Geometric Quantization for the Moduli Space of Vector Bundles with Parabolic Structure

Georgios D. Daskalopoulos and Richard A. Wentworth

Abstract. We initiate a study of the geometric quantization of Chern-Simons gauge theory on Riemann surfaces with punctures. We construct a moduli space of flat connections using weighted Sobolev spaces, and then by analogy with the compact case, we construct a line bundle over this moduli space. The line bundle exists only for certain holonomies which correspond in a one-to-one way with representations. When the bundle does exist, its holomorphic sections reproduce the space of states defined by Segal.

1991 Mathematics Subject Classification: 32G13; 14D20, 14H60, 32G07.

1. Introduction

This paper was first written in June 1991 and revised in February 1992. The motivation derived from the link, first elucidated by Witten [Wi1], between Chern-Simons theory, conformal field theory, and the moduli spaces of flat connections on Riemann surfaces. This point of view was further developed in the Oxford notes [Ox] by Atiyah, Hitchin, and Segal. Our work was an attempt to understand and make rigorous some of the ideas presented in these references.

The main mathematical question was to prove a factorization theorem for certain non-abelian versions of level k theta functions. This goal was achieved in [D-W2, D-W3], where a proof of the Verlinde formula for SU(2) theta functions can also be found. This paper has remained unpublished; however, since some of the results are quoted without proof in the above references, we have decided that it might be worth having this initial work appear in the literature, even with the substantial delay.

We now outline the organization of the paper and the main results: in Section 2, we review some standard results concerning spaces of $\mathbf{G}$ representations of the fundamental group of a Riemann surface with boundaries. These spaces admit natural complex structures via the theorem of Mehta and Seshadri which identifies them with moduli spaces of vector bundles with *parabolic structure*. We review this work in some detail. Moreover, we show how these spaces may be used to form a correspondence between moduli spaces of vector bundles of different degrees. This is essentially the notion of a *Hecke correspondence* (cf. [N-Rn]). For simplicity, we shall throughout the paper restrict ourselves to the case of a single boundary component. All of our results can be generalized to the case of many components, with certain important modifications. Details will be presented elsewhere (see [D-W3]).

In Section 3, we give the gauge theoretic description of these representation spaces. We achieve this by pushing the boundary to infinity and requiring exponential decay both on connections and gauge transformations. This technique was first used by Taubes [T1] in the setting of four dimensional topology.

To be more precise, let ∇_0 be a base flat connection on a trivial $\mathbf{G}$-bundle over the Riemann surface Σ obtained from $\overline{\Sigma}$ by removing a point (the "puncture"). Assume that at infinity (i.e. a disk about the point) ∇_0 has the form $A_0 = i\alpha d\theta$ with holonomy $\exp(2\pi i\alpha)$. Let $\mathcal{A}_{\mathrm{F},\delta}$ denote the space of flat connections $\nabla_0 + A$ such that both A and $\nabla_0 A$ have exponential decay, and let $\mathcal{G}_\delta$ denote the bundle automorphisms g with $\nabla_0 g$, $\nabla_0^2 g$ exponentially decaying. The main theorem of Section 3 is the following (see Theorem 3.8):

Theorem A. *The space of equivalence classes of flat connections $\mathcal{A}_{\mathrm{F},\delta}/\mathcal{G}_\delta$ is homeomorphic to the space of equivalence classes of $\mathbf{G}$-representations of $\pi_1(\Sigma)$ with value at infinity conjugate to $\exp(2\pi i\alpha)$.*

Having identified the spaces, we investigate their topological structure in Section 4. Since we are interested in constructing line bundles, we focus on the low dimensional homotopy and cohomology groups. The results are obtained by means of transversality arguments of the type used in [D-U]. The analytic description of the moduli spaces allows us to define a candidate Ω for the curvature two-form of our line bundle. The form does not, however, always define an integral class, and in Theorem 4.13, we show that for $k \in \mathbb{Z}$, $k\Omega/2\pi$ is integral if and only if $ik\alpha$ is in the coroot lattice of $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}$.

By contrast, on the quotient $\mathcal{F}_{\alpha,\delta} = \mathcal{A}_{\mathrm{F},\delta}/\mathcal{G}_{0,\delta}$, where $\mathcal{G}_{0,\delta}$ denotes the subgroup of automorphisms which are the identity at infinity, the form $k\Omega/2\pi$ is always integral. In Section 5, we show how to use the cocycle (see (5.1)) to construct a line bundle $\tilde{L}_\alpha^{\otimes k}$ with connection $\tilde{\nabla}$ and curvature $-ik\Omega$ over $\mathcal{F}_{\alpha,\delta}$. The question of integrality of Ω on the quotient $\mathcal{A}_{\mathrm{F},\delta}/\mathcal{G}_\delta$ may now be recast as follows: the residual gauge group $\mathcal{G}_\delta/\mathcal{G}_{0,\delta}$ is, for generic holonomies, a maximal torus $\mathbf{T} \subset \mathbf{G}$. The infinitesimal action of $\mathbf{T}$ is given via $\tilde{\nabla}$. The line bundle and connection push down to a bundle $L_\alpha^{\otimes k}$ precisely when this action may be exponentiated. This gives the following correspondence between holonomies and representations (see Theorem 6.1):

Theorem B. *Fix a maximal torus $\mathbf{T} \subset \mathbf{G}$. There is a one-to-one correspondence between characters $\lambda : \mathbf{T} \to U(1)$ invariant under the center $\mathbb{Z}_n$ of $\mathbf{G}$ and elements $i\alpha \in$ Lie $\mathbf{T}$ such that $k\Omega/2\pi$ represents an integral class on $\mathcal{F}_{\alpha,\delta}/\mathbf{T}$.*

Invariance under the center is required for a free action on the base (recall that we are restricting to the case of one puncture).

Finally, in Section 6, we show how the line bundles constructed in the above manner solve a problem in geometric quantization. Namely, consider the compact surface $\overline{\Sigma}$. Let $\overline{\mathcal{A}}_s$ denote the smooth stable connections on a trivial $\mathbf{G}$-bundle over $\overline{\Sigma}$, and let $\overline{\mathcal{G}}^{\mathbb{C}}$ denote the group of smooth complex automorphisms. The determinant line bundle $\Delta \to \overline{\mathcal{A}}_s$ descends via the action of $\overline{\mathcal{G}}^{\mathbb{C}}$ to generate the Picard group of the moduli space $\overline{\mathcal{A}}_s/\overline{\mathcal{G}}^{\mathbb{C}}$. $\Delta^{\otimes k}$ may be regarded as a "quantum" line bundle at level k over the symplectic manifold $\overline{\mathcal{A}}_s$, and the quotient is the corresponding quantum bundle over the moduli space.

Instead of taking the $\overline{\mathcal{G}}^{\mathbb{C}}$ invariant sections, one may choose a highest weight λ and consider the multiplicity space (cf. [G-S1])

$$\mathcal{V}_\lambda = \mathrm{Hom}_{\overline{\mathcal{G}}^{\mathbb{C}}}\left(V_\lambda^*, H^0(\overline{\mathcal{A}}_s, \Delta^{\otimes k})\right) ,$$

where V_λ^* is the (dual) representation space of λ. The homomorphisms are required to intertwine the action of $\overline{\mathcal{G}}^{\mathbb{C}}$ on H^0 with the action on V_λ obtained by evaluation at the puncture. This is essentially Segal's definition of the space of states in [Ox]. We prove the following (see Theorem 6.6):

Theorem C. *Assume* $\mathbf{G} = SU(2)$ *and* $g > 3$. *Fix a dominant weight* $\lambda : \mathbf{T} \to U(1)$ *invariant under the center, and let* α *correspond to* λ *as in Theorem B above. Then* $\mathcal{V}_\lambda \simeq H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}, L_\alpha^{\otimes k}\right)$.

We shall show in [D-W1] that $\mathcal{V}_\lambda$ is zero for all but finitely many λ.

In addition to the papers which motivated this study back in 1991, there has since been a plethora of work on topics related to parabolic bundles, geometric quantization, and the Verlinde formula. We cannot hope to compile an exhaustive collection of references, and so we shall content ourselves instead with briefly mentioning some of those articles having a more or less close connection with the present work and its sequels [DW1-3]: For proofs of the Mehta-Seshadri theorem [Me-Se] in the spirit of Donaldson and Uhlenbeck-Yau, see [Biq], [Po], and [Si1-2]. For more on the moduli space of parabolic bundles on Riemann surfaces, see [Ba], [Bd], [Bd-Hu], [Bs-Rm], [Fu-St], [Ni], [Ns-St]. Geometric quantization of the moduli space of vector bundles, first studied in [Ax] and [Ax-DP-W] (see also [A], [Ox], [Wi1]), has since been considered by many authors. For some of these ideas, see [Bis], [Ch], [J-W1], [R-S-W], and [V-V]. The Verlinde dimension formula [V] was the motivation for this work and many, many others. For a description of the formula in the context of stable bundles, see [B]. Various aspects, partial proofs, and proofs of the formula can be found in [Bv-L], [Be], [Be-Sz], [Don2], [DW2-3], [Fa], [G-P], [J-K], [J-W2], [K-N-Rm], [N-R], [Sz1-2], [Te], [Th1-2], [T-U-Y], [Wi2], [Z]. Finally, we would like to mention the article [Bv] where the reader will find a beautiful exposition of some of the recent developments in this subject.

Acknowledgements. We have benefited from conversations with Scott Axelrod, Aaron Bertram, Dan Freed, Bob Friedman, Lisa Jeffrey, Gordana Matic, Roberto Silvotti, and Karen Uhlenbeck. Our thanks go to them and to the MSRI, where the majority of this work was completed. We would also like to thank Jonathan Poritz for pointing out a mistake in the earlier version of the proof of Proposition 3.5. G.D. is supported in part by NSF grant DMS-9504297; R.W. is supported in part by NSF grant DMS-9503635 and a Sloan Fellowship.

2. Moduli of G-bundles over punctured Riemann surfaces

In this section we introduce the basic geometric object we shall be concerned with throughout this paper. This is the moduli space of vector bundles on a Riemann surface with punctures. For bundles on a compact surface, one has to restrict to an appropriate subspace of the space of vector bundles, namely the *stable* bundles, in order to obtain a "good" moduli space. The famous theorem of Narasimhan and Seshadri states that when stability is taken into account, the moduli space is just the equivalence classes of irreducible unitary representations of the fundamental group.

For vector bundles on Riemann surfaces with punctures, stability must be replaced with Seshadri's notion of *parabolic stability.* That this is the correct generalization follows from the theorem of Mehta and Seshadri which again states that the moduli space, now of parabolic stable bundles, is isomorphic to the space of irreducible representations of the fundamental group of the non-compact surface.

2.1. Representations of the fundamental group

Let $\overline{\Sigma}$ be a compact Riemann surface of genus $g > 0$ with fixed coordinate disk $(\tilde{D}, z)$, i.e. $z : \tilde{D} \to D$ is a complex analytic isomorphism with the unit disk $D \subset \mathbb{C}$. Let $p = z^{-1}(0)$, and denote by Σ the non-compact surface $\overline{\Sigma} \setminus \{p\}$. We may define new coordinates on $\tilde{D}^* = \tilde{D} \setminus \{p\}$: set $w = -\log z$. Then

$$w : \tilde{D}^* \longrightarrow \mathbb{C}\big/(\tau, \theta) \sim (\tau, \theta + 2\pi)$$

maps $\tilde{D}^*$ analytically to the semi-infinite cylinder

$$C = \big\{(\tau, \theta) : \tau \geq 0, 0 \leq \theta \leq 2\pi\big\}\big/(\tau, 0) \sim (\tau, 2\pi)\ .$$

If we use polar coordinates (r, ϕ) on D^*, then the map $w \circ z^{-1} : D^* \to \mathbb{C}$ is simply $\tau = -\log r$, $\phi = \theta$. We also make a choice of base point $x_0 \in \tilde{D}^* \subset \Sigma$ and set $(\tau_0, \theta_0) = w(x_0)$. Finally, we denote by Σ_0 the Riemann surface with boundary: $\Sigma_0 = \Sigma \setminus \{x : \tau(x) > \tau_0\}$.

Let $\mathbf{G} = \mathrm{SU}(n)$, and let P be the trivial principal $\mathbf{G}$-bundle on Σ, E the vector bundle associated to P via the standard representation. Let $\mathcal{A}$ denote the space of all C^∞ $\mathbf{G}$-connections on P. We identify $\mathcal{A} \simeq \Omega^1(\Sigma, \mathfrak{g})$ and endow it with the obvious Fréchet topology on $\Omega^1(\Sigma, \mathfrak{g})$. Let $\mathcal{G}$ denote the group of C^∞ gauge transformations on P, endowed with the obvious Fréchet topology induced by the inclusion $\mathrm{Map}(\Sigma, \mathbf{G}) \subset \mathrm{Map}(\Sigma, \mathfrak{gl}(n, \mathbb{C}))$. Let $\mathcal{G}_0 \subset \mathcal{G}$ be the subgroup of $\mathcal{G}$ preserving the base point x_0. The groups $\mathcal{G}_0, \mathcal{G}$ act smoothly on $\mathcal{A}_F$, the flat connections in $\mathcal{A}$. Let $\mathcal{M}_0 = \mathcal{A}_F/\mathcal{G}_0$, $\mathcal{M} = \mathcal{A}_F/\mathcal{G}$ be the quotient spaces.

Define the evaluation map $\mathrm{e}_0 : \mathcal{G} \to \mathbf{G}$ which takes $g \mapsto g(x_0)$. In view of the exact sequence of groups $1 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \xrightarrow{\mathrm{e}_0} \mathbf{G} \longrightarrow 1$, we can identify $\mathbf{G} \simeq \mathcal{G}/\mathcal{G}_0$. Thus the group $\mathbf{G}$ acts on $\mathcal{M}_0$, and we have the identifications $\mathcal{A}_F/\mathcal{G} \simeq \mathcal{M}_0/\mathbf{G} \simeq \mathcal{M}$. The spaces $\mathcal{M}, \mathcal{M}_0$ are called the *moduli spaces of flat connections* and *based flat connections* on P, respectively.

We choose generators $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ for $\pi_1(\Sigma, x_0)$ whose intersection numbers in the quotient $H_1(\Sigma, \mathbb{Z})$ satisfy $\#a_i \cdot b_j = \delta_{ij}$, $\#a_i \cdot a_j = \#b_i \cdot b_j = 0$. Given $\nabla \in \mathcal{A}_F$, let $\rho = \rho(\nabla)$ be the corresponding holonomy homomorphism. This defines a map $\mathcal{A}_F \to \mathrm{Hom}\,(\pi_1(\Sigma, x_0), \mathbf{G}) \simeq \mathbf{G}^{2g}$ which is $\mathcal{G}_0$-invariant, where the last identification is from $\{a_i, b_i\} \to \{A_i, B_i\}$. It is well known that this induces a homeomorphism of $\mathcal{M}_0 \simeq \mathcal{A}_F/\mathcal{G}_0$ with $\mathrm{Hom}\,(\pi_1(\Sigma, x_0), \mathbf{G}) \simeq \mathbf{G}^{2g}$. This proves

Lemma 2.1. *$\mathcal{M}_0$ is a smooth manifold diffeomorphic to $\mathbf{G}^{2g}$.*

Let c denote the loop $\prod_{i=1}^g [a_i, b_i]$ in $\pi_1(\Sigma, x_0)$ and let $q : \mathcal{M}_0 \to \mathbf{G}$ be the map measuring the holonomy around c. We clearly have

Lemma 2.2. *Under the identification $\mathcal{M}_0 \simeq \mathbf{G}^{2g}$ the map q corresponds to the map $\{A_1, \ldots, A_g, B_1, \ldots, B_g\} \longrightarrow \prod_{i=1}^g [A_i, B_i]$.*

The next proposition summarizes some important properties of the map q in the case $\mathbf{G} = \mathrm{SU}(2)$ which we shall need later on (see Theorem 4.1):

Proposition 2.3. *For $\mathbf{G} = SU(2)$, the map q has the following properties:*

(i) q is surjective ;
(ii) $\{$critical points of $q\}$=$\{$reducible connections$\}$;
(iii) $\{$reducible connections$\} \subset q^{-1}(I)$;
(iv) $q^{-1}(pt.) \setminus \{$reducible connections$\}$ is a smooth manifold of dimension $6g-3$;
(v) $q : \mathcal{M}_0 \setminus q^{-1}(I) \to \mathbf{G} \setminus \{I\}$ is a trivial fiber bundle.

Proof. For (i)-(iv) see [Ak-M], Proposition 2.1 and Corollary 2.2. The result (v) follows from (i)-(iii), since q is clearly proper and the base is contractible. □

We return to the general case $\mathbf{G} = \mathrm{SU}(n)$. Let $\mathbf{G} \cdot a$ denote the adjoint orbit of a in $\mathbf{G}$, and let $\mathcal{M}_{0,a}$ be the subspace of irreducible connections in $q^{-1}(\mathbf{G} \cdot a)$. Clearly, $\mathcal{M}_{0,a}$ is preserved by the action of $\mathbf{G}$, so we set $\mathcal{M}_a = \mathcal{M}_{0,a}/\mathbf{G}$. The next Lemma follows immediately from the definition of the map q:

Lemma 2.4. *(i) $\mathcal{M}_{0,a}$ may be identified with the space of irreducible representations $\rho : \pi_1(\Sigma, x_0) \to \mathbf{G}$ satisfying $\rho(c) \in \mathbf{G} \cdot a$. (ii) $\mathcal{M}_a$ may be identified with the quotient of the space in (i) by the adjoint action of $\mathbf{G}$.*

Lemma 2.5. *For every $a \in \mathbf{G}$, $PU(n) \to \mathcal{M}_{0,a} \to \mathcal{M}_a$ is a principal $PU(n)$ bundle.*

Proof. The group $\mathrm{PU}(n) = \mathrm{SU}(n)/\mathbb{Z}_n$ acts freely on $\mathcal{M}_{0,a}$. The Lemma follows from a general theorem on group actions (cf. [V], Lemma 2.9.11). □

Let $\mathcal{F}_a$ denote the subspace of irreducible flat connections in $q^{-1}(a)$ over $a \in \mathbf{G}$. Then $\mathcal{F}_a$ is a smooth manifold of dimension $(2g-1)(n^2-1)$. Let $\mathbf{P}_a$ denote the normalizer of a in $\mathbf{G}$. For example, for $\mathbf{G} = \mathrm{SU}(2)$, $\mathbf{P}_a$ is a maximal torus for $a \neq \pm I$, and $\mathbf{P}_{\pm I} = \mathbf{G}$. Then $\mathbf{P}_a/\mathbb{Z}_n$, the normalizer modulo the center of $\mathbf{G}$, acts freely on $\mathcal{F}_a$ and we have an isomorphism $\mathcal{M}_a \simeq \mathcal{F}_a/\mathbf{P}_a$. Moreover, for every $a \in \mathbf{G}$, the principal bundle of Lemma 2.5 restricts to a principal bundle

$$\mathbf{P}_a/\mathbb{Z}_n \longrightarrow \mathcal{F}_a \stackrel{\pi}{\longrightarrow} \mathcal{M}_a \tag{2.6}$$

Fibration (2.6) will be of fundamental importance in the subsequent sections. We finish this section with the following fairly standard

Lemma 2.7. *Let ∇ be a flat connection of holonomy $a \in \mathbf{G}$ around p. Fix $i\alpha \in \mathfrak{g}$ with $a = \exp(2\pi i\alpha)$. Then we can find $g \in \mathcal{G}_0$ such that over $\tilde{D}^*$ we can write (in the coordinates w) $\nabla^g = d + i\alpha d\theta$.*

A similar result holds for the group $\mathrm{U}(n)$. We shall refer to the form of the connection $d + i\alpha d\theta$ as *temporal gauge.*

Proof. As a first step, we show that for flat connections we can find a smooth gauge transformation g such that for $\tau \geq \tau_0$, $\nabla^g = d + A(\theta)d\theta$. Indeed, write

$$\nabla = d + a(\tau,\theta)d\theta + b(\tau,\theta)d\tau$$

and let g be a solution to the ODE: $dg/d\tau + b(\tau,\theta)g = 0$, $g(\tau_0) = I$. Such a solution exists for all $\tau \geq \tau_0$ and is invertible, hence also $g^{-1}dg/d\tau + g^{-1}bg = 0$. Since b is periodic in θ we extend the solution to the semi-infinite cylinder C. Then we extend g to all of Σ as an element of $\mathcal{G}_0$ by a cut-off function. We have

$$\begin{aligned}\nabla^g &= d + g^{-1}dg + g^{-1}agd\theta + g^{-1}bgd\tau \\ &= d + \left(g^{-1}\frac{\partial g}{\partial\theta} + g^{-1}ag\right)d\theta + \left(g^{-1}\frac{\partial g}{\partial\tau} + g^{-1}bg\right)d\tau \\ &= d + A(\tau,\theta)d\theta \ , \text{ for } \tau \geq \tau_0 \ .\end{aligned}$$

By assumption, the curvature $F_\nabla = 0$ so $\partial A/\partial\tau = 0$, and this completes the proof of the first step. In order to make A independent of θ, we notice that on the circle any connection is gauge equivalent to constant one. We extend the gauge transformation to Σ, constant on $\tau \geq \tau_0$. □

2.2. The theorem of Mehta and Seshadri

In this section we review some fundamental results involving complex structures on the spaces $\mathcal{M}_a$ from our point of view. The standard references are [Me-Se], [Se], [Si1], [Si2]. We begin by stating the fundamental theorem:

Theorem 2.8. ([Me-Se], Theorem 5.3) *Let $\mathcal{M}_a$ denote the space of isomorphism classes of irreducible flat connections on P with holonomy around the point p conjugate to $a \in \mathbf{G}$. If all the eigenvalues of a are of the form $\exp(2\pi i \alpha_i)$, where the α_i are rational numbers, then $\mathcal{M}_a$ has the structure of a complex manifold of dimension $(n^2-1)(g-1) +$ dim $\mathbf{G}/\mathbf{P}_a$, where $\mathbf{P}_a$ is the normalizer of a in $\mathbf{G}$.*

Let us briefly review this theorem from our point of view. First, recall (cf. Lemma 2.7) that given $[\nabla]$ an equivalence class of flat unitary connections on E of holonomy $a \in \mathrm{U}(n)$ around p we can choose a unitary frame $e_1, \ldots, e_n$ over the punctured disk $\tilde{D}^*$ with respect to which $\nabla = d + i\hat{\alpha} d\theta$ where $\hat{\alpha}$ is the matrix $\mathrm{diag}(\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ and $0 \leq \hat{\alpha}_1 \leq \cdots \leq \hat{\alpha}_n < 1$. Let $\bar{\partial}_\nabla$ be the corresponding $\bar{\partial}$-operator. Notice that if we define a new basis $\{f_i\}$ by $f_i = |z|^{\hat{\alpha}_i} e_i$, then the f_i are holomorphic with respect to $\bar{\partial}_\nabla$. Let $D \times \mathbb{C}^n \to D$ be the trivial bundle over the disk with the trivial $\bar{\partial}$-operator. Let $\bar{E} \to \bar{\Sigma}$ denote the bundle obtained by gluing E with $D \times \mathbb{C}^n$ via the f_i's, and let $\bar{\partial}_{\bar{E}}$ denote the resulting $\bar{\partial}$-operator on $\bar{E}$. It is easy to check that $\bar{E}$ is a holomorphic bundle of Chern class $l = -\sum_{i=1}^n \hat{\alpha}_i$. (Note that the sum of the $\hat{\alpha}_i$'s is always an integer.)

The f_i's determine a complete flag of the fiber $\bar{E}_p \simeq \mathbb{C}^n$. In general, given a holomorphic bundle $\bar{E}$ of degree l, a choice of flag $Fl\ (\overline{E}) \in \mathbf{G}/\mathbf{P}_a$ of the fiber $\overline{E}_p \simeq \mathbb{C}^n$ is called a *quasi-parabolic structure*. Furthermore, the pair $(Fl\ (\overline{E}); \hat{\alpha}_1, \ldots, \hat{\alpha}_n)$, where $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ are weights associated to $Fl\ (\overline{E})$, is called a *parabolic structure* on $\overline{E}$, and $\overline{E}$ is called a *parabolic bundle* (see also [Se]).

Let $\mathcal{B}$ denote the space of holomorphic structures on $\overline{E}$, $\mathcal{K}$ the group of complex automorphisms of $\overline{E}$, and $\mathcal{K}_p$ its subgroup of *parabolic automorphisms*, i.e. the elements of $\mathcal{K}$ preserving the flag $Fl\ (\overline{E})$. Given a holomorphic structure on $\overline{E}$, let us denote also by $\overline{E}$ the resulting holomorphic bundle. Let $\overline{F} \subset \overline{E}$ be a holomorphic subbundle of rank m. The flag $Fl\ (\overline{E})$ induces by restriction a flag $Fl\ (\overline{F})$ on $\overline{F}_p$ with associated weights $0 \leq \hat{\alpha}'_1 \leq \cdots \leq \hat{\alpha}'_m < 1$. The next definition is essential:

Definition 2.9. ([Me-Se], 1.11 and 1.13) Given a subbundle $\overline{F}$ of $\overline{E}$ we define the *parabolic degree* of $\overline{F}$ by $\mathrm{pardeg}(\overline{F}) = \deg(\overline{F}) + \sum_{i=1}^m \hat{\alpha}'_i$. The bundle $\overline{E}$ is called *parabolic stable* if for any subbundle $\overline{F}$ of $\overline{E}$, $\mathrm{pardeg}(\overline{F}) < 0$.

Henceforth, we shall denote by $\mathcal{B}_{p.s.}$ the subspace of all parabolic stable holomorphic structures. The space $\mathcal{B}_{p.s.}$ is an open submanifold of $\mathcal{B}$, and it is thus naturally (via the fixed complex structure on $\overline{\Sigma}$) an infinite dimensional complex manifold. The group $\mathcal{K}_p$ is an infinite dimensional complex Lie group acting on $\mathcal{B}_{p.s.}$ in a holomorphic way. Therefore, the quotient $\mathcal{B}_{p.s.}/\mathcal{K}_p$ is naturally a complex manifold whose tangent space at $[\bar{\partial}_{\bar{E}}] \in \mathcal{B}_{p.s.}/\mathcal{K}_p$ can be identified with $H^1\left(\bar{\Sigma}, End^+\bar{E}\right)$, where $End^+\bar{E}$ is the sheaf of holomorphic sections of End $\bar{E}$ preserving the flag at p. It is useful, however, to have the following slightly different description of $\mathcal{B}_{p.s.}/\mathcal{K}_p$. First, note that given $g \in \mathcal{K}$, the value g_p at the point p acts on the space of flags $\mathbf{G}/\mathbf{T}_a \simeq \mathbf{G}^{\mathbb{C}}/\mathbf{B}_a^+$. Let $\mathcal{P}$ denote the subspace of $\mathcal{B}\times\mathbf{G}/\mathbf{T}_a$ consisting of pairs $(\bar{\partial}_{\bar{E}}, f)$, where $\bar{\partial}_{\bar{E}}$ is parabolic stable with respect to the image of the flag f under the projection $\mathbf{G}/\mathbf{T}_a \to \mathbf{G}/\mathbf{P}_a$ and the weights $\hat{\alpha}_i$. The action of the group $\mathcal{K}$ preserves $\mathcal{P}$, and we denote by $\mathcal{P}/\mathcal{K}$ the quotient space. Then we have the following

Proposition 2.10. *The space $\mathcal{P}/\mathcal{K}$ is homeomorphic to a $\mathbf{P}_a/\mathbf{T}_a$ bundle over $\mathcal{B}_{p.s.}/\mathcal{K}_p$. In particular, if all of the weights are distinct, then $\mathcal{P}/\mathcal{K}$ is homeomorphic to $\mathcal{B}_{p.s.}/\mathcal{K}_p$.*

Returning to our equivalence class $[\nabla]$, the basis $\{f_i\}$ along with the weights $\{\hat{\alpha}_i\}$ determine a flag $f \in \mathbf{G}/\mathbf{P}_a$ as in [M-S], p. 211, and ∇ and $\{f_i\}$ determine a holomorphic structure $\bar{\partial}_{\bar{E}}$. The theorem of Mehta and Seshadri states that if ∇ is irreducible then $\bar{\partial}_{\bar{E}}$ is parabolic stable with respect to f. In light of Proposition 2.10, we have a map

$$\mu : [\nabla] \longmapsto [\bar{\partial}_{\bar{E}}] \tag{2.11}$$

which identifies $\mathcal{B}_{p.s.}/\mathcal{K}_p$ with the space of flat irreducible unitary connections on E of holonomy conjugate to a. Therefore, $\mathcal{M}_a$ is a complex manifold provided we can fix the determinant (recall that we are taking $\mathbf{G} = \mathrm{SU}(n)$ rather than $\mathrm{U}(n)$). Indeed, let $J_l(\overline{\Sigma})$ be the degree l component of the Picard group of $\overline{\Sigma}$, and let

$$\det : \mathcal{B}_{p.s.}/\mathcal{K}_p \longrightarrow J_l(\overline{\Sigma}) \tag{2.12}$$

denote the determinant map associating to a given vector bundle $\overline{E} \in \mathcal{B}_{p.s.}/\mathcal{K}_p$ its determinant line bundle $\det \overline{E} = \bigwedge^n \overline{E} \in J_l(\overline{\Sigma})$. The map det is a holomorphic map of full rank whose fiber may be identified with the space $\mathcal{M}_a$. This defines a complex structure on $\mathcal{M}_a$. The details behind the identification (2.11) are, of course, quite intricate, and we refer to [Me-Se], [Si1], and [Si2] for elaboration. In Section 4, we shall make use of this description of $\mathcal{M}_a$ to extract some useful topological information.

2.3. Correspondence varieties

In this subsection, we would like to show how the spaces $\mathcal{M}_a$ form a correspondence between different moduli spaces of rank 2 bundles over $\overline{\Sigma}$. This is an elaboration of the concluding remarks in [Me-Se] (see also [Be]); this description, however, will be crucial for us in Section 6, so we explain it in some detail. Throughout this section, we set $\mathbf{G} = \mathrm{SU}(2)$.

Let $\mathcal{M}(2,-1)$ denote the moduli space of semistable vector bundles of rank 2 and fixed determinant of degree -1 over the compact Riemann surface $\overline{\Sigma}$. In fact, we shall assume the determinant is $\mathcal{O}(-p)$, where p is the puncture. Assume $a \in \mathbf{G}$, $a \neq \pm I$, $\mathbf{T}_a$ the maximal torus in $\mathbf{G}$ containing a, and parameterize the parabolic weights $0 < \alpha < 1/2$.

Proposition 2.13. *For the choice of weights above, $\mathcal{B}_{p.s.}$ is equal to $\mathcal{B}_s$, the set of stable holomorphic structures on $\overline{E}$, and $\mathcal{M}_a \simeq \mathcal{B}_s \times_{\mathcal{K}} \mathbf{G}/\mathbf{T}_a$. Moreover, there exists a holomorphic projection $p_1 : \mathcal{M}_a \to \mathcal{M}(2,-1)$ which realizes $\mathcal{M}_a$ as a holomorphic $\mathbb{P}^1 \simeq \mathbf{G}/\mathbf{T}_a$ bundle over $\mathcal{M}(2,-1)$.*

Proof. Let $\overline{E}$ be a rank two parabolic stable bundle over $\overline{\Sigma}$ with determinant $\mathcal{O}(-p)$. If $\overline{F} \subset \overline{E}$ is a parabolic subbundle, then $\operatorname{pardeg} \overline{F} < 0$. In particular, $\deg \overline{F} < 0$, and since $\deg \overline{F}$ is an integer, $\deg \overline{F} \leq -1$. In any case, $\mu(\overline{F}) = \deg \overline{F} = -1 < -1/2 = \mu(\overline{E})$, and the first claim follows. The map p_1 is just projection onto the first factor in Proposition 2.10. □

We next construct a map onto the degree zero moduli space. Let $\mathcal{M}(2,0)$ denote the moduli space of semistable rank two vector bundles over $\overline{\Sigma}$ with trivial determinant, and let $\mathcal{M}_s(2,0)$ be the Zariski open subset consisting of stable bundles. Suppose $\overline{E}$ is a parabolic stable bundle. In particular, we have seen that $\overline{E}$ is stable with determinant $\mathcal{O}(-p)$. Moreover, the choice of a quasi-parabolic structure σ at p defines, via evaluation at the point p, a natural map

$$\overline{E}^* \longrightarrow \mathcal{C}_p \longrightarrow 0\ , \tag{2.14}$$

where $\mathcal{C}_p$ denotes the torsion sheaf obtained by extending $\overline{E}^*_p/\sigma^*$ by zero. Let $\overline{F}$ be the kernel of (2.14). It is easy to verify that $\overline{F}$ has trivial determinant and is semistable. We define $\overline{F} = p_0(\overline{E})$ to be its equivalence class in $\mathcal{M}(2,0)$. All in all, we have shown

Proposition 2.15. (cf. [Me-Se]) *The spaces $\mathcal{M}_a$ for $a \neq \pm I$ form a correspondence*

$$\begin{array}{ccccc} & & \mathcal{M}_a & & \\ & {\scriptstyle p_0} \swarrow & & \searrow {\scriptstyle p_1} & \\ \mathcal{M}(2,0) & & & & \mathcal{M}(2,-1) \end{array}$$

between $\mathcal{M}(2,0)$ and $\mathcal{M}(2,-1)$.

Proposition 2.16. *Assume $g > 3$. Then $\mathcal{U}_a = p_0^{-1}(\mathcal{M}_s(2,0))$ in $\mathcal{M}_a$ is a Zariski open set with complement of codimension at least 2.*

Proof. Under the natural biholomorphism $\mathcal{M}(2,-1) \simeq \mathcal{M}(2,1)$ obtained by taking duals, it suffices to show that the image of $\mathcal{U}_a$ in $\mathcal{M}(2,1)$ is Zariski open with complement of codimension at least 2. If however $\overline{E} \in \mathcal{M}_a$, then its dual $\overline{E}^*$ is stable of degree 1, and therefore $\overline{E}^*$ is (0,1)-semistable in the sense of Narasimhan and Ramanan (cf. [N-Rn] Definition 5.1). Moreover, if $\overline{E}$ does not belong to $\mathcal{U}_a$, then it is easy to see that $\overline{E}^*$ is not (0,1)-stable. Hence, $\mathcal{M}_a \setminus \mathcal{U}_a$ is mapped by duality inside the variety of (0,1)-semistable and not (0,1)-stable holomorphic bundles. By the same computation as that in the proof of Proposition 5.4 of [N-Rn], the codimension of this variety is at least $g-2$, which proves the assertion. □

Next, we will proceed by analogy with Proposition 2.10 to construct another $\mathbb{P}^1$-bundle over $\mathcal{M}(2,0)$. Let $\overline{\mathcal{A}}_s$ denote the stable holomorphic structures on a trivial rank 2 bundle on $\overline{\Sigma}$ and denote by $\overline{\mathcal{G}}^{\mathbb{C}}$ the group of complex automorphisms. Then we have

Proposition 2.17. *For $a \neq \pm I$, let $\mathcal{N}_a = \overline{\mathcal{A}}_s \times_{\overline{\mathcal{G}}^{\mathbb{C}}} \mathbf{G}/\mathbf{T}_a$, where $\overline{\mathcal{G}}^{\mathbb{C}}$ acts on $\mathbf{G}/\mathbf{T}_a$ by evaluation at p. Then the natural projection $pr_1 : \mathcal{N}_a \to \mathcal{M}_s(2,0)$ is a holomorphic $\mathbb{P}^1$ bundle.*

The next Theorem will be important in Section 6.

Theorem 2.18. *There is a holomorphic embedding $\varphi : \mathcal{N}_a \to \mathcal{M}_a$ onto $\mathcal{U}_a$ with holomorphic inverse ψ which makes the following diagram commute*

$$\begin{array}{ccc} \mathcal{N}_a & \xrightarrow{\varphi} & \mathcal{M}_a \\ \Big\downarrow{\scriptstyle pr_1} & & \Big\downarrow{\scriptstyle p_0} \\ \mathcal{M}_s(2,0) & \subset & \mathcal{M}(2,0) \end{array}$$

Proof. We define our map φ first. Let $(\overline{F}, f) \in \mathcal{N}_a$. This means $\overline{F}$ is a stable rank 2 bundle with trivial determinant and $f \in \mathbf{G}/\mathbf{T}_a$ can be used to define a line in $\overline{F}_p$, the fiber at p, via a fixed identification $\mathbb{P}(\overline{F}_p) \simeq \mathbf{G}/\mathbf{T}_a$. Let $\mathcal{C}_p$ denote the torsion sheaf obtained as the cokernel of the natural map $0 \to \mathcal{O}(-p) \to \mathcal{O}$. By Grothendieck duality, we can identify

$$\mathrm{Ext}^1(\mathcal{C}_p, \overline{F}) \simeq H^0\left(\overline{\Sigma}, \overline{F}^* \otimes \omega_{\overline{\Sigma}}^* \otimes \mathcal{C}_p\right)^* \simeq \overline{F}_p \ ,$$

and any f as above defines an extension

$$0 \longrightarrow \overline{F} \longrightarrow \overline{E}^* \longrightarrow \mathcal{C}_p \longrightarrow 0 \qquad (2.19)$$

with $\overline{E}^*$ locally free. Moreover, by evaluation at the point p, extension (2.19) naturally defines a line in $\overline{E}_p^*$ which gives a quasi-parabolic structure σ on $\overline{E}$. In fact, it is clear that for a choice of weights as before, $\overline{E}$ is parabolic stable (cf. Proposition 2.13). We then set $\varphi(\overline{F}, f) = (\overline{E}, \sigma)$. One can verify that φ is indeed a morphism. Moreover, as $p_0(\overline{E}) = \overline{F}$, the diagram of Theorem 2.18 commutes and the image of φ is $\mathcal{U}_a$. Therefore, we only need construct an inverse morphism $\psi : \mathcal{U}_a \to \mathcal{N}_a$. This can be done as follows: an element of $\mathcal{U}_a$ consists of a stable bundle $\overline{E}$ of determinant $\mathcal{O}(-p)$ and a quasi-parabolic structure σ at p. Again, as in the definition of the map p_0, σ defines a projection of $\overline{E}^*$ to a torsion sheaf $\mathcal{C}_p$ supported at p and an exact sequence (2.19). But this defines a stable bundle $\overline{F}$ with a flag f, dual to the kernel of the map $\overline{E}^* \to \mathcal{C}_p \to 0$ restricted to p. We set $\psi(\overline{E}) = (\overline{F}, f)$. It is easily verified that ψ is a morphism and an inverse to φ. □

3. The L^2_δ-theory

We shall now take a different approach and describe the spaces $\mathcal{F}_a$ and $\mathcal{M}_a$ via weighted Sobolev spaces. Although much of the material in this section is quite standard (cf. [T1], [T2], [M], and [A-P-S]), the L^2_δ description of these spaces is the key to describing the quantum line bundle in Section 5 using the cocycle.

3.1. Flat connections and the gauge group

We continue to consider the punctured Riemann surface $\Sigma = \overline{\Sigma} \setminus \{p\}$ with fixed coordinate $w = (\tau, \theta) : \tilde{D}^* \to C$ where C is the semi-infinite cylinder from Section 2.1. In addition, we make a choice of metric K on Σ compatible with the complex structure of Σ and such that on $\tilde{D}^*$, $K = w^*(K_C)$, where $K_C = d\tau^2 + d\theta^2$ is the standard flat metric on C. It is important to extend τ to a C^∞-function defined on the whole surface Σ. We achieve this by setting $\tau = 0$ outside a small tubular neighborhood of $\partial\Sigma_0$ in Σ_0.

For $\delta \in \mathbb{R}$, we define the weighted L^p spaces of sections of E, denoted $L^p_\delta(E)$, as the completion of the space of compactly supported sections $C^\infty_0(E)$ in the norm

$$\|\sigma\|_{L^p_\delta} = \left\{ \int_\Sigma e^{\tau\delta} |\sigma|^p \right\}^{1/p} .$$

Given also a positive integer k we define the weighted Sobolev space $L^p_{k,\delta}(E)$ as the completion of $C^\infty_0(E)$ in the norm

$$\|\sigma\|_{L^p_{k,\delta}} = \left\{ \int_\Sigma e^{\tau\delta} \left(|\nabla_0^{(k)}\sigma|^p + \cdots + |\nabla_0\sigma|^p + |\sigma|^p \right) \right\}^{1/p} ,$$

where ∇_0 is a base connection in temporal gauge with holonomy $a = \exp(2\pi i\alpha)$ (see Lemma 2.7). It is not difficult to see that the spaces $L^p_{k,\delta}$ do not change if we replace ∇_0 with any other connection which agrees with ∇_0 on the cylinder C at infinity.

The weighted Sobolev spaces admit multiplication and embedding theorems similar to those for compact manifolds. Therefore, all the gauge theory developed for compact manifolds (e.g. as in [F-U]) extends quite easily to the non-compact case provided we replace the standard Sobolev spaces by the weighted ones.

Following [T1], Section 7, or [M], Section 3, we make the following definitions: let ∇_0 be a base connection in temporal gauge with holonomy a. We define our space of connections $\mathcal{A}_\delta = \nabla_0 + L^2_{1,\delta}\,(T^*\Sigma \otimes \mathfrak{g}_E)$. To define the appropriate gauge group we first consider the following space of sections of $\mathfrak{gl}(E)$. Define

$$\mathcal{R} = \left\{ \varphi \in L^2_{2,\mathrm{loc.}}(\mathfrak{gl}(E)) : \|\nabla_0 \varphi\|_{L^2_{1,\delta}} < \infty \right\} ,$$

and let $\mathcal{H}$ denote the subspace of $\mathcal{R}$ consisting of *harmonic* sections, i.e. $\mathcal{H} = \left\{ \varphi \in \mathcal{R} : e^{-\tau\delta}\nabla_0^* e^{\tau\delta}\nabla_0\varphi = 0 \right\}$, where $\nabla^* = *\nabla*$ is the formal L^2 adjoint.

There is another space of sections closely related to $\mathcal{H}$. Let $\nabla_{0,\partial}$ denote ∇_0 restricted to the bundle $\mathfrak{gl}(E)|_{\partial\Sigma_0}$. Let $\ker\nabla_{0,\partial}$ denote the space of parallel sections of $\mathfrak{gl}(E)|_{\partial\Sigma_0}$ with respect to $\nabla_{0,\partial}$. Since over $\partial\Sigma_0$, $\nabla_{0,\partial} = d/d\theta + i\alpha d\theta$, $a = \exp(2\pi i\alpha)$, $\ker\nabla_{0,\partial}$ can be identified with the normalizer of a in $\mathfrak{gl}(n,\mathbb{C})$. The next two propositions contain all the important properties of the gauge group we shall need.

Proposition 3.1. (cf. [M], Theorem 3.1) *There is a direct sum decomposition* $\mathcal{R} = L^2_{2,\delta}(\mathfrak{gl}(E)) \oplus \mathcal{H}$. *Moreover, there is a well-defined map* $r : \mathcal{R} \to \ker\nabla_{0,\partial}$ *given by* $r(\varphi)(\theta) = \lim_{\tau\to\infty}\varphi(\tau,\theta)$. *The map* r *satisfies* $r^{-1}(0) = L^2_{2,\delta}(\mathfrak{gl}(E))$ *and* $r : \mathcal{H} \to \ker\nabla_{0,\partial}$ *is an isomorphism.*

Proposition 3.2. (cf. [M], Appendix) *The norm*

$$\|\varphi\|^2_{\mathcal{R}} = \|\nabla_0\varphi\|^2_{L^2_{1,\delta}} + \int_{\partial\Sigma_0} |r(\varphi)|^2$$

gives $\mathcal{R}$ *a Banach space structure in which the natural projection onto* $L^2_{1,\delta}(\mathfrak{gl}(E))$, $\mathcal{H}$ *are continuous. Moreover, pointwise multiplication* $\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ *is well-defined and continuous, and* $r : \mathcal{R} \to \ker\nabla_{0,\partial}$ *is a continuous linear map.*

We now make the definitions

$$\mathcal{G}_\delta = \{\varphi \in \mathcal{R} : \varphi\varphi^* = I, \det\varphi = 1\} \ , \quad \mathcal{G}_{0,\delta} = \{\varphi \in \mathcal{G}_\delta : r(\varphi) = I\}$$

It is not difficult to see that $\mathcal{G}_\delta, \mathcal{G}_{0,\delta}$ are Banach Lie groups with Lie algebras

$$\mathrm{Lie}(\mathcal{G}_\delta) = \{\varphi \in \mathcal{R} : \varphi^* = -\varphi\} \ , \quad \mathrm{Lie}(\mathcal{G}_{0,\delta}) = \left\{\varphi \in L^2_{1,\delta} : \varphi^* = -\varphi\right\} \ .$$

Moreover, $\mathcal{G}_{0,\delta}$ is a closed, normal subgroup of $\mathcal{G}_\delta$ and $\mathcal{G}_\delta/\mathcal{G}_{0,\delta}$ may be identified with the normalizer $\mathbf{P}_a$ of a in $\mathbf{G}$. The groups $\mathcal{G}_\delta, \mathcal{G}_{0,\delta}$ act on the space $\mathcal{A}_\delta$ of connections by push forward, the usual way. The action is well-defined and smooth.

Given a connection $\nabla = \nabla_0 + A \in \mathcal{A}_\delta$, let $F_\nabla = \nabla_0 A + \frac{1}{2}[A,A]$ denote the curvature of ∇. It is easy to verify that the map $\nabla \mapsto F_\nabla$ is smooth from $\mathcal{A}_\delta$ to $L^2_\delta\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right)$. Let $\mathcal{A}_{\mathrm{F},\delta}$ denote the subspace of irreducible flat connections. The groups $\mathcal{G}_\delta, \mathcal{G}_{0,\delta}$ preserve $\mathcal{A}_{\mathrm{F},\delta}$. Finally, we denote by $\mathcal{F}_{\alpha,\delta} = \mathcal{A}_{\mathrm{F},\delta}/\mathcal{G}_{0,\delta}$, $\mathcal{M}_{\alpha,\delta} =$

$\mathcal{A}_{F,\delta}/\mathcal{G}_\delta$, the quotient spaces. Then $\mathcal{M}_{\alpha,\delta} = \mathcal{F}_{\alpha,\delta}/\mathbf{P}_a$ under the identification $\mathbf{P}_a \simeq \mathcal{G}_\delta/\mathcal{G}_{0,\delta}$. We conclude this subsection with the following useful

Proposition 3.3. *The space $\mathcal{G}_{0,\delta}$ is connected and path connected.*

Proof. Being a Banach manifold, $\mathcal{G}_{0,\delta}$ is clearly locally path connected, hence the path components and components are identical. It suffices then to find a dense subset of $\mathcal{G}_{0,\delta}$ which is connected. Actually, we shall do this for $\mathcal{G}^{\mathbb{C}}_{0,\delta}$, the complex gauge group – this is sufficient, since the quotient $\mathcal{G}^{\mathbb{C}}_{0,\delta}/\mathcal{G}_{0,\delta}$ is clearly a contractible space.

Let $\overline{\mathcal{G}}^{\mathbb{C}}_1$ be the group of smooth maps $\overline{\Sigma} \to \mathbf{G}^{\mathbb{C}}$ which are the identity in some open neighborhood of p. Denote by $\overline{\mathcal{G}}^{\mathbb{C}}_0$ the group of smooth maps $\Sigma \to \mathbf{G}^{\mathbb{C}}$ which are the identity at p. $\overline{\mathcal{G}}^{\mathbb{C}}_0$ is nothing but the based complex gauge group for connections on a trivial $\mathbf{G}$ bundle over $\overline{\Sigma}$, and since $\mathbf{G}$ is simply connected, $\overline{\mathcal{G}}^{\mathbb{C}}_0$ is path connected. We claim that $\overline{\mathcal{G}}^{\mathbb{C}}_1$ is a dense, connected subset of $\mathcal{G}^{\mathbb{C}}_{0,\delta}$. First, density: the group of smooth compactly supported maps from Σ to $\mathfrak{gl}(n,\mathbb{C})$ is dense in $L^2_{2,\delta}$ by definition. Any $g \in \mathcal{G}^{\mathbb{C}}_{0,\delta}$ may be written as $1+(-1+g)$ with $-1+g \in L^2_{2,\delta}$ (see Proposition 3.1). Hence, there exist smooth, compactly supported $\tilde{g}_j \to -1+g$ as $j \to \infty$. For large j, $1+\tilde{g}_j$ is invertible, therefore

$$g_j = \frac{1+\tilde{g}_j}{\det(1+\tilde{g}_j)^{1/n}} \longrightarrow g$$

and $\{g_j\}$ is a sequence in $\overline{\mathcal{G}}^{\mathbb{C}}_1$ (here we are using the multiplication theorems). This proves that $\overline{\mathcal{G}}^{\mathbb{C}}_1$ is dense. Second, we show that $\overline{\mathcal{G}}^{\mathbb{C}}_1$ is path connected, hence connected. Given $g \in \overline{\mathcal{G}}^{\mathbb{C}}_1$, since $\overline{\mathcal{G}}^{\mathbb{C}}_0$ is path connected we can find a smooth path γ_t in $\overline{\mathcal{G}}^{\mathbb{C}}_0$ with $\gamma_0 = I, \gamma_1 = g$. We want to push γ_t into $\overline{\mathcal{G}}^{\mathbb{C}}_1$. But this is easily done: choose a small neighborhood U of the origin on which $\exp : \mathfrak{g}^{\mathbb{C}} \to \mathbf{G}^{\mathbb{C}}$ is a diffeomorphism. For a sufficiently small disk B about p, we may assume $\gamma_t(z) \in U$ for all $z \in B$ and $0 \le t \le 1$. Hence, we may write $\gamma_t(z) = \exp \Gamma_t(z)$, where Γ_t is a smooth map $B \to U$, $\Gamma_0 \equiv 0$. Choosing B sufficiently small, we may also assume $\Gamma_1 \equiv 0$. Now choose a smooth cut-off function $\eta : B \to [0,1]$ which is one in a neighborhood of ∂B and zero in a neighborhood of the origin. Define $\tilde{\gamma}_t$ to be equal to γ_t on $\Sigma \setminus B$, and $\tilde{\gamma}_t(z) = \exp(\eta(z)\Gamma_t(z))$ for $z \in B$. Then $\tilde{\gamma}_t$ is clearly smooth and in $\overline{\mathcal{G}}^{\mathbb{C}}_1$. Moreover, $\tilde{\gamma}_0 \equiv I$ and $\tilde{\gamma}_1 = g$, so this gives the desired path and completes the proof. □

3.2. Application of the index theorem

Our next goal is to show that the spaces $\mathcal{F}_{\alpha,\delta}, \mathcal{M}_{\alpha,\delta}$ are finite dimensional manifolds of given dimension. To do this we will have to digress a bit to study the

Fredholm properties of the $\bar{\partial}$-operator defined on the $L^2_{k,\delta}$ spaces via the index theorem of Atiyah, Patodi, and Singer.

Let ∇ be a flat connection, and let us denote also by ∇ the associated operator on $\mathfrak{g}$-valued forms. Let

$$\delta_\nabla : L^2_{1,\delta}\left(T^*\Sigma \otimes \mathfrak{g}_E\right) \longrightarrow L^2_\delta\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \oplus L^2_\delta(\mathfrak{g}_E) \tag{3.4}$$

denote the operator $\delta_\nabla = \left(\nabla, e^{-\tau\delta}\nabla^* e^{\tau\delta}\right)$. As we shall see presently, the kernel of the operator δ_∇ is closely related to the tangent spaces of $\mathcal{F}_{\alpha,\delta}, \mathcal{M}_{\alpha,\delta}$ at the point $[\nabla]$. We first prove

Proposition 3.5. *There exists a* $\delta_0 > 0$ *such that for every* $\delta \in (0, \delta_0)$*, the operator* δ_∇ *is a bounded, Fredholm operator of index* $2(g-1)(n^2-1)+\dim(\mathbf{G}/\mathbf{P}_a)$*. Moreover,* $\dim\operatorname{coker}\delta_\nabla = \dim \mathbf{P}_a$*.*

Proof. First, the cokernel of δ_∇ is the kernel of the operator

$$(\delta_\nabla)^* : L^2_{1,\delta}(\mathfrak{g}_E) \oplus L^2_{1,\delta}\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \longrightarrow L^2_\delta\left(T^*\Sigma \otimes \mathfrak{g}_E\right)$$
$$(\delta_\nabla)^*(\varphi, \omega) = \nabla\varphi + e^{-\tau\delta}\nabla^* e^{\tau\delta}\omega\ .$$

Since ∇ is flat, $\ker(\delta_\nabla)^*$ consists of pairs (φ,ω) such that $\nabla\varphi = 0$, $e^{-\tau\delta}\nabla^* e^{\tau\delta}\omega = 0$. Since φ, ω are exponentially decaying the only possibility is $\varphi = 0$, $*\omega \in e^{-\tau\delta}\mathbf{P}_a$, hence $\dim\operatorname{coker}\delta_\nabla = \dim \mathbf{P}_a$.

Next, we observe that it suffices to prove the result for the operator $\tilde{\delta}_\nabla$ defined by composing δ_∇ with the continuous inclusion

$$L^2_\delta\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \oplus L^2_\delta(\mathfrak{g}_E) \hookrightarrow L^2_{\delta/2}\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \oplus L^2_{\delta/2}(\mathfrak{g}_E)\ .$$

Indeed, the kernels of δ_∇ and $\tilde{\delta}_\nabla$ coincide and δ_∇ has closed range if $\tilde{\delta}_\nabla$ does. Moreover, the cokernels are isomorphic under the continuous injection

$$L^2_{\delta/2}\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \oplus L^2_{\delta/2}(\mathfrak{g}_E) \hookrightarrow L^2_\delta\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \oplus L^2_\delta(\mathfrak{g}_E)\ ,$$

coming from multiplication by $e^{-\tau\delta/2}$. This is clear, since by the previous paragraph, if $\psi = (0,\omega) \in$ coker δ_∇, then $e^{\tau\delta}\psi$ is bounded, so in particular $e^{\tau\delta/2}\psi$ is in $L^2_{\delta/2}$.

We also may assume without loss of generality that $\nabla = \nabla_0$ is a connection in temporal gauge (see Lemma 2.7). Thus, locally near the puncture $\nabla = d + i\alpha d\theta$. This is because for general ∇, $\tilde{\delta}_\nabla - \tilde{\delta}_{\nabla_0}$ is a multiplication operator by an element of $L^2_{1,\delta}$. Since the inclusion $L^2_{1,\delta} \hookrightarrow L^4_{\delta/2}$ is compact, and the multiplication

$L^4_{\delta/2} \times L^4_{\delta/2} \to L^2_{\delta/2}$ is continuous, we see that $\tilde{\delta}_\nabla - \tilde{\delta}_{\nabla_0}$ is in fact a compact operator. Therefore, $\tilde{\delta}_\nabla$ is Fredholm if and only if $\tilde{\delta}_{\nabla_0}$ is Fredholm, and $\operatorname{index} \delta_\nabla = \operatorname{index} \tilde{\delta}_\nabla = \operatorname{index} \tilde{\delta}_{\nabla_0}$. Let

$$\tilde{\delta}^0_{\nabla_0} : L^2_{1,\delta}\left(T^*\Sigma \otimes \mathfrak{g}_E\right) \longrightarrow L^2_{\delta/2}\left(\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E\right) \oplus L^2_{\delta/2}(\mathfrak{g}_E)$$

denote the operator (∇_0, ∇_0^*). Since all the derivatives of τ are in L^∞, the operator $\tilde{\delta}_{\nabla_0} - \delta^0_{\nabla_0}$ is a multiplication operator by a function in L^∞, hence by the compactness theorems for weighted Sobolev spaces $\tilde{\delta}_{\nabla_0} - \tilde{\delta}^0_{\nabla_0}$ is a compact operator. Then $\tilde{\delta}_{\nabla_0}$ is Fredholm if and only if $\tilde{\delta}^0_{\nabla_0}$ is Fredholm, and again they have the same indices.

Via the identification $T^*\Sigma \otimes \mathfrak{g}_E \simeq T^*\Sigma^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}$, $\bigwedge^2 T^*\Sigma \otimes \mathfrak{g}_E \oplus \mathfrak{g}_E \simeq \mathfrak{g}_E^{\mathbb{C}}$, the operator $\tilde{\delta}^0_{\nabla_0}$ is nothing but the formal L^2-adjoint of the $\bar{\partial}$-operator associated to ∇_0:

$$\bar{\partial}^*_{\nabla_0} : L^2_{1,\delta}\left(T^*\Sigma^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}\right) \longrightarrow L^2_{\delta/2}(\mathfrak{g}_E^{\mathbb{C}}) \tag{3.6}$$

Following Atiyah, Patodi, and Singer, we can define the same operator $\bar{\partial}^*_{\nabla_0}$ on the modified spaces:

$$\bar{\partial}^*_{\nabla_0} : L^2\left(T^*\Sigma^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}\right) \longrightarrow L^2_{\text{ext.}}(\mathfrak{g}_E^{\mathbb{C}}) \tag{3.6$'$}$$

By $L^2_{\text{ext.}}(\mathfrak{g}_E^{\mathbb{C}})$ we mean the space of extended L^2-sections of $\mathfrak{g}_E^{\mathbb{C}}$ as defined in [A-P-S], Proposition 3.11.

At this point, we require δ_0 to be less than the first positive eigenvalue of $\nabla_{0,\partial}$ on $E|_{\partial\Sigma_0}$, and let $\delta \in (0, \delta_0)$. It follows in exactly the same way as in [A-P-S], Proposition 3.11, that the kernel of (3.6) is the same as the kernel of (3.6$'$) and hence is finite dimensional (see also Lemma 3.12 below). We claim also that the cokernels are the same. Indeed, the cokernel of (3.6) may be identified with

$$\left\{\varphi \in L^2_{\delta/2}(\mathfrak{g}_E^{\mathbb{C}}) : e^{-\tau\delta}\bar{\partial}_{\nabla_0} e^{2\tau\delta}\varphi = 0\right\}$$

which, under the isometry $e^{\tau\delta/4} : L^2_{\delta/2} \to L^2$, corresponds to

$$\left\{\varphi \in L^2(\mathfrak{g}_E^{\mathbb{C}}) : \bar{\partial}_{\nabla_0} e^{\tau\delta/4}\varphi = 0\right\} .$$

The cokernel of (3.6$'$) is the space $\left\{\psi \in L^2_{\text{ext.}}(\mathfrak{g}_E^{\mathbb{C}}) : \bar{\partial}_{\nabla_0}\psi = 0\right\}$. We assert that the map $\varphi \mapsto \psi = \varphi e^{\tau\delta/4}$ is an isomorphism. This follows as in [A-P-S] by the following argument: let $\varphi \in L^2(\mathfrak{g}_E^{\mathbb{C}})$, $\bar{\partial}_{\nabla_0}(e^{\tau\delta/4}\varphi) = 0$. Expand φ near the boundary $\varphi(\tau, \theta) = \sum_\lambda f_\lambda(\tau)\varphi_\lambda(\theta)$ where $\varphi_\lambda(\theta)$ form an orthonormal basis of eigenfunctions of $\nabla_{0,\partial}$. Since $\bar{\partial}_{\nabla_0}(e^{\tau\delta/4}\varphi) = 0$, we have

$$\frac{d}{d\tau}\left(e^{\tau\delta/4} f_\lambda(\tau)\right) + \lambda\left(e^{\tau\delta/4} f_\lambda(\tau)\right) = 0 ,$$

hence $\varphi(\tau,\theta) = \sum_\lambda e^{-(\lambda+\delta/4)\tau} f_\lambda(0)\varphi_\lambda(\theta)$. Since $\varphi \in L^2$, the summation is over $\lambda \geq 0$, hence $\varphi(\tau,\theta) = e^{-\tau\delta/4}\sum_{\lambda\geq 0} e^{-\lambda\tau} f_\lambda(0)\varphi_\lambda(\theta)$. But then the image $\psi(\tau,\theta) = e^{\tau\delta/4}\varphi(\tau,0) = \sum_{\lambda\geq 0} e^{-\lambda\tau} f_\lambda(0)\varphi_\lambda(\theta)$ belongs to the cokernel of (3.6′). The inverse map is constructed in the same way.

By the Atiyah, Patodi, Singer Index Theorem (see [A-P-S], Theorem 3.10), $\bar\partial^*_{\nabla_0}$ is Fredholm, hence so is $\tilde\delta^0_{\nabla_0}$, and $\operatorname{index}\tilde\delta^0_{\nabla_0} = \operatorname{index}(\bar\partial^*_{\nabla_0})$. Moreover,

$$\operatorname{index}(\bar\partial^*_{\nabla_0}) = -(n^2-1)\int_\Sigma c_1 - \frac{h}{2}\ , \quad h = \dim(\mathbf{P}_a \otimes \mathbb{C})\ .$$

Hence, since $\int_\Sigma c_1$ is the Euler characteristic of $\Sigma = 1-2g$, we have

$$\operatorname{index}(\bar\partial^*_{\nabla_0}) = (n^2-1)(2g-1) - \dim \mathbf{P}_a = 2(g-1)(n^2-1) + \dim(\mathbf{G}/\mathbf{P}_a)\ .$$

□

Let us denote by $H^1_{\delta,\nabla}$ the kernel of the operator δ_∇ of (3.4). According to Proposition 3.5, $H^1_{\delta,\nabla}$ is a vector space of dimension $(2g-1)(n^2-1)$. Let $\mathcal{H}$ be the space of harmonic 0-forms, and let $r_0 : \mathcal{H} \to H^1_{\delta,\nabla}$ be the map $r_0(\varphi) = \nabla\varphi$. Let $H^1_{\delta+,\nabla} = H^1_{\delta,\nabla}/\operatorname{im} r_0$. The importance of the spaces $H^1_{\delta,\nabla}, H^1_{\delta+,\nabla}$ is illustrated in the theorem below.

Theorem 3.7. *(i) $\mathcal{F}_{\alpha,\delta}$ is a smooth manifold of dimension $(2g-1)(n^2-1)$. Moreover, for $[\nabla] \in \mathcal{F}_{\alpha,\delta}$, $T_{[\nabla]}\mathcal{F}_{\alpha,\delta} \simeq H^1_{\delta,\nabla}$. (ii) $\mathcal{M}_{\alpha,\delta}$ is a smooth manifold of dimension $2(g-1)(n^2-1) + \dim(\mathbf{G}/\mathbf{P}_a)$. Moreover, for $[\nabla] \in \mathcal{M}_{\alpha,\delta}$, $T_{[\nabla]}\mathcal{M}_{\alpha,\delta} \simeq H^1_{\delta+,\nabla}$. (iii) $\mathcal{M}_{\alpha,\delta}$ naturally has the structure of an almost complex manifold.*

Proof. The proof of (i) and (ii) follows from standard arguments (cf. [Ko]) using our Proposition 3.5 and Theorem 3.4 of [M] (see also Sections 7 and 8 of [T1] for a very similar situation). In order to prove (iii) we proceed as follows: Given $[\nabla] \in \mathcal{M}_{\alpha,\delta}$, lift to a representative $\nabla \in \mathcal{A}_{\mathrm{F},\delta}$ and let $\bar\partial^*_\nabla$ denote the operator associated to ∇ as in (3.6′). Let $\sigma_\nabla : H^1_{\delta,+} \to \ker\bar\partial^*_\nabla$ be the linear map defined as follows: Given $X \in H^1_{\delta,+} \simeq H^1_\delta/\operatorname{im} r_0$, lift to

$$\alpha_X \in H^1_\delta \subset L^2_{1,\delta}\left(T^*\Sigma \otimes \mathfrak{g}_E\right) \subset L^2\left(T^*\Sigma \otimes \mathfrak{g}_E\right)$$

perpendicularly in the L^2-norm to the subspace $\operatorname{im} r_0 \subset H^1_\delta$. Let β_X be the image of α_X under the natural isomorphism $L^2\left(T^*\Sigma \otimes \mathfrak{g}_E\right) \simeq L^2\left(T^*\Sigma^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}\right)$, and let $\gamma_X = \sigma_\nabla(X)$ be the L^2-projection of β_X in $\ker\bar\partial^*_\nabla$. We claim that σ_∇ is an isomorphism. Indeed, as both spaces have the same dimension, it suffices to check that σ_∇ is injective. Assume therefore that $\gamma_X = 0$. Then $\beta_X = \bar\partial_\nabla\psi$, for some $\psi \in L^2_{\mathrm{ext.}}\left(\mathfrak{g}_E^{\mathbb{C}}\right)$, and hence $\alpha_X = \nabla\varphi$, $\varphi = \psi - \psi^*$. But since $0 = e^{-\tau\delta}\nabla^* e^{\tau\delta}\alpha_X = e^{-\tau\delta}\nabla^* e^{\tau\delta}\nabla\varphi$, we obtain that $\varphi \in \mathcal{H}$, and therefore $\alpha_X \in \operatorname{im} r_0$. By the assumption that α_X is perpendicular to the latter, $\alpha_X = 0$, and σ_∇ is therefore an isomorphism.

The almost complex structure on $\mathcal{M}_{\alpha,\delta}$ is now defined by pulling back the complex structure from $\ker \bar{\partial}^*_\nabla$ to $T_{[\nabla]}\mathcal{M}_{\alpha,\delta}$ via σ_∇. This completes the proof of Theorem 3.7. □

3.3. Equivalence of the two descriptions

We shall now show that the L^2_δ description produces the same moduli spaces as the ones described in Section 2 via representations of the fundamental group. Moreover, the almost complex structure on $\mathcal{M}_{\alpha,\delta}$ defined in Theorem 3.7 coincides with that of Mehta and Seshadri (cf. Section 2.2). We start by showing

Theorem 3.8. *There is a natural* $\mathbf{P}_a \simeq \mathcal{G}_\delta/\mathcal{G}_{0,\delta}$ *diffeomorphism* $\Phi : \mathcal{F}_{\alpha,\delta} \to \mathcal{F}_a$. *Moreover,* Φ *induces a diffeomorphism* $\mathcal{M}_{\alpha,\delta} \to \mathcal{M}_a$.

Proof. The construction of Φ proceeds as follows: let $[\nabla] \in \mathcal{F}_{\alpha,\delta}$. First, choose a smooth representative $\nabla \in \mathcal{A}_{\mathrm{F},\delta}$ such that over the punctured disk $\nabla = d + i\alpha d\theta$. This can be achieved as in Lemma 2.7 by noticing that if in the ODE in the proof of 2.7, $b \in L^2_{1,\delta}$, then $g \in \mathcal{G}_\delta$; we then use g multiplied by the inverse of its value at ∞ (cf. Propositions 3.1 and 3.2). We set $\Phi[\nabla]$ to be equal to the class of ∇ in $\mathcal{M}_0$. Clearly, $\Phi[\nabla] \in \mathcal{F}_a$. We claim that Φ is surjective. Let $[\hat{\nabla}] \in \mathcal{F}_a$. Using Lemma 2.2, choose a representative $\hat{\nabla} = d + A_s$ such that on the disk $\tilde{D}^*$, $A_s = i\alpha d\theta$. Let $A = \hat{\nabla} - \nabla_0$, and observe that A is compactly supported. Hence, $\nabla_0 + A \in \mathcal{A}_\delta$ and $\Phi[\nabla_0 + A] = [\hat{\nabla}]$. To show that Φ is injective, let $\Phi[\nabla_1] = \Phi[\nabla_2]$. Choose representatives $\nabla_1 = \nabla_0 + A_1$, $\nabla_2 = \nabla_0 + A_2$ as before. By assumption, $\nabla_2 = \nabla_1^g$ where $g \in \mathcal{G}_0$. Hence, over $\tilde{D}^*$ and via the cylindrical coordinates (τ, θ) we obtain $\partial g/\partial\tau = 0$,$\nabla_{0,\partial} g = 0$. The above equations together with the fact that g at the base point is I imply $g|_{\tilde{D}^*} \equiv I$, hence $g \in \mathcal{G}_{0,\delta}$. Thus $[\nabla_1] = [\nabla_2]$ in $\mathcal{F}_{\alpha,\delta}$, proving that Φ is injective. We have shown Φ to be a bijection – we must verify that Φ and Φ^{-1} are smooth. The smoothness of Φ follows from the fact that the holonomy map is smooth, plus the commutativity of the diagram:

$$\begin{array}{ccccc} & & \mathcal{A}_{\mathrm{F},\delta} & \xrightarrow{\text{hol.}} & \mathrm{Hom}\,(\pi_1(\Sigma), \mathbf{G}) \\ & & \Big\downarrow P & & \Big\downarrow \\ \mathcal{A}_{\mathrm{F},\delta}/\mathcal{G}_{0,\delta} & = & \mathcal{F}_{\alpha,\delta} & \xrightarrow{\Phi} & \mathcal{F}_a \subset \mathcal{M}_0 \end{array}$$

The smoothness of Φ^{-1} is obtained as follows: let $\{\rho_\xi\}$ be any smooth family in $\mathrm{Hom}\,(\pi_1(\Sigma), \mathbf{G})$, and let $\{A_\xi\}$ be any smooth family in $\mathcal{A}_{\mathrm{F},\delta}$ such that holonomy $A_\xi = \rho_\xi$. Then $\Phi^{-1}(\rho_\xi) = P(A_\xi)$, and hence is smooth in ξ (recall that the C^∞-structure on $\mathcal{M}_0$, and hence also that on $\mathcal{F}_a$, was defined via the identification $\mathcal{M}_0 \simeq \mathrm{Hom}\,(\pi_1(\Sigma), \mathbf{G})$. In order to complete the proof of Theorem 3.8, we need to check the equivariance of Φ. Let $\phi \in \mathbf{P}_a \simeq \mathcal{G}_\delta/\mathcal{G}_{0,\delta}$. Let g_ϕ be the unique harmonic gauge transformation satisfying $r(g_\phi) = \phi$. Then

$$\Phi(\phi \cdot [\nabla]) = \Phi\,[\nabla^{g_\phi}] = \Phi\left[\nabla^{(g_\phi \cdot \phi^{-1}) \cdot \phi}\right] = \Phi[\nabla]^\phi = \phi \cdot \Phi[\nabla]\ .$$

The diffeomorphism $\mathcal{M}_{\alpha,\delta} \simeq \mathcal{M}_a$ follows. □

To prepare for the proof of the equivalence of almost complex structures, recall from Section 2.2 that a flat unitary connection ∇ gives rise to a flag f. Let us denote by $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) the endomorphisms of the fiber $\bar{E}_p$ which are upper triangular (resp. lower triangular) with respect to the flag, and by $\mathfrak{u}^+$ and $\mathfrak{u}^-$ we mean the strictly upper and lower triangular endomorphisms, respectively.

Let $\mathcal{S}$ denote the sheaf of meromorphic sections of $End\bar{E} \otimes \Omega^1$ with at most a simple pole at the parabolic point p. We denote also by $\mathcal{S}_0$ the subsheaf of traceless endomorphisms, and by $\mathcal{S}^-$ and $\mathcal{S}_0^-$ those with residue strictly lower triangular at p with respect to the fixed flag f. We have the following

Lemma 3.9. *Hermitian conjugation gives a complex anti-linear injection* $\ker \bar{\partial}_\nabla^* \hookrightarrow H^0\left(\bar{\Sigma}, \mathcal{S}_0^-\right)$.

Proof. The hermitian conjugate of an element of $\ker \bar{\partial}_\nabla^*$ is an L^2 holomorphic (1,0) form ω with values in End E. We must show ω extends as a meromorphic form on $\bar{\Sigma}$ with residue strictly lower triangular. Near the point p we write $\omega(z) = \omega_{ij}(z) dz \otimes f_i \otimes f_j^*$, where ω_{ij} are holomorphic functions on the punctured disk $\tilde{D}^*$ and $\{f_i\}$ is the basis of holomorphic sections introduced in Section 2.2. Writing in terms of the orthonormal basis $\{e_i\}$, $\omega(z) = |z|^{\hat{\alpha}_i - \hat{\alpha}_j} \omega_{ij}(z) dz \otimes e_i \otimes e_j^*$, hence

$$\|\omega\|_{L^2}^2 \sim \int |dz|^2 |\omega_{ij}|^2 |z|^{2(\hat{\alpha}_i - \hat{\alpha}_j)} < +\infty \ .$$

Since $|\hat{\alpha}_i - \hat{\alpha}_j| < 1$, the ω_{ij} may have at most a simple pole at $z = 0$. If $\hat{\alpha}_i - \hat{\alpha}_j > 0$, the residue may be arbitrary, but if $\hat{\alpha}_i - \hat{\alpha}_j \leq 0$, the residue must vanish. This proves the Lemma. □

Lemma 3.10. *Serre duality gives a complex linear isomorphism* $H^0\left(\bar{\Sigma}, \mathcal{S}^-\right) \xrightarrow{\sim} H^1\left(\bar{\Sigma}, End^- \bar{E}\right)^*$.

Proof. Since $End\bar{E}$ is locally free we have the exact sequence

$$0 \longrightarrow End\bar{E} \otimes \mathcal{O}(-p) \longrightarrow End\bar{E} \longrightarrow \mathfrak{gl}^{\mathbb{C}} \longrightarrow 0 \ .$$

Restricting to lower triangular endomorphisms, we have via evaluation $End^-\bar{E} \to \mathfrak{b}^- \to 0$. Let $\mathcal{Q}$ denote the kernel of the sequence above. Then we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Q} & \longrightarrow & End^{-}\bar{E} & \longrightarrow & \mathfrak{b}^{-} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & End\bar{E}\otimes\mathcal{O}(-p) & \longrightarrow & End\bar{E} & \longrightarrow & \mathfrak{gl}^{\mathbb{C}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathfrak{u}^{+} & = & \mathfrak{u}^{+} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the isomorphism $\mathcal{Q} \simeq End\bar{E}\otimes\mathcal{O}(-p)$ is easily proven. Looking at the exact (dual) sequence in cohomology, we have

$$\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow & & & & & & \\
\mathfrak{b}^{+} & \xleftarrow{\jmath} & H^1(\bar{\Sigma},\mathcal{Q})^* & \longleftarrow & H^1(\bar{\Sigma}, End^{-}\bar{E})^* & \longleftarrow & 0 \\
\uparrow & & \| & & & & \\
\mathfrak{gl}^{\mathbb{C}} & \xleftarrow{res} & H^1\left(\bar{\Sigma}, End\bar{E}\otimes\mathcal{O}(-p)\right)^* & & & & \\
\uparrow & & & & & & \\
\mathfrak{u}^{-} & & & & & & \\
\uparrow & & & & & & \\
0 & & & & & &
\end{array}$$

By Serre duality, $H^1\left(\bar{\Sigma}, End\bar{E}\otimes\mathcal{O}(-p)\right)^*$ may be identified with the holomorphic sections of $End\bar{E}\otimes\mathcal{O}(p)\otimes\Omega^1$, and the map to $\mathfrak{gl}^{\mathbb{C}}$ is just the residue. Requiring the residue to be strictly lower triangular identifies $H^0(\bar{\Sigma},\mathcal{S}^-)$ with $\ker(\jmath)\simeq H^1(\bar{\Sigma}, End^-\bar{E})^*$. This proves the Lemma. □

Combining Lemmas 3.9 and 3.10 with the identification (cf. Section 2.2)

$$T_{[\bar{\partial}_{\bar{E}}]}\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right) \simeq H^1\left(\bar{\Sigma}, End^{+}\bar{E}\right) \xrightarrow{*} H^1\left(\bar{\Sigma}, End^{-}\bar{E}\right)^*$$

and the holomorphic inclusion $\mathcal{S}_0^- \hookrightarrow \mathcal{S}^-$, we obtain

Proposition 3.11. *The injection* $\ker\bar{\partial}^*_{\nabla} \hookrightarrow T_{[\bar{\partial}_{\bar{E}}]}\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right)$ *is complex linear.*

Now let $\Psi = \Phi^{-1}$ in Theorem 3.8. If $\mathcal{A}_F^a$ denotes the space of smooth traceless hermitian flat connections on E with holonomy around p conjugate to a, then we clearly can lift Ψ to a map $\tilde{\Psi} : \mathcal{A}_F^a \to \mathcal{M}_{\alpha,\delta}$. Given $\nabla\in\mathcal{A}_F^a$, the map $a \to a^{0,1}$ induces a linear isomorphism between $\ker\bar{\partial}^*_{\nabla}$ in $(3.6')$ and space

$$\mathcal{W} = \left\{ a \in L^2\left(T^*\Sigma\otimes\mathfrak{g}_E\right) : \nabla a = 0\ ,\ \nabla * a = 0 \right\}\ .$$

Recall the map σ_∇ from Theorem 3.7. We have the following

Lemma 3.12. *Under the identification $\mathcal{W} \simeq \ker \bar\partial^*_\nabla$, the restriction of Ψ_* to $\mathcal{W}$ corresponds to σ_∇^{-1}.*

Proof. Let $a \in \mathcal{W}$. By our choice of δ and the equality $e^{\tau\delta} = |z|^{-\delta}$, it is evident from the proof of Lemma 3.9 above that $a^{0,1} \in L^2_{1,\delta}\left(T^*\Sigma^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}\right)$, hence $a \in L^2_{1,\delta}\left(T^*\Sigma \otimes \mathfrak{g}_E\right)$ (see also, [A-P-S], Proposition 3.11). But $\sigma_\nabla^{-1}(a)$ is nothing but the class of a in $H^1_{\delta,+}$, which by definition of $\tilde\Psi$ equals $\Psi_*(a)$. □

Finally, note that the map μ from (2.11) lifts to a map $\tilde\mu : \mathcal{A}^a_F \to \mathcal{B}_{p.s.}/\mathcal{K}_p$. It is clear from Lemmas 3.9, 3.10, and 3.12 that the following diagram commutes:

$$\begin{array}{ccccccccc}
 & \sigma_\nabla^{-1} & \ker\bar\partial^*_\nabla & \xrightarrow{*} & H^0\left(\bar\Sigma, \mathcal{S}_0^-\right) & \longrightarrow & H^0\left(\bar\Sigma, \mathcal{S}^-\right) \\
 & \swarrow & \| & & & & \| \\
T_{[\nabla]}\mathcal{M}_{\alpha,\delta} & & \mathcal{W} & & & & H^1\left(\bar\Sigma, End^-\bar E\right)^* \\
\| & \tilde\Psi_* & \bigcap & & \tilde\mu_* & & \|* \\
T_{[\nabla]}\mathcal{M}_{\alpha,\delta} & \longleftarrow & T_\nabla\mathcal{A}^a_F & \text{——} & \text{——} & \longrightarrow & T_{[\bar\partial_{\bar E}]}\mathcal{B}_{p.s.}/\mathcal{K}_p \\
\| & & \downarrow & & & & \| \\
T_{[\nabla]}\mathcal{M}_{\alpha,\delta} & \longleftrightarrow & T_{[\nabla]}\mathcal{M}_a & \longleftarrow & \text{——} & \longrightarrow & T_{[\bar\partial_{\bar E}]}\mathcal{B}_{p.s.}/\mathcal{K}_p
\end{array}$$

By Proposition 3.11 we now have

Theorem 3.13. *With respect to the almost complex structure defined on $\mathcal{M}_{\alpha,\delta}$ in Theorem 3.7 and the complex structure on $\mathcal{M}_a$ defined via the Mehta-Seshadri theorem, the map $\Phi : \mathcal{M}_{\alpha,\delta} \to \mathcal{M}_a$ is biholomorphic.*

4. Topological results

We now examine the topological properties of the spaces we have defined. The key idea is the interplay between the different descriptions. The analytic description will allow us to define a closed, integral two-form on the spaces $\mathcal{F}_{\alpha,\delta}$ which will serve as the curvature for the pre-quantum line bundle. Topological restrictions will prevent the line bundle descending, except in special cases.

4.1. The topology of the moduli spaces

In this section we analyze the topology of the spaces $\mathcal{M}_a \simeq \mathcal{F}_a/\mathbf{P}_a$ for $a \in \mathbf{G}$. The case $\mathbf{G} = \mathrm{SU}(2)$ is rather simple and essentially follows via Proposition 2.3 from the results of Atiyah and Bott [A-B]. For higher rank, there is not such a simple description, and instead we will make use of certain transversality results developed in [D-U]. In this general case, we have to make the additional assumption that the genus of Σ is strictly greater than two.

To begin, let us assume that $\mathbf{G} = \mathrm{SU}(2)$. Recall from Proposition 2.3 that the map $q : \mathcal{M}_0 \to \mathbf{G}$ is a fibration away from the identity. Therefore, the fiber $\mathcal{F}_a$ of q over $a \neq I$ is homotopy equivalent to $\mathcal{F}_{-I}$. By the results of Atiyah and Bott [A-B], $\pi_1(\mathcal{F}_{-I}) = 0$, $\pi_2(\mathcal{F}_{-I}) = \mathbb{Z}$, and therefore the same is true for $\mathcal{F}_a$, $a \neq I$. Looking at the long exact sequence in homotopy for the fibration (2.6) for $a \neq \pm I$, we obtain

Theorem 4.1. *Suppose* $\mathbf{G} = SU(2)$*,* $a \in \mathbf{G}$*, and* $a \neq \pm I$*. Then* $\mathcal{M}_a$ *is connected and simply connected, and* $\pi_2(\mathcal{M}_a)$ *is free abelian of rank 2.*

We wish to define a pair of generators for $\pi_2(\mathcal{M}_a) \simeq H_2(\mathcal{M}_a, \mathbb{Z})$, $a \neq \pm I$, explicitly. Let β be the image of a generator of $\pi_2(\mathcal{F}_a) \simeq \mathbb{Z}$ in $\pi_2(\mathcal{M}_a)$ under the map π_*. Let γ be an element of $\pi_2(\mathcal{M}_a)$ whose image under the boundary homomorphism ∂ is a generator of $\pi_1(\mathbf{T}_a/\mathbb{Z}_2) \simeq \mathbb{Z}$ and which is perpendicular to β with respect to the dual pairing. This defines β and γ up to sign. We fix the signs to be compatible with the complex structure on $\mathcal{M}_a$. Recall the map $\varphi : \mathcal{N}_a \to \mathcal{U}_a \subset \mathcal{M}_a$ of principal $\mathbb{P}^1$ bundles over $\mathcal{M}_s(2,0)$ (see Theorem 2.18). Let $\overline{\beta}, \overline{\gamma}$ denote the generators of $\pi_2(\mathcal{N}_a)$ defined via projection onto the first and second factors, respectively. Note that $(pr_1)_*\overline{\beta}$ generates the free part of $\pi_2(\mathcal{M}(2,0)) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$. The torsion in $\pi_2(\mathcal{M}(2,0))$ implies another important fact: if $[\mathbb{P}^1]$ denotes the fundamental class of the fiber in the fibration $\mathbb{P}^1 \xrightarrow{\imath} \mathcal{N}_a \xrightarrow{pr_1} \mathcal{M}(2,0)$, then in $H_2(\mathcal{N}_a, \mathbb{Z})$, $\imath_*[\mathbb{P}^1] = 2\overline{\gamma}$. We now prove

Proposition 4.2. *Assume* $a \neq \pm I$*, and* $g > 3$*. Then the map* φ *induces an isomorphism* $\varphi_* : \pi_2(\mathcal{N}_a) \to \pi_2(\mathcal{M}_a)$ *given by* $\overline{\beta} \to \beta$*,* $\overline{\gamma} \to \gamma$*.*

Proof. By Theorem 2.18 and Proposition 2.16, φ clearly gives rise to an isomorphism $\varphi_* : \pi_2(\mathcal{N}_a) \simeq H_2(\mathcal{N}_a, \mathbb{Z}) \to \pi_2(\mathcal{M}_a) \simeq H_2(\mathcal{M}_a, \mathbb{Z})$. In addition, we have the homotopy equivalences $\mathcal{N}_a \sim B(\overline{\mathcal{G}}_p^{\mathbb{C}}/\mathbb{Z}_2)$ and $\mathcal{M}_a \sim B(\mathcal{K}_p/\mathbb{Z}_2)$ (see Propositions 2.10 and 2.17). Therefore, tensoring with the rationals, there is an isomorphism $\varphi_* : H_2(B\overline{\mathcal{G}}_p^{\mathbb{C}}, \mathbb{Q}) \to H_2(B\mathcal{K}_p, \mathbb{Q})$. According to [A-B], Section 2 and [Ni], Proposition 3.2, the $\mathbb{P}^1$ fibrations $B\overline{\mathcal{G}}_p^{\mathbb{C}}$ and $B\mathcal{K}_p$ over $B\overline{\mathcal{G}}^{\mathbb{C}}$ and $B\mathcal{K}$, respectively, are trivial in rational homology. Moreover, it easy to see that $H_2(B\overline{\mathcal{G}}^{\mathbb{C}}, \mathbb{Q}) \simeq H_2(B\overline{\mathcal{G}}_0^{\mathbb{C}}, \mathbb{Q})$, and similarly for $B\mathcal{K}$ and $B\mathcal{K}_0$. Hence, φ_* defines a map

$$\varphi_* : H_2(B\overline{\mathcal{G}}_0^{\mathbb{C}}, \mathbb{Q}) \oplus H_2(\mathbf{G}/\mathbf{T}_a, \mathbb{Q}) \longrightarrow H_2(B\mathcal{K}_0, \mathbb{Q}) \oplus H_2(\mathbf{G}/\mathbf{T}_a, \mathbb{Q}) \ .$$

By the definition of φ, this map preserves the summands. Now from [A-B], Section 2, $\overline{\beta}$ is half the generator of $\pi_2(B\overline{\mathcal{G}}_0^{\mathbb{C}})$, and by the Mehta-Seshadri theorem (cf. [Si1] and [Si2]), $B\mathcal{K}_0$ is homotopy equivalent to $B\mathcal{G}_{0,\delta} \simeq \mathcal{F}_{\alpha,\delta}$. The assertion that $\overline{\beta} \to \beta$ follows from Theorem 3.8, our definition of β, and the fact that φ_*

is an isomorphism on integral homology. The Proposition then follows from the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \to & \pi_2(\mathcal{F}_a) & \to & \pi_2(\mathcal{F}_a/\mathbf{T}_a) & \to & \pi_1(\mathbf{T}_a/\mathbb{Z}_2) & \to & 0 \\ & & \| & & \| & & & & \\ 0 & \leftarrow & \pi_2(B\mathcal{K}_o) & \leftarrow & \pi_2(B\mathcal{K}_p) & \leftarrow & \pi_2(\mathbf{G}/\mathbf{T}_a) & \leftarrow & 0\,. \end{array}$$

□

We return now to the general case $\mathbf{G} = \mathrm{SU}(n)$, $g \geq 3$. We shall analyze the topology of $\mathcal{M}_a$ via the techniques developed in [D-U]. We first recall from Section 2.2 that $\mathcal{M}_a$ can be identified with the fiber of the determinant map $\det\ : \mathcal{B}_{p.s.}/\mathcal{K}_p \to J_l(\overline{\Sigma})$, where $\mathcal{B}_{p.s.}$ denotes the subspace of parabolic stable holomorphic structures on $\overline{E}$ and $\mathcal{K}_p$ is the group of parabolic automorphisms of $\overline{E}$. The next proposition parallels [D-U], Theorem 2.4.

Proposition 4.3. *The inclusion $\mathcal{B}_{p.s.}$ in $\mathcal{B}$ is an homotopy equivalence up to dimension* $2(n-1)(g-2)+1$.

Proof. The proof is similar to the one in [D-U] with the only difference being the existence of a parabolic structure. From Theorem 2.4 in [D-U], the inclusion of $\mathcal{B}_{p.s.}$ is $\mathcal{B}$ is an homotopy equivalence up to an integer d where

$$d + 2\left\{(1-g)m(n-m) + ml - nc\right\} < 0\,.$$

Here l is the degree of $\overline{E}$ and c is the degree of a destabilizing subbundle of rank m. The above inequality has to be satisfied for every possible such subbundle of $\overline{E}$, i.e. for every integer m between 1 and $n-1$ and every integer c such that $c+\sum_{i=1}^m \alpha_i' \geq 0$, where $\{\alpha_i'\}$ is a subcollection of the weights $\{\alpha_i\}$. In other words, the inclusion of $\mathcal{B}_{p.s.}$ in $\mathcal{B}$ is an homotopy equivalence if

$$d < 2\left\{(g-1)m(n-m) + mn\left(\frac{c}{m} - \frac{l}{n}\right)\right\}$$

for all m, c as above. But since $\operatorname{pardeg} E = l + \sum_{i=1}^n \alpha_i = 0$, we obtain $-l/n = -\sum_{i=1}^n \alpha_i/n$. On the other hand, $\operatorname{pardeg} F = c + \sum_{i=1}^m \alpha_i' \geq 0$, hence $c/m \geq -\sum \alpha_i'/m$. Therefore,

$$\begin{aligned} &\left\{(g-1)m(n-m) + mn\Big(\frac{c}{m} - \frac{l}{n}\Big)\right\} \\ &\qquad \geq \left\{(g-1)m(n-m) + mn\left(-\frac{1}{m}\sum_{i=1}^m \alpha_i' + \frac{1}{n}\sum_{i=1}^n \alpha_i\right)\right\} \\ &\qquad = \left\{(g-1)m(n-m) + mn\left(\frac{1}{m}\sum_{i=1}^{n-m} \alpha_i'' - \frac{n-m}{mn}\sum_{i=1}^m \alpha_i'\right)\right\} \\ &\qquad > \left\{(g-1)m(n-m) - m(n-m)\right\} = (g-2)m(n-m)\,. \end{aligned}$$

Since both quantities are integral,

$$(g-1)m(n-m)+mn\left(\frac{c}{m}-\frac{l}{m}\right)\geq (g-2)m(n-m)+1\ .$$

The right hand side is minimized at $m=1$, and the proposition follows. □

Therefore, for $g\geq 3$, $n\geq 3$, the inclusion $\mathcal{B}_{p.s.}\hookrightarrow\mathcal{B}$ is an homotopy equivalence, at least up to dimension 5. This is sufficient to compute the low dimensional homotopy and cohomology groups of the moduli space $\mathcal{M}_a$ that we are interested in.

From the Proposition 4.3, we obtain, as in the ordinary case [D-U],

$$\pi_i\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right)\simeq\pi_i\left(B(\mathcal{K}_p/\mathbb{C}^*)\right)\ ;\ i=0,1,2.$$

We are going first to compute $\pi_i(B\mathcal{K})$; $i=0,1,2$. Let $\mathcal{K}$ denote as before the full complex gauge group of $\overline{E}$. Then the relation between $B\mathcal{K}_p$ and $B\mathcal{K}$ is given in the following fibration (cf. [Ni], eq. (3.2))

$$\mathbf{G}/\mathbf{P}_a\longrightarrow B\mathcal{K}_p\longrightarrow B\mathcal{K}\tag{4.4}$$

(observe that if $\mathbf{N}_a$ denotes the normalizer of a in $\mathrm{U}(n)$, and $\mathbf{P}_a$ the normalizer in $\mathbf{G}=\mathrm{SU}(n)$, then $\mathrm{U}(n)/\mathbf{N}_a\simeq\mathbf{G}/\mathbf{P}_a$). Since $\pi_1(\mathbf{G})=\pi_2(\mathbf{G})=0$, we obtain

$$\pi_1(\mathbf{G}/\mathbf{P}_a)\simeq\pi_0(\mathbf{G}/\mathbf{P}_a)\simeq 0\ ,\quad \pi_2(\mathbf{G}/\mathbf{P}_a)\simeq\pi_1(\mathbf{P}_a)\ .$$

Thus, via the long exact sequence in homotopy associated to (4.4) we obtain for $i=0,1$, $\pi_i(B\mathcal{K}_p)\simeq\pi_i(B\mathcal{K})$, whereas for $i=2$ we have the short exact sequence

$$0\longrightarrow\pi_1(\mathbf{P}_a)\longrightarrow\pi_2(B\mathcal{K}_p)\longrightarrow\pi_2(B\mathcal{K})\longrightarrow 0\tag{4.5}$$

Finally, in order to compute $\pi_i(B\overline{\mathcal{K}}_p)$; $i=0,1,2$, we consider the fibrations

$$B\mathbb{C}^*\longrightarrow B\mathcal{K}_p\longrightarrow B\left(\mathcal{K}_p/\mathbb{C}^*\right)\tag{4.6}$$

induced from $B\mathbb{C}^*\to B\mathcal{K}\to B\left(\mathcal{K}/\mathbb{C}^*\right)$. Via the long exact sequence in homotopy associated to (4.6) and using (4.4) we obtain the short exact sequence

$$0\longrightarrow\pi_1(\mathbf{P}_a)\longrightarrow\pi_2(B\mathcal{K}_p)\longrightarrow\pi_2\left(B(\mathcal{K}_p/\mathbb{C}^*)\right)\longrightarrow 0\ .$$

In particular, since the rank of $\pi_2\left(B(\mathcal{K}/\mathbb{C}^*)\right)$ is one (cf. [A-B], Section 2), we obtain $\operatorname{rank}\pi_2\left(B(\mathcal{K}_p/\mathbb{C}^*)\right)=\operatorname{rank}\pi_1(\mathbf{P}_a)+1$. Summarizing, we have shown that

$$\begin{aligned}\pi_0\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right)&=0\ ,\\ \pi_1\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right)&\simeq H_1\left(\overline{\Sigma},\mathbb{Z}\right)\ ,\\ \operatorname{rank}\pi_2\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right)&=\operatorname{rank}\pi_1(\mathbf{P}_a)+1\ .\end{aligned}$$

In order to compute $\pi_i(\mathcal{M}_a)$ for $i=0,1,2$, we consider the long exact sequence in homotopy associated to the fibration det. Since $J_l(\overline{\Sigma})$ is a torus and det induces an isomorphism $\det_* : \pi_1\left(\mathcal{B}_{p.s.}/\mathcal{K}_p\right) \simeq \pi_1(B\mathcal{K}) \to \pi_1(J_l\left(\overline{\Sigma}\right))$ (cf. [A-B], Section 9), we obtain $\pi_0(\mathcal{M}_a) = \pi_1(\mathcal{M}_a) = 0$, and $\operatorname{rank}\pi_2(\mathcal{M}_a) = \operatorname{rank}\pi_1(\mathbf{P}_a) + 1$. Combining this with Theorem 4.1, we can now state

Theorem 4.7. *Assume that either $\mathbf{G} = SU(2)$, $g \geq 2$, or $\mathbf{G} = SU(n)$, $n \geq 3$, $g \geq 3$. Then the moduli space $\mathcal{M}_a$ is connected, simply connected, and $\operatorname{rank}\pi_2(\mathcal{M}_a) = r + 1$, where $r = \operatorname{rank}\pi_1(\mathbf{P}_a)$.*

Corollary 4.8. *The second cohomology of $\mathcal{M}_a$ is free of rank $r + 1$.*

Proof. Since $\mathcal{M}_a$ is simply connected, $H^2(\mathcal{M}_a, \mathbb{Z})$ is torsion free. The rest follows from Theorem 4.7 and the Hurewicz isomorphism. □

As in the case of SU(2), we would like to understand explicitly the generators of $H^2(\mathcal{M}_a, \mathbb{Z})$; hence, we need to understand the topology of the space $\mathcal{F}_a$. By the long exact sequence in homotopy associated to (2.6) we obtain

$$0 \longrightarrow \pi_2(\mathcal{F}_a) \xrightarrow{\pi_*} \pi_2(\mathcal{M}_a) \xrightarrow{\partial} \pi_1(\mathbf{P}_a/\mathbb{Z}_n) \longrightarrow \pi_1(\mathcal{F}_a) \longrightarrow \pi_1(\mathcal{M}_a) = 0\ . \tag{4.9}$$

Since $\operatorname{rank}\pi_2(\mathcal{M}_a) = \operatorname{rank}(\mathbf{P}_a) + 1$ and $\operatorname{rank}\pi_1(\mathbf{P}_a/\mathbb{Z}_n) = \operatorname{rank}\pi_1(\mathbf{P}_a)$, we clearly have $\operatorname{rank}\pi_2(\mathcal{F}_a) = 1$ and $\operatorname{rank}\pi_1(\mathcal{F}_a) = 0$. Let β be the image of a generator of Free $\pi_2(\mathcal{F}_a)$ in $\pi_2(\mathcal{M}_a) \simeq H_2(\mathcal{M}_a, \mathbb{Z})$ under the map π_*. Let $\gamma_1, \ldots, \gamma_r$ be the elements of Free $\pi_2(\mathcal{M}_a) \subset \pi_2(\mathcal{M}_a) \simeq H_2(\mathcal{M}_a, \mathbb{Z})$ such that their image under ∂ form a set of generators for Free $\pi_1(\mathbf{P}_a)$. Then we have shown

Corollary 4.10. *Assume $\mathbf{G} = SU(n)$, $n \geq 3$, $g \geq 3$. The free part of the group $H_2(\mathcal{M}_a, \mathbb{Z})$ is generated by the elements $\beta, \gamma_1, \ldots, \gamma_r$ as above, where $r = \operatorname{rank}\pi_1(\mathbf{P}_a)$.*

Remark 4.11. Analogous results to 4.7, 4.8, and 4.10 hold also for the spaces $\mathcal{F}_a/\mathbf{T}_a$, where $\mathbf{T}_a$ is a maximal torus in $\mathbf{P}_a$.

4.2. Integrality of the symplectic form

In this subsection, we shall consider the class of a certain closed two-form Ω on the spaces $\mathcal{F}_{\alpha,\delta}$, $\mathcal{M}_{\alpha,\delta}$. This is in preparation for the next section, where we shall construct a line bundle over $\mathcal{F}_{\alpha,\delta}$ with a connection whose curvature is $-i\Omega$. Whether the connection pushes down to $\mathcal{M}_{\alpha,\delta}$ amounts to the question of integrality of the class Ω on $\mathcal{M}_{\alpha,\delta}$.

On the space $\mathcal{A}_{\mathrm{F},\delta}$, defined as in Section 3.1 with respect to some base connection ∇_0 which at infinity is in temporal gauge, we define the two-form

$$\Omega_\nabla(\beta_1, \beta_2) = \frac{1}{2\pi}\int_\Sigma \operatorname{Tr}\left(\beta_1 \wedge \beta_2\right)\ .$$

This is well-defined, since by the inclusion $L^2_{1,\delta} \hookrightarrow L^2$ the integral is convergent. Ω is a closed two-form, and integration by parts shows that it is degenerate in the $\mathcal{G}_{0,\delta}$ directions. Hence, Ω descends to a closed form on $\mathcal{F}_{\alpha,\delta}$, and the normalization has been chosen such that $\Omega/2\pi$ is a representative of an integral class in $H^2(\mathcal{F}_{\alpha,\delta}, \mathbb{Z})$.

Recall that for $a \in \mathbf{G}$, $a = \exp(2\pi i\alpha)$, we denote by $\mathbf{P}_a$ the normalizer of a and by $\mathbf{T}_a$ some choice of maximal torus in $\mathbf{P}_a$. Suppose β is an arbitrary tangent vector to $\mathcal{F}_{\alpha,\delta}$ and $\phi_1, \phi_2 \in \mathrm{Lie}\,\mathbf{T}_a$. Then letting $\tilde{\phi}_1, \tilde{\phi}_2$ denote the unique harmonic extensions satisfying $r(\tilde{\phi}_i) = \phi_i$, $i = 1, 2$, we have

$$\begin{aligned}
\Omega_\nabla(\beta + \nabla\tilde{\phi}_1, \nabla\tilde{\phi}_2) &= \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\left((\beta + \nabla\tilde{\phi}_1) \wedge \nabla\tilde{\phi}_2\right) \\
&= -\frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\left(\nabla(\beta + \nabla\tilde{\phi}_1) \wedge \tilde{\phi}_2\right) + \frac{1}{2\pi}\int_{\partial\Sigma} \mathrm{Tr}\left(\nabla\tilde{\phi}_1 \cdot \tilde{\phi}_2\right) \\
&= \frac{1}{2\pi}\int_{\partial\Sigma} \mathrm{Tr}\left([A_0, \phi_1]\phi_2\right) = \frac{1}{2\pi}\int_{\partial\Sigma} \mathrm{Tr}\left(A_0[\phi_1, \phi_2]\right) = 0\ ,
\end{aligned}$$

since ϕ_1 and ϕ_2 commute. In the above computation, we have written $\nabla_0 = d + A_0$, and by the integral over $\partial\Sigma$ we mean the usual limiting procedure $\tau \to \infty$.

We conclude from the above that Ω descends as a closed two-form on $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$. We now have the following

Theorem 4.12. *For $k \in \mathbb{Z}$, the form $k\Omega/2\pi$ on $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$. defines an integral class if and only if $ik\alpha$ is in the coroot lattice. Thus, a must have the form $a = \mathrm{diag}(e^{2\pi i\lambda_1/k}, \ldots, e^{2\pi i\lambda_n/k})$, where $\lambda_1 + \cdots + \lambda_n = 0$ and all the λ_i's are integers.*

We should point out that this is the form of the holonomy matrix which is important in the geometric quantization of Chern-Simons theory (cf. [A]). We shall have more to say about the consequences of this in Section 6 and in [D-W1].

Proof. By the simple connectivity of $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ and the Hurewicz isomorphism, it suffices to evaluate $k\Omega$ on the generators $\beta, \gamma_1, \ldots, \gamma_r$ from Proposition 4.10 (cf. also Remark 4.11 and note that in this case, $r = n-1$). On β, $k\Omega/2\pi$ is integral by the choice of normalization. For the γ_i's, we use the following explicit realization: fix a winding number $\phi = \mathrm{diag}(\phi_1, \ldots, \phi_n)$, where ϕ_i's are all integers, and $i\phi$ is in the integer lattice of $\mathrm{Lie}\,\mathbf{T}_a$. Choose m, p integers, $1 \le p \le n$, and let ϵ_p denote the $n \times n$ matrix with a single entry 1 in the p-th row, p-th column. Set

$$f(t) = \exp\left\{2\pi it\phi + 2\pi itm\left(\frac{1}{n}\cdot I - \epsilon_p\right)\right\}\ .$$

This defines a map $[0, 1] \to \mathbf{T}_a$ which descends to a closed loop on $\mathbf{T}_a/\mathbb{Z}_n$, and every such loop can be obtained from loops of this form. Fix a point $[\nabla] \in \mathcal{F}_{\alpha,\delta}$, with representative $\nabla_0 + A$. We write $\nabla_0 = d + A_0$ and take $A \in L^2_{1,\delta}(\nabla_0)$. Let $A_s = A_0 + A$. Then $A_s^f = f^{-1}A_s f$ defines a closed loop in $\mathcal{F}_{\alpha,\delta}$ which projects to a the point $[\nabla] \in \mathcal{M}_a$. By definition of the boundary homomorphism in (4.9) we may find a disk D_{A_s} with $\partial D_{A_s} = \mathrm{im}\left(A_s^f\right)$. The projection of D_{A_s} in $\mathcal{M}_a$ is then a sphere and is the realization of the element of $\pi_2(\mathcal{M}_a)$ corresponding to ϕ. Let $\tilde{d}$ denote the trivial connection on the unit disk $D \subset \mathbb{C}$ and A the map $D \to D_{A_s}$. Then

$$\begin{aligned}\frac{k}{2\pi}\int_{D_{A_s}} \Omega &= \frac{k}{4\pi^2}\int_D\int_\Sigma \operatorname{Tr}\left(\tilde{d}A\wedge\tilde{d}A\right) = \frac{k}{8\pi^2}\int_\Sigma\int_{\partial D}\operatorname{Tr}\left(A\tilde{d}A\right)\\ &= \frac{k}{8\pi^2}\int_\Sigma\int_0^1 dt\,\operatorname{Tr}\left(f^{-1}A_s f[f^{-1}A_s f, f^{-1}f']\right)\\ &= \frac{k}{8\pi^2}\int_\Sigma\int_0^1 dt\,\operatorname{Tr}\left(f'f^{-1}[A_s, A_s]\right)\end{aligned}$$

Since A_s is flat, $[A_s, A_s] = -2d(A + A_0)$. Integrating by parts, and using the fact that $A_0 = i\alpha d\theta = i\,\operatorname{diag}(\alpha_1, \ldots, \alpha_n)d\theta$ at infinity,

$$\frac{k}{2\pi}\int_{D_{A_s}}\Omega = \frac{-k}{4\pi^2}\int_0^1 dt\int_{\partial\Sigma}\operatorname{Tr}\left(f'f^{-1}A_0\right) = k\left\{\sum_{i=1}^n \alpha_i\phi_i + \frac{m}{n}\sum_{i=1}^n \alpha_i - m\alpha_p\right\}$$

The middle term vanishes, since $i\alpha \in \mathfrak{g}$. By choosing different ϕ_i's, m, and p, we see that the expression above is an integer for all such choices only when $k\alpha_p \in \mathbb{Z}$ for $p = 1, \ldots, n$. This completes the proof of the theorem. □

5. Construction of the line bundle

We now proceed to define a line bundle with a connection 1-form over the spaces $\mathcal{F}_{\alpha,\delta}$ via a cocycle defined on the space of connections. The topological restrictions of Section 4.2 will prevent this line bundle from descending to $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$, $\mathbf{T}_a \subset \mathbf{P}_a$, except in certain cases. We make this precise in Theorem 5.8, which is the main result of this section.

5.1. The cocycle and connection

Fix a base connection $\nabla_0 = d + A_0$ where A_0 is in temporal gauge with associated holonomy matrix $i\alpha$. Let $\mathcal{G}_{0,\delta}$ be the gauge group defined as in Section 3.1. We define $\tilde{\mathcal{G}}_{0,\delta}$ to be the space of smooth paths γ in $\mathcal{G}_{0,\delta}$ which are the identity at an endpoint; hence we take $\gamma : [0, 1] \to \mathcal{G}_{0,\delta}$ with $\gamma(0) = I$. The evaluation map $e_1 : \tilde{\mathcal{G}}_{0,\delta} \to \mathcal{G}_{0,\delta}$ which takes γ to $\gamma(1)$ is smooth, and by Proposition 3.3 it is surjective. We define a map

$$C_{\nabla_0} : L^2_{1,\delta}(\mathfrak{g}_E) \times \tilde{\mathcal{G}}_{0,\delta} \longrightarrow \mathrm{U}(1) \tag{5.1}$$

$$C_{\nabla_0}(A,\tilde g)=\exp\left\{\frac{i}{4\pi}\int_\Sigma \mathrm{Tr}\left((A+A_0)dgg^{-1}\right)-\frac{i}{12\pi}\int_{\Sigma\times[0,1]}\mathrm{Tr}\left(\tilde d\tilde g\tilde g^{-1}\right)^3\right\}\ .$$

where $g=e_1(\tilde g)$ and $\tilde d=d+d/dt$. The terms in the exponential are well-defined: the first integral is convergent because A, dgg^{-1} are in L^1 and L^2, and A_0 is in L^∞; the second integral is convergent, since for each $t\in[0,1]$, $d\tilde g\tilde g^{-1}$ is in L^2, and $(d\tilde g/dt)\tilde g^{-1}$ is in L^∞.

The sum in the exponential of (5.1) is formally the integral of the Chern-Simons form over $\Sigma\times[0,1]$. As was shown in [R-S-W] for the case of SU(2) (see also [Do-Sa] and [Mick]), this expression gives rise to a cocycle in the case of smooth connections on the compact surface $\overline\Sigma$ defining a line bundle with a natural connection which pushes down to a holomorphic line bundle over the moduli space. The point of our construction, as will be shown, is that the same technique may be used for the moduli space of vector bundles with parabolic structure. We have the following

Lemma 5.2. *The $U(1)$-valued function defined in (5.1) depends only on $e_1(\tilde g)$ and therefore descends to a smooth map $\mathcal{A}_\delta\times\mathcal{G}_{0,\delta}\to U(1)$.*

Proof. Fix $\tilde g$, $e_1(\tilde g)=g$. Any other extension can be gotten from $\tilde g$ by multiplying by $h\in\tilde{\mathcal{G}}_{0,\delta}$ satisfying $e_1(h)=I$. Then

$$\begin{aligned}C_{\nabla_0}(A,\tilde gh)&=C_{\nabla_0}(A,\tilde g)\exp\left\{\frac{i}{12}\pi\int_{\Sigma\times[0,1]}\mathrm{Tr}\left(\tilde d(\tilde gh)h^{-1}\tilde g^{-1}\right)^3\right.\\&\qquad\left.-\frac{i}{12\pi}\int_{\Sigma\times[0,1]}\mathrm{Tr}\left(\tilde d\tilde g\tilde g^{-1}\right)^3\right\}\\&=C_{\nabla_0}(A,\tilde g)\exp\left\{\frac{i}{12\pi}\int_{\Sigma\times[0,1]}\mathrm{Tr}\left(\tilde dhh^{-1}\right)^3\right.\\&\qquad\left.+\frac{i}{4\pi}\int_{\Sigma\times[0,1]}\tilde d\,\mathrm{Tr}\left(\tilde d\tilde g\tilde g^{-1}\tilde dhh^{-1}\right)\right\}\end{aligned}$$

The second term in the exponential clearly vanishes upon integration by parts. By the embedding theorems, we know that h extends as a continuous map $\overline\Sigma\times S^1\to\mathbf{G}$. The first integral is $2\pi i\times$ the integral of the generator of $H^3(\mathbf{G},\mathbb{Z})$ over the image $h(\overline\Sigma\times S^1)$, and is therefore an integer (cf. [Mick]). The exponential is thus always one, and the lemma follows. □

By a straightforward computation along the lines indicated in the above lemma, we also have

Lemma 5.3. *The map $C_{\nabla_0}(A,g)$ satisfies the cocycle condition, i.e. for any $g,h\in\mathcal{G}_{0,\delta}$, $C_{\nabla_0}(A,gh)=C_{\nabla_0}(A^g,h)C_{\nabla_0}(A,g)$.*

Hence, we may use C_{∇_0} to define a twisted principal U(1) bundle $\mathcal{U}=\mathcal{A}_{\mathrm{F},\delta}\times_{\mathcal{G}_{0,\delta}}$ U(1) over $\mathcal{F}_{\alpha,\delta}=\mathcal{A}_{\mathrm{F},\delta}/\mathcal{G}_{0,\delta}$, where the map $\mathcal{U}\to\mathcal{F}_{\alpha,\delta}$ is just projection onto the first factor. We define the line bundle $\tilde{L}_\alpha \to \mathcal{F}_{\alpha,\delta}$ to be the one associated to $\mathcal{U}$ via the standard representation.

We now define a connection on $\tilde{L}_\alpha$. Let ω denote the one-form on $\mathcal{A}_\delta$ given by the expression $\omega_{\nabla_0+A}(\beta) = \frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\,((A+A_0)\wedge\beta)$. The integral makes sense, as before, by the inclusions $L^2_{1,\delta} \hookrightarrow L^1$, $L^2_{1,\delta} \hookrightarrow L^2$ and the fact that $A_0 \in L^\infty$. By adding the Maurer-Cartan form, ω trivially induces a connection $\hat{\omega}$ on $\mathcal{A}_{\mathrm{F},\delta} \times \mathrm{U}(1)$.

Lemma 5.4. *$\hat{\omega}$ descends to a connection on $\mathcal{U}$.*

Proof. We must show that $\hat{\omega}$ is invariant under the twisting by $\mathcal{G}_{0,\delta}$. It suffices to show that for vectors V vertical to the $\mathcal{G}_{0,\delta}$ action, $\hat{\omega}(V) = L_V\hat{\omega} = 0$. The vertical tangent space is generated by vectors of the form $(\nabla\phi, dC_{\nabla_0}(\nabla\phi,\phi))$, $\phi \in L^2_{1,\delta}$. A simple computation shows that $\hat{\omega}$ on the vector V associated to ϕ is

$$\begin{aligned}
\hat{\omega}(V) &= \omega(V) + d\log C_{\nabla_0}(\nabla\phi,\phi) \\
&= \frac{i}{4\pi}\int_\Sigma \mathrm{Tr}\,((A+A_0)\wedge d_{A+A_0}\phi) + \frac{i}{4\pi}\int_\Sigma \mathrm{Tr}\,((A+A_0)\wedge d\phi) \\
&= \frac{i}{2\pi}\int_\Sigma \mathrm{Tr}\,((A+A_0)\wedge d\phi) + \frac{i}{4\pi}\int_\Sigma \mathrm{Tr}\,(\phi[A+A_0, A+A_0])
\end{aligned}$$

Since $0 = d(A+A_0) + \frac{1}{2}[A+A_0, A+A_0]$ and ϕ is exponentially decaying, the above is

$$= \frac{i}{2\pi}\int_\Sigma \mathrm{Tr}\,(A+A_0\wedge d\phi) - \frac{i}{2\pi}\int_\Sigma \mathrm{Tr}\,(\phi d(A+A_0)) = 0\ .$$

upon integration by parts. From this, it follows that $L_V\hat{\omega} = i_V d\hat{\omega} = i_V\Omega = 0$ by that result of Section 4.2. This completes the proof of the lemma. □

We summarize the results of Lemmas 5.2 to 5.4 with the following

Proposition 5.5. *The trivial line bundle $\mathcal{A}_{\mathrm{F},\delta} \times \mathbb{C}$ with connection $\hat{\omega}$ pushes down to an hermitian line bundle $\tilde{L}_\alpha \to \mathcal{F}_{\alpha,\delta}$ and connection $\tilde{\nabla}$ with curvature $-i\Omega$.*

Actually, in what follows, we shall always be interested in an arbitrary power of $\tilde{L}_\alpha$, $\tilde{L}_\alpha^{\otimes k}$, where $k \in \mathbb{Z}$.

5.2. Pushing down the line bundle

In this subsection, we wish to go one step further; namely, we would like to obtain a line bundle with connection $L_\alpha^{\otimes k} \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ by pushing down our construction of the previous section via the action of the torus $\mathbf{T}_a$. Clearly, there is an obstruction to this, for if the bundle $\tilde{L}_\alpha^{\otimes k}$ with connection $\tilde{\nabla}$ pushed down, the curvature $-ik\Omega$ would have to be $2\pi i\times$ an integral class. By Theorem 4.12, we know that this occurs precisely when $ik\alpha$ is in the coroot lattice.

There is a natural action of $\mathbf{T}_a$ on sections of $\tilde{L}_\alpha^{\otimes k}$; a section is to be regarded as a complex valued function ψ on $\mathcal{A}_{\mathrm{F},\delta}$ satisfying the cocycle condition

$$\psi(A^g) = C_{\nabla_0}(A,g)^k \psi(A) \tag{5.6}$$

for $g \in \mathcal{G}_{0,\delta}$. If $t \in \mathbf{T}_a$, define $t \cdot \psi(A) = \psi(A^t)$. It is easily verified that $t \cdot \psi$ still satisfies (5.6), and hence is a section of $\tilde{L}_\alpha^{\otimes k}$.

Now consider a tangent vector $V = \nabla\phi$ at $\nabla \in \mathcal{A}_{\mathrm{F},\delta}$, vertical with respect to the $\mathbf{T}_a$ action. A simple computation shows that the connection 1-form evaluated on this vector is $-ik\omega_\nabla(V) = -k\,\mathrm{Tr}(\phi\alpha)$. Under the assumption that $ik\alpha$ is a coroot, this exponentiates to a representation $\lambda : \mathbf{T}_a \to \mathrm{U}(1)$. Note that λ is invariant under the center $\mathbb{Z}_n$ of $\mathbf{G}$ (see Theorem 6.1), and so it descends to a representation of $\mathbf{T}_a/\mathbb{Z}_n$. The sections of $\tilde{L}_\alpha^{\otimes k}$ which are λ-covariant with respect to the $\mathbf{T}_a/\mathbb{Z}_n$ action, i.e. those which satisfy $t \cdot \psi = \lambda(t)\psi$, then automatically satisfy $\tilde{\nabla}_V \psi = 0$ for any vertical vector V. Conversely, any section satisfying this condition is λ-covariant.

We therefore define a line bundle $L_\alpha^{\otimes k} \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ by declaring its sheaf of sections to be

$$\Gamma\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right) = \Gamma\left(\mathcal{F}_{\alpha,\delta}, \tilde{L}_\alpha^{\otimes k}\right)^\lambda , \tag{5.7}$$

where the right hand side denotes the λ-covariant smooth sections of $\tilde{L}_\alpha^{\otimes k}$ (cf. [Ax]). The connection ∇ on $L_\alpha^{\otimes k}$ may be defined from the connection $\tilde{\nabla}$ on $\tilde{L}_\alpha^{\otimes k}$ by the equation $\nabla_X \psi = \tilde{\nabla}_{\tilde{X}} \psi$, where $\tilde{X}$ is any vector which projects to X via the map $\mathcal{F}_{\alpha,\delta} \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$. The λ-covariance guarantees that this is well-defined, and it is easy to see that ∇ is a connection on $L_\alpha^{\otimes k}$ with curvature $-ik\Omega$. To summarize, we have the

Theorem 5.8. *Suppose $ik\alpha$ is in the coroot lattice of $\mathbf{G}$ and λ is the associated character $\mathbf{T}_a \to U(1)$. There exists an hermitian line bundle $L_\alpha^{\otimes k} \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ with connection ∇ and curvature $-ik\Omega$ whose global smooth sections are given by (5.7), where the superscript λ denotes the subspace of sections satisfying $t \cdot \psi = \lambda(t)\psi$ for all $t \in \mathbf{T}_a$.*

6. A problem in geometric quantization

In this final section, we would like to address the issue of holomorphic structure on the line bundles we have constructed. We shall show that in the cases where the line bundles descend to the space $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$, the space of holomorphic sections form the multiplicity space of a problem in geometric quantization, at least in the case $\mathbf{G} = \mathrm{SU}(2)$. The multiplicities are associated to some representation λ of the group $\mathbf{G}$. The holonomy matrices $i\alpha \in \mathfrak{g}$ correspond to the λ's in a one to one way. We begin by making this relationship between holonomies and representations explicit.

Theorem 6.1. *Fix $k \in \mathbb{Z}$ and a maximal torus $\mathbf{T} \subset \mathbf{G}$. Then there is a one-to-one correspondence between characters $\lambda : \mathbf{T} \to U(1)$ which are invariant under the center and elements $i\alpha \in \operatorname{Lie}\mathbf{T}$ such that $k\Omega/2\pi$ defines an integral class on $\mathcal{F}_{\alpha,\delta}/\mathbf{T}$.*

Proof. By Theorem 4.12, $k\Omega/2\pi$ is an integral class on $\mathcal{F}_{\alpha,\delta}/\mathbf{T}$ if and only if $k\alpha = \lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where the λ_i's are integers. The coroot $i\lambda$ induces a character of $\mathbf{T}$ as follows: an arbitrary element $t \in \mathbf{T}$ may be written $t = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$, $\sum \theta_j = 0$. Hence we define $\lambda(t) = \exp\left(i \sum \theta_j \lambda_j\right)$. The θ_j's are determined only modulo 2π, but that does not change the definition of $\lambda(t)$, since the λ_j's are integers. It is also easy to verify that the condition $\sum \lambda_j = 0$ ensures that for t in the center of $\mathbf{G}$, $\lambda(t) = 1$.

Conversely, suppose λ is a character of $\mathbf{T}$. Write λ as

$$t = \operatorname{diag}(t_1, \ldots, t_n) \longrightarrow \lambda(t) = t_1^{\lambda_1} \cdots t_n^{\lambda_n} ,$$

where the λ_i's are integers. The multi-index $(\lambda_1, \ldots, \lambda_n)$ is only determined up to addition of multiples of $(1, \ldots, 1)$. If we require λ to be invariant under the center, we have $\sum \lambda_j = 0 \mod n$. Therefore, the index $(\lambda_1, \ldots, \lambda_n)$ is completely determined by the requirement $\sum \lambda_j = 0$. Now the matrix $i\alpha = \operatorname{diag}(i\lambda_1/k, \ldots, i\lambda_n/k)$ is an element of $\operatorname{Lie}\mathbf{T}$, and by Theorem 4.12, $k\Omega/2\pi$ defines an integral class on $\mathcal{F}_{\alpha,\delta}/\mathbf{T}$. This completes the proof of Theorem 6.1. □

Example 6.2. For the sake of clarity, let us dwell for a moment on the case $\mathbf{G} = \mathrm{SU}(2)$. A representation $\lambda : \mathbf{T} \to \mathrm{U}(1)$ is characterized by a half-integer $j \in \frac{1}{2}\mathbb{Z}$, the *spin* of the representation, by writing $\lambda_1 - \lambda_2 = 2j$. Then as above, if $t = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$, $\lambda(t) = \exp\left(2i(j\theta)\right)$. Requiring λ to be invariant under the center means that j must in fact be an integer. If we take $\lambda_1 + \lambda_2 = 0$, then $\lambda_1 = j$, $\lambda_2 = -j$, and the corresponding holonomy matrix is $\alpha = \operatorname{diag}(j/k, -j/k)$. We wish to emphasize that $k\alpha$ corresponds to the spin j of the representation; the spin is required to be integral to guarantee invariance under the center.

In the case where $k\Omega/2\pi$ is an integral class in $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$, we have constructed in Theorem 5.8 a line bundle $L_\alpha^{\otimes k} \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ with connection and curvature $-ik\Omega$. By Theorems 3.7 and 3.13, it is easy to see that Ω is of type (1,1). The line bundle $L_\alpha^{\otimes k}$ therefore inherits a holomorphic structure compatible with the connection. We wish to identify the space $H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right)$ of global holomorphic sections of $L_\alpha^{\otimes k}$.

At this point, we restrict ourselves to the case $\mathbf{G} = \mathrm{SU}(2)$. The reason for this is that we have need of the correspondences between moduli spaces discussed in Section 2.3, and these were only defined for rank two. Therefore, *for the rest of the paper* we let $\mathbf{G} = \mathrm{SU}(2)$. In addition, so that the maps we use are defined off a set of high codimension, we assume $g > 3$.

Let us recall the construction of Section 2.3. We let $\overline{\mathcal{A}}_s$ denote the smooth stable connections on a trivial $\mathbf{G}$-bundle over $\overline{\Sigma}$ and $\overline{\mathcal{G}}^{\mathbb{C}}$ the group of complex automorphisms. Choose a non-negative integer λ. Regarding λ as the spin of a representation, we have the associated representation space V_λ of dimension $2\lambda+1$.

As in Example 6.2, we may also regard λ as a character $\lambda : \mathbf{T}_a \to \mathrm{U}(1)$. Then we have the natural holomorphic bundles, $\Delta^{\otimes k}$ and B_λ over $\overline{\mathcal{A}}_s \times \mathbf{G}/\mathbf{T}_a$; Δ is the trivial extension of the determinant bundle over $\overline{\mathcal{A}}_s$, and B_λ is the trivial extension of the Borel-Weil-Bott bundle over $\mathbf{G}/\mathbf{T}_a$ determined by the character λ. Using a computation similar to that of Theorem 4.12, we find $c_1(B_\lambda) = 2\lambda$, and V_λ may be identified with $H^0(\mathbf{G}/\mathbf{T}_a, B_\lambda)$ via the Borel-Weil-Bott theorem (cf. [G-S2]). If λ is invariant under the center $\mathbb{Z}_2$ of $\mathbf{G}$, then the center acts trivially on the total space of the line bundle $\Delta^{\otimes k} \otimes B_\lambda \to \overline{\mathcal{A}}_s \times \mathbf{G}/\mathbf{T}_a$, and we may take the quotient by $\overline{\mathcal{G}}^{\mathbb{C}}$ to obtain a holomorphic bundle $S_\lambda \to \mathcal{N}_a$. Let α be the matrix corresponding to λ as in Theorem 6.1. In the case where $a = \exp(2\pi i\alpha) \neq \pm I$, we have by Theorem 2.18 a holomorphic embedding $\varphi : \mathcal{N}_a \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ with an inverse ψ on the image. This gives rise to

Proposition 6.3. *For $a \neq \pm I$, the bundles $\psi^* S_\lambda$ and $L_\alpha^{\otimes k}$ over the image of φ have the same first Chern class.*

Proof. As in Proposition 4.2, let us denote the generators of $H^2(\mathcal{N}_a, \mathbb{Z}) \simeq \pi_2(\mathcal{N}_a)$ by $\overline{\beta}$, $\overline{\gamma}$. The natural symplectic form $\overline{\Omega}$ on $\overline{\mathcal{A}}_s$ descends to a representative of the class $\overline{\beta}$. Using, for example, the Quillen connection on $\Delta^{\otimes k}$, one has $c_1(\Delta^{\otimes k}) = ik\overline{\Omega}/2\pi$ (cf. [Q], [Don1]). On the other hand, the Chern class of B_λ is 2λ, and because of the torsion class in $\pi_2(\mathcal{M}(2,0))$, we have $c_1(S_\lambda) = k\overline{\beta} + \lambda\overline{\gamma}$ (see the discussion preceding Proposition 4.2). The result now follows from Proposition 4.2 and (the proof of) Theorem 4.12. □

Proposition 6.4. *Let α correspond to λ as in Theorem 6.1. Let $\mathbf{T}_a$ be a maximal torus containing* $\exp(2\pi i\alpha)$ *and a point $a \neq \pm I$. Then there is an isomorphism $H^0(\mathcal{N}_a, S_\lambda) \simeq H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right)$.*

Proof. In the case $\exp(2\pi i\alpha) \neq \pm I$, we may take $a = \exp(2\pi i\alpha)$. Then the manifold $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ is compact and simply connected, so by the exponential sequence, bundles are classified by their first Chern classes. By Theorem 2.18 and Proposition 6.3, $\psi^* S_\lambda$ and $L_\alpha^{\otimes k}$ agree outside a set of codimension ≥ 2, and so their extensions are holomorphically equivalent. Now, since ψ is a biholomorphism outside a set of codimension ≥ 2, the sections of S_λ are in one-to-one correspondence with sections of $\psi^* S_\lambda$. Notice that the above argument shows that line bundles on $\mathcal{N}_a$ are in fact classified by their first Chern classes. In the case $\exp(2\pi i\alpha) = I$, we have from Proposition 2.17 and Theorem 3.8 that $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ is biholomorphic to $\mathcal{N}_a$, hence the Proposition holds in this case again by a Chern class computation. In the case $\exp(2\pi i\alpha) = -I$, we argue as follows: by the theorem of Narasimhan-Seshadri, $\mathcal{M}(2,-1)$ is naturally diffeomorphic to $\mathcal{F}_{-I}/\mathrm{SO}(3)$ (cf. [A-B], Section 6). By Theorem 3.8, $\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ is then diffeomorphic to a $\mathbb{P}^1$-bundle over $\mathcal{M}(2,-1)$ and thus inherits a natural complex structure. The construction of Section 2.3 produces a holomorphic map $\varphi : \mathcal{N}_a \to \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a$ as before, and the Chern class computation above proves the Proposition in this case as well. □

We can now easily identify the space of global holomorphic sections of our bundle $L_\alpha^{\otimes k}$ in terms of the sections of $\Delta^{\otimes k}$ and the character λ. Let $i\alpha \in \mathfrak{g}$ be the holonomy matrix corresponding to λ via Theorem 6.1. We shall denote by $H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right)$ the infinite dimensional space of global holomorphic sections of $\Delta^{\otimes k}$ over $\overline{\mathcal{A}}_s$. Following Guillemin and Sternberg [G-S1], we define the λ-*covariant subspace* of $H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right)$, $\mathcal{V}_\lambda = \mathrm{Hom}_{\overline{\mathcal{G}}^{\mathbb{C}}}\left(V_\lambda^*, H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right)\right)$ as follows: first, notice that the gauge group $\overline{\mathcal{G}}^{\mathbb{C}}$ acts on V_λ by evaluating the gauge transformation at the puncture p, and on $H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right)$ by its standard action (cf. [Q], [Don1]). Then $\mathcal{V}_\lambda$ consists of homomorphisms that intertwine the $\overline{\mathcal{G}}^{\mathbb{C}}$ action. Thus, $\mathcal{V}_\lambda$ is equivalent to the $\overline{\mathcal{G}}^{\mathbb{C}}$ invariant elements of $H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right) \otimes V_\lambda$, and so

$$\begin{aligned}\mathcal{V}_\lambda &= \left\{H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right) \otimes V_\lambda\right\}^{\overline{\mathcal{G}}^{\mathbb{C}}} = \left\{H^0\left(\overline{\mathcal{A}}_s, \Delta^{\otimes k}\right) \otimes H^0\left(\mathbf{G}/\mathbf{T}_a, B_\lambda\right)\right\}^{\overline{\mathcal{G}}^{\mathbb{C}}} \\ &= H^0\left(\overline{\mathcal{A}}_s \times \mathbf{G}/\mathbf{T}_a, \Delta^{\otimes k} \otimes B_\lambda\right)^{\overline{\mathcal{G}}^{\mathbb{C}}} = H^0\left(\mathcal{N}_a, S_\lambda\right) \simeq H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right) ,\end{aligned}$$

by Proposition 6.4. We can now state our main result as follows:

Theorem 6.6. *For $\mathbf{G} = SU(2)$ and $g > 3$, let $L_\alpha^{\otimes k}$ be the holomorphic line bundle constructed in Section 5, λ the character corresponding to α via Theorem 6.1. Then $\mathcal{V}_\lambda = H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right)$.*

We conclude with a few remarks. First, our λ-covariant subspace $\mathcal{V}_\lambda$ is essentially Segal's definition of the space of states (or conformal blocks) associated to a Riemann surface with boundary labeled by λ [Ox]. Therefore, our Theorem 6.6 exhibits a quantum line bundle on the moduli space of vector bundles with parabolic structure constructed via the Chern-Simons functional and whose space of holomorphic sections reproduces Segal's space. Notice also that the restriction that λ be invariant under the center is no restriction at all: since $-I$ acts trivially on the determinant bundle Δ, $\mathcal{V}_\lambda$ vanishes identically if λ is not invariant. Second, it should not be difficult to verify that all of the higher cohomology of $L_\alpha^{\otimes k}$ vanishes and that therefore $\dim H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right)$ is independent of the underlying complex structure. This would involve computing the canonical bundle and using a vanishing theorem as in [H]. Third, it would be interesting to generalize Theorem 6.6 to higher rank and low genus. For this it is necessary to construct the correct correspondence variety as in Section 2.3. Finally, we note that $H^0\left(\mathcal{F}_{\alpha,\delta}/\mathbf{T}_a, L_\alpha^{\otimes k}\right)$ may naturally be identified with holomorphic sections of a vector bundle over the moduli space $\mathcal{M}(2,0)$. This is seen by pushing forward $L_\alpha^{\otimes k}$ via the fibration

$$\begin{array}{ccc}\mathbf{G}/\mathbf{T}_a & \longrightarrow & \mathcal{F}_{\alpha,\delta}/\mathbf{T}_a \\ & & \downarrow \\ & & \mathcal{M}(2,0)\end{array}$$

from Proposition 2.15. We shall discuss the implications of this point of view in detail in [D-W1].

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