Energy of Harmonic Maps and Gardiner's Formula

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Abstract. It is shown that the usual first variational formula for the energy of a harmonic map (or equivariant harmonic map) with respect to the conformal structure on a two dimensional domain extends to case of nonpositively curved metric space targets. As applications, we recover Gardiner's formula for the variation of the Hubbard-Masur differential and a proof of the existence and uniqueness of quadratic differentials realizing a pair of measured foliations that fill a surface.

1. Introduction

Energy minimizing maps to singular spaces have been useful in a variety of problems related to groups actions (cf. [**GrS, KS1, KS2, KS3, J2**]). In a series of papers, Mike Wolf showed that many of the important ideas and results in Teichmüller and Thurston theory could be interpreted and proved using the notion of a harmonic map to an $\mathbb{R}$-tree (cf. [**W1, W2, W3**] and also [**DDW1, DDW2**] – for a survey of the subject, see [**DW**]). In this note we give another such application.

Let us first fix notation. We denote by Σ a closed, compact, oriented surface, and by $\mathbf{T}(\Sigma)$ the Teichmüller space of equivalence classes of conformal structures on Σ. We will use σ to denote a Riemannian metric on Σ (or sometimes just its conformal class) and $[\sigma]$ the point in $\mathbf{T}(\Sigma)$ corresponding to σ. Let X be a length space that is nonpositively curved (NPC) in the sense of Alexandrov (see [**KS1**] or [**BH**]), and let $\mathrm{Iso}(X)$ denote its isometry group.

Let $\rho : \pi_1(\Sigma) \to \mathrm{Iso}(X)$ be a homomorphism. In [**KS1**], Korevaar and Schoen developed the Sobolev theory of ρ-equivariant finite energy maps (or L^2_1-maps) $u : (\widetilde{\Sigma}, \sigma) \longrightarrow X$, where $\widetilde{\Sigma}$ is the universal cover of Σ, and by abuse of notation σ is the pullback to $\widetilde{\Sigma}$ of the metric σ on Σ (see also [**J2**]). If ρ is *proper* (see [**KS2**, §2] for the definition) then there exists a ρ-equivariant map which has minimal energy among all such maps [**KS2**, Theorem 2.1.3]. We call these energy minimizers *harmonic maps*. The energy $E(u, \sigma)$ of such a map depends only on the conformal structure and is a diffeomorphism invariant (provided the diffeomorphism fixes ρ up to conjugation). Thus we can make the following

1991 *Mathematics Subject Classification.* Primary: 58E20; Secondary: 32G15, 20E08.
Key words and phrases. Harmonic maps, Teichmüller space, measured foliation.
Supported in part by NSF grants DMS-0204496 and DMS-0505512.

Definition 1.1. Given a proper homomorphism $\rho : \pi_1(\Sigma) \to \mathrm{Iso}(X)$, let $E_\rho : \mathbf{T}(\Sigma) \to \mathbb{R}^+ = [0, +\infty)$ be defined by the energy $E_\rho[\sigma] = E(u, \sigma)$, where σ is in the class $[\sigma]$ and u is a ρ-equivariant energy minimizing map $u : (\widetilde{\Sigma}, \sigma) \to X$.

The observation of this article is that the well-known formula for the derivative of E_ρ continues to hold in this more general setting. To state this precisely, we recall that the directional derivatives of an L^2_1-map u give a symmetric L^1-tensor $\pi_{ij}(u)$. The $(2,0)$ part $\Phi = \Phi_{zz}dz^2$ of π_{ij} on $(\widetilde{\Sigma}, \sigma)$ is called the *Hopf differential* of u. In terms of a local conformal coordinate $z = x_1 + ix_2$ and π_{ij},

$$\Phi_{zz} = \frac{1}{4}\left\{\pi_{11} - \pi_{22} - 2i\pi_{12}\right\} \ . \tag{1.1}$$

It was noted by Schoen (cf. [**S**, p. 154]) that when u is harmonic Φ is actually holomorphic, and indeed it is the lift of a holomorphic quadratic differential on (Σ, σ), which we also denote by Φ. Next, recall from Kodaira-Spencer theory that tangent vectors to the space of conformal structures on Σ are given by Beltrami differentials $\mu = \mu^z_{\bar z}(\partial/\partial z) \otimes d\bar z$. The natural pairing between quadratic differentials and Beltrami differentials is given by

$$\langle \Phi, \mu \rangle = \int_\Sigma \Phi_{zz}\mu^z_{\bar z}|dz|^2 \ . \tag{1.2}$$

With this understood, we now state

Theorem 1.2. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Iso}(X)$ be proper, and let E_ρ be defined as in Definition 1.1. Then E_ρ is differentiable on $\mathbf{T}(\Sigma)$. If σ_t, $-1 \le t \le 1$, is a differentiable family of metrics on Σ with Beltrami differential μ at $t = 0$, and Φ is the Hopf differential of a ρ-equivariant energy minimizer $u : (\widetilde{\Sigma}, \sigma_0) \to X$, then*

$$\left.\frac{d}{dt}\right|_{t=0} E_\rho[\sigma_t] = -4\,\mathfrak{Re}\langle \Phi, \mu \rangle \ .$$

In the case where X is a smooth Riemannian manifold, this formula has some history. Wolf [**W4**] provides a derivation and refers to earlier notes of Schoen, as well as [**T1, T2, J1**]. The referee suggests that the earliest computation of this sort may be due to Douglas (cf. [**D**, eq. (12.29)]). Since the pairing between holomorphic quadratic differentials and harmonic Beltrami differentials is nondegenerate, an immediate consequence of this theorem is the existence of (weakly) minimal surfaces in X (cf. [**SY, SU**]). Let us call a map *conformal* if it is harmonic with vanishing Hopf differential. Then we have

Corollary 1.3. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Iso}(X)$ be proper. If E_ρ is a proper function on $\mathbf{T}(\Sigma)$ then there exists a conformal structure σ on Σ and a conformal ρ-equivariant harmonic map $u : (\widetilde{\Sigma}, \sigma) \to X$.*

Properness of E_ρ in the context of convex cocompact representations has been addressed in [**GW**], and this can be used to deduce the properness of the action of the mapping class group on the moduli space of representations. Here we show how two other well-known facts follow from this result. Let F be a measured foliation on Σ (for background see [**FLP**]). The Hubbard-Masur theorem [**HM**] asserts that for any given conformal structure σ on Σ there is a unique holomorphic quadratic differential Φ_F such that F is measure equivalent to the horizontal trajectory of Φ_F

(alternatively, the vertical trajectory of $-\Phi_F$). The *extremal length* of F is defined by (see [**GM**])

$$\text{ext}_F[\sigma] = \int_\Sigma |\Phi_F| \ . \tag{1.3}$$

This gives a well-defined function $\text{ext}_F : \mathbf{T}(\Sigma) \to \mathbb{R}^+$. Now let $\widetilde{F}$ be the lift of F to $\widetilde{\Sigma}$. Then the leaf space T_F of $\widetilde{F}$ along with the metric induced by the transversal measure has the structure of an $\mathbb{R}$-tree (see [**W1**]), which is a particular example of an NPC space. Moreover, $\pi_1(\Sigma)$ acts in the obvious way by isometries, and this action is proper. Hence, there is a $\pi_1(\Sigma)$-equivariant harmonic map $u : (\widetilde{\Sigma}, \sigma) \to T_F$ with Hopf differential Φ. In [**W3**] it was shown that $\Phi = -\Phi_F$ and moreover, the energy

$$E(u, \sigma) = 2 \int_\Sigma |\Phi| = 2 \int_\Sigma |\Phi_F| = 2 \, \text{ext}_F[\sigma] \ . \tag{1.4}$$

Hence, Theorem 1.2 implies

THEOREM 1.4 (Gardiner's formula, [**G1, G2**]). *For any measured foliation F, ext_F is differentiable on $\mathbf{T}(\Sigma)$. If σ_t, $-1 \leq t \leq 1$, is a differentiable family of conformal structures on Σ with Beltrami differential μ at $t = 0$, and Φ_F is the Hubbard-Masur differential for F at σ_0, then*

$$\left. \frac{d}{dt} \right|_{t=0} \text{ext}_F[\sigma_t] = 2 \, \mathfrak{Re} \langle \Phi_F, \mu \rangle \ .$$

As a second application, consider the problem of realizing a *pair* of measured foliations as the vertical and horizontal trajectories of a single quadratic differential. We recall the following

DEFINITION 1.5. A pair F_+, F_- of measured foliations on Σ is called filling if for any third measured foliation G

$$i(F_+, G) + i(F_-, G) \neq 0 \ ,$$

where $i(\cdot, \cdot)$ denotes the intersection number.

It is relatively straightforward to see that the vertical and horizontal trajectories of a quadratic differential are filling (cf. [**GM**, Lemma 5.3]). We will show in Section 3 that the converse is a consequence of Corollary 1.3.

THEOREM 1.6. *A pair F_+, F_- of measured foliations on Σ is filling if and only there is a conformal structure σ and a holomorphic quadratic differential Φ on (Σ, σ) such that F_+ and F_- are measure equivalent to the vertical and horizontal foliations of Φ, respectively. Moreover, the point $[\sigma] \in \mathbf{T}(\Sigma)$ is uniquely determined, and for each $\sigma \in [\sigma]$ the quadratic differential Φ is also unique.*

COROLLARY 1.7. *If F_+, F_- are filling, then there is a conformal structure σ and a holomorphic quadratic differential Φ on (Σ, σ) such that*

$$i(F_+, F_-) = \int_\Sigma |\Phi| \ .$$

We note that Theorem 1.6 was essentially proven in [**GM**]. The strategy there was to minimize the product $\text{ext}_{F_+}[\sigma] \, \text{ext}_{F_-}[\sigma]$ for $\sigma \in \mathbf{T}(\Sigma)$, which occurs along a Teichmüller geodesic. The approach taken below (which is similar to that in [**W4**]) amounts to minimizing the sum $\mathrm{ext}_{F_+}[\sigma]+\mathrm{ext}_{F_-}[\sigma]$, which by (1.4) may be regarded as half the energy of the equivariant harmonic map $(\widetilde{\Sigma},\sigma)\to X=T_{F_+}\times T_{F_-}$. Minimizing the energy with respect to $[\sigma]$ gives an analytic proof of Theorem 1.6. Moreover, we will show that the minimum is unique. Indeed, if $[\sigma]$ is a minimum then for any other conformal structure the equivariant harmonic map to X factors through the Teichmüller map to $(\widetilde{\Sigma},\sigma)$.

Acknowledgements. Thanks to Vladimir Fock and Bill Goldman for discussions. I am also grateful to Scott Wolpert and Mike Wolf for the invitation to the Ahlfors-Bers Colloquium, and to the referee for several important comments and corrections.

2. First Variational Formula

Let $\sigma=\sum_{i,j=1}^2\sigma_{ij}dx_i\otimes dx_j$ be a Riemannian metric on Σ. As is customary, the inverse of σ is denoted by σ^{ij}. For convenience, we will continue to denote by σ the Riemannian metric on the universal cover $\widetilde{\Sigma}$ of Σ. Let X be an NPC space as in the introduction. Let $u:(\widetilde{\Sigma},\sigma)\to X$ be an L^2_1-map in the sense of Korevaar and Schoen. Then there is a well-defined integrable directional energy tensor $\pi_{ij}=\pi_{ij}(u)$ associated to u (see [**KS1**, §2.3]). The energy of u is given by

$$E(u,\sigma)=\frac{1}{2}\int_\Sigma |du|^2=\frac{1}{2}\int_\Sigma \pi_{ij}\sigma^{ij}\sqrt{\det\sigma_{ij}}\,dx_1dx_2\ . \tag{2.1}$$

Note that the integrand, a priori defined on $\widetilde{\Sigma}$, descends to Σ. Now suppose we are given a differentiable family of metrics σ_t, $-1\leq t\leq 1$, $\dot\sigma=(d\sigma_t/dt)|_{t=0}$. Then we have constants $C(t)\to 1$ as $t\to 0$, such that

$$C^{-1}(t)E(u,\sigma_0)\leq E(u,\sigma_t)\leq C(t)E(u,\sigma_0)\ . \tag{2.2}$$

In particular, u has finite energy with respect to all σ_t provided it has finite energy for some t. Moreover, we have

Lemma 2.1. *Let u be an L^2_1-map with σ_0-Hopf differential Φ. Also, let μ be the Beltrami differential of the family σ_t at $t=0$. Then*

$$\lim_{t\to 0}\frac{1}{t}\left(E(u,\sigma_t)-E(u,\sigma_0)\right)=-4\,\mathfrak{Re}\langle\Phi,\mu\rangle\ .$$

Proof. Since the directional energy tensor $\pi_{ij}(u)$ is in L^1, the result follows from the usual calculation (for smooth targets) and dominated convergence (see (2.4) below). □

Now we assume that $\rho:\pi_1(\Sigma)\to X$ is proper. For each metric in the family σ_t we have a ρ-equivariant harmonic map $u_t:(\widetilde{\Sigma},\sigma_t)\to X$ with directional energy tensors $\pi_{ij}(u_t)$.

Lemma 2.2. *As $t\to 0$, $\pi_{ij}(u_t)\to\pi_{ij}(u_0)$ weakly in L^1.*

Proof. We have

$$\begin{aligned}
E(u_0,\sigma_0)&\leq E(u_t,\sigma_0) &&\text{since } u_0 \text{ is } \sigma_0\text{-minimizing}\\
&\leq C(t)E(u_t,\sigma_t) &&\text{by (2.2)}\\
&\leq C(t)E(u_0,\sigma_t) &&\text{since } u_t \text{ is } \sigma_t\text{-minimizing}\\
&\leq C^2(t)E(u_0,\sigma_0) &&\text{by (2.2) again}
\end{aligned}$$

Since $C(t) \to 1$ as $t \to 0$, it follows that $E(u_t, \sigma_0) \to E(u_0, \sigma_0)$ as $t \to 0$. The result now follows from [**KS2**, Theorem 3.9]. □

We also have

LEMMA 2.3. *Let σ_t and u_t be as above. Let Φ be the σ_0-Hopf differential of u_0, and let μ be as in Lemma 2.1. Then*

$$\lim_{t\to 0} \frac{1}{t}\left(E(u_t, \sigma_t) - E(u_t, \sigma_0)\right) = -4\,\mathfrak{Re}\langle \Phi, \mu\rangle \ .$$

PROOF. For this, we elaborate on the calculation in Lemma 2.1. First, since the energy is conformally invariant, it suffices to restrict to traceless variations of the metric. Set

$$f_t^{ij} = \frac{1}{t}\left(\sigma_t^{ij}\sqrt{\det(\sigma_t)_{ij}} - \sigma_0^{ij}\sqrt{\det(\sigma_0)_{ij}}\right)$$

with $f_t^{ij} \to f_0^{ij}$ uniformly. Choose local conformal coordinates so that the metric $(\sigma_0)_{ij} = g\delta_{ij}$ where $g^2 = \det(\sigma_0)_{ij}$. Then we compute $f_0^{ij} = -g^{-1}\dot\sigma_{ij}$, and so

$$\pi_{ij}(u_0) f_0^{ij} = -g^{-1}\left(\dot\sigma_{11}(\pi_{11} - \pi_{22}) + 2\dot\sigma_{12}\pi_{12}\right) \ . \tag{2.3}$$

A simple computation shows that the correspondence between Beltrami differentials and traceless symmetric 2-tensors representing variations of the metric is given by: $2g\mu_{\bar z}^{z} = \dot\sigma_{11} + i\dot\sigma_{12}$ (cf. the discussion following Proposition 2.10 of [**DW**]). Hence, by the definition of the Hopf differential (1.1),

$$\begin{aligned} 4\,\mathfrak{Re}(\Phi_{zz}\mu_{\bar z}^{z}) &= \frac{1}{2g}\,\mathfrak{Re}(\pi_{11} - \pi_{22} - 2i\pi_{12})(\dot\sigma_{11} + i\dot\sigma_{12}) \\ &= \frac{1}{2g}\left(\dot\sigma_{11}(\pi_{11} - \pi_{22}) + 2\dot\sigma_{12}\pi_{12}\right) \\ &= -\tfrac{1}{2}\pi_{ij}(u_0) f_0^{ij} \qquad \text{by (2.3).} \end{aligned} \tag{2.4}$$

Using this and (1.2),

$$\begin{aligned} &\lim_{t\to 0}\left\{\frac{1}{t}\left(E(u_t, \sigma_t) - E(u_t, \sigma_0)\right) + 4\,\mathfrak{Re}\langle\Phi, \mu\rangle\right\} \\ &\quad = \frac{1}{2}\lim_{t\to 0}\int_\Sigma \left\{\pi_{ij}(u_t) f_t^{ij} - \pi_{ij}(u_0) f_0^{ij}\right\} dx_1 dx_2 \\ &\quad = \frac{1}{2}\lim_{t\to 0}\int_\Sigma \left\{\pi_{ij}(u_t)(f_t^{ij} - f_0^{ij}) + (\pi_{ij}(u_t) - \pi_{ij}(u_0)) f_0^{ij}\right\} dx_1 dx_2 \end{aligned}$$

The second term above vanishes, since by Lemma 2.2, $\pi_{ij}(u_t) \to \pi_{ij}(u_0)$ weakly. By (2.2), $\pi_{ij}(u_t)$ is uniformly bounded in L^1, so the first term is bounded by a constant times $\sup|f_t^{ij} - f_0^{ij}|$. Hence, it also vanishes as $t \to 0$. □

PROOF OF THEOREM 1.2. Since u_t is minimizing for σ_t we have

$$E(u_t, \sigma_t) - E(u_0, \sigma_0) \le E(u_0, \sigma_t) - E(u_0, \sigma_0) \ .$$

Since u_0 is minimizing for σ_0 we have

$$E(u_t, \sigma_t) - E(u_0, \sigma_0) \ge E(u_t, \sigma_t) - E(u_t, \sigma_0) \ .$$

By Lemmas 2.1 and 2.3 we have

$$\limsup_{t\downarrow 0} \frac{1}{t}\left\{E(u_t,\sigma_t)-E(u_0,\sigma_0)\right\} \leq -4\,\mathfrak{Re}\langle \Phi,\mu\rangle$$

$$\liminf_{t\downarrow 0} \frac{1}{t}\left\{E(u_t,\sigma_t)-E(u_0,\sigma_0)\right\} \geq -4\,\mathfrak{Re}\langle \Phi,\mu\rangle$$

Taken together, we conclude

$$\lim_{t\downarrow 0} \frac{1}{t}\left\{E(u_t,\sigma_t)-E(u_0,\sigma_0)\right\} = -4\,\mathfrak{Re}\langle \Phi,\mu\rangle\ .$$

A similar argument applies for the limit as $t\uparrow 0$. This completes the proof. □

3. Realizing Pairs of Measured Foliations

Let F_+, F_- be measured foliations on Σ, and let $T_\pm = T_{F_\pm}$ be their associated $\mathbb{R}$-trees. Set $X = T_+ \times T_-$. Then X is an NPC space with a proper action $\rho : \pi_1(\Sigma) \to \mathrm{Iso}(X)$ coming from the diagonal action of $\pi_1(\Sigma)$ on T_+ and T_-. Let E_ρ be the associated energy functional defined in Definition 1.1. In this section we prove the following

Theorem 3.1. *If F_+, F_- are filling, then E_ρ is proper.*

Let us introduce some more notation. Let $\mathcal{C}(\Sigma)$ denote the set of isotopy classes of simple, closed, essential curves on Σ. If we are given a conformal structure σ, we let $\ell_\sigma(\gamma)$ denote the length of the geodesic in the class $\gamma\in\mathcal{C}(\Sigma)$ with respect to hyperbolic metric on Σ in the conformal class of σ. If $\rho : \pi_1(\Sigma)\to \mathrm{Iso}(X)$ where X is an NPC space, we let $\ell_X(\gamma)$ denote the translation length of $\rho(\gamma)$ as an isometry of X. Note that $\rho(\gamma)$ is ambiguous up to conjugation, but the translation length is well-defined. Finally, let **MCG** denote the mapping class group of Σ. Then $\mathbf{M}(\Sigma) = \mathbf{T}(\Sigma)/\mathbf{MCG}$ is the Riemann moduli space of curves. If σ is a conformal structure on Σ with $[\sigma]\in\mathbf{T}(\Sigma)$, we denote the associated point in $\mathbf{M}(\Sigma)$ by $[[\sigma]]$. Given this, we continue now with the

Proof of Theorem 3.1. The proof, which follows along the lines of [**SU, SY**], proceeds by contradiction. Suppose F_+ and F_- are filling but that E_ρ fails to be proper. Then we can find a divergent sequence $\{[\sigma_j]\}$, $[\sigma_j]\in\mathbf{T}(\Sigma)$, and a constant B such that $E_\rho[\sigma_j]\leq B$ for all j. By divergent we mean that the sequence eventually leaves every compact subset. We first note the following

Lemma 3.2. *Let $\{[\sigma_j]\}$ be a divergent sequence in $\mathbf{T}(\Sigma)$. Then* either *there is $\gamma\in\mathcal{C}(\Sigma)$ such that $\ell_{\sigma_j}(\gamma)\to 0$ along some subsequence;* or *there exist distinct curves $\gamma_j\in\mathcal{C}(\Sigma)$ and a constant C such that $\ell_{\sigma_j}(\gamma_j)\leq C$ along some subsequence.*

Proof. If the associated sequence $[[\sigma_j]]\in\mathbf{M}(\Sigma)$ is divergent, then by the Mumford-Mahler compactness theorem we can find $\gamma_j\in\mathcal{C}(\Sigma)$ such that $\ell_{\sigma_j}(\gamma_j)\to 0$ along a subsequence. After passing to a further subsequence, at least one of the two possibilities in the conclusion of the lemma holds. We may therefore assume $\{[[\sigma_j]]\}$ lies in a compact subset of $\mathbf{M}(\Sigma)$. Hence, there exist $g_j\in\mathbf{MCG}$ such that $g_j[\sigma_j]\to[\sigma_\infty]\in\mathbf{T}(\Sigma)$ (again after perhaps passing to a subsequence). Since the initial sequence $\{[\sigma_j]\}$ is divergent, it must be the case that infinitely many of the g_j are distinct. It then follows by [**McP**, eq. (2.13)] that we can find $\gamma\in\mathcal{C}(\Sigma)$ such that infinitely many $\gamma_j = g_j^{-1}(\gamma)$ are distinct. On the other hand, for large j,

$$\ell_{\sigma_j}(\gamma_j) = \ell_{g_j(\sigma_j)}(\gamma)\leq 2\ell_{\sigma_\infty}(\gamma)\ .$$

In particular, the second possibility in the conclusion of the lemma holds. This completes the proof. □

Continuing with the proof of the theorem, since $E_\rho[\sigma_j] \le B$, the energy minimizers

$$u_j : (\widetilde{\Sigma}, \sigma_j) \to X$$

are uniformly Lipschitz (cf. [**KS1**, Theorem 2.4.6]). Hence, there is a constant, which we also denote by B, such that

$$\ell_X(\gamma) \le B\ell_{\sigma_j}(\gamma) \tag{3.1}$$

for all $\gamma \in \mathcal{C}(\Sigma)$. Now consider Lemma 3.2. Since F_+ and F_- are filling, we have

$$\ell^2_X(\gamma) = \ell^2_{T_+}(\gamma) + \ell^2_{T_-}(\gamma) = i^2(F_+, \gamma) + i^2(F_-, \gamma) \ne 0 \tag{3.2}$$

for every $\gamma \in \mathcal{C}(\Sigma)$. So the first possibility in the conclusion of Lemma 3.2 is ruled out by (3.1). We therefore assume there are distinct γ_j with $\ell_{\sigma_j}(\gamma_j) \le C$. By (3.1) there is a constant C' such that

$$\ell_X(\gamma_j) \le C' \ . \tag{3.3}$$

As in (3.2) we also have

$$\ell^2_X(\gamma_j) = i^2(F_+, \gamma_j) + i^2(F_-, \gamma_j) \ . \tag{3.4}$$

Since the γ_j are distinct there are positive numbers $r_j \to 0$ and a measured foliation G such that (after passing to a subsequence) $r_j\gamma_j \to G$ in the Thurston topology. Using (3.3) and (3.4) we conclude that

$$0 = \lim_{j\to\infty} r_j\, i(F_+, \gamma_j) = i(F_+, G)$$
$$0 = \lim_{j\to\infty} r_j\, i(F_-, \gamma_j) = i(F_-, G) \ .$$

This contradicts the assumption that F_+, F_- are filling, and so completes the proof. □

Proof of Theorem 1.6. Assuming Theorem 3.1 above, it follows from Corollary 1.3 that there is a conformal structure σ for which the Hopf differential of an energy minimizer $u : (\widetilde{\Sigma}, \sigma) \to X$ vanishes. But u is simply the product of energy minimizers $u_\pm$ to $T_\pm$ with Hopf differentials Φ_{F_+} and Φ_{F_-}, respectively. The vanishing of the Hopf differential of u implies $\Phi_{F_+} + \Phi_{F_-} = 0$. Hence, $\Phi = \Phi_{F_+} = -\Phi_{F_-}$ has vertical foliation equivalent to F_+, and horizontal foliation equivalent to F_-. Moreover, by (1.4),

$$E_\rho[\sigma] = 2\,\mathrm{ext}_{F_+}[\sigma] + 2\,\mathrm{ext}_{F_-}[\sigma] = 4\int_\Sigma |\Phi| \ . \tag{3.5}$$

The uniqueness part is proven in [**GM**, Theorem 3.1]. Here we give an analytic proof. For this, it suffices to show that E_ρ has a unique critical point. Suppose $[\sigma] \in \mathbf{T}(\Sigma)$ is a critical point of E_ρ, and let Φ be the Hopf differential of the conformal harmonic map $u : (\widetilde{\Sigma}, \sigma) \to X$. If we endow $\widetilde{\Sigma}$ with the (singular) conformal metric $4|\Phi|$, then $(\widetilde{\Sigma}, 4|\Phi|)$ is an NPC space. Let $[\sigma_1] \ne [\sigma]$. Then there is a unique harmonic map $v : (\widetilde{\Sigma}, \sigma_1) \longrightarrow (\widetilde{\Sigma}, 4|\Phi|)$ equivariant with respect to the natural action of $\pi_1(\Sigma)$ (in fact, though we will not need this, the quotient map $v : (\Sigma, \sigma_1) \to (\Sigma, 4|\Phi|)$ is precisely the Teichmüller map (cf. [**Ku**]) with terminal differential 4Φ). Since T_+ and T_- are the vertical and horizontal leaf spaces for

Φ, it is clear that u is an isometric, totally geodesic embedding $(\widetilde{\Sigma}, 4|\Phi|) \hookrightarrow X$. In particular, the pull-back by u of a convex function on X is convex on $(\widetilde{\Sigma}, 4|\Phi|)$. Since v is harmonic, it follows that the pull-back of a convex function on X by $u_1 = u \circ v$ is subharmonic on $(\widetilde{\Sigma}, \sigma_1)$ (cf. [**KS1**]). By a version of Ishihara's theorem (which is valid in this case; see [**DW**, Theorem 3.8]) we conclude that u_1 is harmonic. But then if μ is the Beltrami differential of v,

$$\begin{aligned} E_\rho[\sigma_1] = E(u_1) = E(v) &= \int_\Sigma (|v_z|^2 + |v_{\bar{z}}|^2) 4|\Phi(v(z))||dz|^2 \\ &= 4\int_\Sigma (1 + |\mu|^2)|\Phi(v(z))||v_z|^2|dz|^2 \\ &\geq 4\int_\Sigma |\Phi(v(z))||v_z|^2|dz|^2 \\ &= 4\int_\Sigma |\Phi| = E_\rho[\sigma] \qquad \text{by (3.5)}, \end{aligned}$$

with equality if and only if $\mu \equiv 0$, i.e. v is conformal. Since v is equivariantly homotopic to the identity and $[\sigma_1] \neq [\sigma]$, the latter is not possible. Hence, $E_\rho[\sigma_1] > E_\rho[\sigma]$. If $[\sigma_1]$ were another critical point of E_ρ, then the same argument would imply $E_\rho[\sigma] > E_\rho[\sigma_1]$; contradiction. □

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