|^2 - c|\nabla d\Omega||F_A|^2\end{aligned}$$

The last inequality follows from completion of square. □

Given this, we can repeat the argument in [17, Lemma 3.1] to prove Theorem 2.3.

2.2.2. *Method II.* Everything is local, so we assume connections are on the trivial bundle in $\mathbb{R}^n$. Uhlenbeck's "good gauge" theorem states:

Theorem 2.5 ([25, Thm. 1.3])**.** *Fix $n/2 < p < n$. There is $\varepsilon_0 > 0$ and a constant c_n such that if $A \in L_1^p$ is a connection on $B_1(0)$ and $f_{n/2}(x, 1) < \varepsilon_0$, then A is gauge equivalent to a connection (also denoted A) satisfying:*

(1) $d^*A = 0$;
(2) $*A$ *vanishes on* $\partial B_1(0)$;
(3) $\|A\|_{L_1^{n/2}} \leq c_n f_{n/2}(0, 1)$;
(4) $\|A\|_{L_1^p} \leq c_n \|F_A\|_{L^p}$.

We will also need

Lemma 2.6. *There is $\varepsilon(n) > 0$ such that if A is a connection on $B_1(0)$ satisfying $\|A\|_{L^n} \leq \varepsilon(n)$ and items (i) and (ii) of the Theorem, then item (iv) holds for all p, $n/2 \leq p < n$.*

The following result will allow us to go from L^2 estimates to L^p estimates. Let $L^p(x, r) := L^p(B_r(x))$.

Theorem 2.7. *There are positive constants κ_n, r_0 and for every for every $2 \leq p < n$, C_p, with the following significance: Suppose A is a solution to* (1.1)*, and $f_{n/2}(x, r) \leq \kappa_n$ for $r \leq r_0$. Then*

$$f_p(x, r/2) \leq C_p \, f_2(x, r)$$

Proof. Rescale to take $r = 1$. Use Theorem 2.5 and Lemma 2.6 for $p = 2$ to find a gauge where: $d^*A = 0$, and

$$\|A\|_{L_1^2(x,1)} \leq C\|F_A\|_{L^2(x,1)} = C' f_2(x, 1) \tag{2.9}$$

Now write the equation for the laplacian of A as:

$$\Delta A + \{A, dA\} + \{A, A, A\} = d_A^* F_A = *(F_A \wedge d\Omega)$$

$$(\Delta + 1)A + \{A, dA\} + \{A, A, A\} = *(dA \wedge d\Omega) \tag{2.10}$$

where the brackets indicate multilinear expressions. Let $\mathscr{L}$ be the linear operator acting on A on the left hand side of (2.10). Note that $L_1^{n/2} \hookrightarrow L^n$, so $[A, A] \in L^{n/2}$, and both dA and $[A, A]$ are small in $L^{n/2}$. We also have $L_1^p \times L^{n/2} \hookrightarrow L_{-1}^p$. Hence, we see that $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$ is a perturbation of $\mathscr{L}_0 := \Delta + 1 : L_1^p \to L_{-1}^p$ by $\mathscr{L}_1 : L_1^p \to L_{-1}^p$ of small norm. As in [24, p. 6], a Meyers type interior estimate for $\mathscr{L}_0$ implies one for $\mathscr{L}$:

$$\|u\|_{L_1^p(x,1/2)} \leq C_p(\|u\|_{L_1^2(x,1)} + \|\mathscr{L}u\|_{L_{-1}^p(x,1)}) \tag{2.11}$$

where $u = A$. Now using (2.9), the L^p_{-1} norm of the right hand side of (2.10) is bounded by $f_2(x, 1)$ for $p = 2n/(n-2) > 2$. The estimate (2.11) then gives an improved L^p_1 bound on A for p slightly bigger than 2. Reiterating this argument, we get L^p_1 bounds on A for any $p < n$. □

Bootstrapping (2.10) gives the estimate:

$$\sup_{y \in B_{r/2}(x)} r^2 |F_A(y)| \le C_n\, f_2(x, r) \tag{2.12}$$

Let us fill in some details. First, notice that for $n/2 \le p < n$, $L^p_1 \times L^p_1 \hookrightarrow L^p$. Moreover, $L^p_1 \times L^p \hookrightarrow L^q$, with $q \to n$ as $p \to n$. Hence, from (2.10) and the L^p-elliptic estimate for the Laplacian, we get that $A \in L^p_{2,loc}$, for $n/2 < p < n$. Again applying multiplication theorems, we get that $\Delta A \in L^p_1$, and hence, $A \in L^p_{3,loc}$. This implies A is $C^{1,\alpha}$, and the estimate follows.

There is one more step:

Lemma 2.8. *Suppose $4\rho < r_0$, $f_2(\xi, 4\rho) = \varepsilon < \varepsilon_0$. Moreover, assume $f_{n/2}(x, r) \le \kappa_n$ for some $r < \rho$. Then:*

$$f_{n/2}(x, r/2) \le C_n \varepsilon$$
$$\sup_{y \in B_{r/4}(x)} r^2 |F_A(y)| \le K_n \varepsilon$$

Proof. Apply Theorem 2.7 with $p = n/2$, and use (2.12). □

Notice that this Lemma says that once both $f_{n/2}$ and f_2 are sufficiently small, then $f_{n/2}$ is even smaller than expected. Now Theorem 2.1 and Uhlenbeck's continuity method argument [24, proof of Thm. 1.6] gives the proof of Theorem 2.3.

2.3. **Proof of Theorem 1.2.** This follows from Theorems 2.1 and 2.3 as in the Yang-Mills case (see [17, 25]).

3. Rectifiability of the blow-up locus

The results in this section are all local. We will fix a sequence of Ω-YM connections A_i over $B_{1+\delta_0} := \{x \in \mathbb{R}^n : |x| < 1 + \delta_0\} \subset \mathbb{R}^n$ with $\|F_{A_i}\|_{L^2(B_{1+\delta_0})}$ uniformly bounded and look at the convergence over $B =: B_1$. Here, $\delta_0 > 0$ is fixed, and $B_{1+\delta_0}$ is endowed with any fixed smooth metric with volume form dV. We assume the standard coordinates are geodesic normal with respect to the metric. Define

$$\Sigma = \{x \in B : \lim_{r \to 0^+} \liminf_i r^{4-n} \int_{B_r(x)} |F_A|^2 dV \ge \epsilon_0^2\}. \tag{3.1}$$

From the results in the previous section, we only know that Σ is a closed subset of B with locally finite $(n-4)$-Hausdorff measure. We will show that Σ has better structure by generalizing the result in [22]; namely, we prove Theorem 1.4.

The proof closely follows the arguments in [14, 22]. The monotonicity formula obtained in Theorem 2.1 is a key component.

3.1. **Elementary properties.** By passing to a subsequence, we can assume

(1) up to gauge transformations, A_i converges to A_∞ locally away from Σ;
(2) $\mu_i := |F_{A_i}|^2 dV$ converges weakly to μ as a sequence of Radon measures, i.e. for any compact supported continuous function f, we have
$$\lim_i \mu_i(f) = \mu(f).$$

By Fatou's lemma, we have
$$\mu = |F_{A_\infty}|^2 dV + \nu \tag{3.2}$$
for some nonnegative Radon measure ν, which is called the *defect measure.*

Lemma 3.1. *The following properties hold:*

(1) For a.e. $0 < r \ll 1$, $\lim_i \mu_i(B_r(x)) = \mu(B_r(x))$;
(2) $r^{4-n}\mu(B_r(x))$ is increasing with r. In particular, the function
$$\Theta^{n-4}(\mu, x) = \lim_{r\to 0+} r^{4-n}\mu(B_r(x))$$
is well-defined, and it is called the energy density of μ at x. Furthermore, Θ^{n-4} is upper semi-continuous and $\mathcal{H}^{n-4}$ approximately continuous at $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$.
(3) $x \in \Sigma$ if and only if $\Theta^{n-4}(\mu, x) \geq \epsilon_0^2$;
(4) for $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$,
$$\limsup_{r\to 0} r^{4-n} \int_{B_r(x)} |F_{A_\infty}|^2 dV = 0.$$

Proof. (1) follows from the elementary fact that $\mu(\partial B_r(x)) = 0$ for a.e. $0 < r \ll 1$. The first part of (2) now follows from (1) and the fact that $r^{4-n}\mu_i(B_r(x))$ increases as r increases. The upper semicontinuity follows directly from the monotonicity formula. The $\mathcal{H}^{n-4}$ approximate continuity property follows as in [22, Lemma 3.2.2] (see also [14, p. 803]). For (3), suppose $\Theta^{n-4}(\mu, x) \geq \epsilon_0^2$, obviously, $x \notin \Sigma$. Now suppose $x \in \Sigma$, if $\Theta^{n-4}(\mu, x) < \epsilon_0^2$, by (1), $\mu_i(B_r(x)) < \epsilon_0^2$ for $0 < r \ll 1$. By ϵ-regularity, A_i converges smoothly near x which implies $x \notin \Sigma$. This is a contradiction. For (4), see [22, p. 222]. $\square$

Remark 3.2. From this, we know $\Sigma = \{x \in B : \Theta^{n-4}(\mu, x) \geq \epsilon_0^2\}$, which recovers the statement that Σ a closed subset of B of finite $(n-4)$-dimensional Hausdorff measure. Furthermore, Σ is intrinsically associated to μ.

In the following, we always denote
$$\pi(\mu) = \Sigma. \tag{3.3}$$
We also define
$$\mathrm{Sing}(A_\infty) = \{x \in B : \limsup_{r\to 0} r^{4-2n} \int_{B_r(x)} |F_{A_\infty}|^2 > 0\} \tag{3.4}$$

Lemma 3.3. *The following holds*

(1) $\Sigma = Supp(\nu) \cup \mathrm{Sing}(A_\infty)$;
(2) ν is absolutely continuous with respect to the $(n-4)$ Hausdorff measure on Σ. In particular, $\nu = \Theta(x)\mathcal{H}^{n-4}_\Sigma$ where
$$\epsilon_0^2 \leq \Theta(x) \leq C = C(\delta_0, n) \sup_i \|F_{A_i}\|_{L^2(B_{1+\delta_0})}$$
for $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$.

Proof. For (1), suppose $x \notin \Sigma$, we know $\Theta(\mu, x) < \epsilon_0^2$. By ϵ-regularity, A_i converges smoothly near x which implies $\nu = 0$ near x and A_∞ is smooth near x. Suppose $x \in \Sigma$, if $x \notin Supp(\nu)$, then

$$\lim_{r\to 0} r^{4-n} \int_{B_r(x)} |F_{A_\infty}|^2 = \Theta(\mu, x) \geq \epsilon_0^2.$$

i.e. $x \in \mathrm{Sing}(A_\infty)$. For (2), by Theorem 2.1 we know that

$$r^{4-n}\mu(B_r(x)) \leq \delta_0^{4-n}\mu(B_{\delta_0}(x))$$

which implies μ is absolutely continuous with respect to the $(n-4)$-Hausdorff measure. In particular, we have

$$\mu|_\Sigma = \Theta(x)\mathcal{H}^{n-4}_\Sigma.$$

for some measurable function $\Theta(x)$. Since

$$\lim_{r\to 0} r^{4-n} \int_{B_x(r)} |F_{A_\infty}|^2 \,\mathrm{dVol} = 0$$

for $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$, we know

$$\nu(x) = \Theta(x)\mathcal{H}^{n-4}_\Sigma$$

for $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$. The conclusion follows from the density estimate above and the classical fact that

$$2^{4-n} \leq \limsup_{r\to 0} \frac{\mathrm{Vol}_{\mathcal{H}^{n-4}}(\Sigma \cap B_r(x))}{r^{n-4}} \leq 1$$

for $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$. □

3.2. **Tangent cone measures.** Fix $x_0 \in B$, define

$$\tau_\lambda : B_{\delta_0}(x_0) \to B_{\delta_0}(x_0) : x_0 + \xi \mapsto x + \lambda\xi$$

For $E \subset B_{\delta_0}(x_0)$ measurable, let

$$\mu_\lambda(E) = \lambda^{4-n}\mu(\tau_\lambda(E))$$

In this section we prove the following (cf. [22, Lemma 3.2.1])

Proposition 3.4. *For any $\lambda_j \downarrow 0$ there is a Radon measure η such that (after passing to a subsequence) $\mu_{\lambda_j} \to \eta$ weakly. Moreover, η is a* cone measure*, in the sense that*

$$\lambda^{4-n}\eta(\lambda E) = \eta(E)$$

for any $\lambda > 0$ and $E \subset B_{\delta_0}(x_0)$ measurable.

Proof. Let $ds_\lambda^2 = \lambda^{-2}\tau_\lambda^* ds^2$ be the pull-back metric and dV_λ the associated volume form. Similarly, let $A_{i,\lambda} = \tau_\lambda^* A_i$. We also pull back the hermitian structure. Then:

$$F_{A_{i,\lambda}} = \tau_\lambda^* F_{A_i} \quad ; \quad |F_{A_{i,\lambda}}|^2(x) = \lambda^4 |F_{A_i}|^2(\tau_\lambda(x))$$

The weak convergence of $\mu_{\lambda_i} \to \eta$, for some Radon measure η, follows from the monotonicity. Notice that since

$$\sigma^{4-n}\mu(B_\sigma(x_0)) \leq \rho^{4-n}\mu(B_\rho(x_0))$$

we have

$$\sigma^{4-n}\eta(B_\sigma(x_0)) = \Theta(\mu, x_0)$$

We wish to show η is a cone measure. For this it suffices to show that for any radially invariant function $\phi \geq 0$,

$$\sigma^{4-n} \int_{B_\sigma(x)} \phi \, d\eta = \rho^{4-n} \int_{B_\rho(x)} \phi \, d\eta \tag{3.5}$$

for all σ, ρ (cf. [22], top of p. 225). By a diagonalization argument we may assume

$$|F_{A_{i,\lambda}}|^2 \, dV_{\lambda_i} \longrightarrow \eta$$

weakly. To prove (3.5), note that

$$\begin{aligned} \sigma^{4-n} \int_{B_\sigma(x)} \phi |F_{A_{i,\lambda_i}}|^2 \, dV_{\lambda_i} - \rho^{4-n} \int_{B_\rho(x)} \phi |F_{A_{i,\lambda_i}}|^2 \, dV_{\lambda_i} \\ = \int_\sigma^\rho ds \, \frac{d}{ds} \left\{ s^{4-n} \int_{B_s(x)} \phi |F_{A_{i,\lambda_i}}|^2 \, dV_{\lambda_i} \right\} \\ = \int_\sigma^\rho ds \, \frac{d}{ds} \left\{ s^{4-n} \int_{B_1(x)} \phi |F_{\tau_s^* A_{i,\lambda_i}}|^2 \, \tau_s^* dV_{\lambda_i} \right\} \end{aligned} \tag{3.6}$$

Now $s^{4-n} \tau_s^* dV_{\lambda_i} = (1 + O(s^2 \lambda_i)) dV_0$, so

$$\frac{d}{ds} (s^{4-n} \tau_s^* dV_{\lambda_i}) \longrightarrow 0$$

uniformly as $\lambda_i \to 0$. Since F_{A_i} has uniformly bounded L^2-norm, this term vanishes. It suffices to estimate the term coming from

$$\frac{d}{ds} F_{\tau_s^* A_{i,\lambda_i}} = d_{\tau_s^* A_{i,\lambda_i}} \partial_s (\tau_s^* A_{i,\lambda_i})$$

At this point we can assume A_{i,λ_i} is in radial gauge, i.e. $\imath_{\partial_r} A_{i,\lambda_i} = 0$. Then

$$\imath_{\partial_r} F_{A_{i,\lambda_i}} = \partial_r A_{i,\lambda_i}$$

and so

$$\partial_s (\tau_s^* A_{i,\lambda_i}) = r \imath_{\partial_r} F_{\tau_s^* A_{i,\lambda_i}}$$

It follows that

$$\frac{d}{ds} (\phi |F_{\tau_s^* A_{i,\lambda_i}}|^2) = 2 \langle d_{\tau_s^* A_{i,\lambda_i}} (r \imath_{\partial_r} F_{\tau_s^* A_{i,\lambda_i}}), \phi \, F_{\tau_s^* A_{i,\lambda_i}} \rangle$$

Integrating by parts, we see that (3.6) is bounded by a constant times the integral of

$$r^{4-n} |\imath_{\partial_r} F_{A_{i,\lambda_i}}| |F_{A_{i,\lambda_i}}|$$

over $B_\rho(x)$, where the constant depends on ϕ, $d\phi$, and $d\Omega$. By Theorem 2.1 we have

$$\int_{B_\rho(x)} r^{4-n} |\imath_{\partial_r} F_{A_{i,\lambda_i}}|^2 dV_{\lambda_i} \longrightarrow 0$$

and so the result follows. $\square$

Remark 3.5. An alternative argument follows [15, Lemma 4.1.4]. In order to show η is a cone measure, it suffices to show that for any compactly supported function ψ over B we have

$$\frac{d}{ds} (s^{4-n} (\tau_s^* \eta)(\psi)) = 0.$$

To prove this, note that

$$\begin{aligned}\frac{d}{ds}(s^{4-n}(\tau_s^*\eta)(\psi)) &= \frac{d}{ds}(s^{4-n}\int_{\mathbb{R}^n}\psi_s d\eta)\\ &= -s^{3-n}\int_{\mathbb{R}^n}((n-4)\psi_s + s^{-1}x\cdot(\nabla\psi)_s)dV\end{aligned}$$

where $\psi_s(x) = \psi(x/s)$ and $(\nabla\psi)_s(x) = (\nabla\psi)(x/s)$. So it suffices to show that

$$\int_{\mathbb{R}^n}((n-4)\psi_s + s^{-1}x\cdot(\nabla\psi)_s)dV = 0.$$

From the proof of Theorem 2.1, we have

$$\begin{aligned}&\left|\int_M |F_A|^2(x\cdot\nabla\psi + (n-4)\psi + \psi O(r^2))dV\right|\\ \leq&\left|-2\sup|d\Omega|\int_M \psi r|\iota_{\partial_r}F_A||F_A|dV + \int_M 4\psi' r|\iota_{\partial_r}F_A|^2 dV + \int_M O(r^2)\psi|F_A|^2dV\right|.\end{aligned}$$

for any Ω-YM connection A over (M,g) and compactly supported function ψ. We plug in $(A,\psi) = (A_{i,\lambda_i},\psi_s)$ and get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}|F_{A_{i,\lambda_i}}|^2(s^{-1}x\cdot(\nabla\psi)_s + (n-4)\psi_s + \psi_s O(r^2))dV\right|\\ \leq&\left|-2\sup|d\Omega_i|\int_{\mathbb{R}^n}\psi_s r|\iota_{\partial_r}F_{A_i^\lambda}||F_{A_i^\lambda}|dV + \int_M 4\psi_s' r|\iota_{\partial_r}F_{A_i^\lambda}|^2dV + \int_{\mathbb{R}^n}O(r^2)\psi_s|F_{A_i^\lambda}|^2dV\right|\end{aligned}$$

By taking limits the right hand side vanishes, and this gives

$$\int_{\mathbb{R}^n}((n-4)\psi_s + s^{-1}x\cdot(\nabla\psi)_s)d\eta = 0.$$

Here, since the base metric converges smoothly to the flat metric on $\mathbb{R}^n$, the $O(r^2)$ term vanishes in the limit.

Now we fix a tangent measure η. Define

$$L_\eta := \{x\in\mathbb{R}^n : \Theta^{n-4}(\eta,x) = \Theta^{n-4}(\eta,0) = \Theta^{n-4}(\mu,x_0)\}.$$

The following can be deduced from the monotonicity formula and the dimension reduction argument of Federer (cf. [15, p. 27]).

Lemma 3.6. *For any $y\in L_\eta$, η is invariant in the direction of y. In particular, L_η is a linear subspace of $\mathbb{R}^n$. Furthermore,* $\dim L_\eta \leq n-4$.

Define

$$\Sigma_j := \{x\in\Sigma : \dim L_\eta \leq j \text{ for all the tangent measures } \eta \text{ at } x\}.$$

Then we have

Proposition 3.7. *There exists a filtration which consists of closed subsets*

$$\Sigma_0\subset\Sigma_1\subset\cdots\subset\Sigma_{n-4} = \Sigma$$

with the Hausdorff dimension satisfying $\dim(\Sigma_j)\leq j$.

3.3. **Results parallel to stationary harmonic maps and Yang-Mills connections.** The following *geometric lemma* can be obtained by directly replacing the energy density associated to the harmonic map with Θ^{n-4} in [15] or the Yang-Mills case in [22]

Lemma 3.8. *Suppose $\Theta^{n-4}(\mu,\cdot)$ is $\mathcal{H}^{n-4}$ approximately continuous at $x \in \Sigma$. For any $0 < r \ll 1$, there exists $n-4$ points $x_1^r, \cdots x_{n-4}^r$ with*

- *$\Theta^{n-4}(\mu, x_i^r) \geq \Theta^{n-4}(\mu, x) - \epsilon_r$ where $\epsilon_r \to 0$ as $r \to 0$;*
- *$d(x_1, x) \geq rs$ and $d(x_i, x + span\{x_1 - x \cdots, x_{n-4} - x\}) \geq rs$ for some $s \in (0,1)$ independent of r.*

Given the geometric lemma, we have the existence of *weak tangent planes* as follows

Proposition 3.9. *For any point $x \in \Sigma$ and any $\delta > 0$, there exists $r_x > 0$ and a tangent plane $L \in \mathrm{Gr}(\mathbb{R}^n, n-4)$ so that $\mu(B_r(x) \setminus L_{\delta r}) = 0$ where $L_{\delta r}$ denotes the δr neighborhood of L in $\mathbb{R}^n$.*

As a corollary, this implies the *null projection* property.

Proposition 3.10. *Suppose $E \subset \Sigma$ is a purely $(n-4)$-unrectifiable set, then*

$$\mathrm{Vol}_{\mathcal{H}^{n-4}}(P_V(E)) = 0$$

for any orthogonal projections $P_V : \mathbb{R}^n \to V \in \mathrm{Gr}(\mathbb{R}^n, n-4)$.

3.4. **Positive projection density.** The argument for the following is the same as [15] and [22]. We will only point out where the change is necessary and refer the reader there for more details.

Proposition 3.11. *For $\mathcal{H}^{n-4}$ a.e. points $x \in \Sigma$,*

$$\lim_{r\to 0} \frac{\mathrm{Vol}_{\mathcal{H}^{n-4}}(P_V(\Sigma \cap B_r(x)))}{\alpha(n-4)r^{n-4}} \geq \frac{1}{2}$$

for some projection $P_V : \mathbb{R}^n \to V \in \mathrm{Gr}(\mathbb{R}^n, n-4)$.

Proof. Otherwise, we can find a point $x_0 \in \Sigma$ so that

$$\limsup_{r} r^{4-n} \int_{B_r(x_0)} |F_{A_\infty}|^2 = 0$$

and $\Theta^{n-4}(\mu,\cdot)$ is approximately continuous at $x_0 \in \Sigma$ but

$$\lim_{r\to 0} \frac{\mathrm{Vol}_{\mathcal{H}^{n-4}}(P_V(\Sigma \cap B_r(x_0)))}{\alpha(n-4)r^{n-4}} < \frac{1}{2}.$$

In particular, the tangent measure of μ at x_0 takes the form $\Theta^{n-4}(x_0)\mathcal{H}^{n-4}_{\mathbb{R}^{n-2}}$ for some $\mathbb{R}^{n-2} \subset \mathbb{R}^n$. Recall that from the diagonalization argument we assume

$$\mu_{\lambda_i} \rightharpoonup \Theta^{n-4}(x_0)\mathcal{H}^{n-4}_{\mathbb{R}^{n-4}}.$$

Define

$$\alpha_{\lambda_i} = \sum_{\alpha=1}^{n-2} |\iota_{\partial_\alpha} F_{A_{i,\lambda_i}}|^2 \,\mathrm{dVol}$$

We know that for any fixed $\delta > 0$ and i large, $\alpha_{\lambda_i}(B_{3/2}) \leq \delta$. Now we define

$$\mathscr{F}_{\lambda_i} : (\mathbb{R}^{n-4} \times 0) \times (0,1) \to \mathbb{R}$$

$$\mathscr{F}_{\lambda_i}(x,\epsilon) = \int_{B_2^n} |F_{A_{i,\lambda_i}}|^2(x+y)\psi_\epsilon(y_1)\phi^2(y_2) \,\mathrm{dVol}_y$$

Here, $y = (y_1, y_2) \subset \mathbb{R}^{n-4} \times \mathbb{R}^4$, $\psi_\epsilon(y_1) = \epsilon^{4-n}\psi(y_1/\epsilon)$ where ψ is a nonnegative compactly supported function on the unit ball in $\mathbb{R}^4$ with integral being 1, while ϕ is smooth and compactly supported on the unit ball in $\mathbb{R}^{n-4}$. To simplify the notation, we will denote $F := F_{A_{i,\lambda_i}}$, $\partial_\alpha = \frac{\partial}{\partial y_\alpha}$ and ∇_α as the covariant derivatives. Viewing $|F|$ as a function of y, we have

$$\begin{aligned}\partial_\alpha |F|^2 &= -2\operatorname{Tr}(\nabla_\alpha F_{\gamma\beta} F^{\gamma\beta}) \\ &= 4\operatorname{Tr}(\nabla_\gamma F_{\beta\alpha} F^{\gamma\beta}) \\ &= 4\partial_\gamma \operatorname{Tr}(F_{\beta\alpha} F^{\gamma\beta}(x+y)) \pm 4(\iota_{\partial_\alpha} F, *(F\wedge\Omega)).\end{aligned}$$

For any $1 \le \alpha \le n-4$, we have

$$\begin{aligned}\frac{\partial}{\partial x_\alpha}\mathscr{F}_{\lambda_i} &= \int_{B_2^n} \frac{\partial}{\partial x_\alpha}(|F|^2(x+y))\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y \\ &= \int_{B_2^n} \partial_\alpha |F|^2(x+y)\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y \\ &= \int_{B_2^n} 4\partial_\gamma \operatorname{Tr}(F_{\beta\alpha}F^{\gamma\beta})(x+y)\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y \\ &\quad \pm \int_{B_2^n} 4(\iota_{\partial_\alpha}F, *(F\wedge\Omega)))(x+y)\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y \\ &= \sum_{\gamma=n-4}^{n} \int_{B_2^n} 4\operatorname{Tr}(F_{\beta\alpha}F^{\gamma\beta})\psi_\epsilon(y_1)\frac{\partial}{\partial y_\gamma}\phi^2(y_2)\,\mathrm{dVol}_y \\ &\quad \pm \int_{B_2^n} 4(\iota_{\partial_\alpha}F, *(F\wedge\Omega)))\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y \\ &\quad + \sum_{\gamma=1}^{n-4} 4\frac{\partial}{\partial x_\gamma}\int_{B_2^n} \operatorname{Tr}(F_{\beta\alpha}F^{\gamma\beta})(x+y)\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y\end{aligned}.$$

This implies $\nabla\mathscr{F}_{\lambda_i} = \vec{f}_{\lambda_i} + \operatorname{div}\vec{G}_{\lambda_i}$, where

$$\begin{aligned}(\vec{f}_{\lambda_i})_\alpha &= \sum_{\gamma=n-4}^{n} \int_{B_2^n} 4\operatorname{Tr}(F_{\beta\alpha}F^{\gamma\beta})\psi_\epsilon(y_1)\frac{\partial}{\partial y_\gamma}\phi^2(y_2)\,\mathrm{dVol}_y \\ &\quad \pm \int_{B_2^n} 4(\iota_{\partial_\alpha}F, *(F\wedge\Omega)))\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y\end{aligned}$$

and

$$(\vec{G}_{\lambda_i})^\gamma_\alpha = \int_{B_2^n} 4\operatorname{Tr}(F_{\beta\alpha}F^{\gamma\beta})(x+y)\psi_\epsilon(y_1)\phi^2(y_2)\,\mathrm{dVol}_y\,.$$

Here the divergence of $\vec{G}_{\lambda_i}$ is taken for each vector component of $\vec{G}_{\lambda_i}$. Since $\alpha_{\lambda_i} \rightharpoonup 0$, we know that for any $\delta > 0$,

$$\|\vec{f}_{\lambda_i}\|_{L^2(B_2^{n-4})} + \|\vec{G}_{\lambda_i}\|_{L^2(B_2^{n-4})} \le \delta$$

for i sufficient large and λ sufficiently small. Given this, by [15, Lemma 4.2.10] we know for any δ_1 there exist constants $C_{\lambda_i}(\epsilon)$

$$\|\mathscr{F}_{\lambda_i}(\cdot,\epsilon) - C_{\lambda_i}(\epsilon)\|_{L^1(B_2^{n-2})} \le \delta_1.$$

Letting $\epsilon \to 0$, we have for some constants C_{λ_i},

$$\left| \int_{B_2^{n-4}} |F_{A_{i,\lambda_i}}|^2(a, y_2)\phi^2(y_2)dy_2 - C_i^\lambda \right| \le \delta_1$$

when i large. As in [14, 22], this then implies $\lim C_{\lambda_i} = \Theta^{n-4}(\mu, x_0)$. It then follows as in those references that the projection from $\mathbb{R}^n \to \mathbb{R}^{n-4} \times 0$ will give a contradiction. $\square$

3.5. **Proof of Theorem 1.4.** Now we are ready to finish the proof for Theorem 1.4 as in [14, 22]. By the Besicovitch-Federer decomposition theorem, we can write $\Sigma = \Sigma^r \cup \Sigma^u$, where Σ^r is $(n-4)$-rectifiable while Σ^u is purely $(n-4)$-unrectifiable. Furthermore, if $\Sigma^u \neq \emptyset$, then $\mathrm{Vol}_{\mathcal{H}^{n-4}}(\Sigma^u) > 0$. By Proposition 3.10, we know

$$\mathrm{Vol}_{\mathcal{H}^{n-4}}(P_V(\Sigma^u \cap B_r(x))) = 0$$

while by Proposition 3.11, we have

$$\mathrm{Vol}_{\mathcal{H}^{n-4}}(P_V(\Sigma^u \cap B_r(x))) > 0$$

for $0 < r \ll 1$. This is a contradiction. In particular, this implies $\mathrm{Vol}_{\mathcal{H}^{n-4}}(\Sigma^u) = 0$, and so $\Sigma^u = \emptyset$. Thus, Σ is $(n-4)$-rectifiable.

4. Weak compactification of the moduli space of smooth Ω-Yang-Mills connections

In this section, we will study the compactification of the moduli space of smooth Ω-YM connections on a fixed bundle E with bounded L^2 norm of curvature over (M, g). We denote the moduli space as

$$\mathcal{A}_{\Omega,c} := \{A \in \mathcal{A} : d_A^*(F_A + *(F_A \wedge \Omega)) = 0, \int_M |F_A|^2 \le c\}$$

Given a sequence $A_i \in A_{\Omega,c}$, by passing to a subsequence, we can assume $|F_{A_i}|^2\,\mathrm{dVol}$ converges to μ a sequence of Radon measures, and modulo gauge transformations, A_i converges to A outside $\pi(\mu)$. Define $\overline{\mathcal{A}_{\Omega,c}}$ to be the space of such pairs (A, μ).

Definition 4.1. Given a sequence $(A_i, \mu_i) \in \overline{\mathcal{A}_{\Omega,c}}$, we say A_i converges to a finite energy Ω-YM connection (A_∞, μ_∞) if

(1) μ_i converges to μ_∞ weakly as a sequence of Radon measures;
(2) up to gauge transforms, A_i converges to A_∞ outside $\pi(\mu_\infty)$.

Theorem 4.2. *$\overline{\mathcal{A}_{\Omega,c}}$ is weakly sequentially compact in the sense that every sequence $\{(A_i, \mu_i)\}$ in $\overline{\mathcal{A}_{\Omega,c}}$ sub-converges to some $(A_\infty, \mu_\infty) \in \overline{\mathcal{A}_{\Omega,c}}$.*

Proof. Given a sequence $(A_i, \mu_i) \in \overline{\mathcal{A}_{\Omega,c}}$, by assumption, for each i, we can find a sequence of $\{A_{ij}\}_j$ so that $\mu_{ij} = |F_{A_{ij}}|^2\,\mathrm{dVol}$ converges to μ_i weakly as a sequence of Radon measures. By a diagonal sequence argument, we can assume μ_{ij} and μ_i both converge weakly to μ_∞ as sequences of Radon measures. The following now is needed to guarantee the existence of the limit of A_i

$$\limsup_i \pi(\mu_i) \subset \pi(\mu_\infty). \tag{4.1}$$

Suppose this is not true. By passing to a subsequence, there exists a sequence of points $x_i \in \pi(\mu_i)$ which converges to $x_\infty \notin \pi(\mu_\infty)$. In particular, we have for $0 < r < dist(x_\infty, \pi(\mu_\infty))$

$$\mu_\infty(\partial B_r(x_\infty)) = 0,$$

which implies $r^{4-n}\mu_i(B_r(x_i)) \leq \epsilon_0/2$, for r sufficiently small. This, of course, contradicts with the assumption that $x_i \in \pi(\mu_i)$. Given this, up to gauge transforms, we can assume A_i sub-converges to A_∞ outside $\pi(\mu_\infty)$ smoothly. Indeed, a priori, we only know that A_i converges to A_∞ outside a closed subset $\widetilde{\Sigma} \subset M \setminus \pi(\mu_\infty)$ of Hausdorff codimension at 4 set. However, since we already know that $\mu_\infty|_{M\setminus\pi(\mu_\infty)} = |F_{A_\infty}|^2 \,\mathrm{dVol}$, by Lemma 3.1, we know

$$r^{4-n}\mu_i(B_r(x)) \leq \epsilon_0/2$$

for i large. This implies that A_i converges to A_∞ smoothly over $B_r(x)$. In particular, we know $\widetilde{\Sigma} = \emptyset$, i.e. A_i sub-converges to A_∞ smoothly outside $\pi(\mu_\infty)$. Now by a diagonal sequence argument again, we can assume A_{ij} sub-converges to A_∞ smoothly outside $\pi(\mu_\infty)$. The sequential compactness follows. □

Remark 4.3.
- For general finite energy Ω-YM connections on a fixed bundle over M, or even YM connections, we do not know whether we can take a limit or not due to lack of control of $\mathrm{Sing}(A_i)$. It is very crucial to assume they all come from limits of smooth connections here.
- The compactness we obtain here is very weak due to the fact that the limiting bundles E_∞ are not known to be isometric to $E|_{M\setminus\Sigma}$. This does, however, hold in the case of Hermitian-Yang-Mills connections over general complex manifolds (see Corollary 7.4)

5. Singularity formation

5.1. **Bubbling connections at a generic point.** Using the proof of Proposition 3.11, the argument in [22, Prop. 4.1.1] for the case of Yang-Mills connections gives

Proposition 5.1. *Fix a point $x \in \Sigma$ so that*

- *the tangent plane of Σ at x exists uniquely;*
- *$\Theta^{n-4}(\mu, \cdot)$ is $\mathcal{H}^{n-4}$-Hausdorff continuous at x ;*
- $\limsup_r r^{4-n} \int_{B_r} |F_{A_\infty}|^2 = 0.$

By passing to a subsequence, up to gauge transforms, A_{i,λ_i} converges to a Ω_x-YM connection B_∞ over $\mathbb{R}^n$ with $\mathbb{R}^n = T_x\Sigma \times (T_x\Sigma)^\perp$ satisfying $\iota_v F_{B_\infty} = 0$, for any $v \in T_x\Sigma$.

Following [22], we call B_∞ a *bubbling connection* of the sequence $\{A_i\}$ at x.

5.2. **Tangent cones of the limits.** Denote $(A^\lambda_\infty, \mu^\lambda_\infty) = \lambda^*(A_\infty, \mu_\infty)$ where $\lambda : B_{\lambda^{-1}\delta_0}(x) \to B_{\delta_0}(x)$.

Proposition 5.2. *By passing to a subsequence,*

- *μ^λ_∞ converges to a cone measure η;*
- *up to gauge transforms, A^λ_∞ converges to A^c_∞ outside*

$$\pi(\eta) = \{x \in \mathbb{R}^n : \Theta^{n-4}(\eta, x) \geq \epsilon_0^2\}$$

which is scaling invariant. Furthermore, $\iota_{\partial_r} F_{A^c_\infty} = 0$.

Proof. The first statement follows from Proposition 3.4. Given this, it follows the same as Theorem 4.2 that

$$\limsup_{\lambda} \pi(\mu_\infty^\lambda) \subset \pi(\eta).$$

Now up to gauge transforms, we can assume A_∞^λ sub-converges to A_∞^c smoothly outside $\pi(\eta)$. It follows from the monotonicity formula that $\iota_{\partial_r} F_{A_\infty^c} = 0$, outside $\pi(\eta)$. Since η is a cone measure, we know also $\pi(\eta)$ is also a cone. $\square$

We call (A_∞^c, η) a *tangent cone* of (A_∞, μ_∞) at the point x. A priori, we donot know whether it is unique or not since this involves a choice of the subsequence.

Remark 5.3. In [22], the tangent cones of general stationary Yang-Mills connections are shown to exist where the stationary condition is needed for the monotonicity formula. Here as long as we know (A_∞, μ_∞) comes from the limit of smooth connections, it already has a monotonicity property that suffices for use.

5.3. **Ω-ASD instantons and calibrated geometries.** Given the analytic results above, it is straightforward to see that the results in [22] hold for general Ω-ASD instantons without assuming Ω to be closed. More precisely, we assume (A_∞, μ_∞) is an finite energy Ω-ASD instanton which comes from the limit of a sequence of smooth Ω-ASD instantons with uniformly bounded L^2 norm on curvature. We also write

$$\mu_\infty = |F_{A_\infty}|^2 \,\mathrm{dVol} + \Theta^{n-4}(x)\mathcal{H}^{n-4}_\Sigma$$

as before. Similar to Proposition 4.2.1 in [22], the following holds

Proposition 5.4. *A bubbling connection B_∞ of (A_∞, μ_∞) at $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$ is a Ω_x-ASD instanton. In particular, Ω_x induces a volume form of Σ at x.*

This implies the following, as pointed out in the Yang-Mills case in [22, p. 242, Remark 5]). The proof is exactly the same.

Theorem 5.5. *For the limiting connection (A_∞, μ_∞)*

- $\dfrac{1}{8\pi^2}\Theta^{n-4}(x)$ *is integer valued at $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$;*
- *Ω restricts to a volume form of $T_x\Sigma$ at $\mathcal{H}^{n-4}$ a.e. $x \in \Sigma$.*

6. Removable Singularities

In this section, using the main results in [20] we generalize the removable singularity theorem for stationary Yang-Mills fields in [21] to the case of Ω-YM connections. The argument closely follows [20, Theorem 10]. Below we will denote by A an Ω-YM connection defined on the trivial bundle over $M \setminus \Sigma$, where $M = [-4, 4]^n$ endowed with a smooth Riemannian metric, Ω is a smooth $(n-4)$-form on M, and Σ is a closed subset of U of finite $(n-4)$-dimensional Hausdorff measure.

Theorem 6.1. *If $\sup_{x\in M} \sup_\sigma f_2(x, r)$ is sufficiently small, then for any $B_r(x) \subset \Omega$, there exists a gauge transform g over $B_r(x) \setminus \Sigma$ so that $g(A)$ extends to a smooth connection over $B_r(x)$.*

Proof. Denote $f = |F_A|$. It suffices to show that f satisfies

$$-\Delta f + \alpha\frac{|df|^2}{f} - c|F_A|^2 f \leq Cf \tag{6.1}$$

over $M \setminus \Sigma$ for some $\alpha > 0$. Indeed, given (6.1), by [20, Thm. 9] we know that $f \in L^\infty([-1,1]^n)$. Now the existence of the gauge transformation follows from [20, App. C, Thm. 19]. It remains to show that f satisfies the inequality (6.1). By (2.8) we have

$$\begin{aligned}-\frac{1}{2}\Delta|F_A|^2 &= -|\nabla_A F_A|^2 + (\nabla_A^*\nabla_A F_A, F_A)\\ &= -|\nabla_A F_A|^2 + (\{F_A, F_A\}, F_A) + (\{R_g, F_A\}, F_A) + (\{d\Omega, \nabla_A F_A\}, F_A)\end{aligned}$$

which implies

$$\begin{aligned}&-\frac{1}{2}\Delta|F_A|^2 + |\nabla_A F_A|^2 + |d_A F_A|^2 + |d_A^* F_A|^2\\ \leq&(\{F_A, F_A\}, F_A) + (\{R_g, F_A\}, F_A) + (\{d\Omega, \nabla_A F_A\}, F_A) + |d\Omega \wedge F_A|^2\\ \leq&C|F_A|^3 + C_\epsilon|F_A|^2 + \epsilon|\nabla_A F_A|^2\end{aligned}$$

where the last line follows from Hölder's inequality, and $0 < \epsilon \ll 1$ is to be determined later. This then implies

$$-\frac{1}{2}\Delta|F_A|^2 + (1-\epsilon)(|\nabla_A F_A|^2 + |d_A F_A|^2 + |d_A^* F_A|^2) - C|F_A|^3 \leq C_\epsilon|F_A|^2. \tag{6.2}$$

Now the improved Kato inequality (see [20, Thm. 5]) gives

$$|\nabla_A F_A|^2 + |d_A F_A|^2 + |d_A^* F_A|^2 \geq \frac{n}{n-1}|d|F_A||^2.$$

Combined with (6.2) this gives

$$-\frac{1}{2}\Delta|F_A|^2 + (1-\epsilon)\frac{n}{n-1}|d|F_A||^2 - C|F_A|^3 \leq C_\epsilon|F_A|^2.$$

Substituting $f = |F_A|$ and $u = |F_A|^2$, we have

$$-\frac{1}{2}\Delta f^2 + \frac{(1-\epsilon)n}{n-1}|df^2|^2 - Cuf \leq C_\epsilon f^2.$$

A straightforward calculation now shows

$$-\Delta f + \left(\frac{(1-\epsilon)n}{n-1} - 1\right)\left|\frac{df}{f}\right|^2 - Cu \leq C_\epsilon f.$$

Choose ϵ so that $\alpha = \dfrac{(1-\epsilon)n}{n-1} - 1 > 0$, and (6.1) follows. □

7. Hermitian-Yang-Mills connections over general complex manifolds

7.1. **Improvement of the analytic results.** In this section, we will generalize Tian's holomorphic cycle theorem for Hermitian-Yang-Mills connections over Kähler manifolds [22, Thm. 4.3.3] to the case of Hermitian manifolds. More precisely, we fix A_i to be a sequence of HYM connections over an m-dimensional Hermitian manifold (X, ω) with $\|F_{A_i}\| \leq C$. These are *not* Yang-Mills connections in general. As before, let

$$\Sigma = \{x \in B : \lim_{r\to 0^+}\liminf_i r^{4-2m}\int_{B_x(r)}|F_A|^2 \geq \epsilon_0^2\}.$$

Then we can assume

- $\mu_i := |F_{A_i}|^2 \,\mathrm{dVol} \rightharpoonup \mu = |F_{A_\infty}|^2 \,\mathrm{dVol} + \nu$ where $\mathrm{supp}(\nu)$ is equal to the pure complex codimension 2 part of Σ;
- up to gauge transforms, A_i sub-converges to A_∞ outside Σ.

Remark 7.1. Strictly speaking, without assuming the Hermitian-Einstein constant vanishes, i.e. $\sqrt{-1}\Lambda F_A = 0$, HYM connections are not exactly Ω-ASD instantons in the sense of (1.3), where $\Omega = \omega^{m-2}/(m-2)!$. But it is projectively Ω-ASD connections in the sense that

$$*(F_A^{\perp} \wedge \Omega) = -F_A^{\perp}$$

where $F_A^{\perp} = F_A - \mu \,\mathrm{Id}\, \omega$ satisfying $F_A^{\perp} \wedge \omega^{m-1} = 0$. It is straightforward to see that the results for Ω-YM connections holds for this case by using the same argument. There is another way to see this. By the Bochner-Kodaira-Nakano identity (see [5, Theorem 1.1]), we have

$$d_A^* F_A = \rho F_A$$

for some $\rho = \rho([\Lambda, \partial\omega], [\Lambda, \bar{\partial}\omega])$, for which the same arguments as for Ω-YM connections apply. The results in the previous sections hold in this case.

The following can be deduced easily from [1, Thm. 2].

Proposition 7.2. *(1) E_∞ can be extended uniquely as a reflexive sheaf $\mathcal{E}_\infty$ over M. For any local section $s \in \mathcal{E}_\infty$, $\log^+ |s|^2 \in H^1_{loc} \cap L^\infty_{loc}$. Furthermore, A_∞ can be extended to be defined over $M \setminus \mathrm{Sing}(\mathcal{E}_\infty)$. In particular, $\mathrm{Tr}(F_{A_\infty} \wedge F_{A_\infty})$ is closed across Σ, thus the current*

$$c_2(\Sigma) = \lim_{j_i} \mathrm{Tr}(F_{A_{j_i}} \wedge F_{A_{j_i}}) - \mathrm{Tr}(F_{A_\infty} \wedge F_{A_\infty})$$

is closed.

(2) $\Sigma = \mathrm{Sing}(\mathcal{E}_\infty) \cup \cup_k \Sigma_k$ is a complex subvariety of M and

$$c_2(\Sigma) = \sum m_k [\Sigma_k]. \tag{7.1}$$

In particular, $\nu = \sum m_k \mathcal{H}^{2n-4}_{\Sigma_k}$ where Σ_k are the irreducible pure codimension 2 components of Σ and

$$\mu_\infty = |F_{A_\infty}|^2 \,\mathrm{dVol} + \sum_k m_k \mathcal{H}^{2n-4}_{\Sigma_k}. \tag{7.2}$$

Proof. For (1), locally by replacing ω with any Kähler metric, it does not change the fact that $\|F_{A_\infty}\|_{L^2_{loc}} < \infty$. By Theorem 2 in [1], we know that E_∞ can be extended uniquely as a reflexive sheaf $\mathcal{E}_\infty$ over M. Furthermore, for any local section $s \in \mathcal{E}_\infty$, $\log^+ |s|^2 \in H^1_{loc}$. Then the local L^∞ bound follows from Moser iteration. Given this, one can directly repeat the proof for Proposition 1 in [1] to extend A_∞ by extending the metric H_∞ locally. Now we use Simpson's trick to show the closedness of $\mathrm{Tr}(F_{A_\infty} \wedge F_{A_\infty})$ (see [19, p. 71]). By proceeding with stratum of $\mathrm{Sing}(\mathcal{E}_\infty)$ which has codimension at least 6, we can choose a point $x \in \mathrm{Sing}(\mathcal{E}_\infty)$ which is smooth at $x \in \mathrm{Sing}(\mathcal{E}_\infty)$. Let ψ be a smooth $(n-5)$-form which is compactly supported near x.

- Suppose ψ has vanishing constant coefficients. We can choose a family of cut-off function ϕ_ϵ which vanishes over an ϵ-neighborhood of x and $d(\phi_\epsilon \psi)$ is uniformly bounded. In particular, we have

$$\int_M \mathrm{Tr}(F_{A_\infty} \wedge F_{A_\infty}) \wedge d\psi = \lim_{\epsilon \to 0} \int_M \mathrm{Tr}(F_{A_\infty} \wedge F_{A_\infty}) \wedge d(\phi_\epsilon \psi) = 0.$$

- In general, since $\mathrm{Sing}(\mathcal{E}_\infty)$ has codimension at least 6, we know that $\psi = \sum_i dx_i \wedge \omega_i$, where x_i are defining coordinates for $\mathrm{Sing}(\mathcal{E}_\infty)$. Now $\psi - \sum_i d(x_i\omega_i)$ vanishes along $\mathrm{Sing}(\mathcal{E}_\infty)$ and satisfies $d(\psi - \sum_i d(x_i\omega_i)) = d\psi$. By the special case above, we know

$$\int_M \mathrm{Tr}(F_{A_\infty} \wedge F_{A_\infty}) \wedge d\psi = 0.$$

Now we prove (2). We first show $\mathrm{Sing}(\mathcal{E}_\infty) \cup \cup_k \Sigma_k \subset \Sigma$. From the above, we know $\mathrm{Sing}(A_\infty) \subset \mathrm{Sing}(\mathcal{E}_\infty)$. It remains to show that $\mathrm{Supp}(\nu)$ is a pure codimension 2 subvariety of M. Indeed, we know Σ is calibrated by $\omega^{m-2}/(m-2)!$, which implies $T_x\Sigma$ is a complex analytic subspace of T_xM. Given this, it follows from part (1) and Theorem 5.5 that $c_2(\Sigma)$ is a closed integral current. Then by King's theorem [12] we can express $c_2(\Sigma)$ in the form (7.1) for some integers m_k and pure codimension 2 subvarieties Σ_k of M. This implies $\Sigma \subset \mathrm{Sing}(\mathcal{E}_\infty) \cup \cup_k \Sigma_k$., through which the top pure codimension 2 parts are identified. For the other direction, suppose not, there exists a point $x \in \mathrm{Sing}(\mathcal{E}_\infty)$ with $\Theta^{n-4}(\mu_\infty, x) = 0$. As Theorem 4.2, we can conclude that $r^{4-2n}\mu_i(B_r(x)) < \epsilon_0/2$, for i large and r small. This implies that A_i sub-converges to A_∞ smoothly near x, which gives a contradiction. In sum, we have $\Sigma = \mathrm{Sing}(\mathcal{E}_\infty) \cup \cup_k \Sigma_k$. □

Remark 7.3.
- It follows by exactly the same argument that Proposition 7.2 (1) holds for general admissible Hermitian-Yang-Mills connections over complex Hermitian manifolds, i.e. smooth Hermitian-Yang-Mills connections defined away from a closed Hausdorff codimension 4 set.
- It is straightforward to see that the proof for the closedness part holds for general finite energy Ω-YM connections with mild singularities; for example, when the singular set can be stratified by smooth manifolds of real codimension at least 6. In general, it is conjectured that the set of essential singularities of finite energy Ω-ASD instantons when Ω is closed has Hausdorff codimension at least 6 (see [22]).

Corollary 7.4. *As a smooth bundle, $E_\infty|_{M\setminus\Sigma} \cong E|_{M\setminus\Sigma}$. In particular, we can assume there exists a sequence of bundle isometries $\Phi_{j_i} : E_\infty \to E|_{M\setminus\Sigma}$ so that $\Phi_{j_i}^* A_{j_i}$ locally converges to A_∞ smoothly away from Σ.*

Given this, let E be a Hermitian bundle over a compact Hermitian manifold (M, ω). Denote $\overline{M_{HYM,c}}$ to be the space of limits of smooth Hermitian-Yang-Mills connections on E with L^2 norm of curvature bounded by c *mod gauge* (smooth wherever the connections are smooth). We give $\overline{M_{HYM,c}}$ a topology by specifying a basis of open neighborhood as $\mathcal{U}_{\vec{\epsilon},\phi}([A,\mu])$ consisting of $[(A',\mu')] \in \overline{M_{HYM,c}}$ satisfying
- A' lies in the ϵ_1 neighborhood of A outside a ϵ_1 neighborhood of $\pi(\mu)$;
- $|\mu(\phi) - \mu'(\phi)| < \epsilon_2$.

Here $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$ with $\epsilon_i > 0$ for $i = 1, 2$ and ϕ is a continuous and bounded function.

Remark 7.5. When $m = 2$, this topology coincides exactly with the topology in the case of four dimensional manifolds (see [6, Section 4.4]).

Given this, we have the following improved version of Theorem 4.2

Theorem 7.6. *$\overline{M_{HYM,c}}$ is a first countable sequentially compact Hausdorff space.*

By Proposition 7.2, the moduli space can be also viewed as consisting of pairs $(A_\infty, \mathcal{C}^{an})$ mod gauge where $\mathcal{C}^{an} = \sum_k m_k\Sigma_k$ is a integer linear combination of pure codimension two subvarities of X. Later we will not make a difference between them.

7.2. **HYM connections over a class of balanced manifolds of Hodge-Riemann type.** Now we assume (M,ω) is an m-dimensional compact balanced Hermitian manifold of Hodge-Riemann type as defined in [3, Def. 2.7]. This means we can write

$$\omega^{m-1} = \omega_0 \wedge \Omega_0$$

where ω_0 is a strictly positive $(1,1)$ form, Ω_0 is of type $(m-2,m-2)$, and

(1) $d\omega^{m-1} = 0$;
(2) $d\Omega_0 = 0$;
(3) for any $p+q=2$, there exists a pointwise Q-orthogonal decomposition

$$\Lambda^{p,q} = \mathbb{C}\omega_0 \oplus P^{p,q}$$

where $P^{p,q} = \{\alpha \in \Lambda^{p,q} : \alpha \wedge \omega_0 \wedge \Omega_0 = 0\}$;
(4) $Q(\alpha,\beta) := (\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} * (\alpha \wedge \overline{\beta} \wedge \Omega_0)$ is positive definite on $P^{p,q}$.

In this case, a uniform bound for the L^2 norm of curvature of all the smooth irreducible Hermitian-Yang-Mills connections is automatic by the following observation.

Lemma 7.7. *Given any HYM connection A on E,*

$$\int_X |F_A|^2 \frac{\omega^m}{m!} \leq C$$

where $C = C(c(E), \omega_i)$.

Proof. By conditions (3) and (4) we have

$$\int_X |F_A|^2 \frac{\omega^{m-1}}{(m-1)!} \leq C_1 (\int_X \mathrm{Tr}(F_A \wedge F_A) \wedge \Omega_0 + C_2 \int_X |f|^2 \omega_0 \wedge \omega_0 \wedge \Omega_0)$$

where $F_A^{\perp} = F_A - f\,\mathrm{Id}\,\omega_0$. Here

$$f = \mu \frac{\frac{\omega^n}{n!}}{\omega_0 \wedge \omega_0 \wedge \Omega_0}$$

In particular, we have

$$\int_X |F_A|^2 \frac{\omega^n}{n!} \leq C_1 (\int_X F_A \wedge F_A \wedge \Omega_0 + C_2 \mu^2 \int_X \frac{\frac{\omega^n}{n!}}{\omega_0 \wedge \omega_0 \wedge \Omega_0} \frac{\omega^n}{n!}). \tag{7.3}$$

The result follows. $\square$

In this case, we denote the compactification of the moduli space of HYM connections mod gauge as $\overline{M_{HYM}}$ by choosing c large.

Theorem 7.8. *On a unitrary bundle over a compact balanced Hermitian manifold (X,ω) of Hodge-Riemann type, $\overline{M_{HYM}}$ is a first countable sequentially compact Hausdorff space.*

Now we would like to give an important class of balanced metrics of Hodge-Riemann type, which comes from *multipolarizations*. Namely, for any positive $(1,1)$ forms $\omega_0, \cdots, \omega_{m-2}$ on a compact complex manifold X so that

$$\begin{aligned} \frac{\omega^{m-1}}{(m-1)!} &= \omega_0 \wedge \cdots \wedge \omega_{m-1} \\ d(\omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{m-2}) &= 0 \\ d(\omega_1 \wedge \cdots \wedge \omega_{m-2}) &= 0 \end{aligned} \tag{7.4}$$

then by the main result in [23] we get a balanced Hermitian metric ω of Hodge-Riemann type by setting $\Omega_0 = \omega_1 \wedge \cdots \wedge \omega_{m-2}$.

Corollary 7.9. *On a unitrary bundle over a compact balanced Hermitian manifold (X, ω) satisfying (7.4), $\overline{M_{HYM}}$ is a first countable sequentially compact Hausdorff space.*

In particular, this gives the following

Corollary 7.10. *On a unitrary bundle over a compact Kähler manifold (X, ω), $\overline{M_{HYM}}$ is a first countable sequentially compact Hausdorff space.*

Remark 7.11. When (X, ω) is a projective algebraic manifold, i.e. $\omega = c_1[L]$ for some line bundle L, it is known that $\overline{M^*_{HYM}}$, which denotes the closure of the space of irreducible HYM connections with *fixed determinants* in $\overline{M_{HYM}}$, admits a complex structure coming from the algebraic geometric side. The induced complex structure makes it an algebraic space (see [10]). We will explain how it can be generalized to the case of multipolarizations in the following by using the same argument in [10] and the algebraic geometric results in [11].

7.3. $\overline{M^*_{HYM}}$ **for multipolarizations.** In this section, we fix (E, H) to be a unitary vector bundle over a compact complex Hermitian manifold (X, ω) so that

$$\frac{\omega^{m-1}}{(m-1)!} = \omega_0 \wedge \cdots \wedge \omega_{m-2}$$

where $[\omega_i]$ are all ample classes, i.e. $[\omega_i] = c_1(L_i)$ for some ample line bundles L_i. Set $\Omega_0 = \omega_1 \wedge \cdots \wedge \omega_{m-2}$. As mentioned above, we can view the moduli space $\overline{M^*_{HYM}}$ consisting of pairs $(A_\infty, \mathcal{C}^{an})$ mod gauge. It is a sequentially compact Hausdorff space. Using the argument in [10], we briefly explain how a complex structure could be given to $\overline{M^*_{HYM}}$ to make it an algebraic space.

7.3.1. *Moduli space of semistable torsion free sheaves via multipolarizations.* In this section, we will recall the construction for the compactification of the moduli space of semistable sheaves with given numerical classes and fixed determinant. We refer the readers to [11] for more details. Recall that the space of slope semistable sheaves having the same Chern classes as E over (X, ω) is bounded, i.e. if we fix $\mathcal{O}(1)$ to be any polarization of X, for fixed k large enough, for any $\mathcal{E}$, we have $H^i(X, \mathcal{E}(k)) = 0$, for $i > 1$, and $\mathcal{E}(k)$ is globally generated. Let

$$\mathcal{H} = \mathbb{C}^{\oplus \tau(k)} \otimes \mathcal{O}(-k)$$

where τ denotes the Hilbert polynomial of $\mathcal{E}$. Now we know for k fixed large enough, all such sheaves can be viewed as points $[q : \mathcal{H} \to \mathcal{E}]$ in $Quot(\mathcal{H}, \tau)$ by choosing an isomorphism $\mathbb{C}^{\oplus \tau(k)} \cong H^0(X, \mathcal{E}(k))$. Here $Quot(\mathcal{H}, \tau)$ denotes the space of points given by surjective maps $q : \mathcal{H} \to \mathcal{E}$, where the Hilbert polynomial of $\mathcal{E}$ is equal to τ_E, modulo the equivalence: $q : \mathcal{H} \to \mathcal{E}$ and $q' : \mathcal{H} \to \mathcal{E}'$ are equivalent if and only if there exists an isomorphism $f \circ q = q'$, i.e. $\ker(q) = \ker(q')$. Furthermore, there exists a universal quotient

$$q_{\mathcal{U}} : \mathcal{O}_{Quot(H, \tau_E)} \otimes \mathcal{H} \to \mathcal{U}.$$

over $Quot(H, \tau_E) \times X$ which restricts to the natural quotient at each point $[q]$. Now we denote $R^{\mu ss}$ as the subscheme of $Quot(\mathcal{E}, \mathcal{H})$ consisting of elements $[q : \mathcal{H} \to \mathcal{E}]$ so that

- $\mathcal{E}$ is semistable;
- $\det(\mathcal{E}) = \mathcal{J}$;
- $\mathcal{E}$ has the same numerical classes as $\mathcal{E}$;

- q induces an isomorphism between $\mathbb{C}^{\oplus\tau(k)}$ and $H^0(X, \mathcal{E}(k))$.

Define $\mathcal{Z}$ as the weak normalization of the reduction of $R^{\mu ss}$. Denote

$$q_{\tilde{\mathcal{U}}} : \mathcal{O}_{Quot(H,\tau_E)} \otimes \mathcal{H} \to \tilde{\mathcal{U}}$$

as the pull-back of the universal quotient $[q_{\mathcal{U}}]$ to $\mathcal{Z} \times X$. Consider the class

$$u_{n-1} = -\operatorname{rank}(E) c_1(L_1) \cdots c_1(L_{n-1}) + \chi(c_1(L_1) \cdots c_1(L_{n-1}).c(E))[\mathcal{O}_x]$$

where $x \in X$ is a fixed point. Now consider the line bundle

$$\mathcal{L}_{n-1} := \lambda_{\tilde{\mathcal{U}}}(u_{n-1})$$

of which the higher power is a semi-ample line bundle over $\mathcal{Z}$. Then one can form a formal GIT quotient as

$$M^{\mu ss} := \operatorname{Proj}(\oplus_{k \geq 0} H^0(\mathcal{Z}, \mathcal{L}_{n-1}^{\nu N})^{\mathsf{SL}})$$

for some N. The conclusion is that this is a projective scheme with certain universal properties and the natural surjective map $\pi : \mathcal{Z} \to M^{\mu ss}$ collapses the SL orbits and $\pi(q) = \pi(q')$ only if the sheaves $\mathcal{E}$ and $\mathcal{E}'$ associated to q and q' share the same graded sheaf $\operatorname{Gr}^{HNS}(\mathcal{E}) \cong \operatorname{Gr}^{HNS}(\mathcal{E}')$ and $\mathcal{C}(\mathcal{E}) = \mathcal{C}(\mathcal{E}')$. When $\dim X = 2$, the converse holds.

7.3.2. *Complex structure on $\overline{M^*_{HYM}}$ induced from a continuity map $\overline{\Phi}$.* Given a stable unitary bundle over $(E, H, \bar{\partial}_A)$ over (X, ω), the most general version of the Donaldson-Uhlenbeck-Yau theorem states that there exists a complex gauge transformation g so that the unitary connection given by $(H, g(\bar{\partial}_A))$ is a HYM connection that is unique up to unitary gauge transformations. Now this can be generalized to the case of stable reflexive sheaf using the notion of *admissible* HYM connections (i.e. finite energy on the smooth locus). Suppose $[q] \in Quot$ represents a semistable torsion free sheaf $\mathcal{E}$. We can take the graded sheaf $\operatorname{Gr}^{HNS}(\mathcal{E})$ associated to a Harder-Narasimhan-Seshadri filtration of $\mathcal{E}$. From this we can extract canonical algebraic data as

$$((\operatorname{Gr}^{HNS}(\mathcal{E}))^{**}, \mathcal{C}(\mathcal{E}))$$

from which the first factor gives a unique admissible HYM connection $A(\mathcal{E})$. Here

$$\mathcal{C}(\mathcal{E}) = \sum m_k^{alg} \Sigma_k$$

where Σ_k is a pure codimension two subvariety of X and

$$m_k^{alg} = h^0(\Delta, ((\operatorname{Gr}^{HNS}(\mathcal{E}))^{**}/\operatorname{Gr}^{HNS}(\mathcal{E}))|_\Delta).$$

Here Δ is a generic holomorphic transverse slice of Σ_k.

Definition 7.12. We define $\overline{M^s}$ to be the closure of $(M^s)^{wn}$ in $M^{\mu ss}$ where $(M^s)^{wn}$ denotes the weak normalization of M^s.

Then we have

Theorem 7.13. *There exists a continuous map*

$$\overline{\Phi} : \overline{M^s} \to \overline{M^*_{HYM}}$$

which restricts to the natural map

$$\Phi : (M^s)^{wn} \to (M^*_{HYM})^{wn}.$$

More precisely, suppose $[q : \mathcal{H} \to \mathcal{E}]$ represents a point in $\overline{M^s}$, then $\overline{\Phi}([\mathcal{E}]) = (A(\mathcal{E}), \mathcal{C}(\mathcal{E}))$.

We very briefly explain how the proof is done and refer the reader to [10] for more details. We fix a sequence of smooth HYM connections $\{A_i\}$ on E which sub-converges to $(A_\infty, \mathcal{C}^{an})$. By the boundedness, we can put $\mathcal{E}_i = (E, \bar{\partial}_{A_i})$ in a fixed Quot scheme and thus obtain an algebraic limit which can behave badly in general. More precisely, by fixing k large and choosing an L^2 orthonormal basis for $H^0(X, \mathcal{E}_i(k))$, we get a sequence of elements $[q_i]$ in the corresponding Quot scheme. Then we can take an algebraic limit $[q_\infty]$ of $[q_i]$ in the Quot scheme. As in [10, Sec. 4], it can be concluded that q_∞ induces a sheaf inclusion $\mathcal{F}_\infty^{alg} \to \mathcal{E}_\infty$ which is an isomorphism outside some codimension two subvariety. In particular, $\mathcal{E}_\infty = (\mathcal{F}_\infty^{alg})^{**}$. Using the argument in [10, Sec. 4.3], the singular Bott-Chern formula applied to the filtration of $\mathcal{H}$ induced by $[q_\infty]$ gives $\mathcal{C}(\mathcal{F}_\infty^{alg}) = \mathcal{C}$. In particular, as in [10], this gives that the map $\overline{\Phi}$ is continuous. Given this, since all the essential algebraic geometric results [11] used in [10] are done for multipolarizations, it is straightforward to adapt the corresponding statements in [10] to the case of multipolarizations to obtain the following

Theorem 7.14. *There exists a complex structure on $\overline{M_{HYM}^*}$ which makes $\overline{M_{HYM}^*}$ an algebraic space so that the natural map $\overline{\Phi} : \overline{M^s} \to \overline{M_{HYM}^*}$ is an algebraic morphism.*

References

1. Shigetoshi Bando and Yum-Tong Siu, *Stable sheaves and Einstein-Hermitian metrics*, Geometry and analysis on complex manifolds, World Sci. Publ., River Edge, NJ, 1994, pp. 39–50. MR 1463962
2. Xuemiao Chen and Song Sun, *Reflexive sheaves, Hermitian-Yang-Mills connections, and tangent cones*, Invent. Math. (2021), 1–57.
3. Xuemiao Chen and Richard Wentworth, *The nonabelian Hodge correspondence for balanced Hermitian metrics of Hodge-Riemann type*, (2021), preprint.
4. E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuyts, *First-order equations for gauge fields in spaces of dimension greater than four*, Nuclear Phys. B **214** (1983), no. 3, 452–464. MR 698892
5. Jean-Pierre Demailly, *Complex analytic and differential geometry*, Online notes.
6. S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1990, Oxford Science Publications. MR 1079726
7. Simon Donaldson and Ed Segal, *Gauge theory in higher dimensions, II*, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 1–41. MR 2893675
8. Simon Donaldson and Richard Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), Oxford Univ. Press, Oxford, 1998, pp. 31–47. MR 1634503
9. Daniel S. Freed and Karen K. Uhlenbeck, *Instantons and four-manifolds*, Mathematical Sciences Research Institute Publications, vol. 1, Springer-Verlag, New York, 1984. MR 757358
10. Daniel Greb, Benjamin Sibley, Matei Toma, and Richard Wentworth, *Complex algebraic compactifications of the moduli space of Hermitian-Yang-Mills connections on a projective manifold*, To appear in Geometry and Topology (2018), http://arxiv.org/abs/1810.00025.
11. Daniel Greb and Matei Toma, *Compact moduli spaces for slope-semistable sheaves*, Algebr. Geom. **4** (2017), no. 1, 40–78. MR 3592465
12. James R. King, *The currents defined by analytic varieties*, Acta Math. **127** (1971), no. 3-4, 185–220. MR 393550
13. Jun Li and Shing-Tung Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 560–573. MR 915839
14. Fang-Hua Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*, Ann. of Math. (2) **149** (1999), no. 3, 785–829. MR 1709303
15. Fanghua Lin and Changyou Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. MR 2431658
16. Yoichi Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 449–476. MR 946247

17. Hiraku Nakajima, *Compactness of the moduli space of Yang-Mills connections in higher dimensions*, J. Math. Soc. Japan **40** (1988), no. 3, 383–392. MR 945342
18. Peter Price, *A monotonicity formula for Yang-Mills fields*, Manuscripta Math. **43** (1983), no. 2-3, 131–166. MR 707042
19. Carlos Tschudi Simpson, *Systems of Hodge bundles and uniformization*, ProQuest LLC, Ann Arbor, MI, 1987, Thesis (Ph.D.)–Harvard University. MR 2636035
20. Penny Smith and Karen Uhlenbeck, *Removability of a codimension four singular set for solutions of a Yang-Mills-Higgs equation with small energy*, preprint (2018), http://arxiv.org/abs/1811.03135.
21. Terence Tao and Gang Tian, *A singularity removal theorem for Yang-Mills fields in higher dimensions*, J. Amer. Math. Soc. **17** (2004), no. 3, 557–593. MR 2053951
22. Gang Tian, *Gauge theory and calibrated geometry. I*, Ann. of Math. (2) **151** (2000), no. 1, 193–268. MR 1745014
23. V. A. Timorin, *Mixed Hodge-Riemann bilinear relations in a linear context*, Funktsional. Anal. i Prilozhen. **32** (1998), no. 4, 63–68, 96. MR 1678857
24. Karen K. Uhlenbeck, *A priori estimates for Yang-Mills fields*, Unpublished manuscript.
25. ———, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982), no. 1, 31–42. MR 648356
26. Alex Waldron, *Uhlenbeck compactness for Yang-Mills flow in higher dimensions*, arXiv:1812.10863 (2018).

Department of Mathematics, University of Maryland, College Park, MD 20742, USA
Email address: `xmchen@umd.edu`
Email address: `raw@umd.edu`