

A DONALDSON-UHLENBECK-YAU THEOREM FOR NORMAL VARIETIES AND SEMISTABLE BUNDLES ON DEGENERATING FAMILIES

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ABSTRACT. In this paper, we first prove a Donaldson-Uhlenbeck-Yau theorem over projective normal varieties smooth in codimension two. As a consequence we deduce the polystability of (dual) tensor products of stable reflexive sheaves, and we give a new proof of the Bogomolov-Gieseker inequality, along with a precise characterization of the case of equality. This also improves several previously known algebro-geometric results on normalized tautological classes. We study the limiting behavior of semistable bundles over a degenerating family of projective normal varieties. In the case of a family of stable vector bundles, we study the degeneration of the corresponding HYM connections and these can be characterized from the algebro-geometric perspective. In particular, this proves another version of the singular Donaldson-Uhlenbeck-Yau theorem for the normal projective varieties in the central fiber. As an application, we apply the results to the degeneration of stable bundles through the deformation to projective cones, and we explain how our results are related to the Mehta-Ramanathan restriction theorem.

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1. INTRODUCTION

Two fundamental results in the study of stable bundles over compact Kähler and projective algebraic manifolds are the Donaldson-Uhlenbeck-Yau (DUY) theorem and the Mehta-Ramanathan (MR) restriction theorem. The DUY theorem proves the existence of a Hermitian-Einstein (HE) metric on any slope stable vector bundle \mathcal{E} over a compact Kähler manifold (Y, ω) [8, 32, 9]. The associated Chern connections are called Hermitian-Yang-Mills (HYM) connections. The result was generalized by Bando and Siu to stable reflexive sheaves via the introduction of *admissible* metrics [1]. Important consequences include the polystability of tensor products of stable reflexive sheaves, as well as the Bogomolov-Gieseker inequality and a characterization of the case of equality in terms of projectively flat bundles. It moreover gives equivalent characterizations of reflexive sheaves with nef normalized tautological class. We refer to [24] for a more detailed discussion of this.

Suppose now that (Y, ω) is a Hodge manifold, i.e. $[\omega] = c_1(L)$ for some ample line bundle $L \rightarrow Y$. Let \mathcal{F} be a semistable (resp. stable) bundle over Y . The MR restriction theorem states for k sufficiently large, $\mathcal{F}|_V$ is semistable (resp. stable) for generic $V \in \mathbb{P}(H^0(Y, L^k))$ [22, 10]. This result was instrumental in Donaldson's proof of the DUY theorem in higher dimensions [9]. We refer to [17] for a discussion of some other important consequences of the restriction theorem.

The first result of this paper is a version of the DUY and Bando-Siu theorems for stable reflexive sheaves over projective normal varieties Y which are smooth in codimension 2. A motivation for a generalization to the singular case comes from the corresponding consequences mentioned above, just as the smooth case.

Next we study the behavior of semistable bundles on degenerating families, which gives us another version of the DUY theorem. The motivation here is to give a strategy for an analytic proof of the MR theorem. The idea goes as follows. Suppose \mathcal{F} is a stable bundle over a smooth projective manifold (Y, ω, L) , with $V \subset Y$ as above. Deform Y to the projective normal cone (see Section 5 for details). The naive guess is that the HYM connection on Y deforms to a HYM connection on the cone, which by the symmetry of the cone should have implications for the (semi)stability of the restriction. There is price to pay here due to the fact that we have introduced singularities on the base. Nevertheless, assuming the restriction theorem holds on the cone, then by a continuity argument one can conclude that the restriction $\mathcal{F}|_V$ is semistable (resp. stable). Working backwards, in the case where

the restriction theorem actually does hold, one can show that the sheaf defined by the limiting HYM connection is indeed of cone type, in the obvious sense. However, there are plenty of examples that tell us that this is *not* always true, i.e. the limiting HYM connection does not have to be a cone in general. Also, as in the singular DUY theorem, a technical difficulty is that we introduced a singularity to the base, and the known analytic results about admissible HYM connections on smooth manifolds are not known to hold across the singular set in general.

Given the discussion above, a closely related question naturally arises: what is the structure of the limiting HYM connection in general? This leads to the study of the behavior of semistable bundles over a degenerating family of smooth varieties with a normal central fiber. It turns out that an algebraic geometric result can be constructed in this setting using known techniques, and this can in turn be used to characterize the analytic degeneration. In the special case of a deformation to the projective cone, this picture is closely related to the MR restriction theorem. Indeed, we show that the limiting HYM connection on the cone restricts to the HYM connection on $\mathcal{F}|_V$, if the latter is assumed to be stable. Thus, the HYM connection on $\mathcal{F}|_V$ is obtained from the HYM connection on \mathcal{F} by deforming the Kähler metric on the base.

1.1. Main results. We now state more precisely the results sketched above.

1.1.1. Singular DUY theorem. In this section, we assume that (Y, ω) is a normal projective variety smooth in codimension 2, with $Y \subset \mathbb{P}^N$ and $\omega = \omega_{FS}|_{Y^{reg}}$. Slope stability of reflexive sheaves on Y with respect to ω can be defined as in the nonsingular case, and this agrees with the algebro-geometric notion of slope stability with respect to the hyperplane section D on Y (see Section 2.2). With this understood, the first main result is the following.

Theorem 1.1 (SINGULAR DUY). *Let \mathcal{F} be a stable reflexive sheaf over (Y, ω) . Then there exists an admissible HE metric on \mathcal{F} that is unique up to a constant scaling. Moreover, for any local holomorphic section s of \mathcal{F} , $\log^+ |s|^2 \in W_{loc}^{1,2} \cap L_{loc}^\infty$.*

As a corollary of Theorem 1.1, we have (see Corollary 2.17),

Corollary 1.2. *Given \mathcal{F}_1 and \mathcal{F}_2 stable reflexive sheaves over (Y, ω) , then $(\text{Sym}^k \mathcal{F}_1)^{**}$, $(\wedge^k \mathcal{F}_1)^{**}$, $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{**}$ are all polystable.*

As in the case of smooth manifolds, Theorem 1.1 gives rise to an analytic proof of the Bogomolov-Gieseker inequality of Miyaoka [23, Cor. 4.7], as well as a characterization of the case of equality. For the definitions of Chern classes and the pairing below, see Section 2.2.

Corollary 1.3 (BOGOMOLOV-GIESEKER INEQUALITY). *Let $[H]$ be the class of a hyperplane in $Y \subset \mathbb{P}^N$. Suppose $\mathcal{F} \rightarrow Y$ is a stable reflexive sheaf. Then*

$$(2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2) \cdot [H]^{n-2} \geq 0 .$$

Moreover, the equality holds if and only if \mathcal{F} is projectively flat outside $\text{Sing}(\mathcal{F})$.

Remark 1.4. We emphasize that the inequality in Corollary 1.3 is well known. For projective varieties it can be reduced to the surface case by using the MR restriction theorem [23]. More recently, it was extended to Kähler spaces [33]. The novelty in the statement above is the characterization of the case of equality.

In Section 3, we use Theorem 1.1 to improve results of Höring-Peternell on normalized tautological classes.

1.1.2. *Degeneration.* Let $p : \mathcal{X} \rightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be a flat family of projective varieties so that $X_t := p^{-1}(t)$ is smooth for $t \neq 0$ and X_0 is normal. A family $\{\mathcal{E}_t\}_{t \neq 0}$ of semistable torsion-free sheaves $\mathcal{E}_t \rightarrow X_t$ is called “algebraic” if we can find a coherent sheaf $\mathcal{E} \rightarrow \mathcal{X}$, such that $\mathcal{E}|_{X_t} \simeq \mathcal{E}_t$. We say $\mathcal{E} \rightarrow \mathcal{X}$ is a semistable degeneration of $\{\mathcal{E}_t\}_{t \neq 0}$ if furthermore \mathcal{E} is flat over \mathbb{P}^1 (cf. [17, p. 34]), and $\mathcal{E}_0 = \mathcal{E}|_{X_0}$ is torsion-free and semistable. Recall that for a semistable sheaf, $\text{Gr}(\mathcal{E}_0)$ and $\mathcal{C}(\mathcal{E}_0)$ denote the associated graded sheaf of the Jordan-Hölder filtration of \mathcal{E}_0 , and its associated codimension 2 cycle, respectively (see Definition 4.3). On the algebraic side, we prove the following.

Theorem 1.5. *A semistable degeneration always exists for any algebraic family $\{\mathcal{E}_t\}$ where \mathcal{E}_t are semistable bundles for all $t \neq 0$. Furthermore, for any two semistable degenerations \mathcal{E} and \mathcal{E}' , $\text{Gr}(\mathcal{E}_0)^{**} = \text{Gr}(\mathcal{E}'_0)^{**}$. If X_0 is smooth in codimension 2, then we also have $\mathcal{C}(\mathcal{E}_0) = \mathcal{C}(\mathcal{E}'_0)$.*

Next, assume that $\{\mathcal{E}_t\}$ are stable vector bundles for $t \neq 0$, and let A_t be the corresponding admissible HYM connections. Known analytic results tell us that we can take a gauge theoretical limit for sequences $t_i \rightarrow 0$. The limiting data consists of two parts: a smooth HYM connection A_∞ defined outside some analytic subvariety Σ of X_0 , and the so-called blow-up locus $\Sigma_b = \sum_k m_k^{an} \Sigma_k$, which is an integral linear combination of pure codimension 2 subvarieties (see Section 4.2 for more details). Let \mathcal{E}_∞ be the reflexive sheaf defined by A_∞ over X_0^{reg} .

Theorem 1.6. *Fix \mathcal{E} to be any degeneration of $\{\mathcal{E}_t\}_t$ where \mathcal{E}_t are stable vector bundles for $t \neq 0$ and \mathcal{E}_0 is torsion-free semistable. Then:*

- (I) \mathcal{E}_∞ can be extended to be a semistable reflexive sheaf $\overline{\mathcal{E}_\infty}$ over X_0 . Furthermore, for any local section s of $\overline{\mathcal{E}_\infty}$, $\log^+ |s|^2 \in W_{loc}^{1,2} \cap L_{loc}^\infty$;
- (II) $\text{Gr}(\overline{\mathcal{E}_\infty})^{**} = \text{Gr}(\mathcal{E}_0)^{**}$. In particular, if X_0 is smooth in codimension 2, then $\overline{\mathcal{E}_\infty} = \text{Gr}(\mathcal{E}_0)^{**}$;
- (III) If X_0 is smooth in codimension 2, then $\mathcal{C}(\mathcal{E}_0) = \Sigma_b$.

In particular, this gives the following version of DUY theorem

Corollary 1.7 (SINGULAR DUY FOR DEGENERATIONS). *Given \mathcal{E} a semistable degeneration of $\{\mathcal{E}_t\}_t$ where \mathcal{E}_t are stable vector bundles and \mathcal{E}_0 is torsion-free semistable. Then there exists an admissible HE metric on \mathcal{E}_0^{**} .*

1.1.3. *Relation to the Mehta-Ramanathan restriction theorem.* As another application of the results above, in Section 5, we study the degeneration of a semistable bundle through the deformation to projective cones of smooth divisors, and we show how our results are related to the MR restriction theorem. In a sense, the result obtained here (see Theorem 5.7 and Corollaries 5.8) describes the general picture when the restriction theorem might fail, i.e. the nongeneric setting. Even when the restriction theorem holds for a given stable bundle, it does not, of course, apply to the restriction of HYM connections in general. Corollary 5.10, however, can be viewed as an analytic version of the restriction theorem: the deformation to the projective cone gives a way to interpolate to the HYM connection of the restriction.

Remark 1.8. Another interesting question is whether we can use the HYM connections on the projective cones to recover the original HYM connection on \mathcal{F} via a limit as the degree increases. This is motivated by the observation that high degree hypersurfaces can be chosen to cover the ambient variety as the degree $\rightarrow \infty$. If indeed this can be made rigorous it could lead to an analytic proof of the Mehta-Ramanathan restriction theorem. We leave this for future study.

1.2. **Sketch of the proofs.** We first need definitions of Chern classes. Since we are working over singular varieties, Chern classes are naturally defined in corresponding homology groups following [2]. The first Chern class can be always defined, and the second Chern class is defined when the base variety is smooth in codimension 2. Corresponding Chern numbers, e.g. *slopes*, can be computed algebraically. The technical difficulty here is to show that the curvature of the Chern connection for the admissible Hermitian metric can indeed be used to compute Chern numbers as in the Chern-Weil theory of the smooth case. This boils down to the fact that the corresponding curvature terms indeed define closed currents across the singular set, as well as an application of Poincaré duality in the ambient smooth manifold (see Proposition 2.6). It is here where we need to work with normal complex analytic subvarieties of smooth ones with induced metrics.

Given the above, the idea for the proof of the singular DUY theorem is to use gauge theoretical methods going back to Donaldson [8], Uhlenbeck-Yau [32] and Bando-Siu [1]. We use Hironaka's theorem on resolution of singularities to simultaneously resolve \mathcal{F} and Y and obtain a holomorphic vector bundle $\widehat{\mathcal{F}}$ on a smooth resolution $\widehat{Y} \rightarrow Y$. Since Y is projective, it can be shown quite easily that $\widehat{\mathcal{F}}$ is stable with respect to a small perturbation ω_ϵ of the pullback of ω . In particular, by the DUY theorem we have a family of HYM connections A_ϵ on $\widehat{\mathcal{F}}$. After passing to a subsequence we may find an Uhlenbeck limit A_∞ of A_ϵ over the regular part Y^{reg} of Y . Then one

can show that A_∞ defines a reflexive sheaf isomorphic to the original \mathcal{F} , so the conclusion follows. This is due to a theorem of Siu and a use of MR restriction theorem.

The proof of the existence of a semistable degeneration of semistable bundles is a slight generalization of Langton's results ([19]) by following the argument in [17]. Now we explain about the uniqueness. For two different semistable degenerations, by restricting to a flat family of high degree curves, one can get the uniqueness of Gr^{**} . For the blow-up locus, we need to assume X_0 is smooth in codimension 2 so that we can restrict our family to a family of high degree smooth surfaces, and thus get uniqueness of the algebraic blow-up locus \mathcal{C} . This relies crucially on the known results for the compactification of semistable torsion-free sheaves over smooth projective surfaces.

Assume the family above consists of stable torsion-free sheaves \mathcal{E}_t . We can take a gauge theoretical limit of the corresponding HYM connections A_t and get an admissible HYM connection A_∞ and the blow-up locus Σ_b which is an integer linear combination of pure codimension 2 subvarieties of X_0 . Since in this case, we deal with normal projective varieties in general, the fact that A_∞ defines a semistable reflexive sheaf over X_0 is very technical. Assuming this, since we are working in the projective case, we can realize \mathcal{E}_∞ as the double dual of some sheaf represented by an element in a certain Quot scheme which naturally carries the information of the blow-up locus. This transforms our problem into a similar problem which can be dealt with as the uniqueness properties of the semistable degenerations.

Let \mathcal{E}_∞ be the reflexive sheaf A_∞ defines over X_0^{reg} . Now we first explain why \mathcal{E}_∞ can be extended to be a reflexive sheaf over X_0 . To solve these technical issues, the projectivity plays a key role in that we can construct a lot of sections of the twisted bundle $\mathcal{E}_\infty(k)$ for k fixed but large. Indeed, twisting by the pullback of the positive line bundle on Y , we know that the bundles $\mathcal{E}_t(k)$ have many sections. We wish to take limits to produce nontrivial sections of $\mathcal{E}_\infty(k)$. Using the HYM condition, the construction proceeds once one proves that the Sobolev constant of (X_t, ω_t) has a uniform bound which follows from a result of Leon Simon.

Given the above, we want to show that \mathcal{E}_∞ can be extended as a coherent reflexive sheaf over X_0 . This follows from an induction argument (cf. [25]). We start with one limiting section s : it will generate a rank 1 saturated subsheaf \mathcal{L} of \mathcal{E}_∞ . Furthermore, by the Remmert-Stein theorem it can be extended to a rank 1 reflexive sheaf over Y . Now by increasing k , we can find another section that generates a higher rank coherent subsheaf of \mathcal{E}_∞ , since by the Riemann-Roch theorem $\mathcal{L}(k)$ cannot contain all the limiting sections for k large. This now gives us a rank 2 saturated coherent subsheaf of \mathcal{E}_∞ which can also be extended as a coherent sheaf over X_0 . Continuing in this way, the result is obtained. Furthermore, one can show the limiting sections generate \mathcal{E}_∞ outside some codimension 2 subvariety. Thus the extension

follows, the fact that $\overline{\mathcal{E}_\infty}$ is semistable again follows from certain regularity results, which enable us to do integration by parts.

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2. SINGULAR DUY THEOREM

2.1. Admissible HYM connections. Let (Y, ω) be a normal subvariety of \mathbb{P}^N with $\omega = \Omega_{FS}|_{Y^{reg}}$. Here we write $Y = Y^{reg} \cup Y^s$, where Y^{reg} (resp. Y^s) denotes the smooth (resp. singular) locus of Y . Following [1], we will use the following definition.

Definition 2.1. Let $\mathcal{F} \rightarrow Y^{reg} \setminus \Sigma$ be a holomorphic vector bundle, where Σ is a closed set of Y^{reg} of Hausdorff real codimension ≥ 4 . A Hermitian metric h on \mathcal{F} is called *admissible* if

- $\sqrt{-1}\Lambda_\omega F_h$ is bounded;
- $\int_Y |F_h|^2 < \infty$,¹

where F_h denotes the curvature of the Chern connection of the pair (\mathcal{F}, h) defined on the smooth locus of $\mathcal{F} \rightarrow Y^{reg}$. If furthermore,

$$(2.1) \quad \sqrt{-1}\Lambda_\omega F_h = \mu \text{Id}$$

for some constant μ , then h is called a HE metric and the connection A is called an admissible HYM connection.

2.2. Chern classes. We need definitions of Chern classes of coherent sheaves on singular varieties. It seems there is not a standard definition, but we are only interested in ch_1 and ch_2 , and for these different methods likely give the same answer with our assumptions. Most importantly, we wish to compare these Chern classes with the currents defined by admissible metrics. It will be useful to use the approach of Baum-Fulton-MacPherson [2]. Recall the notation of the previous section, although here we choose an arbitrary embedding $Y \subset M$, where M is smooth and projective. As will be clear in the following, we will define the first Chern class in general and the second Chern class only when $\dim Y^s \leq n - 3$. (Co)homology groups will be taken with complex coefficients. Let $\mathcal{F} \rightarrow Y$ be a coherent analytic sheaf. Choose a locally free resolution on $M \supset Y$:

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow J_*\mathcal{F} \longrightarrow 0,$$

and denote the complex of \mathcal{F}_i 's by \mathcal{F}_\bullet . Let

$$\text{ch}^*(\mathcal{F}_\bullet) := \sum_{i=0}^{\ell} (-1)^i \text{ch}^*(\mathcal{F}_i),$$

¹The integration is understood to be taken over the smooth locus Y^{reg} .

where $\text{ch}^*(\mathcal{F}_i) \in H^*(M)$ is the Chern character. Since $j_*\mathcal{F}$ is supported on Y , $\text{ch}^*(\mathcal{F}_\bullet)$ defines a class in $H^*(M, M \setminus Y)$ (cf. [30, p. 285]). We set:

$$\text{ch}_Y^M(\mathcal{F}) := A(\text{ch}^*(\mathcal{F}_\bullet)) \in H_*(Y) .$$

Here, $A : H^*(M, M \setminus Y) \rightarrow H_*(Y)$ is Alexander duality. Define:

$$(2.2) \quad \tau(\mathcal{F}) = \text{Td}(M) \cap \text{ch}_Y^M(\mathcal{F}) .$$

Then $\tau(\mathcal{F})$ is independent of the embedding $Y \subset M$.

In the case where \mathcal{F} is the restriction of a vector bundle on M , then $\tau(\mathcal{F}) = \text{ch}^*(\mathcal{F}) \cap \tau(Y)$, where $\tau(Y) := \tau(\mathcal{O}_Y)$ ([2, p. 116 (4)]). Note that $\tau_n(Y) = [Y]$, the fundamental class of Y ([2, p. 129]). Motivated by this, we make the following

Definition 2.2. Chern classes $\text{ch}_i(\mathcal{F}) \in H_{2n-2i}(Y)$, $i = 0, 1, 2$, are defined by

$$(2.3) \quad \begin{aligned} \text{ch}_0(\mathcal{F}) &= \tau_n(\mathcal{F}) = \text{rk}(\mathcal{F})[Y] , \\ \text{ch}_1(\mathcal{F}) &= \tau_{n-1}(\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot \tau_{n-1}(Y) . \\ \text{ch}_2(\mathcal{F}) &= \tau_{n-2}(\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot \tau_{n-2}(Y) - \text{ch}_1(\mathcal{F}) \cdot \tau_{n-1}(Y) . \end{aligned}$$

Let us explain the last term on the right hand side of the expression for $\text{ch}_2(\mathcal{F})$. Since $\text{codim}(Y^s) \geq 3$,

$$H_{2n-2i}(Y) \simeq H_{2n-2i}^{\text{BM}}(Y^{\text{reg}}) , \quad i = 0, 1, 2 ,$$

where $H_*^{\text{BM}}(Y^{\text{reg}})$ denotes the Borel-Moore, or locally finite, homology. Using Poincaré duality $H_p^{\text{BM}}(Y^{\text{reg}}) \simeq H^{2n-p}(Y^{\text{reg}})$ and the cap product:

$$H^k(Y^{\text{reg}}) \otimes H_p^{\text{BM}}(Y^{\text{reg}}) \longrightarrow H_{p-k}^{\text{BM}}(Y^{\text{reg}})$$

gives well-defined intersection pairings

$$\text{ch}_1(\mathcal{F}) \cdot \tau_{n-1}(Y) , \quad \text{ch}_1(\mathcal{F}) \cdot \text{ch}_1(\mathcal{F}) \in H_{2n-4}(Y) .$$

Another consequence of the assumption $\text{codim}(Y^s) \geq 3$ means we have a homomorphism

$$(2.4) \quad H_{2n-2i}(Y) \simeq H_{2n-2i}^{\text{BM}}(Y^{\text{reg}}) \longrightarrow [H_c^{2n-2i}(Y^{\text{reg}})]^* , \quad i = 0, 1, 2 .$$

If h is an admissible metric on a reflexive coherent sheaf \mathcal{F} in the sense of Definition 2.1, then the Chern-Weil forms $\text{ch}_1(\mathcal{F}, h)$ and $\text{ch}_2(\mathcal{F}, h)$ for the Chern connection of (\mathcal{F}, h) also give elements of $[H_c^{2n-2i}(Y^{\text{reg}})]^*$ via integration. We wish to prove:

Proposition 2.3. *For $i = 0, 1, 2$, the image of the classes $\text{ch}_i(\mathcal{F})$ under the map (2.4) coincide with the classes of $\text{ch}_i(\mathcal{F}, h)$.*

Let $\widehat{M} \supset \widehat{Y} \xrightarrow{\pi} Y$ be an embedded resolution of singularities of Y , $\hat{j} : \widehat{Y} \hookrightarrow \widehat{M}$, and let $\widehat{\mathcal{F}} = \pi^*\mathcal{F}$. We furthermore let $\widetilde{\mathcal{F}}_\bullet = \pi_1(\widehat{\mathcal{F}})$. Since:

$$0 \longrightarrow \mathcal{F} \longrightarrow \pi_*(\widehat{\mathcal{F}}) \longrightarrow T \longrightarrow 0 ,$$

where $\text{supp}(T) \subset Y^s$, it follows that $\tau_i(\mathcal{F}) = \tau_i(\tilde{\mathcal{F}}_\bullet)$ for $i \geq n - 2$ if Y^s has codimension ≥ 3 . Moreover, by the naturality of the homological Todd class we have: $\tau(\tilde{\mathcal{F}}_\bullet) = \pi_*\tau(\hat{\mathcal{F}})$. Hence,

$$(2.5) \quad \tau_i(\mathcal{F}) = \pi_*\tau_i(\hat{\mathcal{F}}), \quad i \geq n - 2.$$

Let $\hat{\mathcal{F}}_\bullet$ be a locally free resolution of $\hat{\mathcal{F}}$ on the smooth variety \hat{Y} . Let $\hat{\rho}: \hat{U} \rightarrow \hat{Y}$ be a holomorphic retraction of an open neighborhood $\hat{U} \subset \hat{M}$ to \hat{Y} .

Lemma 2.4. $\tau(\hat{\mathcal{F}}) = \text{ch}^*(\hat{\rho}^*\hat{\mathcal{F}}_\bullet) \cap \tau(\hat{Y})$.

Proof. Since \hat{Y} is smooth, we have:

$$\begin{aligned} \tau(\hat{\mathcal{F}}) &= \text{Td}(\hat{M}) \cap \text{ch}_{\hat{Y}}^{\hat{M}}(\hat{\mathcal{F}}) \\ &= \text{Td}(\hat{M}) \cap A(\text{ch}^*(\hat{\rho}^*(\hat{\mathcal{F}}_\bullet)) \cup \text{ch}^*(\hat{j}_*\mathcal{O}_{\hat{Y}})) \\ &= \text{Td}(\hat{M}) \cap \text{ch}^*(\hat{\rho}^*(\hat{\mathcal{F}}_\bullet)) \cap A(\text{ch}^*(\hat{j}_*\mathcal{O}_{\hat{Y}})) \\ &= \text{ch}^*(\hat{\rho}^*(\hat{\mathcal{F}}_\bullet)) \cap (\text{Td}(\hat{M}) \cap \text{ch}_{\hat{Y}}^{\hat{M}}(\mathcal{O}_{\hat{Y}})) \\ &= \text{ch}^*(\hat{\rho}^*(\hat{\mathcal{F}}_\bullet)) \cap \tau(\hat{Y}). \end{aligned}$$

The first to second (resp. second to third) lines is in [30, p. 291 (resp. p. 288)]. \square

Proof of Proposition 2.3. Let Ω be a smooth, compactly supported, closed form on Y^{reg} of degree $2n - 2i$, $i = 0, 1, 2$, representing a class in $H_c^{2n-2i}(Y^{reg})$. We may assume that $\text{supp}\Omega$ is contained in the smooth locus of \mathcal{F} . Choose connections A_\bullet on the bundles in the resolution $\hat{\rho}^*(\hat{\mathcal{F}}_\bullet)$, and let $\text{ch}^*(\hat{\rho}^*\hat{\mathcal{F}}_\bullet, A_\bullet)$ denote the alternating sums of Chern-Weil forms. Then on the support of Ω (or $\pi^*\Omega$), $\text{ch}^{n-i}(\hat{\rho}^*\hat{\mathcal{F}}_\bullet, A_\bullet)$ and $\text{ch}_{n-i}(\mathcal{F}, h)$ differ by a smooth exact form. We will denote the images of classes by the map (2.4) with the same notation. Now, on the one hand, from (2.5), we have

$$\tau_i(\mathcal{F})([\Omega]) = \pi_*\tau_i(\hat{\mathcal{F}})([\Omega]) = \tau_i(\hat{\mathcal{F}})(\pi^*[\Omega]) = \pi^*[\Omega] \cap \tau_i(\hat{\mathcal{F}}).$$

On the other hand, from Lemma 2.4,

$$\begin{aligned} \pi^*[\Omega] \cap \tau_i(\hat{\mathcal{F}}) &= \pi^*[\Omega] \cap \left(\text{ch}^*(\hat{\rho}^*\hat{\mathcal{F}}_\bullet) \cap \tau(\hat{Y}) \right)_i \\ &= \sum_{j=i}^n \left(\pi^*[\Omega] \cup \text{ch}^{j-i}(\hat{\rho}^*\hat{\mathcal{F}}_\bullet) \right) \cap \tau_j(\hat{Y}) \\ &= \sum_{j=i}^n \left[\pi^*\Omega \wedge \text{ch}^{j-i}(\hat{\rho}^*\hat{\mathcal{F}}_\bullet, A_\bullet) \right] \cap \tau_j(\hat{Y}). \end{aligned}$$

Since $\tau_n(\hat{Y}) = [\hat{Y}]$, we have for each case:

$$\tau_n(\mathcal{F})([\Omega]) = \text{rk}(\mathcal{F}) \int_Y \Omega$$

$$\begin{aligned}\tau_{n-1}(\mathcal{F})([\Omega]) &= \text{rk}(\mathcal{F})\tau_{n-1}(Y)([\Omega]) + \int_Y \Omega \wedge \text{ch}_1(\mathcal{F}, h) \\ \tau_{n-2}(\mathcal{F})([\Omega]) &= \text{rk}(\mathcal{F})\tau_{n-2}(Y)([\Omega]) + \tau_{n-1}(Y)([\Omega \wedge \text{ch}_1(\mathcal{F}, h)]) \\ &\quad + \int_Y \Omega \wedge \text{ch}_2(\mathcal{F}, h)\end{aligned}$$

Comparing with Definition 2.2, this completes the proof. \square

We also note the following.

Lemma 2.5. *The forms $\text{ch}_1(\mathcal{F}, h)$, $\text{ch}_1^2(\mathcal{F}, h)$, and $\text{ch}_2(\mathcal{F}, h)$ define closed currents on Y as elements of the dual space of the restriction to Y of smooth forms on M .*

Proof. Since the curvature is in L^2 , the integrals are well defined. The fact that the currents are closed follows exactly as in [28, Sec. 6.5.3]. \square

To define (semi)stability of \mathcal{F} we need a notion of degree. Using Lemma 2.5, this can be defined analytically in the usual way. In the projective case, let $[\alpha] \in A_1(Y)$ be the cycle defined by intersecting hyperplanes in $M \subset \mathbb{P}^N$. By choosing a representative in general position, $[\alpha]$ uniquely defines a homology class in $H_2(Y^{reg}) \simeq H_c^{2n-2}(Y^{reg})$. Via (2.4), we define:

$$(2.6) \quad \text{deg}_\alpha(\mathcal{F}) := \text{ch}_1(\mathcal{F}) \cdot [\alpha] \in \mathbb{Q}.$$

Similarly, we define the slope of \mathcal{F} to be $\mu_\alpha(\mathcal{F}) = \text{deg}_\alpha(\mathcal{F})/\text{rk}(\mathcal{F})$. Then a reflexive sheaf \mathcal{F} is *stable* (resp. *semistable*) if $\mu_\alpha(\mathcal{S}) < \mu_\alpha(\mathcal{F})$ (resp. \leq) for all coherent subsheaves $\mathcal{S} \subset \mathcal{F}$ with $0 < \text{rk } \mathcal{S} < \text{rk } \mathcal{F}$.

Proposition 2.6. *Let $\omega = \omega_{FS}|_{Y^{reg}}$. The following holds:*

$$\text{deg}_\alpha(\mathcal{F}) = \int_Y \text{ch}_1(\mathcal{F}, h) \wedge \omega^{n-1}.$$

Proof. It follows from Proposition 2.3 that for general representatives α ,

$$\text{deg}_\alpha(\mathcal{F}) = \int_\alpha \text{ch}_1(\mathcal{F}, h).$$

We write $\alpha = H_1 \cap \cdots \cap H_{n-1} \cap Y$, where H_i are generic smooth divisors in the class of $\mathcal{O}_M(1)$ on M , and α is in the smooth locus of \mathcal{F} . Choose a Poincaré dual $\text{PD}(H_1)$ supported in a small neighborhood of the H_1 . Write: $\omega - \text{PD}(H_1) = dB$, where B is smooth. Then by Lemma 2.5,

$$\int_Y \text{ch}_1(\mathcal{F}, h) \wedge d(B \wedge \omega^{n-2}) = 0$$

and

$$\int_Y \text{ch}_1(\mathcal{F}, h) \wedge \omega^{n-1} = \int_Y \text{ch}_1(\mathcal{F}, h) \wedge \text{PD}(H_1) \wedge \omega^{n-2}.$$

Continuing in this way, we have

$$\int_Y \text{ch}_1(\mathcal{F}, h) \wedge \omega^{n-1} = \int_Y \text{ch}_1(\mathcal{F}, h) \wedge \text{PD}(H_1) \wedge \cdots \wedge \text{PD}(H_{n-1}).$$

Since we have assumed that $\text{PD}(H_1) \wedge \cdots \wedge \text{PD}(H_{n-1})$ is supported in the smooth locus of \mathcal{F} , the usual Poincaré-Lelong formula gives

$$\begin{aligned} & \int_Y \text{ch}_1(\mathcal{F}, h) \wedge \text{PD}(H_1) \wedge \cdots \wedge \text{PD}(H_{n-1}) \\ &= \int_{H_1 \cap Y} \text{ch}_1(\mathcal{F}, h) \wedge \text{PD}(H_2) \wedge \cdots \wedge \text{PD}(H_{n-1}) \\ &= \int_{H_1 \cap H_2 \cap Y} \text{ch}_1(\mathcal{F}, h) \wedge \text{PD}(H_3) \wedge \cdots \wedge \text{PD}(H_{n-1}) \\ &\quad \vdots \\ &= \int_{\alpha} \text{ch}_1(\mathcal{F}, h) . \end{aligned}$$

□

Corollary 2.7. *If h is an admissible HE metric on \mathcal{F} , i.e. a solution to (2.1), then*

$$\mu = \frac{2\pi}{\text{rk } \mathcal{F} \cdot \text{vol}(Y)} \int_Y \text{ch}_1(\mathcal{F}, h) \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{2\pi}{(n-1)! \text{vol}(Y)} \mu_{\alpha}(\mathcal{F}) .$$

Furthermore, by [11, Ex. 18.3.6], we have the following, which explains why we have defined the Chern classes in the way we did.

Corollary 2.8 (ASYMPTOTIC RIEMANN-ROCH FORMULA). *The Hilbert polynomial of \mathcal{F} over $(Y, \mathcal{O}_Y(1))$ is given by*

$$P_{\mathcal{F}}(k) = \text{rk}(\mathcal{F})(a_1 k^n + a_2(\mu(\mathcal{F}) + \deg(Y))k^{n-1}) + O(k^{n-2})$$

for some universal constants a_1, a_2 .

2.3. Singular Bott-Chern formula. In the following, we fix any admissible metric on $\mathcal{O}_Y(-k)$. Assuming $\dim Y^s \leq n-3$, the argument in [26] gives the following singular Bott-Chern formula based on our definition of ch_2 . Recall that associated to a torsion-free sheaf \mathcal{F} is a cycle $\mathcal{C}(\mathcal{F})$ consisting of the irreducible codimension 2 pieces of the support of $\mathcal{F}^{**}/\mathcal{F}$, with multiplicities.

Proposition 2.9 (SINGULAR BOTT-CHERN FORMULA). *Assume that \mathcal{F} is a torsion-free sheaf which admits an admissible HYM connection A_{∞} over Y where $\dim Y^s \leq n-3$. Given an exact sequence*

$$(2.7) \quad 0 \longrightarrow \ker \phi \longrightarrow \mathcal{O}_Y(-k)^{\oplus N} \xrightarrow{\phi} \mathcal{F} \longrightarrow 0 ,$$

then

$$\text{ch}_2(A_{\infty}) + \text{ch}_2(\ker \phi) - \text{ch}_2(\mathcal{O}(-k)^{\oplus N}) = \mathcal{C}(\mathcal{F})$$

in $[H_c^{2n-4}(Y^{\text{reg}})]^*$.

Proof. From the exact sequences (2.7) and

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{**} \longrightarrow T \longrightarrow 0$$

and the naturality of the homological Todd class, we have

$$\tau(\mathcal{F}^{**}) = \tau(\mathcal{F}) + \tau(T) = \tau(\mathcal{O}_Y(-k)^{\oplus N}) - \tau(\ker \phi) + \tau(T) .$$

By Proposition 2.3, $\text{ch}_2(A_\infty) = \text{ch}_2(\mathcal{F}^{**})$, and since T is supported in codimension 2, $\tau_{n-1}(T) = 0$, and $\tau_{n-2}(T) = \mathcal{C}(\mathcal{F})$ (cf. [2, Remark, p. 17]). The result follows. \square

2.4. Proof of Singular DUY theorem. By Hironaka's resolution of singularities, we may choose a projective resolution of singularities $p : \widehat{\mathbb{P}}^N \rightarrow \mathbb{P}^N$ with the strict transform \widehat{Y} of Y being smooth. We may also assume that $\widehat{\mathcal{F}} = ((p|_{\widehat{Y}})^* \mathcal{F})^{**}$ is locally free. Let θ be any fixed Kähler metric on $\widehat{\mathbb{P}}^N$ and $\omega_\epsilon = (p^* \omega_{FS} + \epsilon \theta)|_{\widehat{Y}}$. We sometimes use \widehat{Y}_ϵ to emphasize that \widehat{Y} is endowed with the Kähler metric ω_ϵ . We have the following elementary observation.

Lemma 2.10. $\mu_\alpha(\mathcal{F}) = \lim_{\epsilon \rightarrow 0} \mu_\epsilon(\widehat{\mathcal{F}})$.

Proof. As in Section 2.2, $\text{ch}_1(\mathcal{F})$ defines an element of $H^2(Y^{reg})$. Then $\text{ch}_1(\widehat{\mathcal{F}}) = p^* \text{ch}_1(\mathcal{F}) + \sum_i a_i E_i$, where E_i are the exceptional divisors of the blow-up maps. The result follows from the definition (2.6) of degree. \square

Proposition 2.11. $(\widehat{\mathcal{F}}, \omega_\epsilon)$ is stable for $0 < \epsilon \ll 1$.

Proof. Write

$$\omega_\epsilon^{n-1} = \omega^{n-1} + \sum_i c_i^\epsilon \alpha_i ,$$

where α_i is the wedge product of $(p^* \omega_{FS}|_{\widehat{Y}}, \theta|_{\widehat{Y}}$ and c_i^ϵ are constants depending on ϵ and $1 \leq i \leq n-1$. Furthermore, $c_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. To prove stability of $\widehat{\mathcal{F}}$ we need to show that for any nontrivial proper subsheaf $\widehat{\mathcal{S}} \subset \widehat{\mathcal{F}}$, $\mu_\epsilon(\widehat{\mathcal{S}}) < \mu_\epsilon(\widehat{\mathcal{F}})$, where μ_ϵ denotes the slope of a sheaf with respect to the Kähler class $[\omega_\epsilon]$. Let $\mathcal{S} = p_* \widehat{\mathcal{S}}$. We have

$$\mu_\epsilon(\widehat{\mathcal{S}}) = \frac{1}{\text{rk}(\widehat{\mathcal{S}})} \int_{\widehat{Y}} c_1(\det \widehat{\mathcal{S}}) \wedge \omega_\epsilon^{n-1} = \mu(\mathcal{S}) + \frac{1}{\text{rk}(\widehat{\mathcal{S}})} \sum_i c_i^\epsilon \int_{\widehat{Y}} c_1(\det \widehat{\mathcal{S}}) \wedge \alpha_i .$$

Let $f(\epsilon, \widehat{\mathcal{S}}) = \mu_\epsilon(\widehat{\mathcal{F}}) - \mu_\epsilon(\widehat{\mathcal{S}})$. Then

$$\begin{aligned} f(\epsilon, \widehat{\mathcal{S}}) &= \mu(\mathcal{F}) - \mu(\mathcal{S}) \\ &+ \sum_i c_i^\epsilon \left(\frac{1}{\text{rk}(\widehat{\mathcal{F}})} \int_{\widehat{Y}} c_1(\det \widehat{\mathcal{F}}) \wedge \alpha_i - \frac{1}{\text{rk}(\widehat{\mathcal{S}})} \int_{\widehat{Y}} c_1(\det \widehat{\mathcal{S}}) \wedge \alpha_i \right) . \end{aligned}$$

It suffices to show that for ϵ small enough, one has $f(\epsilon, \widehat{\mathcal{S}}) > 0$ for any such $\widehat{\mathcal{S}}$ as above. In order to prove this, we first note that there exists $C > 0$ so

that $\mu(\mathcal{F}) - \mu(\mathcal{S}) \geq C > 0$, for any $\mathcal{S} = p_*\widehat{\mathcal{S}}$, since \mathcal{F} is stable over (Y, ω) . So it suffices to show that there exists a constant C' so that

$$\frac{1}{\text{rk}(\widehat{\mathcal{S}})} \int_{\widehat{Y}} c_1(\det \widehat{\mathcal{S}}) \wedge \alpha_i \leq C' .$$

To show this, take a line bundle \mathcal{L} over \widehat{Y} that is ample enough so that we have an inclusion as $\widehat{\mathcal{F}} \hookrightarrow \mathcal{L}^{\oplus l}$, for some $l > 0$. In particular, $\widehat{\mathcal{S}}$ can be viewed as a subsheaf of $\mathcal{L}^{\oplus l}$, which implies $\det(\widehat{\mathcal{S}})$ can be viewed as a subsheaf of $\wedge^{\text{rk}(\widehat{\mathcal{S}})}(\mathcal{L}^{\oplus l})$. In particular, since $\int_D \alpha_i \geq 0$ for any divisor D , we have

$$\frac{1}{\text{rk}(\widehat{\mathcal{S}})} \int_{\widehat{Y}} c_1(\det \widehat{\mathcal{S}}) \wedge \alpha_i \leq \int_{\widehat{Y}} c_1(\mathcal{L}) \wedge \alpha_i .$$

The conclusion follows. \square

Given the above, the idea is to take limits of the HYM connections A_ϵ on $\widehat{\mathcal{F}}$ over $(\widehat{Y}, \omega_\epsilon)$. We need the following

Lemma 2.12. *There exist C independent of ϵ so that*

$$\int_{\widehat{Y}} |F_{A_\epsilon}|^2 d\text{Vol}_\epsilon \leq C .$$

Proof. Indeed, we have

$$\int_{\widehat{Y}} |F_{A_\epsilon}|^2 d\text{Vol}_\epsilon \leq \int_{\widehat{Y}} \text{Tr}(F_{A_\epsilon} \wedge F_{A_\epsilon}) \wedge \frac{\omega_\epsilon^{n-2}}{(n-2)!} + c\mu_\epsilon^2 \text{Vol}(\widehat{Y}_\epsilon) ,$$

where c is some dimensional constant. Given this, the result follows from Lemma 2.10. \square

Fix any sequence $A_i := A_{\epsilon_i}$ with $\epsilon_i \rightarrow 0^+$ as $i \rightarrow \infty$. Set

$$\Sigma = \{x \in Y^{reg} : \lim_{r \rightarrow 0} \liminf_i r^{4-2n} \int_{B_x(r)} |F_{A_i}|^2 d\text{Vol}_{\epsilon_i} > 0\} ,$$

which is a complex subvariety of Y^{reg} of codimension at least 2 (see [31, Thm. 4.3.3]) and can be decomposed as

$$\Sigma = \Sigma_b \cup \text{Sing}(A_\infty)$$

where Σ_b denotes the pure codimension 2 part of Σ . By passing to a subsequence we can assume

- as a sequence of Radon measures over \widehat{Y} ,

$$\mu_i := |F_{A_i}|^2 d\text{Vol}_i \rightharpoonup \mu_\infty = |F_{A_\infty}|^2 d\text{Vol} + \nu ,$$

where $\nu|_{Y^{reg}} = \sum_k 8\pi^2 m_k \Sigma_k$. Here Σ_k denotes the pure codimension 2 components of Σ and $m_k \in \mathbb{Z}$ is usually called the analytic multiplicity of Σ_k .

- outside Σ , up to gauge transformation, A_i converge to A_∞ locally smoothly.

Using Siu's extension theorem [29], combined with [1, Thm. 2], we have the following results regarding the extension property.

Proposition 2.13. *Let \mathcal{F}_∞ be the holomorphic vector bundle on $Y^{reg} \setminus \Sigma$ defined by $\bar{\partial}_{A_\infty}$. Then $\iota_*\mathcal{F}_\infty$ is a reflexive coherent sheaf on Y where $\iota : Y^{reg} \setminus \Sigma \rightarrow Y$ denotes the natural inclusion map. Furthermore, for any local section s of \mathcal{F}_∞ , $\log^+ |s|^2 \in W_{loc}^{1,2} \cap L_{loc}^\infty$.*

Proof. The extension can be done in two steps:

- (1) by Theorem 2 in [1], F can be extended to be a coherent reflexive sheaf $\mathcal{F}_\infty \rightarrow Y^{reg}$;
- (2) by [29, Thm. 5], \mathcal{F}_∞ can be further extended as a coherent reflexive sheaf over Y ; this is because Y^s has codimension ≥ 3 and Y is normal.

Now we prove the second part regarding local sections. For this, one can directly repeat the argument in [1, Thm. 2] with known results from [27]. Let $y \in Y^s$. By choosing a local coordinate of \mathbb{P}^N at y , we can assume $Y \subset B^l \times B^{N-l}$ and $p^{-1}(t)$ is a smooth surface for generic $t \in p(Y)$. Here $y = (0, 0)$ and $p : B^l \times B^{N-l} \rightarrow B^{N-l}$ denotes the natural projection. Furthermore, we can assume the metric on $B^l \times B^{N-l}$ is flat. In the following, we will use $A \lesssim B$ (resp. $A \gtrsim B$) to denote $A \leq cB$ (resp. $A \geq cB$) for some constant c . For generic $t \in p(Y)$ as above, denote $Y_t = p^{-1}(t) \cap Y$, $f_t = \log^+ |s|^2|_{Y_t}$ and $F_t = F_{A_\infty}|_{Y_t}$. We know that

$$\Delta f_t \geq -|F_t|.$$

Let χ be a cut-off function in B^l supported in a small neighborhood of 0. Now we can multiply the above inequality by $\chi^2 f_t$ and do integration by parts to get

$$\int_{Y_t} |\nabla(\chi f_t)|^2 \leq \epsilon \int_{Y_t} |\chi f_t|^2 + \epsilon^{-1} \int_{Y_t} \chi^2 |F_t|^2 + \int_{Y_t} |\nabla \chi|^2 f_t^2$$

for any $0 < \epsilon \ll 1$. By the Sobolev inequality in [27, Thm. 18.6], we could conclude

$$\int_{Y_t} |\nabla(\chi f_t)|^2 \lesssim \int_{Y_t} |\nabla \chi|^2 f_t^2 + \int_{Y_t} \chi^2 |F_t|^2.$$

From this, we could integrate it with respect to t , and have that for any $K \subset p(Y)$ compact

$$\int_{p^{-1}(K)} |\nabla'(\chi f_t)|^2 \lesssim \int_{p^{-1}(K)} |\nabla \chi|^2 f_t^2 + \int_{p^{-1}(K)} \chi^2 |F_t|^2$$

where ∇' denotes the total derivatives in the fiber direction of the projection p . Now we can take finitely many such projections p to cover all the derivatives, thus could conclude $\log^+ |s|^2 \in W_{loc}^{1,2}$. Suppose s is a local section of $\overline{\mathcal{E}_\infty}$. By [21, eq. (5.5)], we have a Sobolev inequality for any function in $W^{1,2}(Y^{reg})$. Since $\Delta \log^+ |s|^2 \geq -\mu(\mathcal{E}_\infty)$, by Moser iteration applied to

$\eta^2 \log^+ |s|^2$ where η is a local cut-off function near the point in \mathbb{P}^N , we know $\log^+ |s|^2 \in L_{loc}^\infty$. \square

Lemma 2.14. *The closure of Σ is a subvariety of Y .*

Proof. Using the Bishop extension theorem [3], to show Σ_b can be extended, it suffices to show that $\Sigma_k \subset M$ has bounded volume. Indeed, since μ_i weakly converges to μ_∞ as a sequence of Radon measures over \widehat{Y} , we have

$$\text{Vol}(\Sigma_k) \leq \limsup_i \int_{\widehat{Y}_{e_i}} |F_{A_i}|^2 \leq C .$$

The extension of $\text{Sing}(A_\infty)$ follows from that \mathcal{F}_∞ can be extended to be global reflexive sheaf over Y and $\text{Sing}(A_\infty) = \text{Sing}(\mathcal{F}_\infty)$. \square

Now we prove the existence of admissible HE metric in the rank 1 case.

Corollary 2.15. *Suppose $\text{rk } \mathcal{F} = 1$. Then there exists an admissible HYM metric on \mathcal{F} .*

Proof. Indeed, by choosing any higher curve $C \subset Y^{reg}$ so that $A_i|_C$ converges to A_∞ smoothly. Then it is easy to see that $\overline{\mathcal{F}_\infty}|_C$ is isomorphic to $\mathcal{F}|_C$. Since C can be chosen to be any higher degree, we know \mathcal{F} and $\overline{\mathcal{F}_\infty}$ must be isomorphic. \square

Corollary 2.16. *$\overline{\mathcal{F}_\infty}$ is semistable.*

Proof. Suppose $\mathcal{S} \subset \overline{\mathcal{F}_\infty}$ is a proper saturated destabilizing subsheaf with $\text{rk } \mathcal{S} = m$ and $\mathcal{L} = \det \mathcal{S}$. Then the sheaf $(\Lambda^m \overline{\mathcal{F}_\infty} \otimes \mathcal{L}^*)^{**}$ admits a global nontrivial section s and

$$\mu := \mu((\Lambda^m \overline{\mathcal{F}_\infty} \otimes \mathcal{L}^*)^{**}) < 0 .$$

By Corollary 2.15, \mathcal{L} admits a HE metric. Applying Proposition 2.13 to $(\Lambda^m \overline{\mathcal{F}_\infty} \otimes \mathcal{L}^*)^{**}$, we know s is bounded. Moreover, we have

$$\Delta |s|^2 = |\nabla s|^2 - \mu |s|^2 \geq 0 .$$

A contradiction to $s \neq 0$ then follows by integration against a cut-off near the singularities Y^s . \square

Proof of Theorem 1.1. Since $\overline{\mathcal{F}_\infty}$ is semistable and \mathcal{F} is stable, by the Mehta-Ramanathan restriction theorem, we can choose a high degree curve $C \subset Y \setminus (Y^s \cup \Sigma)$ so that $\overline{\mathcal{F}_\infty}|_C$ is semistable and $\mathcal{F}|_C$ is stable. By assumption, we know $A_i|_C$ converges to $A|_C$ smoothly up to gauge. By the well-known results regarding moduli space of semistable bundles over curves, we know $\mathcal{F}|_C$ and $\overline{\mathcal{F}_\infty}|_C$ must be S -equivalent, thus isomorphic. Since C can be chosen to be of any high degree, we conclude that \mathcal{F} and $\overline{\mathcal{F}_\infty}$ must be isomorphic (cf. [14, Lemma 5.4]). For uniqueness, suppose h, h' are two admissible HE metrics on \mathcal{F} . Then, as in Corollary 2.16, $\text{id} \in \text{Hom}(\mathcal{F}, \mathcal{F})$ is parallel with respect to the natural metric $h^* \otimes h'$. In particular, if we write $h' = h(g, \cdot)$, then g is a holomorphic endomorphism of \mathcal{F} . By stability,

it therefore must be a constant multiple of the identity, and the conclusion follows. \square

As a direct corollary of this, combined with the regularity results, we have

Corollary 2.17. *If a reflexive sheaf $\mathcal{F} \rightarrow Y$ admits an admissible HE metric, then \mathcal{F} is polystable.*

Proof. Semistability follows as in the proof of Corollary 2.16. Suppose $\mathcal{S} \subset \mathcal{F}$ is a proper stable saturated subsheaf with $\mu(\mathcal{S}) = \mu(\mathcal{F})$. Note that \mathcal{S} is reflexive. Let h' be the admissible HE metric on \mathcal{S} . Now as Corollary 2.16, we know that $\text{id} : \mathcal{S}' \rightarrow \mathcal{F}$ is parallel. This implies there is a holomorphic, [18, Prop. 4.1.7]) implies there is a holomorphic, orthogonal splitting $\mathcal{F} = \mathcal{S} \oplus \mathcal{S}^\perp$ off a set of codimension ≥ 3 in Y^{reg} . As above, the sheaves \mathcal{S} and \mathcal{S}^\perp extend as reflexive sheaves on Y , and the induced metrics are admissible HE. The result now follows by induction on the rank. \square

3. BOGOMOLOV-GIESEKER EQUALITY AND NEF TAUTOLOGICAL CLASSES

We begin this section with the

Proof of Corollary 1.3. Let $Y^s \subset S \subset Y$ be a closed subvariety of codimension at least 3 so that \mathcal{F} is locally free on $Y \setminus S$. Using Proposition 2.3, it follows exactly as in the proof of Proposition 2.6 that

$$\begin{aligned} & (2rc_2(\mathcal{F}) - (r-1)c_1^2(\mathcal{F})) \cdot [H]^{n-2} \\ &= \int_Y (2rc_2(\mathcal{F}, h) - (r-1)c_1^2(\mathcal{F}, h)) \wedge \omega^{n-2} \\ &= \int_{Y \setminus S} (2rc_2(\mathcal{F}, h) - (r-1)c_1^2(\mathcal{F}, h)) \wedge \omega^{n-2}. \end{aligned}$$

The inequality then follows as in the smooth case (cf. [18, Thm. 4.4.7]). Moreover, in the case of equality, $\mathcal{F}|_{Y \setminus S}$ is projectively flat. Hence, there is a representation $\rho : \pi_1(Y \setminus S) \rightarrow \text{PGL}_r(\mathbb{C})$. Since by the codimension assumption, $\pi_1(Y \setminus S) \simeq \pi_1(Y^{reg})$, we obtain from ρ a $\text{PGL}_r(\mathbb{C})$ bundle $P \rightarrow Y^{reg}$, isomorphic (as projective bundles) to $\mathbb{P}(\mathcal{F})$ on $Y \setminus S$. Let $\mathcal{L} \rightarrow P$ be the pull-back of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ under this isomorphism. Again using the codimension of S , we know that \mathcal{L} extends as line bundle over $\pi : P \rightarrow Y^{reg}$. Moreover, \mathcal{L} restricts to $\mathcal{O}(1)$ on each fiber of P . Hence, $\pi_* \mathcal{L} =: \tilde{\mathcal{F}}^*$ is a vector bundle with a projectively flat connection. Since $\tilde{\mathcal{F}}^* \simeq \mathcal{F}^*$ on $Y \setminus S$, and \mathcal{F} is reflexive, we have $\tilde{\mathcal{F}} \simeq \mathcal{F}$ on Y^{reg} . This completes the proof. \square

These arguments can be further used to improve on several previous algebro-geometric results corresponding to the case of the normalized tautological class of \mathcal{F} being nef, and \mathcal{F} pseudo-effective, which we will give definitions below. We start with $Y \subset \mathbb{P}^N$, a projective normal subvariety

that is smooth in codimension 2. Let \mathcal{F} be a reflexive sheaf over Y . Denote $\mathcal{F}^0 = \mathcal{F}|_{Y \setminus (Y^s \cup \text{Sing}(\mathcal{F}))}$. Now we define the *normalized tautological class* by

$$\zeta_{\mathcal{F}} := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F}^0)}(1) \otimes \pi^*(\det(\mathcal{F}^0)^{-1})) \in H^2(\mathbb{P}(\mathcal{F}^0), \mathbb{Z}) ,$$

where $\pi : \mathbb{P}(\mathcal{F}^0) \rightarrow Y \setminus (Y^s \cup \text{Sing}(\mathcal{F}))$ denotes the natural projection. Here we used the algebro-geometric convention $\mathbb{P}(\mathcal{F}^0) := \text{Proj}(\oplus_{k \geq 0} \text{Sym}^k \mathcal{F}^0)$.

Definition 3.1. Given \mathcal{F} as above

- \mathcal{F} is called nef if $\zeta_{\mathcal{F}}|_{\pi^{-1}(V)}$ is nef for any smooth projective subvarieties $V \subset Y \setminus (Y^s \cup \text{Sing}(\mathcal{F}))$ of dimension 1, 2.
- *pseudo-effective* if for any $c > 0$ there exists $j \in \mathbb{N}$ so that for $i > cj$,

$$H^0(X, S^{[i]} \mathcal{F} \otimes \mathcal{O}(jH)) \neq 0$$

for some (and hence any) ample Cartier divisor H . Here $S^{[i]}(\mathcal{F}) := (\text{Sym}^i(\mathcal{F}))^{**}$.

Following the argument as [24, Thm. 4.1], we will prove the following generalization of Corollary 1.3.

Theorem 3.2 (NORMALIZED TAUTOLOGICAL CLASS IS NEF). *Let \mathcal{F} be a reflexive sheaf over Y . The following are equivalent:*

- (1) \mathcal{F} is locally free over Y^{reg} and $\zeta_{\mathcal{F}}$ is nef.
- (2) \mathcal{F} is semistable for some ample divisor A (not necessarily the one from the embedding $Y \subset \mathbb{P}^N$) and

$$(2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2) \cdot [A]^{n-2} = 0 ;$$

- (3) \mathcal{F} admits a filtration: $0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$, where $\mathcal{F}_i/\mathcal{F}_{i-1}$ are projectively flat over Y^{reg} and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) = \mu(\mathcal{F})$.

Proof. Given any sheaf \mathcal{H} , we denote

$$\Delta(\mathcal{H}) := 2rc_2(\mathcal{H}) - (r-1)c_1^2(\mathcal{H}) .$$

We assume (1) holds and prove (2). By definition, we know for any smooth curve C lying in Y^{reg} as the intersections of hypersurfaces in $|\mathcal{O}(A)|$, $\tau_{\mathcal{F}^0}|_C$ is nef, which by the well-known result is equivalent to $\mathcal{F}^0|_C$ being semistable. This implies \mathcal{F} is semistable. Choose S to be a smooth projective surface in $Y \setminus Y^s$ as an intersection of smooth hypersurfaces in $|\mathcal{O}(A)|$ so that $\mathcal{F}|_S$ is locally free and semistable. Since $\tau_{\mathcal{F}}|_S$ is nef by assumption, by the well-known result in the smooth case,

$$\Delta(\mathcal{F}) \cdot [A]^{n-2} = \Delta(\mathcal{F}|_S) = 0 .$$

Suppose (2) holds. We prove (3) by induction on rank, where the rank 1 case follows from Corollary 1.3. Assume the statement holds for rank smaller than $\text{rk}(\mathcal{F})$. We can choose $\mathcal{E}_1 \subset \mathcal{F}$ to be a saturated stable subsheaf with $\mu(\mathcal{E}_1) = \mu(\mathcal{F})$. Then we have the following exact sequence

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

where \mathcal{E}_1 is projectively flat over Y^{reg} by Corollary 1.3, and \mathcal{G} is torsion-free and semistable with $\mu(\mathcal{G}) = \mu(\mathcal{F})$. The statement follows from

- (a) $\Delta(\mathcal{E}_1) \cdot [A]^{n-2} = \Delta(\mathcal{G}) \cdot [A]^{n-2} = \Delta(\mathcal{G}^{**}) \cdot [A]^{n-2} = 0$;
- (b) $\mathcal{G}^{**} = \mathcal{G}$ over Y^{reg} .

Indeed, we can apply the induction to \mathcal{E}_1 and \mathcal{G}^{**} to get such filtrations as in (3), and these naturally give a filtration for \mathcal{F} over Y^{reg} . We first prove (a). Choose a smooth projective surface $S \subset Y^{reg}$ given by the intersection of smooth hypersurfaces in $|\mathcal{O}_Y(A)|$, and so that $\mathcal{E}_1|_S$ and $\mathcal{G}|_S$ are torsion-free and semistable, and $\mathcal{F}|_S$ is locally free and semistable. We have the exact sequence

$$0 \longrightarrow \mathcal{E}_1|_S \longrightarrow \mathcal{F}|_S \longrightarrow \mathcal{G}|_S \longrightarrow 0 .$$

By [17, Corollary 7.3.2], we know that

$$\Delta(\mathcal{E}_1) \cdot [S] = \Delta(\mathcal{F}) \cdot [S] = \Delta(\mathcal{G}) \cdot [S] = 0 .$$

Together with the Bogomolov inequality for $(\mathcal{G}|_S)^{**}$, this implies $\Delta((\mathcal{G}|_S)^{**}) = 0$. Thus $\mathcal{G}|_S = (\mathcal{G}|_S)^{**}$ and $\Delta(\mathcal{G}^{**}) \cdot [S] = 0$. Note this argument in particular implies $\text{supp}(\mathcal{G}^{**}/\mathcal{G})$ has codimension at least 3. But then the argument in [26, Prop. 2.3] shows that $\mathcal{G} = \mathcal{G}^{**}$ on Y^{reg} , and this proves (b). Now suppose (3) holds and we prove (1). Obviously, \mathcal{F} is locally free over Y^{reg} . For any smooth projective subvarieties $V \subset Y^{reg}$ of dimension 1 or 2, we know $\mathcal{F}|_V$ admits a filtration so that the graded factors are projectively flat over V . The conclusion thus follows from the known results in the smooth case. \square

We also have

Theorem 3.3 (PSEUDO-EFFECTIVE). *Suppose \mathcal{F} is a stable reflexive sheaf over Y so that $c_1(\mathcal{F}) \cdot [A]^{n-1} = 0$, for some ample Cartier divisor A and $S^{[l]}\mathcal{F}$ are all indecomposable. If \mathcal{F} is pseudo-effective, then $\mathcal{F}|_{Y^{reg}}$ is flat.*

Proof. By Theorem 1.1, \mathcal{F} carries an admissible HE metric; hence, so does $S^{[l]}(\mathcal{F})$. By Corollary 2.17 and the assumption of indecomposability, $S^{[l]}(\mathcal{F})$ must be stable. By [16, Thm. 1.1] we know that

$$c_1(\mathcal{F}) \cdot [H]^{n-1} = c_2(\mathcal{F}) \cdot [H]^{n-2} = 0 ,$$

and by Theorem 3.2, we conclude that $\mathcal{F}|_{Y^{reg}}$ is flat. \square

Remark 3.4. In [16], $S^{[l]}\mathcal{F}$ is assumed to be stable, and it is concluded that $c_1(\mathcal{F})^2 \cdot H^{n-2} = 0$ and $c_2(\mathcal{F}) \cdot H^{n-2} = 0$. Then by assuming Y is klt and using [12], the authors can pass to a finite Galois cover $\nu : \tilde{Y} \rightarrow Y$ which is étale in codimension 1, and thence conclude that the reflexive pullback of \mathcal{F} is a numerically flat bundle.

4. DEGENERATIONS OF SEMISTABLE BUNDLES

In this section, we fix \mathcal{X} to be a subvariety of $\mathbb{P}^N \times \mathbb{P}^1$ so that

$$p : \mathcal{X} \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$

is a family with $p^{-1}(t)$ smooth for $t \neq 0$ and $p^{-1}(0)$ normal. Denote $X_t = p^{-1}(t)$. Let $\{\mathcal{E}_t\}_{t \neq 0}$ be a family of semistable bundles. We say that $\{\mathcal{E}_t\}_{t \neq 0}$ is an *algebraic family* if we can find a coherent sheaf $\mathcal{E} \rightarrow \mathcal{X}$ so that $\mathcal{E}|_{X_t} \cong \mathcal{E}_t$ for all $t \neq 0$. In the following we shall always consider such algebraic families.

4.1. The algebraic side. We use the hyperplane section $D \subset \mathbb{P}^N$ to define stability of sheaves on the fibers of $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$.

Definition 4.1. A coherent sheaf $\mathcal{E} \rightarrow \mathcal{X}$ is called an algebraic degeneration of $\{\mathcal{E}_t\}_{t \neq 0}$ if \mathcal{E} is flat over \mathbb{P}^1 , $\mathcal{E}|_{X_t} \cong \mathcal{E}_t$ for $t \neq 0$, and $\mathcal{E}_0 := \mathcal{E}|_{X_0}$ is torsion-free. \mathcal{E} is called a semistable degeneration of $\{\mathcal{E}_t\}_{t \neq 0}$ if furthermore \mathcal{E}_0 is torsion-free and semistable.

We need the following simple observation following from the flatness assumption.

Lemma 4.2. *For different semistable degenerations \mathcal{E}' and \mathcal{E} of the same family $\{\mathcal{E}_t\}_{t \neq 0}$, $\det \mathcal{E}'_0 \cong \det \mathcal{E}_0$.*

To state the main results of this section, we make the following definition. Given a semistable sheaf \mathcal{F} over a projective normal variety $Y \subset \mathbb{P}^N$ with a fixed polarization. Let $\text{Gr}(\mathcal{F})$ be the graded sheaf associated to a Jordan-Hölder filtration of \mathcal{F} . Assume that Y is smooth in codimension 2. Let $\mathcal{T} = \text{Gr}(\mathcal{F})^{**}/\text{Gr}(\mathcal{F})$. Then $\dim \text{supp}(\mathcal{T}) \leq n - 2$.

Definition 4.3 (Lemma). The algebraic blow-up locus of \mathcal{F} is defined as

$$(4.1) \quad \mathcal{C}(\mathcal{F}) = \sum m_k^{alg} \Sigma_k,$$

where Σ_k are irreducible codimension 2 subvarieties of Y and m_k^{alg} is equal to $h^0(D, \mathcal{T}|_D)$ where D is a generic transverse slice of Σ_k . In particular, $\mathcal{C}(\mathcal{E})$ is a finite sum.

We refer the readers for detailed proof in Section 2.5 in [13], which is done for smooth projective varieties but can be seen to work here.

4.1.1. Hecke transform and existence of semistable degenerations. In this section, we will prove the first part of Theorem 1.5, i.e. a semistable degeneration always exists which is a slight generalization of [19] using the same argument outlined in [17, p. 59], where the completeness of the moduli space of semistable sheaves over a fixed projective manifold is studied. For the interest of the reader, the extension of a sheaf across a negative divisor is also studied in [4].

We fix \mathcal{E} to be any reflexive sheaf over \mathcal{X} so that $\mathcal{E}|_{X_t} = \mathcal{E}_t$ for $t \neq 0$. This always exists by assumption. Since \mathcal{E} is torsion-free, we know \mathcal{E} always

defines a flat family on \mathcal{X} over \mathbb{P}^1 . We start with the following observation which generalizes [5, Lemma 3.23] by the same argument.

Lemma 4.4. \mathcal{E}_0 is torsion-free.

Proof. Fix any $x \in X_0$. By the assumptions, we can represent \mathcal{E} near x as

$$(4.2) \quad 0 \longrightarrow \mathcal{E}|_U \longrightarrow \mathcal{O}_U^{\oplus n_1} \xrightarrow{\phi} \mathcal{O}_U^{\oplus n_2} .$$

where U is an open neighborhood of x in \mathcal{X} . Restricting this exact sequence to $D = U \cap X_0$, one gets an exact sequence

$$\mathrm{Tor}_1(\mathcal{O}_D, \mathrm{Im}(\phi)) \longrightarrow \mathcal{E}|_D \longrightarrow \mathcal{O}_D^{\oplus n_1} .$$

Since $\mathrm{Im}(\phi)$ is torsion-free, by using the natural resolution for \mathcal{O}_D we have $\mathrm{Tor}_1(\mathcal{O}_D, \mathrm{Im}(\phi)) = 0$. This implies $\mathcal{E}|_D \hookrightarrow \mathcal{O}_D^{\oplus n_1}$. In particular, $\mathcal{E}|_D$ is torsion-free. \square

Given any saturated subsheaf $\underline{\mathcal{S}} \subset \mathcal{E}_0$, i.e. $\mathcal{E}_0/\underline{\mathcal{S}}$ is torsion-free. Let \mathcal{E}' be the sheaf given by the following exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow (\iota_0)_*(\mathcal{E}_0/\underline{\mathcal{S}}) \longrightarrow 0 .$$

Definition 4.5. \mathcal{E}' is called a Hecke transform of \mathcal{E} along $\underline{\mathcal{S}}$.

By repeating exactly the same argument as [4, Lemma 2.4], we have

Lemma 4.6. \mathcal{E}' is reflexive and \mathcal{E}'_0 is torsion-free. Furthermore, there exists a short exact sequence

$$0 \longrightarrow \mathcal{E}_0/\underline{\mathcal{S}} \longrightarrow \mathcal{E}'_0 \longrightarrow \underline{\mathcal{S}} \longrightarrow 0 .$$

Now we start with \mathcal{E} as above. We define a sequence of \mathcal{E}^k inductively by defining $\mathcal{E}^1 = \mathcal{E}$ and \mathcal{E}^{k+1} to be the Hecke transform of \mathcal{E}^k along the maximal destabilizing subsheaf $\underline{\mathcal{F}}_k$ of \mathcal{E}_0^k . For each \mathcal{E}^k , we can associate it with a nonnegative number

$$\beta_k := \mu(\underline{\mathcal{F}}_k) - \mu(\mathcal{E}_0^k) = \mu(\underline{\mathcal{F}}_k) - \mu(\mathcal{E}_0)$$

which measures how far \mathcal{E}^k is from being semistable. Note that $\beta_k = 0$ if and only if \mathcal{E}^k is a semistable degeneration of $\{\mathcal{E}_t\}_{t \neq 0}$. In the following, we denote

$$\underline{\mathcal{G}}_k = \mathcal{E}_0^k / \underline{\mathcal{F}}_k .$$

By definition, we have the following exact sequence

$$(4.3) \quad 0 \longrightarrow \underline{\mathcal{F}}_{k-1} \longrightarrow \mathcal{E}_0^{k-1} \longrightarrow \underline{\mathcal{G}}_{k-1} \longrightarrow 0 ,$$

and by Lemma 4.6, we have

$$(4.4) \quad 0 \longrightarrow \underline{\mathcal{G}}_{k-1} \longrightarrow \mathcal{E}_0^k \longrightarrow \underline{\mathcal{F}}_{k-1} \longrightarrow 0 .$$

Lemma 4.7. β_k decreases and for k large, the following hold:

- $\beta_k = \beta_{k-1}$;
- there are natural isomorphisms $\underline{\mathcal{F}}_k \cong \underline{\mathcal{F}}_{k-1}$ and $\underline{\mathcal{G}}_k \cong \underline{\mathcal{G}}_{k-1}$. In particular, $\mathcal{E}_0^k = \underline{\mathcal{F}}_k \oplus \underline{\mathcal{G}}_k$.

Proof. It follows from (4.4) that $\mu(\underline{\mathcal{F}}_k) \leq \mu(\underline{\mathcal{F}}_{k-1})$ since through the last map in the exact sequence, one has a map $\underline{\mathcal{F}}_k \rightarrow \underline{\mathcal{F}}_{k-1}$. Indeed, this follows from $\underline{\mathcal{F}}_{k-1}$ being semistable if the map is nontrivial. Otherwise, we have a nontrivial map $\underline{\mathcal{F}}_k \rightarrow \underline{\mathcal{G}}_{k-1}$ and the maximal destabilizing sheaf of $\underline{\mathcal{G}}_{k-1}$ has slope strictly smaller than $\underline{\mathcal{F}}_{k-1}$. In particular, we know β_k decreases as k increases, thus must stabilize for k large since it is nonnegative. Now we assume the equality holds $\beta_k = \beta_{k-1}$ i.e. $\mu(\underline{\mathcal{F}}_k) = \mu(\underline{\mathcal{F}}_{k-1})$. Then we must have an injective map $\underline{\mathcal{F}}_k \rightarrow \underline{\mathcal{F}}_{k-1}$. Otherwise, we have an exact sequence

$$0 \longrightarrow \underline{\mathcal{G}}'_k \longrightarrow \underline{\mathcal{F}}_k \longrightarrow \underline{\mathcal{F}}_{k-1}$$

where $\underline{\mathcal{G}}'_k \subset \underline{\mathcal{G}}_k$. This implies $\mu(\underline{\mathcal{F}}_k) < \mu(\underline{\mathcal{F}}_{k-1})$, which is a contradiction. Given this, if the equality holds, we have the following exact sequence by the sequence (4.4)

$$0 \longrightarrow \underline{\mathcal{G}}_{k-1} \longrightarrow \underline{\mathcal{G}}_k = \mathcal{E}_0^k / \underline{\mathcal{F}}_k \longrightarrow \underline{\mathcal{F}}_{k-1} / \underline{\mathcal{F}}_k \longrightarrow 0 ,$$

thus an injective map $\underline{\mathcal{G}}_{k-1} \rightarrow \underline{\mathcal{G}}_k$ for k large. Since $\mu(\underline{\mathcal{G}}_{k-1}) = \mu(\underline{\mathcal{G}}_k)$, the inclusion map must induce an isomorphism in codimension 1, thus $\underline{\mathcal{G}}_k^{**}$ are the same for k large. This then implies $\underline{\mathcal{G}}_{k-1} \cong \underline{\mathcal{G}}_k$ for k large. Thus $\underline{\mathcal{F}}_k \cong \underline{\mathcal{F}}_{k-1}$. \square

Now we are ready to prove the main results by following the argument in [17, p. 59].

Proposition 4.8. *For k larger, \mathcal{E}^k is a semistable degeneration of $\{\mathcal{E}_t\}_t$ for k sufficiently large.*

Proof. Otherwise, by replacing \mathcal{E} with \mathcal{E}^{k_0} for some k_0 large, we assume all the conditions in Lemma 4.7 hold below. Denote $\underline{\mathcal{F}} = \underline{\mathcal{F}}_k$ and $\underline{\mathcal{G}} = \underline{\mathcal{G}}_k$. Let R be the complete DVR associated to the point $0 \in \mathbb{P}^1$, and let \mathcal{X}_R be the fiber over $\text{Spec}(R)$. Denote z to be the generator of the maximal ideal of R and the fibration still as $p : \mathcal{X}_R \rightarrow \text{Spec}(R)$. Abusing notation, we continue to denote by \mathcal{E}^k the restriction of \mathcal{E}^k to \mathcal{X}_R . Then \mathcal{E}^k is semistable on the generic fiber of $\mathcal{X}_R \rightarrow \text{Spec}(R)$. Let $\mathcal{Q}^k = \mathcal{E} / \mathcal{E}^k$. Then $\mathcal{Q}^k|_{X_0} \cong \mathcal{G}$ and by definitions there are short exact sequence

$$0 \longrightarrow \iota_* \underline{\mathcal{G}} \longrightarrow \mathcal{Q}^{k+1} \longrightarrow \mathcal{Q}^k \longrightarrow 0 ,$$

where $\iota : X_0 \rightarrow \mathcal{X}_R$ denote the natural embedding. From this, we conclude that \mathcal{Q}^k is an $R/(z^k)$ -flat quotient of $\mathcal{E}/(z^k)\mathcal{E}$ by [17, Lemma 2.1.3]. In particular, $\text{Spec } R/(z^k) \in \text{Im}(\pi)$ where $\pi : \text{Quot}_{\mathcal{X}_R/R}(\mathcal{E}, P(\underline{\mathcal{G}})) \rightarrow \text{Spec } R$, thus π must be surjective and this gives a flat quotient $\mathcal{E} \rightarrow \mathcal{G}$ in the quot scheme $\text{Quot}_{\mathcal{X}_R/R}(\mathcal{E}, P(\underline{\mathcal{G}}))$ which destabilizes the sheaf on the generic fiber. Contradiction. \square

4.1.2. *Uniqueness of Gr^{**} .* The uniqueness of the double dual of the limiting sheaf in Theorem 1.5 is a direct corollary of the following two lemmas.

Lemma 4.9. *Given any two semistable degenerations \mathcal{E} and \mathcal{E}' of $\{\mathcal{E}_t\}_t$, for l large, for generic family of curves $\mathcal{Y} \subset \mathcal{X}$ defined by $\mathcal{O}(l)$, denote $Y_t = p^{-1}(t) \cap \mathcal{Y}$, then $\mathcal{E}|_{Y_0}$ and $\mathcal{E}'|_{Y_0}$ are both semistable vector bundles for $t \neq 0$. In particular, $\mathcal{E}|_{Y_0}$ and $\mathcal{E}'|_{Y_0}$ are S -equivalent.*

Proof. The first part is a direct corollary of the MR restriction theorem. Now we also know $\mathcal{E}|_{Y_t}$ and $\mathcal{E}'|_{Y_t}$ is semistable for $t \neq 0$. The second conclusion follows from the geometry of a relative version of the moduli space of semistable bundles over curves (see [17, Thm. 4.3.7]). \square

Also we need the following well-known lemma.

Lemma 4.10. *$\text{Gr}(\mathcal{E}_0)^{**} = \text{Gr}(\mathcal{E}'_0)^{**}$ if and only if $\mathcal{E}_0|_Y$ and $\mathcal{E}'_0|_Y$ are S -equivalent for generic curves Y given as a complete intersection of hyper-surfaces in $\mathbb{P}(H^0(X_0, \mathcal{O}(l))^*)$, for l large.*

4.1.3. *Uniqueness of algebraic blow-up locus when $\text{codim } X_0^s \geq 3$.* In this section, we assume $n \geq 3$. Given \mathcal{E} and \mathcal{E}' two semistable degenerations of $\{\mathcal{E}_t\}_t$, assume $\mathcal{C}(\mathcal{E}) \neq \mathcal{C}(\mathcal{E}')$, similar to [14, p. 64], we have

Proposition 4.11. *By restricting to a generic family of smooth surfaces transverse to $\mathcal{C}(\mathcal{E}') \cup \mathcal{C}(\mathcal{E})$, we get two different semistable degenerations of the corresponding restriction of $\{\mathcal{E}_t\}_t$ in terms of the algebraic blow-locus.*

Given this, in the following, we can assume \mathcal{X} to be a family of smooth projective surfaces and two semistable degenerations \mathcal{E} and \mathcal{E}' of $\{\mathcal{E}_t\}$ so that $\mathcal{C}(\mathcal{E}_0) \neq \mathcal{C}(\mathcal{E}'_0)$. To show uniqueness of bubbling sets, we first briefly recall how the moduli space of semistable sheaves with given determinants of numerical classes over $X_0 = p^{-1}(0)$ is constructed. We refer readers to [17] for more details. Using the fact that the set of isomorphism classes of semistable sheaves is bounded, and hence m -regular for some integer m , we can put those sheaves in a fixed Quot scheme. More precisely, denote

$$\mathcal{H} := \mathcal{O}_{X_0}(-m)^{\oplus P(m)},$$

and let $R^{\mu ss} \subset \text{Quot}(\mathcal{H}, P)$ be the locally closed subscheme of all quotients $[q : \mathcal{H} \rightarrow \mathcal{F}]$ such that \mathcal{F} is torsion-free μ -semistable with rank equal to r , determinant equal to Q , and same numerical classes. Furthermore, q induces an isomorphism between V and $H^0(\mathcal{F}(m))$. The group $\text{SL}(V)$ acts on $R^{\mu ss}$ naturally. Now we consider the universal quotient $\tilde{q} : \mathcal{O}_{R^{\mu ss}} \otimes \mathcal{H} \rightarrow \tilde{\mathcal{F}}$, and the line bundle $\mathcal{N} := \lambda_{\tilde{\mathcal{F}}}(u_1(c))$, where $u_1(c)$ is certain sheaf associated to $\mathcal{O}(1)$ and \mathcal{O}_x for some point $x \in X_0$. For some positive integer $\nu > 0$, \mathcal{N}^ν is generated by $\text{SL}(V)$ -invariant global sections. Furthermore, there is an integer $N > 0$ so that

$$\bigoplus_{l \geq 0} H^0(R^{\mu ss}, \mathcal{N}^{lN})^{\text{SL}(V)}$$

is a finitely generated graded ring. Now define

$$M^{\mu_{ss}} := \text{Proj} \left(\bigoplus_{k \geq 0} H^0(R^{\mu_{ss}}, \mathcal{N}^{kN})^{\text{SL}(V)} \right),$$

where $H^0(R^{\mu_{ss}}, \mathcal{N}^{kN})^{\text{SL}(V)}$ denotes the space of sections in $H^0(R^{\mu_{ss}}, \mathcal{N}^{kN})$ invariant under the action of $\text{SL}(V)$. Also, we have a map

$$(4.5) \quad \Phi : R^{\mu_{ss}} \longrightarrow M^{\mu_{ss}}.$$

which collapses the $\text{SL}(V)$ orbits into points. Furthermore, the image of two points $[q_0 : \mathcal{H} \rightarrow \mathcal{E}_0]$ and $[q'_0 : \mathcal{H} \rightarrow \mathcal{E}'_0]$ are the same in $M^{\mu_{ss}}$ if and only if

$$\text{Gr}(\mathcal{E}_0)^{**} = \text{Gr}(\mathcal{E}'_0)^{**}, \text{ and } \mathcal{C}(\mathcal{E}_0) = \mathcal{C}'(\mathcal{E}'_0).$$

We want now to do such constructions for the family \mathcal{X} . Denote

$$\mathcal{H}_{\mathcal{X}} = \underline{\mathbb{C}}^{\oplus P(m)} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-k).$$

Now we define $R_{\mathcal{X}}^{\mu_{ss}} \subset \text{Quot}(\mathcal{H}_{\mathcal{X}}, P)$ to be the space of flat families of torsion-free semistable sheaves over \mathcal{X} with the same numerical classes and Hilbert polynomial P as the sheaves in $M^{\mu_{ss}}$. By repeating the construction above for a family, we have a well-defined object

$$\mathcal{M}_{\mathcal{X}}^{\mu_{ss}} := \text{Proj} \left(\bigoplus_{k \geq 0} H^0(R_{\mathcal{X}}^{\mu_{ss}}, \mathcal{N}_{\mathcal{X}}^{kN})^{\text{SL}(V)} \right),$$

which admits a natural rational map $\Phi_{\mathcal{X}} : R_{\mathcal{X}}^{\mu_{ss}} \dashrightarrow \mathcal{M}_{\mathcal{X}}^{\mu_{ss}}$ over \mathbb{P}^1 .

Proposition 4.12. *For N large, near the central fiber, $\Phi_{\mathcal{X}}$ is well-defined and restricts to Φ on the central fiber.*

Proof. This follows directly from the construction actually. Indeed, the sections used to construct $M^{\mu_{ss}}$ are pulled back from the moduli space of semistable sheaves over a high degree generic curves. In our case, since the family is flat, we can always assume the high degree curve in the central fiber fits into a flat family. Now the conclusion follows from [17, Thm. 4.3.7], which formulates the relative version of moduli space of semistable bundles over curves, and this provides enough sections. \square

Combined with Proposition 4.11, this implies the uniqueness of the bubbling set in Theorem 1.5. Indeed, this follows from that $[\mathcal{E}_0]$ and $[\mathcal{E}'_0]$ give the same point in $M^{\mu_{ss}}$ by continuity. In particular, $\mathcal{C}(\mathcal{E}_0) = \mathcal{C}(\mathcal{E}'_0)$ which is a contradiction to our assumption. We emphasize here by assumption, we are working over a family of smooth surface.

4.2. The analytic side. Now we prove Theorem 1.6. Recall we assume $\mathcal{E}_t \rightarrow X_t$ are stable vector bundles for $t \neq 0$, and we want to study the analytic degeneration of the corresponding HYM connections with respect to the natural Kähler metric $\omega_t = \omega_{FS}|_{X_t}$ on X_t induced from the embedding $\mathcal{X} \rightarrow \mathbb{P}_1 \times \mathbb{P}^N$.

As for this, we always fix a sequence $\mathcal{E}_i := \mathcal{E}_{t_i}$ where $t_i \rightarrow 0$ as $i \rightarrow \infty$. Let A_i be the admissible HYM connection on \mathcal{E}_{t_i} over $(X_i, \omega_i) := (\mathcal{X}_{t_i}, \omega_{t_i})$. By passing to a subsequence, up to gauge transforms, A_i converges to an admissible HYM connection A_∞ away from a complex codimension 2 subvariety $\Sigma \subset X_0^{reg}$. More precisely, take a precompact exhaustion of X_0^{reg} as $U_1 \subset U_2 \subset \dots \subset X_0^{reg}$. Over each U_j , by fixing diffeomorphisms of the base, we can assume ω_i and the complex structure J_i of X_i are all defined on U_j which converge to the complex structure and the Kähler metric of $X_0^{reg} \cap U_j$. Now we can apply Uhlenbeck compactness to get a limit over U_j and a diagonal sequence argument gives the limiting connection A_∞ as well as the bubbling set

$$\Sigma = \{x \in X_0^{reg} : \lim_{r \rightarrow 0} \liminf_i r^{4-2n} \int_{B_r(x)} |F_{A_i}|^2 > 0\}$$

which is a subvariety of X_0^{reg} and decomposes as $\Sigma = \Sigma_b \cup \text{Sing}(A_\infty)$, where Σ_b is of pure codimension 2 (cf. [31]). As a sequence of Radon measures over Y^{reg} ,

$$\mu_i := |F_{A_i}|^2 d\text{Vol}_i \rightharpoonup \mu_\infty = |F_{A_\infty}|^2 d\text{Vol} + \nu,$$

where $\nu|_{Y^{reg}} = \sum_k 8\pi^2 m_k \Sigma_k$ and Σ_k denotes the pure codimension 2 components of Σ . Here $m_k \in \mathbb{Z}$ is called the analytic multiplicity of Σ_k . In the following, abusing notation, we will use $\Sigma_b = \sum_k m_k \Sigma_k$ to include the multiplicity information. Let \mathcal{E}_∞ be the holomorphic vector bundle defined by A_∞ over X_0^{reg} .

4.2.1. *Proof of Theorem 1.6 (I).* Fix k large enough and a sequence $\{s_i\}_j$ of holomorphic sections of $H^0(X_i, \mathcal{E}_i(k))$ with L^2 norms normalized to be 1.

Lemma 4.13. *There exists a constant C so that*

$$\|s_i\|_{W^{1,2}} < C, \quad \|s_i\|_{L^\infty} < C.$$

In particular, by passing to a subsequence, s_i converges locally smoothly to a nontrivial holomorphic section s_∞ away from Σ satisfying $\|s_\infty\|_{L^2} = 1$ and

$$\|s_\infty\|_{W^{1,2}} \leq C, \quad \|s_\infty\|_{L^\infty} \leq C.$$

Proof. The $W^{1,2}$ bound follows from integration of the equation

$$\Delta |s_i|^2 = |\nabla s_i|^2 - \mu |s_i|^2.$$

Since we have a uniform Sobolev constant for X_{t_i} , which is due to the fact that X_{t_i} is a family of minimal submanifold in \mathbb{P}^N (see [27]), the uniform L^∞ bound follows from Moser iteration. Note here that the X_{t_i} are all smooth. \square

Fix k large enough. Choose an orthonormal basis $\{s_i^j\}_j$ for $H^0(X_i, \mathcal{E}_i(k))$. By passing to a subsequence, we can assume s_i^j converges to s_∞^j . Let \mathcal{F}_∞^k be the image sheaf of the map $q_\infty : \mathcal{O}(-k)^{\oplus N_k} \rightarrow \mathcal{E}_\infty$, where $q_\infty = (s_\infty^1, \dots, s_\infty^{N_k})$. Using the argument in [13, p. 54], we have

Lemma 4.14. $\mathcal{F}_\infty^k \subset \mathcal{F}_\infty^{k+1}$. Furthermore, over any fixed compact subset $K \subset Y^{reg}$, the equality holds for k large .

Proposition 4.15. \mathcal{E}_∞ can be extended to a coherent reflexive sheaf $\overline{\mathcal{E}_\infty} \rightarrow X_0$.

Proof. It suffices to prove that for any $r \leq \text{rk}(\mathcal{E}_\infty)$, there exists a *saturated* subsheaf $\mathcal{G}_r \subset \mathcal{E}_\infty$, so that \mathcal{G}_r extends to a reflexive coherent sheaf on Y , and $\text{rk } \mathcal{G}_r = r$. Then for $r = \text{rk}(\mathcal{E}_\infty)$, we must have $\mathcal{G}_r = \mathcal{E}_\infty$ on X_0^{reg} , and the result follows. We prove the existence of \mathcal{G}_r by induction on r .

$r = 1$: Take a nonzero section $\sigma \in H^0(X_0^{reg}, \mathcal{E}_\infty(k))$. Then there is a Weil divisor $Z \subset X_0^{reg}$ so that the image of the injection

$$0 \longrightarrow \mathcal{O}_{X_0^{reg}}(Z) \xrightarrow{\sigma} \mathcal{E}_\infty(k)$$

defines a saturated rank 1 subsheaf $\mathcal{G}_1 \subset \mathcal{E}_\infty$. By the Remmert-Stein extension theorem we know that Z can be extended to a Weil divisor over X_0 , since X_0 is normal. In particular, this tells us that \mathcal{G}_1 can be extended to a coherent sheaf which admits a map to \mathcal{E}_∞ away from X_0^s . This proves the result for $r = 1$.

$r \geq 1$: Assume the statement has been proved for $\text{rk} = r - 1$, and let \mathcal{G}_{r-1} be a saturated subsheaf of \mathcal{E}_∞ which can be extended to be a coherent sheaf over X_0 . Now we look at $\mathcal{E}_\infty/\mathcal{G}_{r-1}$. By the asymptotic Riemann-Roch theorem (see Corollary 2.8), $\mathcal{E}_\infty(k)/\mathcal{G}_{r-1}(k)$ admits a section which generates a rank 1 subsheaf of $\mathcal{E}_\infty(k)/\mathcal{G}_{r-1}(k)$ for k large. Otherwise, $\mathcal{G}_{r-1}(k)$ admits $P(k)$ sections for k large, which is a contradiction. We pick such a section σ of $\mathcal{E}_\infty(k)/\mathcal{G}_{r-1}(k)$. As above, we can assume σ gives a rank 1 saturated coherent subsheaf of $\mathcal{E}_\infty/\mathcal{G}_{r-1}$. Let $\mathcal{G}_r \subset \mathcal{E}_\infty$ be the sheaf which fits into the following exact sequence over X_0^{reg} as

$$0 \longrightarrow \mathcal{G}_{r-1} \longrightarrow \mathcal{G}_r \longrightarrow \mathcal{L}_1 \longrightarrow 0 .$$

Since \mathcal{G}_{r-1} and \mathcal{L}_1 are coherent over X_0 , we know \mathcal{G}_r is also coherent over X_0 . \square

As Lemma 2.14, we have

Corollary 4.16. *The closure of Σ is a subvariety of X_0 .*

Proof. We first prove the pure codimension 2 part can be extended. Take such a component Σ_k . Note for any compact set $K \subset Y^{reg}$,

$$\Sigma_k \cap K \leq \limsup_i \int_{X_i} |F_{A_i}|^2 \leq C$$

where C is independent of K . In particular, Σ_k has bounded volume in \mathbb{P}^N , thus could be extended by Bishop extension theorem. The extension of $\text{Sing}(A_\infty)$ follows from $\text{Sing}(A_\infty) = \text{Sing}(\mathcal{E}_\infty)$ while \mathcal{E}_∞ can be extended to be a coherent sheaf over X_0 by Proposition 4.15. \square

We still denote the closure of Σ and Σ_b in X_0 by Σ and Σ_b . Given the extension property and the asymptotic Riemann-Roch theorem 2.8, we have the following.

Corollary 4.17. *For k large, $\mathcal{F}_\infty^k = \mathcal{F}_\infty^{k+1}$ and $\mu(\mathcal{F}_\infty^k) = \mu(\mathcal{E})$. In particular, the inclusion $\mathcal{F}_\infty^k \subset \mathcal{E}_\infty$ induces a bundle isomorphism outside a codimension 2 subvariety.*

As an easy corollary of this, we have

Proposition 4.18. *For any local section s of $\overline{\mathcal{E}_\infty}$, $\log^+ |s|^2 \in L_{loc}^q$ for any $q > 1$.*

Proof. Since $\overline{\mathcal{E}_\infty}/\mathcal{F}_\infty^k$ is a torsion sheaf, there exists a holomorphic function P so that $Ps \in L_{loc}^2$. Note that

$$\log^+ |s|^2 \leq \log^+ |Ps|^2 + \log^+ \frac{1}{|P|^2}.$$

Since $(\log^+ x)^q \leq C_q(|x|^2 + 1)$ for some constant C_q , we know $(\log^+ |Ps|^2)^q \in L_{loc}^1$ for any $q > 1$. It remains to show $(\log^+ \frac{1}{|P|^2})^q \in L_{loc}^1$. This follows from $(\log^+ x)^q \leq Cx^{\frac{1}{N}}$ for some constant $C = C(q, N)$ and the fact that $1/|P|^{\frac{2}{N}}$ is integrable for N large, which follows from the smooth case by passing to a resolution of singularities. \square

Corollary 4.19. *For any global section s of $\overline{\mathcal{E}_\infty}$, $\log^+ |s|^2 \in W^{1,2}$. In particular, $|s|$ is bounded.*

Proof. This follows from an easy cut-off argument. Indeed, we know

$$\Delta \log^+ |s|^2 \geq -\mu(\overline{\mathcal{E}_\infty}).$$

Take a cut-off function χ_ϵ supported outside an ϵ neighborhood of $Y^s \cup \Sigma$ in \mathbb{P}^N satisfying $|\nabla \chi_\epsilon| < 1/\epsilon$. Now the conclusion follows by multiplying the equation above by $\chi_\epsilon^2 \log^+ |s|^2$, integrating by parts, and letting $\epsilon \rightarrow 0$. Then we get $\log^+ |s|^2 \in W^{1,2}$. Given this, the L^∞ bound follows as Proposition 2.13. \square

As a direct corollary of this, similar to Corollary 2.16, we have

Corollary 4.20. *$\overline{\mathcal{E}_\infty}$ is semi-stable with slope equal to $\mu(\mathcal{F})$.*

4.2.2. *Proof of Theorem 1.6 (II).* The first part about Gr^{**} follows exactly as section 4.1.2 since we know the limiting sheaf is semistable. The second part follows from Corollary 2.17.

4.2.3. *Proof of Theorem 1.6 (III).* Now we assume X_0 is smooth in codimension 2. From above, we know \mathcal{E}_∞ can be extended as a reflexive sheaf $\overline{\mathcal{E}_\infty}$ over X_0 and we also have the following natural maps

$$q_i^k : \mathcal{O}(-k)^{\oplus N_k} \xrightarrow{(s_i^1, \dots, s_i^{N_k})} \mathcal{E}_i$$

By passing to a subsequence, we can take a limit of $\{q_i^k\}_i$ in the corresponding relative Quot scheme, which we denote as

$$q_\infty^{alg,k} : \mathcal{O}(-k)^{\oplus N_k} \longrightarrow \mathcal{F}^{alg,k}.$$

Similar to [13, Sec. 4.2], we have

Lemma 4.21. $q_\infty^{alg,k}$ induces an isomorphism $\phi : \mathcal{F}^{alg,k} \rightarrow \mathcal{F}_\infty^k$ over Y for k large.

Denote $\mathcal{F}_\infty = \cup_k \mathcal{F}_\infty^k$ which is equal to \mathcal{F}_∞^k for any k large. Using Theorem 2.9 and the argument in [26, Sec. 4.3], we have

Proposition 4.22. $\Sigma_b^{alg} = \mathcal{C}(\mathcal{F}_\infty)$.

We also observe

Lemma 4.23. *By restricting to a smooth family of surfaces $\mathcal{Y} \subset \mathcal{X}$ and taking a discrete subsequence of smooth surfaces $Y_i := \mathcal{Y} \cap X_i$ which converges to a smooth surface $Y_\infty \subset X_0$ on the central fiber, we can assume the following*

- (1) $\mathcal{E}_0|_{Y_\infty}$ and $\mathcal{F}_\infty|_{Y_\infty}$ are torsion-free semistable with the same Hilbert polynomial;
- (2) $[\mathcal{H}|_{Y_i} \otimes \mathcal{O}_{Y_i}(-m) \rightarrow \mathcal{E}_i|_{Y_i}]$ converges to $[\mathcal{H}|_{Y_\infty} \otimes \mathcal{O}_{Y_\infty}(-m) \rightarrow \mathcal{F}_\infty|_{Y_\infty}]$ in the corresponding relative Quot scheme.

In particular, $\mathcal{C}(\mathcal{E}_0|_{Y_\infty}) = \mathcal{C}(\mathcal{F}_\infty|_{Y_\infty})$.

Proof. The first statement follows from the restriction theorem and a straightforward calculation using that \mathcal{E}_0 and \mathcal{F}_∞ have the same Hilbert polynomial. For the second statement, we can always take a limit of the sequence $[\mathcal{H}|_{Y_i} \otimes \mathcal{O}_{Y_i}(-m) \rightarrow \mathcal{E}_i|_{Y_i}]$ in the corresponding Quot scheme as $[\mathcal{H}|_{Y_\infty} \otimes \mathcal{O}_{Y_\infty}(-m) \rightarrow \mathcal{F}'_\infty]$; furthermore, \mathcal{F}'_∞ and $\mathcal{E}_0|_{Y_\infty}$ have the same Hilbert polynomial. As [13, Lemma 4.4], there exists a surjective map $\mathcal{F}'_\infty \rightarrow \mathcal{F}_\infty|_{Y_\infty}$ which now has to be an isomorphism since they have the same Hilbert polynomial. The last statement follows from Proposition 4.12 which implies $\mathcal{F}_\infty|_{Y_\infty}$ and $\mathcal{E}_0|_{Y_\infty}$ give the same points in $M_{Y_\infty}^{\mu_{ss}}$ by continuity. \square

Similar to the uniqueness of the cycle in Theorem 1.5, it follows from Proposition 4.11, Lemma 4.23 and Proposition 4.22

Corollary 4.24. $\mathcal{C}(\mathcal{E}_0) = \mathcal{C}(\mathcal{F}_\infty)$ for any semistable degeneration \mathcal{E} of $\{\mathcal{E}_t\}_t$. In particular, $\Sigma_b^{alg} = \mathcal{C}(\mathcal{E}_0)$.

5. THE MEHTA-RAMANATHAN RESTRICTION THEOREM

In this section, as an application, we will study the relationship between the MR restriction theorem of semistable bundles with the degeneration of the semistable bundles through the deformation to the projective cones.

5.1. Deformation to projective cones. Fix (X, L, ω) to be a Hodge manifold, i.e. $[\omega] = c_1(L)$ for some very ample line bundle L and let V be a smooth hypersurface defined by a section in $H^0(X, L)$. We recall the construction of the projective cone. Through the embedding

$$X \longrightarrow \mathbb{P}^N := \mathbb{P}(H^0(X, L)^*),$$

V can be realized as the intersection of X with a hyperplane \mathbb{P}^{N-1} in \mathbb{P}^N . Here \mathbb{P}^N is endowed with the Fubini-Study metric given by the L^2 metric. Using the L^2 metric, we can write $\mathbb{P}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1}$ and a point in \mathbb{P}^N will be denoted by $Z := [Z_1, \dots, Z_{N+1}]$. We also denote $\hat{Z} := [Z_1, \dots, Z_N, 0]$. Now we can take the affine cone $C(V)$ given by V in \mathbb{C}^N and the projective cone $\overline{C(V)}$ is given by adding a copy of V to $C(V)$ at infinity. Denote by ω_0 the induced metric on $\overline{C(V)} \subset \mathbb{P}^N$. In general, $\overline{C(V)}$ is not normal. We have the following well-known lemma (see [15, I-Ex. 3.18, II-Ex. 5.14]).

Lemma 5.1. *The following are equivalent:*

- (1) $\overline{C(V)}$ is normal;
- (2) $H^0(\mathbb{P}^N, \mathcal{O}(k)) \rightarrow H^0(V, \mathcal{O}(k)|_V)$ is surjective for each k ;
- (3) The homogeneous coordinate ring of X in \mathbb{P}^N is an integrally closed domain.

Let $\hat{\mathcal{X}}$ be the blow-up of $X \times \mathbb{P}^1$ along $V \times \{0\}$. Denote $q : \hat{\mathcal{X}} \rightarrow \mathbb{P}^1$ as the natural projection map. We know $q^{-1}(t) = X$ for $t \neq 0$ and $q^{-1}(0) = X \cup_V \overline{N_V}$ where $\overline{N_V}$ denotes the space by adding an infinity copy of V to N_V and X and $\overline{N_V}$ are attached along $V \subset X$ and the infinity section V . It is well-known that we can blow down $\hat{\mathcal{X}}$ by contracting a copy of X in the central fiber to get \mathcal{X} , which now is a flat family of irreducible varieties. Let $p : \mathcal{X} \rightarrow \mathbb{P}^1$ denote the natural projection map and denote $X_t = p^{-1}(t)$ as before. The subtlety is that X_0 might not be normal in general, and so in order to apply the degeneration results we obtained in Section 4 we need to make this extra assumption.

Let \mathcal{L} be the line bundle obtained by first pulling back L over $X \times \mathbb{P}^1$, then pulling back to $\hat{\mathcal{X}}$ and tensoring it with $-\overline{N_V}$, finally pushing down to \mathcal{X} through the blow-down map. We have the following result and examples from [7, 20].

- Lemma 5.2.** (1) *There exists a natural map $\phi : \overline{C(V)} \rightarrow X_0$ which is an isomorphism away from the vertex. It is an isomorphism if and only if $H^0(X, \mathcal{O}(kL)) \rightarrow H^0(V, \mathcal{O}(kL)|_V)$ is surjective for each k .*
- (2) *\mathcal{X} is a flat family and there exists an embedding $\iota : \mathcal{X} \rightarrow \mathbb{P}^N \times \mathbb{P}^1$ where $\mathcal{L} = \iota^* \pi_1^* \mathcal{O}_{\mathbb{P}^N}(1)$. Here $\pi_1 : \mathbb{P}^N \times \mathbb{P}^1 \rightarrow \mathbb{P}^N$ denotes the natural projection map.*

Example 5.3. *In the construction, let X be a Riemann surface of $g \geq 1$ and V be a point p . In this case, one readily checks that $\overline{C(V)} = \mathbb{P}^1$ while X_0 has to be singular since its arithmetic genus is $g \geq 1$.*

Example 5.4. *In the construction, if X is Fano or a simply connected Calabi-Yau manifold, X_0 always coincides with $\overline{C(V)}$.*

Example 5.5. *In the construction, we take $X = \mathbb{P}^n$, $L = \mathcal{O}(k)$ and V to be a degree k smooth hypersurface. In this case, it is known that V must be projectively normal (see [15, Ex. II.8.4]). Furthermore, all the conditions in Lemma 5.2 are satisfied. The central fiber X_0 is normal and $X_0 = \overline{C(V)}$. When $k = 1$, we know $\overline{C(V)} = \mathbb{P}^n$.*

We will use the following notation for the cone $\overline{C(V)}$

- denote the vertex of $\overline{C(V)}$ by $o \in \overline{C(V)}$;
- let $\pi : \overline{C(V)} \setminus \{o\} \rightarrow V$ be the natural surjective map;
- denote $\iota : \overline{C(V)} \setminus \{o\} \hookrightarrow \overline{C(V)}$ the natural embedding.

For use later, we observe the following

Corollary 5.6 (HE metrics on cone-type sheaves). *Let $\mathcal{E} = \iota_*\pi^*(\mathcal{E}|_V)$ be a polystable stable reflexive sheaf over $\overline{C(V)}$, then an admissible HE metric on \mathcal{E} can be given by*

$$h := \frac{|\widehat{Z}|^{2\mu}}{|Z|^{2\mu}} \pi^* \underline{h}$$

where \underline{h} is a HE metric on $\mathcal{E}|_V$ and $\mu = \mu(\mathcal{E})$. In particular, the admissible HE metric on \mathcal{E} restricts to the admissible HE metric on $\mathcal{E}|_V$.

Proof. It suffices to show that h is also an admissible HE metric on \mathcal{E} . Indeed, by definition

$$F_h = \mu \partial \bar{\partial} \ln |Z|^2 \text{Id} - \mu \partial \bar{\partial} \ln |\widehat{Z}|^2 \text{Id} + F_{\pi^* \underline{h}} .$$

Since

$$\omega_0|_{\overline{C(V)}} = \sqrt{-1} \partial \bar{\partial} \ln(1 + |z|^2) = \frac{|z|^2}{1 + |z|^2} \sqrt{-1} \partial \bar{\partial} \ln |z|^2 + \frac{\sqrt{-1} \partial |z|^2 \wedge \bar{\partial} |z|^2}{(|z|^2 + 1)^2 |z|^2} ,$$

where $z = (\frac{Z_1}{Z_{N+1}}, \dots, \frac{Z_N}{Z_{N+1}})$, by a direct computation we have

$$\sqrt{-1} \Lambda_{\omega_0} F_{\pi^* \underline{h}} = \frac{|Z|^2}{|\widehat{Z}|^2} \mu \text{Id} .$$

Similarly,

$$\sqrt{-1} \Lambda_{\omega_0} \partial \bar{\partial} \ln |\widehat{Z}|^2 = \frac{|Z|^2}{|\widehat{Z}|^2} \text{Id} .$$

In particular, we have $\sqrt{-1} \Lambda_{\omega_0} F_h = \mu \text{Id}$. Also it is obvious that the L^2 norm of F_h is bounded over $\overline{C(V)}$, since it has quadratic blow-up when $n \geq 3$, and $\sqrt{-1} F_{\underline{h}} = \mu \text{id} \omega_{FS}|_V$ when $n = 2$. The conclusion follows. \square

5.2. Semistable bundles through deformation to projective cones.

In the following, we will work under the assumption that X_0 constructed above is normal. We will use the induced metrics from the embedding $\mathcal{X} \subset \mathbb{P}^N \times \mathbb{P}^1$ from (3) in Lemma 5.2 for the deformation of the base as projective varieties. Notice that a semistable bundle $\mathcal{F} \rightarrow X$ gives rise to an algebraic family $\{\mathcal{E}_t = \mathcal{F}\}_t$. As a special case of Theorem 1.5, we have

Corollary 5.7. *Assume X_0 is normal. A semistable degeneration of $\{\mathcal{E}_t = \mathcal{F}\}_t$ always exists. For different semistable degenerations \mathcal{E} and \mathcal{E}' ,*

$$\mathrm{Gr}(\mathcal{E}_0)^{**} = \mathrm{Gr}(\mathcal{E}'_0)^{**} ,$$

and when $n \geq 3$, $\mathcal{C}(\mathcal{E}_0) = \mathcal{C}(\mathcal{E}'_0)$.

Assume now that \mathcal{F} is stable, and let A_t be the unique HYM connection on $\mathcal{E}_t = \mathcal{F}$ with respect to the corresponding induced metric. As a special case of the discussion in Section 4.2, by passing to a subsequence, we can assume A_t converges to A_∞ with blow-up locus Σ_b subvarieties of X_0 . Let \mathcal{E}_∞ be the reflexive sheaf over X_0 defined by A_∞ . As a consequence of Theorem 1.6, we have

Corollary 5.8. *Assume X_0 is normal. For each limiting pair (A_∞, Σ_b) ,*

- \mathcal{E}_∞ can be extended to a reflexive sheaf $\overline{\mathcal{E}_\infty}$ over $\overline{C(V)}$. Furthermore, for any $q > 1$; for any global section s , $\log^+ |s|^2 \in W^{1,2} \cap L^\infty$;
- $\mathrm{Gr}(\overline{\mathcal{E}_\infty})^{**} = \mathrm{Gr}(\mathcal{E}_0)^{**}$;
- Assume $n \geq 3$. Then $\Sigma_b = \mathcal{C}(\mathcal{E}_0)$ for any semistable degeneration \mathcal{E} of $\{\mathcal{E}_t = \mathcal{F}\}$ through the deformation to projective cones and $\overline{\mathcal{E}_\infty} = \mathrm{Gr}(\mathcal{E}_0)^{**}$

Remark 5.9.

- In a sense, under the given assumptions, our results produce a canonical object when the restrictions of semistable bundles to smooth hypersurfaces fail to be semistable.
- Using the example $\mathcal{E} = T_{\mathbb{P}^n}$ and $V = \mathbb{P}^{n-1}$, A_∞ does not have a cone property. Indeed, in this example A_∞ will be the original HYM connection because the isomorphism class of $T_{\mathbb{P}^n}$ is \mathbb{C}^* invariant. This is very different from [6].

By replacing L with L^k in the construction and choosing V generic, we know from the MR restriction theorem that $\mathcal{F}|_V$ is stable. In this case, a natural semistable degeneration \mathcal{E} can be constructed so that $\mathcal{E}_0|_{\overline{C(V)} \setminus \{o\}} = \pi^*(\mathcal{E}|_V)$. By Corollary 5.6 and Corollary 5.8, we have the following analytic version of MR restriction theorem for stable bundles

Corollary 5.10. *Assume $X_0 = \overline{C(V)}$ is normal and that $\mathcal{F}|_V$ is stable. Then $\mathcal{F}_\infty = \iota_* \pi^*(\mathcal{F}|_V)$ and $A_\infty|_V$ is the HYM connection on $\mathcal{F}|_V$.*

Proof. The sheaf part is clear. For the connection part, it suffices to notice that for the HE metric constructed on $\iota_* \pi^*(\mathcal{F}|_V)$ in Corollary 5.6, the sheaf is generated by L^2 sections, as is the sheaf defined by the limiting HYM

connection. In particular, the isomorphism between \mathcal{F}_∞ and $\iota_*\pi^*(\mathcal{F}_V)$ is uniformly bounded, thus parallel as in Corollary 2.17. \square

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