A GENERALIZED QUOT SCHEME AND MEROMORPHIC VORTICES

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ABSTRACT. Let X be a compact connected Riemann surface. Fix a positive integer r and two nonnegative integers d_p and d_z. Consider all pairs of the form $(\mathcal{F}\,,f)$, where $\mathcal{F}$ is a holomorphic vector bundle on X of rank r and degree $d_z - d_p$, and

$$f\,:\,\mathcal{O}_X^{\oplus r}\,\longrightarrow\,\mathcal{F}$$

is a meromorphic homomorphism which an isomorphism outside a finite subset of X and has pole (respectively, zero) of total degree d_p (respectively, d_z). Two such pairs $(\mathcal{F}_1\,,f_1)$ and $(\mathcal{F}_2\,,f_2)$ are called isomorphic if there is a holomorphic isomorphism of $\mathcal{F}_1$ with $\mathcal{F}_2$ over X that takes f_1 to f_2. We construct a natural compactification of the moduli space equivalence classes pairs of the above type. The Poincaré polynomial of this compactification is computed.

1. Introduction

Take a compact connected Riemann surface X. Fix positive integers r and d. Consider pairs of the form $(E\,,f)$, where E is a holomorphic vector bundle on X of rank r and degree d, and

$$f\,:\,\mathcal{O}_X^{\oplus r}\,\longrightarrow\,E$$

is an $\mathcal{O}_X$–linear homomorphism which is an isomorphism outside a finite subset of X. This implies that the total degree of zeros of f is d. Two such pairs $(E_1\,,f_1)$ and $(E_2\,,f_2)$ are called equivalent if there is a holomorphic isomorphism

$$\phi\,:\,E_1\,\longrightarrow\,E_2$$

such that $\phi\circ f_1\,=\,f_2$. Such pairs are examples of vortices [BDW], [Br], [BR], [Ba], [EINOS].

For any pair $(E\,,f)$ of the above type, consider the dual homomorphism

$$f^*\,:\,E^*\,\longrightarrow\,(\mathcal{O}_X^{\oplus r})^*\,=\,\mathcal{O}_X^{\oplus r}\,.$$

The quotient $\mathcal{O}_X^{\oplus r}/\mathrm{image}(f^*)$ is an element of the Quot scheme $\mathrm{Quot}(r,d)$ that parametrizes all torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree d. Conversely, given any torsion quotient

$$\mathcal{O}_X^{\oplus r}\,\stackrel{\psi}{\longrightarrow}\,T$$

of degree d, consider the homomorphism

$$\mathcal{O}_X^{\oplus r}\,=\,(\mathcal{O}_X^{\oplus r})^*\,\stackrel{\psi'}{\longrightarrow}\,\mathrm{kernel}(\psi)^*$$

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induced by the inclusion $\mathrm{kernel}(\psi) \hookrightarrow \mathcal{O}_X^{\oplus r}$. The pair $(\mathrm{kernel}(\psi)^*\,,\psi')$ is clearly of the above type. Therefore, the moduli space of equivalence classes of pairs $(E\,,f)$ is identified with the Quot scheme $\mathrm{Quot}(r,d)$.

Here we consider pairs of the form $(E\,,f)$, where E is a holomorphic vector bundle on X of rank r and degree d, and

$$f\,:\,\mathcal{O}_X^{\oplus r}\,\longrightarrow\,E$$

is an $\mathcal{O}_X$–linear meromorphic homomorphism which is an isomorphism outside a finite subset of X. We assume that the total degree of the poles of the meromorphic homomorphism is d_p. This implies that the total degree of the zeros of the meromorphic homomorphism is $d+d_p$. As before, two such pairs $(E_1\,,f_1)$ and $(E_2\,,f_2)$ will be called equivalent if there is a holomorphic isomorphism

$$\phi\,:\,E_1\,\longrightarrow\,E_2$$

such that $\phi\circ f_1\,=\,f_2$. The equivalence classes of pairs can be considered as examples of meromorphic vortices.

We construct a natural compactification of the moduli space of these meromorphic vortices. We compute the Poincaré polynomial of this compactification.

2. Notations and Conventions

Let S be a scheme and $Y\,\longrightarrow\,S$ a smooth projective morphism. Given a coherent sheaf $\mathcal{F}$ on Y flat over S and a numerical polynomial $r(t)$, we denote by $\mathrm{Quot}(\mathcal{F}/S, r(t))$ the Grothendieck Quot scheme over S parametrizing quotients of $\mathcal{F}$ with Hilbert polynomial $r(t)$ [Gr]. There is a universal exact sequence on $\mathrm{Quot}(\mathcal{F}/S, r(t))\times_S Y$

$$0\,\longrightarrow\,\mathcal{K}^{\mathrm{univ}}_{\mathrm{Quot}(\mathcal{F}/S,r(t))}\,\longrightarrow\,\pi_Y^*\mathcal{F}\,\longrightarrow\,\mathcal{Q}^{\mathrm{univ}}_{\mathrm{Quot}(\mathcal{F}/S,r(t))}\,\longrightarrow\,0\,,$$

where $\pi_Y\,:\,\mathrm{Quot}(\mathcal{F}/S, r(t))\times_S Y\,\longrightarrow\,Y$ is the natural projection. Often we will just drop the subscripts and write $\mathcal{K}^{\mathrm{univ}}$ or $\mathcal{Q}^{\mathrm{univ}}$ instead. This construction is well behaved with respect to pull-backs, so let us record the following:

Lemma 2.1. *For any morphism $g\,:\,T\,\longrightarrow\,S$, the base change*

$$\mathrm{Quot}(g^*\mathcal{F}/T, r(t))\,\cong\,\mathrm{Quot}(\mathcal{F}/S, r(t))\times_S T$$

holds.

Proof. This follows by examining the corresponding representable functors. □

We will mostly be interested in the case where $Y\,\longrightarrow\,S$ is a smooth, connected and of relative dimension one, that is a relative curve, and $\mathcal{F}$ is locally free of rank r. Further, we will only consider torsion quotients of rank zero and degree d. This Quot scheme will be denoted by $\mathrm{Quot}(\mathcal{F}/S, d)$. When $r\,=\,1$ and S is a point, then

$$\mathrm{Quot}(\mathcal{O}, d)\,=\,\mathrm{Sym}^d(Y)\,,$$

the d-th symmetric power of Y.

Given an positive integer d by *a partition of length* $k > 0$ *of* d we mean a sequence $\mathbf{P} = (p_1, p_2, \cdots, p_k)$ of non-negative integers with $\sum_{i=1}^{k} p_i = d$. For such a partition define $d(\mathbf{P}) := \sum_{i=1}^{k}(i-1)p_i$. We will write

$$\mathrm{Sym}^{\mathbf{P}}(Y) = \mathrm{Sym}^{p_1}(Y) \times \cdots \times \mathrm{Sym}^{p_r}(Y).$$

3. A relative Quot scheme

Let X be a compact connected Riemann surface. Let $\mathcal{E}$ and $\mathcal{F}$ be two holomorphic vector bundles on X of common rank r. Take a dense open subset $U \subset X$, such that the complement $S := X \setminus U$ is a finite set, and take an isomorphism of coherent analytic sheaves

$$f : \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$$

over U. This homomorphism f will be called *meromorphic* if there is a positive integer n such that f extends to a homomorphism of coherent analytic sheaves

$$\widehat{f} : \mathcal{E} \longrightarrow \mathcal{F} \otimes \mathcal{O}_X(nS) \supset \mathcal{F}$$

over X, where S is the reduced divisor defined by the finite subset S. Note that since the divisor S is effective, we have $\mathcal{F} \subset \mathcal{F} \otimes \mathcal{O}_X(nS)$. Therefore, f is meromorphic if and only if the homomorphism f is algebraic with respect to the algebraic structures on $\mathcal{E}|_U$ and $\mathcal{F}|_U$ given by the algebraic structures on $\mathcal{E}$ and $\mathcal{F}$ respectively.

Take a meromorphic homomorphism f as above. We note that the extension $\widehat{f}$ is uniquely determined by f because f and $\widehat{f}$ coincide over U. The inverse image

$$\mathcal{E}(f) := \widehat{f}^{-1}(\mathcal{F}) \subset \mathcal{E}$$

(recall that $\mathcal{F} \subset \mathcal{F} \otimes \mathcal{O}_X(nS)$) is clearly independent of the choice of n. We note that both $\mathcal{E}(f)$ and $\widehat{f}(\mathcal{E}(f))$ are holomorphic vector bundles on X because they are coherent analytic subsheaves of holomorphic vector bundles. Both of then are of rank r, and the restriction

$$\widehat{f}|_{\mathcal{E}(f)} : \mathcal{E}(f) \longrightarrow \widehat{f}(\mathcal{E}(f)) \tag{3.1}$$

is an isomorphism of holomorphic vector bundles. Define

$$\mathcal{Q}_p(f) := \mathcal{E}/\mathcal{E}(f) \quad \text{and} \quad \mathcal{Q}_z(f) := \mathcal{F}/(\widehat{f}(\mathcal{E}(f))) \tag{3.2}$$

(the subscripts "p" and "z" stand for "pole" and "zero" respectively). We note that both $\mathcal{Q}_p(f)$ and $\mathcal{Q}_z(f)$ are torsion coherent analytic sheaves on X. In particular, their supports are finite subsets of X. From (3.2) it follows that

$$\mathrm{degree}(\mathcal{Q}_p(f)) = \mathrm{degree}(\mathcal{E}) - \mathrm{degree}(\mathcal{E}(f)) \quad \text{and} \tag{3.3}$$

$$\mathrm{degree}(\mathcal{Q}_z(f)) = \mathrm{degree}(\mathcal{F}) - \mathrm{degree}(\widehat{f}(\mathcal{E}(f))).$$

Fix positive integers r, d_p and d_z. Set the domain $\mathcal{E}$ to be the trivial vector bundle $\mathcal{O}_X^{\oplus r}$ of rank r. Consider all triples of the form $(\mathcal{F}, U, f)$, where

- $\mathcal{F}$ is a holomorphic vector bundle on X of rank r,
- U is the complement of a finite subset of X, and
- $f : \mathcal{O}_X^{\oplus r}|_U = \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U$ is a meromorphic homomorphism such that

$$\mathrm{degree}(\mathcal{Q}_p(f)) = d_p \quad \text{and} \quad \mathrm{degree}(\mathcal{Q}_z(f)) = d_z.$$

Since $\widehat{f}|_{\mathcal{E}(f)}$ in (3.1) is an isomorphism, from (3.3) we conclude that

$$\text{degree}(\mathcal{F}) \;=\; d_z - d_p + \text{degree}(\mathcal{O}_X^{\oplus r}) \;=\; d_z - d_p\,. \tag{3.4}$$

Two such triples $(\mathcal{F}_1\,,U_1\,,f_1)$ and $(\mathcal{F}_2\,,U_2\,,f_2)$ will be called *equivalent* if there is a holomorphic isomorphism of vector bundles over X

$$\beta\,:\,\mathcal{F}_1\,\longrightarrow\,\mathcal{F}_2$$

such that

$$\beta\circ(f_1|_{U_1\cap U_2})\;=\;f_2|_{U_1\cap U_2}\,.$$

Therefore, the equivalence class of $(\mathcal{F}\,,U\,,f)$ depends only on $(\mathcal{F}\,,f)$ and it is independent of U. More precisely, $(\mathcal{F}\,,U\,,f)$ is equivalent to $(\mathcal{F}\,,W\,,f|_W)$ for every $W\,\subset\,U$ such that the complement $U\setminus W$ is a finite set.

Let

$$\mathrm{Q}^0\;=\;\mathrm{Q}^0_X(r,d_p,d_z) \tag{3.5}$$

be the space of all equivalence classes of triples of the above form. We will embed Q^0 as a Zariski open subset of a smooth complex projective variety.

Take any triple $(\mathcal{F}\,,U\,,f)$ as above that is represented by a point of Q^0. Consider the short exact sequence

$$0\,\longrightarrow\,\mathcal{E}(f)\,:=\,\text{kernel}(q_p)\,\longrightarrow\,\mathcal{E}\,=\,\mathcal{O}_X^{\oplus r}\,\stackrel{q_p}{\longrightarrow}\,\mathcal{Q}_p(f)\,\longrightarrow\,0\,, \tag{3.6}$$

where q_p denotes the projection to the quotient in (3.2). We also have

$$\mathcal{E}(f)\;=\;\widehat{f}(\mathcal{E}(f))\,\hookrightarrow\,\mathcal{F}$$

(recall that $\widehat{f}|_{\mathcal{E}(f)}$ in (3.1) is an isomorphism). Let

$$0\,\longrightarrow\,\mathcal{F}^*\,\longrightarrow\,\mathcal{E}(f)^* \tag{3.7}$$

be the dual of the above inclusion of $\mathcal{E}(f)$ in $\mathcal{F}$. From (3.6) we have $\text{degree}(\mathcal{E}(f)^*)\;=\;\text{degree}(\mathcal{Q}_p(f))\;=\;d_p$. Therefore, from (3.4) it follows that

$$\text{degree}(\mathcal{E}(f)^*/\mathcal{F}^*)\;=\;\text{degree}(\mathcal{E}(f)^*)-\text{degree}(\mathcal{F}^*)\;=\;d_p+d_z-d_p\;=\;d_z$$

as $\text{degree}(\mathcal{F}^*)\;=\;-\text{degree}(\mathcal{F})$. These imply that we can recover the equivalence class of $(\mathcal{F}\,,f)$ once we know the following two:

- the torsion quotient $\mathcal{Q}_p(f)$ of $\mathcal{O}_X^{\oplus r}$ of degree d_p, and
- the torsion quotient $\mathcal{E}(f)^*/\mathcal{F}^*$ of $\mathcal{E}(f)^*$ of degree d_z.

(It should be clarified that "knowing the torsion quotient $\mathcal{Q}_p(f)$" means knowing the sheaf $\mathcal{Q}_p(f)$ along with the surjective homomorphism $\mathcal{O}_X^{\oplus r}\,\longrightarrow\,\mathcal{Q}_p(f)$; similarly "knowing the torsion quotient $\mathcal{E}(f)^*/\mathcal{F}^*$" means knowing the sheaf $\mathcal{E}(f)^*/\mathcal{F}^*$ along with the surjective homomorphism from $\mathcal{E}(f)^*$ to it.) Indeed, once we know $\mathcal{Q}_p(f)$, we know the kernel $\mathcal{E}(f)$ and hence know $\mathcal{E}(f)^*$; if we know the quotient $\mathcal{E}(f)^*/\mathcal{F}^*$, then we know the subsheaf $\mathcal{F}^*$ of $\mathcal{E}(f)^*$. The dual of this inclusion $\mathcal{F}^*\,\hookrightarrow\,\mathcal{E}(f)^*$, namely the homomorphism

$$\mathcal{E}(f)\,\longrightarrow\,\mathcal{F}\,,$$

gives the meromorphic homomorphism f. In other words, we have the diagram

$$\begin{tikzcd} & 0 \arrow[d] & & & \\ 0 \arrow[r] & \mathcal{E}(f) \arrow[r] \arrow[d, "\widehat{f}"] & \mathcal{O}^{\oplus r} \arrow[r] & \mathcal{Q}_p \arrow[r] & 0 \\ & \mathcal{F} \arrow[d] & & & \\ & \mathcal{Q}_z \arrow[d] & & & \\ & 0 & & & \end{tikzcd}$$

Let $\mathrm{Quot}(r, d_p)$ be the Quot scheme parametrizing the torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree d_p. We have the tautological short exact sequence of coherent analytic sheaves on $X \times \mathcal{Q}(r, d_p)$

$$0 \longrightarrow \mathcal{K}^{\mathrm{univ}} \longrightarrow p_X^* \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{Q}^{\mathrm{univ}} \longrightarrow 0\,, \tag{3.8}$$

where p_X is the projection of $X \times \mathrm{Quot}(r, d_p)$ to X. We write $\mathcal{K} = \mathcal{K}^{\mathrm{univ}}$. Now consider the dual vector bundle

$$\mathcal{K}^* \longrightarrow X \times \mathrm{Quot}(r, d_p) \stackrel{p_Q}{\longrightarrow} \mathrm{Quot}(r, d_p)\,,$$

where p_Q is the natural projection. Using p_Q, we will consider $\mathcal{K}^*$ as a family of vector bundles on X parametrized by $\mathrm{Quot}(r, d_p)$. For any point $y \in \mathrm{Quot}(r, d_p)$, the vector bundle $\mathcal{K}^*|_{X \times \{y\}}$ over X will be denoted by $\mathcal{K}^*_{|y}$. Let

$$\varphi \,:\, \mathrm{Quot}(r, d_p, d_z) \,:=\, \mathrm{Quot}(\mathcal{K}^*/\,\mathrm{Quot}(r, d_p), d_z) \,\longrightarrow\, \mathcal{Q}(r, d_p) \tag{3.9}$$

be the relative Quot scheme over $\mathrm{Quot}(r, d_p)$, for the family $\mathcal{K}^*$, parametrizing the torsion quotients of degree d_z. Therefore, for any point $y \in \mathrm{Quot}(r, d_p)$, the fiber $\varphi^{-1}(y)$ is the Quot scheme parametrizing the torsion quotients of degree d_z of the vector bundle $\mathcal{K}^*_{|y}$.

Both $\mathrm{Quot}(r, d_p)$ and the fibers of φ are irreducible smooth projective varieties. The morphism φ is smooth. Therefore, the projective variety $\mathrm{Quot}(r, d_p, d_z)$ is irreducible and smooth.

Consider Q^0 defined in (3.5). We have a map

$$\eta' \,:\, \mathrm{Q}^0 \longrightarrow \mathrm{Quot}(r, d_p)$$

that sends any triple $(\mathcal{F}\,, U\,, f) \in \mathrm{Q}_0$ to the point representing the quotient $Q_p(f)$ in (3.6). Let

$$\eta \,:\, \mathrm{Q}^0 \longrightarrow \mathrm{Quot}(\mathcal{K}^*, d_z) \,=\, \mathrm{Quot}(r, d_p, d_z) \tag{3.10}$$

be the map that sends any point $\alpha = (\mathcal{F}\,, U\,, f) \in \mathrm{Q}_0$ to the point of $\varphi^{-1}(\eta'(\alpha))$ that represents the quotient $\mathcal{E}(f)^*/\mathcal{F}^*$ in (3.7). This map η is injective because, as observed earlier, the equivalence class of the pair $(\mathcal{F}\,, f)$ can be recovered from the quotient $\mathcal{Q}_p(f)$ of $\mathcal{O}_X^{\oplus r}$ and the quotient $\mathcal{E}(f)^*/\mathcal{F}^*$ of $\mathcal{E}(f)^*$. The image of η is clearly a Zariski open subset of $\mathrm{Quot}(r, d_p, d_z)$.

Let $\bigwedge^r \mathcal{K} \longrightarrow \bigwedge^r p_X^* \mathcal{O}_X^{\oplus r} = p_X^* \mathcal{O}_X$ be the r-th exterior power of the homomorphism in (3.8). Considering it as a family of subsheaves of $\mathcal{O}_X$ of degree $-d_p$ parametrized by $\mathrm{Quot}(r, d_p)$, we have the corresponding classifying morphism

$$\delta_1' : \mathrm{Quot}(r, d_p) \longrightarrow \mathrm{Quot}(1, d_p) = \mathrm{Sym}^{d_p}(X).$$

Let

$$\delta_1 := \delta_1' \circ \varphi : \mathrm{Quot}(r, d_p, d_z) =: Q \longrightarrow \mathrm{Sym}^{d_p}(X) \tag{3.11}$$

be the composition, where φ is constructed in (3.9). Next, consider the tautological subsheaf

$$\mathcal{S} \hookrightarrow (\mathrm{Id}_X \times \varphi)^* \mathcal{K}^*$$

on $X \times \mathrm{Q}$. Let $\bigwedge^r \mathcal{S} \hookrightarrow \bigwedge^r (\mathrm{Id}_X \times \varphi)^* \mathcal{K}^*$ be the r-th exterior power of the above inclusion. Let

$$\delta_2 : \mathrm{Q} \longrightarrow \mathrm{Sym}^{d_z}(X) \tag{3.12}$$

be the morphism that sends any $y \in \mathrm{Q}$ to the scheme theoretic support of the quotient sheaf

$$(\bigwedge^r (\mathrm{Id}_X \times \varphi)^* \mathcal{K}^*_{|\varphi(y)})/(\bigwedge^r \mathcal{S}|_{X \times \{y\}}) \longrightarrow X.$$

Now define the morphism

$$\delta := (\delta_1, \delta_2) : \mathrm{Quot}(r, d_p, d_q) = \mathrm{Q} \longrightarrow \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X), \tag{3.13}$$

where δ_1 and δ_2 are constructed in (3.11) and (3.12) respectively. It can be shown that δ is surjective. In fact, in Section 4 we will construct, and use, a section of δ.

Remark 3.1. Let $\mathcal{M}_X(r, d_z - d_p)$ denote the moduli stack of vector bundles on X of rank r and degree $d_z - d_p$. Since there is a universal bundle over $X \times \mathrm{Quot}(r, d_p, d_q)$, we get a morphism

$$\mathrm{Quot}(r, d_p, d_q) \longrightarrow \mathcal{M}_X(r, d_z - d_p).$$

4. Fundamental group of $\mathrm{Quot}(r, d_p, d_z)$

Proposition 4.1. *The homomorphism between fundamental groups induced by the morphism δ in* (3.13) *is an isomorphism.*

Proof. We will first construct a section of δ. Let

$$D(d_p) \subset X \times \mathrm{Sym}^{d_p}(X)$$

be the divisor consisting of all $(x, \{y_1, \cdots, y_{d_p}\})$ such that $x \in \{y_1, \cdots, y_{d_p}\}$. Then the subsheaf

$$\mathcal{O}_{X \times \mathrm{Sym}^{d_p}(X)}(-D(d_p)) \oplus \mathcal{O}^{\oplus r-1}_{X \times \mathrm{Sym}^{d_p}(X)} \subset \mathcal{O}^{\oplus r}_{X \times \mathrm{Sym}^{d_p}(X)}$$

produces a classifying morphism

$$\theta_1 : \mathrm{Sym}^{d_p}(X) \longrightarrow \mathrm{Quot}(r, d_p). \tag{4.1}$$

Let ξ_1 (respectively, ξ_2) denote the projection of $\mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X)$ to $\mathrm{Sym}^{d_p}(X)$ (respectively, $\mathrm{Sym}^{d_z}(X)$). Like before, $D(d_z) \subset X \times \mathrm{Sym}^{d_p}(X)$ be the divisor consisting of all $(x, \{y_1, \cdots, y_{d_z}\})$ such that $x \in \{y_1, \cdots, y_{d_z}\}$. The subsheaf

$$((\mathrm{Id}_X \times \xi_1)^* (\mathcal{O}_{X \times \mathrm{Sym}^{d_p}(X)}(D(d_p))) \otimes (\mathrm{Id}_X \times \xi_2)^* (\mathcal{O}_{X \times \mathrm{Sym}^{d_z}(X)}(-D(d_z))))$$

$$\oplus(\mathcal{O}^{\oplus r-1}_{X\times \mathrm{Sym}^{d_p}(X)\mathrm{Sym}^{d_z}(X)})^* \subset (\mathrm{Id}_X \times \xi_1)^*(\mathcal{O}_{X\times \mathrm{Sym}^{d_p}(X)}(-D(d_p)) \oplus \mathcal{O}^{\oplus r-1}_{X\times \mathrm{Sym}^{d_p}(X)})^*$$

produces a classifying morphism

$$\theta : \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X) \longrightarrow \mathrm{Quot}(r, d_p, d_q)\,. \tag{4.2}$$

We note that $\varphi \circ \theta = \theta_1$, where φ and θ_1 are the morphisms constructed in (3.9) and (4.1) respectively.

It is straightforward to check that

$$\delta \circ \theta = \mathrm{Id}_{\mathrm{Sym}^{d_p}(X)\times \mathrm{Sym}^{d_z}(X)}\,, \tag{4.3}$$

where δ is constructed in (3.13). In view of this section θ, we conclude that the induced homomorphism between fundamental groups

$$\delta_* : \pi_1(\mathrm{Quot}(r, d_p, d_q)) \longrightarrow \pi_1(\mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X))$$

is surjective (the base points of fundamental groups are suppressed in the notation).

Let

$$U \subset \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X)) \tag{4.4}$$

be the Zariski open subset consisting of all

$$(x\,, y) = (\{x_1\,, \cdots, x_{d_p}\}\,, \{y_1\,, \cdots, y_{d_z}\}) \in \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X))$$

such that the $d_p + d_z$ points $\{x_1\,, \cdots, x_{d_p}\,, y_1\,, \cdots, y_{d_z}\}$ are all distinct, equivalently, the effective divisor $x + y$ is reduced. Let

$$\theta_0 := \theta|_U : U \longrightarrow \mathrm{Quot}(r, d_p, d_z) \tag{4.5}$$

be the restriction of the map θ in (4.2). Also, consider the restriction

$$\delta_0 := \delta|_{\delta^{-1}(U)} : \delta^{-1}(U) \longrightarrow U\,. \tag{4.6}$$

Every fiber of δ_0 is identified with $(\mathbb{P}^{r-1}_{\mathbb{C}})^{d_p} \times (P^{r-1}_{\mathbb{C}})^{d_z}$, where $\mathbb{P}^{r-1}_{\mathbb{C}}$ is the projective space parametrizing the hyperplanes in $\mathbb{C}^r$ and $P^{r-1}_{\mathbb{C}}$ is the projective space parametrizing the lines in $\mathbb{C}^r$ (so $P^{r-1}_{\mathbb{C}}$ parametrizes the hyperplanes in $(\mathbb{C}^r)^*$). From the homotopy exact sequence associated to δ_0 it follows that the induced homomorphism of fundamental groups

$$\delta_{0,*} : \pi_1(\delta^{-1}(U)) \longrightarrow \pi_1(U)$$

is an isomorphism. The variety $\mathrm{Quot}(r, d_p, d_z)$ is smooth, and $\delta^{-1}(U)$ is a nonempty Zariski open subset of it. Therefore, the homomorphism

$$\iota_* : \pi_1(\delta^{-1}(U)) \longrightarrow \pi_1(\mathrm{Quot}(r, d_p, d_z))$$

induced by the inclusion $\iota : \varphi^{-1}(U) \hookrightarrow \mathrm{Quot}(r, d_p, d_z)$ is surjective. Since $\delta_{0,*}$ is an isomorphism, this implies that the homomorphism

$$\theta_{0,*} : \pi_1(U) \longrightarrow \pi_1(\mathrm{Quot}(r, d_p, d_z))$$

induced in θ_0 in (4.5) is surjective. Since θ_0 extends to θ, this immediately implies that the homomorphism

$$\theta_* : \pi_1(\mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X)) \longrightarrow \pi_1(\mathrm{Quot}(r, d_p, d_z)$$

induced in θ in (4.2) is surjective. Since θ_* is surjective, and the composition $\delta_* \circ \theta_*$ is injective (see (4.3)) we conclude that δ_* is also injective. $\square$

5. Cohomology of $\mathrm{Quot}(r, d_p, d_z)$

5.1. **Generalization of a theorem of Bifet.** Let $S_1, S_2, \cdots, S_k$ be a smooth connected projective varieties over $\mathbb{C}$. Fix some line bundles $\mathcal{L}_i$ on $S_i \times X$ of relative degree d_i over S_i. In other words

$$\deg(\mathcal{L}_i|_{s\times X}) = d_i$$

for each point $s \in S_i$. Set $S = S_1 \times \cdots \times S_k$. Let

$$\pi_{S_i\times X} : S \times X \longrightarrow S_i \times X$$

be the natural projection. Define

$$\widetilde{\mathcal{L}}_i := \pi^*_{S_i\times X}\mathcal{L}_i .$$

Let

$$\phi : \mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d) \longrightarrow S$$

be the relative Quot scheme that parametrizes the torsion quotients of degree d. So for any $s = (s_1, \cdots, s_k) \in S$, the fiber $\phi^{-1}(s)$ parametrizes the torsion quotients of $\oplus_{i=1}^k \mathcal{L}_i|_{s_i\times X}$ of degree d. By deformation theory, ϕ is a smooth morphism of relative dimension kd, so $\mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d)$ is smooth of dimension $\dim(S) + kd$. The torus $\mathbb{G}_m^k$ acts on $\mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d)$ via its action on $\bigoplus_{i=1}^k \widetilde{\mathcal{L}}_i$.

For any positive integer p, let $\mathrm{Quot}(\mathcal{L}_i/S_i, p) \longrightarrow S_i$ denote the relative Quot scheme parametrizing the torsion quotients of $\mathcal{L}_i/S_i$ of degree p. So the fiber of $\mathrm{Quot}(\mathcal{L}_i/S_i, p)$ over any $s \in S_i$ parametrizes the torsion quotients of $\mathcal{L}_i|_{s\times X}$ of degree p.

Proposition 5.1. *There is a bijection between the partitions $\mathbf{P} = (p_1, p_2, \cdots, p_k)$ of d of length k and the connected components of the fixed point loci of the $\mathbb{G}_m^k$ action on $\mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d)$. The component corresponding to the partition $\sum_{i=1}^k p_i = d$ is the product of Quot schemes*

$$\mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P}) := \mathrm{Quot}(\mathcal{L}_1/S_1, p_1) \times \mathrm{Quot}(\mathcal{L}_2/S_2, p_2) \times \ldots \times \mathrm{Quot}(\mathcal{L}_k/S_k, p_k)$$

with the obvious structure morphism to S.

Proof. On applies the argument used to prove Lemme 1 in [Bif]. $\square$

As all schemes and morphisms are assumed to be projective it is possible to choose a one parameter subgroup

$$\mathbb{G}_m \hookrightarrow \mathbb{G}_m^k$$

so that

$$\mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d)^{\mathbb{G}_m} = \mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d)^{\mathbb{G}_m^k} .$$

Further, the above one-parameter subgroup can be chosen to be given by an increasing sequence of weights $\lambda_1 < \lambda_2 < \ldots < \lambda_k$.

There is an induced action of $\mathbb{G}_m$ on the tangent space at a fixed point x. The action preserves the normal space to the fixed point locus and we wish to describe the subspace of positive weights.

Take a partition $\mathbf{P} = (p_1, \cdots, p_k)$ of D. As before, let

$$\mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P}) \subset \mathrm{Quot}(\oplus_i\widetilde{\mathcal{L}}_i/S, d)^{\mathbb{G}_m^k}$$

be the connected component corresponding to $\mathbf{P}$. For a point $x \in \mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P})$, its image in S_i will be denoted by x_i. The line bundle $\mathcal{L}_i|_{x_i\times X}$ on X will be denoted by $\mathcal{L}_i^x$. The point x_i is given by the exact sequence

$$0 \longrightarrow \mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i) \longrightarrow \mathcal{L}_i^x \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0$$

where D_i is an effective divisor on X with $\deg D_i = p_i$. The relative tangent bundle for the projection ϕ is

$$T_x \,\mathrm{Quot}(\oplus_i \widetilde{\mathcal{L}}_i/S, d)/S \;=\; \bigoplus_{i,j=1}^{k} \mathrm{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i)\,, \mathcal{O}_{D_j})\,.$$

On the other hand, the relative tangent space to the fixed point locus $\mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P})$ is

$$T_x \,\mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P})/S \;=\; \bigoplus_{i=1}^{k} \mathrm{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i)\,, \mathcal{O}_{D_i})\,.$$

Consequently, the normal bundle N to $\mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P}) \subset \mathrm{Quot}(\oplus_i \widetilde{\mathcal{L}}_i/S, d)$ is

$$N_x \;=\; \bigoplus_{i\neq j} \mathrm{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i\,, \mathcal{O}_{D_j})\,.$$

Also, the subspace of positive weights is

$$N_x^+ \;=\; \bigoplus_{i<j} \mathrm{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i\,, \mathcal{O}_{D_j})$$

because the torus acts on $\mathrm{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i\,, \mathcal{O}_{D_j})$ with weight $\lambda_j - \lambda_i$. It follows that

$$d(\mathbf{P}) \;:=\; \dim N_x^+ \;=\; \sum_{i=1}^{k} (i-1)p_i\,.$$

Proposition 5.2. *The Quot schemes for line bundles*

$$\mathrm{Quot}(\mathcal{L}_i/S_i, p_i) \;=\; \mathrm{Quot}(\mathcal{O}/S_i, p_i) \;=\; \mathrm{Sym}^{p_i}(X) \times S_i\,.$$

The Poincaré polynomial of $\mathrm{Quot}(\oplus_i \widetilde{\mathcal{L}}_i/S, d)$ *is given by*

$$\begin{aligned} P(\mathrm{Quot}(\oplus_i \widetilde{\mathcal{L}}_i/S, d), t) \;&=\; \sum_{\mathbf{P}} t^{2d(\mathbf{P})} P(\mathrm{Quot}(\mathcal{L}_i/S_i, p_i), t) \\ &=\; \sum_{\mathbf{P}} t^{2d(\mathbf{P})} P(S_i, t) P(\mathrm{Sym}^{p_i}(X), t)\,, \end{aligned}$$

where the sum is over all partitions of d of length k.

Proof. The isomorphism $\mathrm{Quot}(\mathcal{O}/S_i, p_i) \xrightarrow{\sim} \mathrm{Quot}(\mathcal{L}_i/S_i, p_i)$ is by tensoring exact sequences with $\mathcal{L}_i$. The second equality is via (2.1).

We need to recall the theorems of [Bia] and [Ki] in our present context. The torus action determines two stratifications of the variety $\mathrm{Quot}(\oplus_i \widetilde{\mathcal{L}}_i/S, d)$. The strata are in bijection with connected components of the fixed point locus which are in turn in bijection with partitions of d of length k. Given such a partition $\mathbf{P}$, its corresponding strata are

$$\mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P})^+ \;:=\; \{x \mid \lim_{t\to 0} t.x \,\in\, \mathrm{Quot}(\mathcal{L}_\bullet/S, \mathbf{P})\}$$

and

$$\mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})^- := \{x \mid \lim_{t\to\infty} t.x \in \mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})\}\,.$$

Both of these stratifications are known to be perfect. There are affine fibrations

$$\mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})^+ \longrightarrow \mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P}) \quad \text{and} \quad \mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})^- \longrightarrow \mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})$$

of relative dimensions $\dim N_x^+$ and $\dim N_x^-$ respectively, where $x \in \mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})$ is an arbitrary closed point. It follows that the codimension of $\mathrm{Quot}(\mathcal{L}_\bullet/S,\mathbf{P})^-$ is $\dim N_x^+$ which gives the above formula for the Poincare polynomial. □

5.2. **The cohomology of** $\mathrm{Quot}(r, d_p, d_z)$**.** In this subsection we describe the Poincaré polynomial of $\mathrm{Quot}(r, d_p, d_z)$. Consider the morphism φ in (3.9). There is a natural action of the torus $\mathbb{G}_m^r$ on the target $\mathrm{Quot}(\mathcal{O}^r, d_p) = \mathcal{Q}(r, d_p)$. This action clearly lifts to the domain $\mathrm{Quot}(r, d_p, d_z)$ for φ.

The previous subsection provides us with a decomposition and an induced formula for the Poincaré polynomial of $\mathrm{Quot}(r, d_p, d_z)$. Let us recall it quickly in the present context. There is a bijection between connected components of fixed point locus and partitions of d_p of length r. Given a partition $\mathbf{P} = (p_1\,, p_2\,, \cdots, p_r)$, the corresponding component of $\mathrm{Quot}(\mathcal{O}^r, d_p)^{\mathbb{G}_m}$ is

$$\begin{aligned}\mathrm{Quot}(\mathcal{O}, p_1) \times \cdots \times \mathrm{Quot}(\mathcal{O}, p_r) &= \mathrm{Sym}^{p_1}(X) \times \cdots \times \mathrm{Sym}^{p_r}(X)\\ &= \mathrm{Sym}^{\mathbf{P}} X.\end{aligned}$$

There are universal divisors $D^{\mathrm{univ}}_{p_i}$ inside $\mathrm{Sym}^{p_i}(X)\times X$. The component of $\mathrm{Quot}(r, d_p, d_z)^{\mathbb{G}_m^r}$ corresponding to $\mathbf{P}$, that is

$$\phi^{-1}(\mathrm{Sym}^{p_1}(X) \times \mathrm{Sym}^{p_2}(X) \times \cdots \times \mathrm{Sym}^{p_r}(X))$$

is then identified with $\mathrm{Quot}(\oplus_i \mathcal{O}_{\mathrm{Sym}^{p_i}(X)\times X}(D^{\mathrm{univ}}_{p_i})/\mathrm{Sym}^{p_i}(X), d_z)$. As the morphism φ in (3.9) is smooth, and smooth morphisms preserve codimension, we obtain the following formula for the Poincaré polynomial:

$$P(\mathrm{Quot}(r, d_p, d_z), t) = \sum_{\mathbf{P}} t^{2d(\mathbf{P})} P(\mathrm{Quot}(\oplus_i \mathcal{O}_{\mathrm{Sym}^{p_i}(X)\times X}(D^{\mathrm{univ}}_{p_i})/\mathrm{Sym}^{p_i}(X), d_z), t). \tag{5.1}$$

To complete the calculation we need to compute the Poincaré polynomials of

$$\mathrm{Quot}(\oplus_i \mathcal{O}_{\mathrm{Sym}^{p_i}(X)\times X}(D^{\mathrm{univ}}_{p_i})/\mathrm{Sym}^{\mathbf{P}}(X), d_z)\,.$$

Once again Proposition 5.2 applies. The connected components of the fixed point loci are in bijection with partitions of d_z of length r. Given a partition $\mathbf{Q} = (q_1\,, \cdots, q_r)$, the corresponding connected component is

$$\mathrm{Quot}(\mathcal{O}_{\mathrm{Sym}^{p_1}(X)\times X}(-D_{p_1})/\mathrm{Sym}^{p_1}(X), q_1)\times\cdots\times\mathrm{Quot}(\mathcal{O}_{\mathrm{Sym}^{p_r}(X)\times X}(-D_{p_r})/\mathrm{Sym}^{p_r}(X), q_r)$$

which is canonically isomorphic to

$$\mathrm{Sym}^{\mathbf{P},\mathbf{Q}} X := \mathrm{Sym}^{p_1}(X) \times \cdots \times \mathrm{Sym}^{p_r}(X) \times \mathrm{Sym}^{q_1}(X) \times \cdots \times \mathrm{Sym}^{q_r}(X)\,.$$

We obtain the following formula:

$$P(\mathrm{Quot}(\oplus_i \mathcal{O}\mathrm{Sym}^{p_i}(X) \times X(D^{\mathrm{univ}}_{p_i})/\mathrm{Sym}^{\mathbf{P}}(X), d_z)) = \sum_{\mathbf{Q}} t^{2d(\mathbf{Q})} P(\mathrm{Sym}^{\mathbf{P},\mathbf{Q}}(X), t)\,.$$

Putting this all together we obtain the following:

Theorem 5.3. *The Poincaré polynomial for* $\mathrm{Quot}(r, d_p, d_z)$ *is*

$$P(\mathrm{Quot}(r, d_p, d_z), t) = \sum_{\mathbf{P}} \sum_{\mathbf{Q}} t^{2[d(\mathbf{P})+d(\mathbf{Q})]} P(\mathrm{Sym}^{\mathbf{P}}(X), t) P(\mathrm{Sym}^{\mathbf{Q}}(X), t)\,,$$

where $\mathbf{P}$ *varies over all partitions of* d_p *of length* r *and* $\mathbf{Q}$ *varies over all partitions of* d_z *of length* r.

Poincaré polynomial of $\mathrm{Sym}^n(X)$ is the coefficient of t^n in

$$\frac{(1+tx)^{2g_X}}{(1-t)(1-tx^2)}\,,$$

where g_X is the genus of X [Ma, p. 322, (4.3)]. Using this and Theorem 5.3 we get an explicit expression for $P(\mathrm{Quot}(r, d_p, d_z), t)$.

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