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Local degeneration of the moduli space of vector bundles and factorization of rank two theta functions. I

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Contents

1 Introduction

In this paper we study the behavior of the moduli space of rank two vector bundles on a compact Riemann surface and its associated "theta functions" as the surface

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degenerates to a reducible curve. More precisely, let Σ_t be an analytic family of compact Riemann surfaces degenerating as $t \to 0$ to a reducible curve Σ_0 with a single *separating* node p_0. Let $\bar{\Sigma}^+ \cup \bar{\Sigma}^-$ denote the normalization of Σ_0, and let $p_0^\pm \in \bar{\Sigma}^\pm$ denote the pre-image of p_0 under the map $\bar{\Sigma}^+ \cup \bar{\Sigma}^- \to \Sigma_0$. Our goal is to "factorize" the moduli space of rank two vector bundles on Σ_t and the space of holomorphic sections of a line bundle on the moduli space and express them in terms of the corresponding objects on $\bar{\Sigma}^\pm$. Let us introduce our theorem by first illustrating what is meant by factorization in a simpler situation.

Consider the case of line bundles of fixed degree, say zero. In this case, the moduli space is a complex torus, $J(\Sigma_t)$, the Jacobian variety of Σ_t. The behavior of $J(\Sigma_t)$ and the theta divisor Θ_t as the surface Σ_t degenerates to Σ_0 may be understood by first studying the abelian differentials on Σ_t (cf. [Fay, Y]). One finds, for example, that the period matrix associated to Σ_t becomes block diagonal over the fiber $t = 0$. The Jacobian therefore degenerates to a product torus $J(\bar{\Sigma}^+) \times J(\bar{\Sigma}^-)$. The analytic family of theta divisors Θ_t has irreducible fibers, except at $t = 0$ where the components are $\Theta^+ \times J(\bar{\Sigma}^-)$ and $J(\bar{\Sigma}^+) \times \Theta^-$. Hence, the space of holomorphic sections of the k-th power of Θ_t may be seen to degenerate to the product of sections of $[\Theta^+]^{\otimes k}$ and $[\Theta^-]^{\otimes k}$. The dimensions of these spaces factorize: $k^g = k^{g_+ + g_-} = k^{g_+} \cdot k^{g_-}$.

We shall prove an analogue of the above result for holomorphic sections of the determinant line bundle Δ over the moduli space $\mathcal{M}$ of rank two semi-stable vector bundles on Σ with fixed trivial determinant. The space $\mathcal{M}$ is the same, by the theorem of Narasimhan and Seshadri, as the space of equivalence classes of flat SU(2) connections on Σ. We call holomorphic sections of Δ^k *rank two theta functions of level k*.

A fundamental difference in this type of factorization for rank two as opposed to rank one theta functions is the appearance of representations of the group SU(2) (cf. [Ox]). Consider the compact surfaces $\bar{\Sigma}^\pm$ with marked points $p_0^\pm$ identified in Σ_0 as a node as before. If $\mathscr{A}_s^\pm$ denotes the infinite dimensional space of stable holomorphic structures on a trivial rank two bundle $\bar{E}^\pm \to \bar{\Sigma}^\pm$ and $\mathscr{G}_\pm^{\mathbb{C}}$ the complex automorphisms, then $\mathscr{G}_\pm^{\mathbb{C}}$ acts on the determinant bundles $\Delta_\pm \to \mathscr{A}_s^\pm$, and the theta functions may be described by the invariant sections $H^0(\mathscr{A}_s^\pm, \Delta_\pm^k)^{\mathscr{G}_\pm^{\mathbb{C}}}$. They are holomorphic sections of a corresponding bundle on the quotient $\mathscr{A}_s^\pm/\mathscr{G}_\pm^{\mathbb{C}}$. By a theorem of Drezet and Narasimhan [Dr-N], these extend uniquely as sections of an invertible sheaf on the compactification of $\mathscr{A}_s^\pm/\mathscr{G}_\pm^{\mathbb{C}}$.

Given an irreducible representation V_λ of SU(2) of dimension $2\lambda + 1$ where λ is a half-integer, we may let $\mathscr{G}_\pm^{\mathbb{C}}$ act on V_λ by evaluation at $p_0^\pm$. We then form the multiplicity space

$$\mathscr{V}_\lambda(\bar{\Sigma}^\pm) = \mathrm{Hom}_{\mathscr{G}_\pm^{\mathbb{C}}}(V_\lambda, H^0(\mathscr{A}_s^\pm, \Delta_\pm^k)) \tag{1.1}$$

where the homomorphisms intertwine the $\mathscr{G}_\pm^{\mathbb{C}}$ action. It was shown in [D-W1] that $\mathscr{V}_\lambda(\bar{\Sigma}^\pm)$ may be identified with holomorphic sections of a particular line bundle $\mathscr{L}_\pm(k, \lambda)$ over the moduli space of parabolic bundles on $\Sigma^\pm$. Note that since the center $\mathbb{Z}_2 \subset \mathscr{G}_\pm^{\mathbb{C}}$ acts trivially on $\mathscr{A}_s^\pm$ and $\Delta_\pm^k$, $\mathscr{V}_\lambda(\bar{\Sigma}^\pm)$ vanishes trivially unless λ is an integer. In [D-W2], we showed that $\mathscr{V}_\lambda(\bar{\Sigma}^\pm)$ also vanishes for $\lambda > k/2$. Strictly speaking, in the latter reference we needed the technical assumption that the genera of $\bar{\Sigma}^\pm$ be > 3.* This requirement is due to the existence of singularities in some of the moduli spaces involved. We are now prepared to state the

* Note added: This assumption will be removed in part II of this paper.

Main Theorem. *Suppose $\bar{\Sigma}^{\pm}$ have genera greater than three. Then the degeneration $\Sigma_t \to \Sigma_0$ defines an isomorphism of vector spaces*

$$H^0(\mathscr{M}, \Delta^k) \simeq \bigoplus_{0 \leq \lambda \leq k/2} \mathscr{V}_\lambda(\bar{\Sigma}^+) \otimes \mathscr{V}_\lambda(\bar{\Sigma}^-) . \tag{1.2}$$

The isomorphism is given explicitly by the composition of the maps (6.15), (6.16), and (6.17). It is also worth mentioning that the higher cohomology of Δ^k for positive k can be shown to vanish; we prove the same result for the bundles $\mathscr{L}_\pm(k, \lambda)$ for $0 \leq \lambda \leq k/2$ (see Theorem 6.9). Thus the Main Theorem is equivalent to the following equality between Euler characteristics of holomorphic line bundles:

$$\chi(\Delta^k) = \sum_{0 \leq \lambda \leq k/2} \chi(\mathscr{L}_+(k, \lambda))\chi(\mathscr{L}_-(k, \lambda)) . \tag{1.3}$$

The method of proof of the Main Theorem is conceptually quite simple. We first restrict the holomorphic sections of Δ^k to certain real (smooth, compact) hypersurfaces $\mathscr{M}_a$ in the moduli space defined by fixing the conjugacy class of the holonomy around the separating cycle. We show that the restriction map induces an isomorphism between $H^0(\mathscr{M}, \Delta^k)$ and $H^0_{\mathrm{CR}}(\mathscr{M}_a, \Delta^k)$, the space of global sections of Δ^k on $\mathscr{M}_a$ satisfying the tangential Cauchy-Riemann equations (CR-*sections*). By degenerating the complex structure on $\mathscr{M}$, we construct a limiting CR-*structure* on $\mathscr{M}_a$, and a limiting CR-line bundle for Δ^k. Then by a semi-continuity theorem for H^0_{CR} analogous to that for ordinary Dolbeault cohomology we show that the dimension of $H^0_{\mathrm{CR}}(\mathscr{M}_a, \Delta^k)$ does not jump at the limit. There we identify the space with the direct sum (1.2), and this establishes the Main Theorem.

It may at first seem unnatural to restrict to one hypersurface $\mathscr{M}_a$. The necessity for doing so stems from two facts: First, the construction of a line bundle over the moduli space of parabolic bundles given in [D-W1] cannot be extended simultaneously for all holonomies; this is due to a simple topological obstruction (see [Fr] for a discussion of this point). Second, it is not clear how to produce an analytic family of complex structures on the whole moduli space $\mathscr{M}$ which extends past $t = 0$. Our construction does this provided the set of points corresponding to connections with central holonomy around the separating cycle is discarded. This set has real codimension three and so does not affect the holomorphic sections. However, because the remaining manifold is open one does not à priori have the semi-continuity results alluded to above.

The purpose of this paper is two-fold: The first is to describe the degeneration of the moduli space of stable bundles, at least locally, as we approach a nodal curve. Our method is similar to the "conic degeneration" of the $\bar{\partial}$-operator considered by Seeley and Singer (cf. [S-S, S, B-S]). The second, motivated by the Main Theorem, is to analyze the complex geometry of the real hypersurfaces $\mathscr{M}_a$ and their associated boundary cohomology groups $H^{p,q}_b = H^{p,q}_b(\mathscr{M}_a, \Delta^k)$.

We present the details as follows: In Sect. 2 it is shown how the based moduli spaces fiber. We give explicit descriptions of the tangent spaces involved in terms of solutions to the *Neumann boundary value problem.* In order to lift holomorphic vector fields, we define connections on the based moduli spaces and investigate the signs of their curvatures in certain canonically defined directions. In Sect. 3 we describe the local degeneration of the moduli spaces of stable bundles. We prove a result similar to that describing the aforementioned behavior of holomorphic differentials near a nodal curve – in the present case, the differentials are coupled to the adjoint bundle $\mathfrak{g}_E$ associated to the underlying holomorphic bundle E. We define an operator D_0 acting on an appropriate Sobolev completion of the smooth sections of $T^*\Sigma_0 \otimes \mathfrak{g}_E$. The kernel of D_0 may be identified with meromorphic 1-forms with simple poles at $p_0^\pm$ and residues lower (resp. upper) triangular with respect to a fixed identification of the fiber $(\mathfrak{g}_E)_{p_0^\pm}$. The diagonal parts of the residues on each component $\bar{\Sigma}^\pm$ are required to be opposite one another (see Proposition 3.12). The tangent space to the moduli space $\mathscr{M}(t)$ (with the induced complex structure from Σ_t) is described by the kernel of the corresponding operator D_t acting on sections of $T^*\Sigma_t \otimes \mathfrak{g}_{E_t}$. The "local degeneration" of the moduli space is then given by the following

(1.4) **Theorem** (see Proposition 3.23, Corollary 3.24, and Proposition 4.13). *The family $\{D_t\}$ of unbounded Fredholm operators is continuous $D_t \to D_0$ in the graph norm.*

Instead of degenerating the moduli space associated to the closed surface Σ_t, one can first restrict to the components $\Sigma_t^\pm$ with boundary and consider the Neumann problem mentioned above. The "local factorization" of the moduli space is the statement that either degenerating the closed surface, or first restricting and then degenerating, are compatible operations. In other words, the solutions to the Neumann problem converge in the kernel of D_0 as well (see Corollary 3.21). The analysis relies on explicit constructions of parametrices for the various operators involved.

In Sect. 4, we restrict our attention to the real hypersurfaces $\mathscr{M}_a$ of $\mathscr{M}$ consisting of equivalence classes of connections with holonomy around the separating cycle conjugate to $a \in \mathrm{SU}(2)\setminus\{\pm I\}$. The complex structure on $\mathscr{M}(t)$ induces a CR-structure on $\mathscr{M}_a(t)$. We show that as $t \to 0$, the family $\mathscr{M}_a(t)$ has a well-defined limit. Precisely,

(1.5) **Theorem** (see Theorem 4.11). *There is a limiting* CR-*structure $\mathscr{M}_a(0)$ on $\mathscr{M}_a$ such that the $\mathscr{M}_a(t)$ form a differentiable family $\mathfrak{M}_a$ of* CR-*manifolds parameterized by the disk.*

Given a CR-manifold, the *Levi form* is a measure of its holomorphic convexity. For the family $\mathfrak{M}_a$ we prove

(1.6) **Theorem** (see Theorem 4.19 and Corollary 4.20). *For sufficiently small choice of the degeneration parameter t, the Levi form of $\mathscr{M}_a(t)$ has everywhere at least two positive and two negative eigenvalues.*

Theorem 1.6 is proven by relating the Levi form of the limiting CR-structure to the curvature of the universal bundle over the Hecke correspondence and using the results of [D-W2]. The indefiniteness of the Levi form, combined with a Hartog's type argument due to Lewy, Andreotti and Grauert, imply the following extension theorem.

(1.7) **Theorem** (see Theorem 4.25). *For a sufficiently small choice of the degeneration parameter t, the restriction map*

$$\rho: H^0(\mathscr{M}(t), \Delta^k) \to H^0_{\mathrm{CR}}(\mathscr{M}_a(t), \Delta^k)$$

is an isomorphism.

Theorem 1.7 reduces the Main Theorem to the problem of identifying $H^0_{CR}(\mathcal{M}_a, \Delta^k)$ in terms of the $\mathcal{V}_\lambda^\dagger$'s. We do this by letting the degeneration parameter $t \to 0$. We first construct a family L^k of CR-line bundles on $\mathfrak{M}_a$. Explicitly, for $t \neq 0$ we set $L(t)^k$ to be the restriction of the determinant line bundle Δ^k on $\mathcal{M}_a(t)$ as a hermitian bundle with connection constructed as in [R-S-W], whereas for $t = 0$ we set $L(0)^k$ to be the hermitian line bundle with connection constructed in [D-W1]. Then we prove in Theorem 6.6 that L^k is a differentiable family of CR-line bundles on $\mathfrak{M}_a$.

By the analogue of semi-continuity for Dolbeault cohomology, which holds also for $\bar{\partial}_b$ cohomology under the convexity properties of $\mathcal{M}_a$ described in Theorem 1.6 (see Theorem 5.8), in order to show that the dimension of $H^0_{CR}(\mathcal{M}_a(t), L(t)^k)$ is constant near $t = 0$, it suffices to show $H_b^{0,1}(\mathcal{M}_a(t), L(t)^k)$ vanishes at $t = 0$. We prove this by first realizing $\mathcal{M}_a(0)$ (with the limiting CR-structure) as the boundary of a compact complex manifold $D(\mathcal{M}_a)$. Indeed, the manifold $D(\mathcal{M}_a)$ is nothing but the holomorphic disk bundle associated to the circle bundle $\mathcal{M}_a \to \mathcal{M}_a^+ \times \mathcal{M}_a^-$, where $\mathcal{M}_a^\pm$ are the moduli spaces of parabolic bundles on $\hat{\Sigma}^\pm$ with weight a. The line bundle $L(0)^k$ extends as a holomorphic bundle on $D(\mathcal{M}_a)$. We then relate the boundary cohomology of $\mathcal{M}_a(0)$ to the absolute cohomology of $D(\mathcal{M}_a)$. We prove

(1.8) **Theorem** (see Theorem 5.11 and Corollary 6.12). *For* $k \geqq 0$,

$$H_b^{0,1}(\mathcal{M}_a(0), L(0)^k) = 0 .$$

The proof of the Main Theorem then follows by decomposing the CR-sections on $\mathcal{M}_a(0)$ in terms of irreducible representations of the circle action and using the results of [D-W1] and [D-W2].

In the past few years a great number of papers on this subject have appeared, all of them treating the problem from a different point of view. We refer the reader to the following articles: [Be, Be-Sz, C, Don, Fa, J-W, N-R, Sz, Th, T-U-Y, Wi, Z]. This list is most probably not complete, and apologies are made for any omissions. For clarity of exposition, we have restricted ourselves in this paper to the simple case of degeneration of a compact surface by pinching a single separating cycle. The more general case will be treated in the sequel.

2 Gluing flat connections

The purpose of this section is to give a description of the gluing of moduli spaces of flat vector bundles. In Subsect. 2.1 we describe the gluing via a fiber product on the level of based moduli spaces (Proposition 2.5 and Corollary 2.6). In Subsect. 2.2 we explain an alternative description of the gluing in terms of the Neumann problem on Riemann surfaces with boundary. Such a description is necessary in order to understand the degeneration of the moduli spaces treated in Sect. 3. Finally, in order to be able to lift vector fields to the based moduli spaces we need to introduce certain connections on the fiber products. This is accomplished via the universal connections of Atiyah and Singer. In Subsect. 2.3 we outline the construction of the connections and investigate the sign of their curvatures.

2.1 Fibrations of moduli spaces

Let $\bar{\Sigma}^+$ and $\bar{\Sigma}^-$ be compact Riemann surfaces of genera g_+, g_- with fixed coordinate disks $(\tilde{D}_+, z_+)$ and $(\tilde{D}_-, z_-)$, respectively. Thus

$$z_\pm : \tilde{D}_\pm \to D$$

are complex analytic isomorphisms with the unit disk $D \subset \mathbb{C}$. Given a complex number t in the punctured disk D^*, we define the compact Riemann surface Σ_t of genus $g = g_+ + g_-$ by gluing $\bar{\Sigma}^+ \setminus \{|z_+| < |t|\}$ and $\bar{\Sigma}^- \setminus \{|z_-| < |t|\}$ via the equation $z_+ z_- = t$. Let $\varepsilon = |t|^{1/2}$, and let C_ε denote the circle in Σ_t defined by the equation $|z_\pm| = \varepsilon$. Then C_ε is a separating cycle, i.e. $\Sigma_t \setminus C_\varepsilon$ consists of two components. We shall refer to the neighborhood $|t| < |z_\pm| < 1$ of C_ε in Σ_t as the *pinching region*. The closure of each component is a Riemann surface with boundary which we shall denote by Σ_t^+ and Σ_t^-, respectively. Clearly,

$$\Sigma_t = \Sigma_t^+ \bigcup_{C_\varepsilon} \Sigma_t^- .$$

Let $C_\varepsilon^\pm = \partial(\Sigma_\varepsilon^\pm)$ with the induced orientations. We give C_ε the orientation from C_ε^+, which is of course opposite to the orientation of C_ε^-. Finally, we choose a base point $x_{0,\varepsilon}$ on $C_\varepsilon \subset \Sigma_t$. Let $p_0^\pm = z_\pm^{-1}(0)$ and denote $\Sigma^\pm = \bar{\Sigma}^\pm \setminus \{p_0^\pm\}$, $\Sigma_0 = \Sigma^+ \cup \Sigma^-$. We also fix a smooth family of diffeomorphisms

$$\chi_t : \Sigma_0 \to \Sigma_t \setminus \{C_\varepsilon\} , \tag{2.1}$$

satisfying $\chi_0 = \mathrm{id}$. Furthermore, we require $\chi_t = \mathrm{id}$ *outside* the pinching region for all t. We shall denote by $\chi_t^\pm$ the restriction of χ_t to $\Sigma_t^\pm$.

For the rest of Subsect. 2.1 fix $t \neq 0$ and set

$$\Sigma = \Sigma_t, \quad \Sigma^\pm = \Sigma_t^\pm, \quad C = C_\varepsilon, \quad x_0 = x_{0,\varepsilon}, \quad \text{and} \quad \chi = \chi_t .$$

Since we are not interested in complex structures in this section, there is no ambiguity in using the same notation for a punctured Riemann surface and a surface with boundary.

Let $\mathbf{G} = SU(2)$, $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}$ the Lie algebra of $\mathbf{G}$, and let $P^\pm$, P be the trivial $\mathbf{G}$-bundles over $\Sigma^\pm$, Σ, and $E^\pm$, E the vector bundles associated via the standard representation. Let $\mathscr{A}^\pm$, $\mathscr{A}$ be the spaces of C^∞ connections and $\mathscr{G}^\pm$, $\mathscr{G}$ the groups of C^∞ gauge transformations on $P^\pm$, P, respectively. Let $\mathscr{G}_0^\pm$, $\mathscr{G}_0$ be the subgroups of $\mathscr{G}^\pm$, $\mathscr{G}$ consisting of based gauge transformations, i.e. those which are the identity at x_0. We topologize all spaces via the C^∞ Fréchet topologies. The groups $\mathscr{G}^\pm$, $\mathscr{G}_0^\pm$, $\mathscr{G}$, $\mathscr{G}_0$ then act smoothly on $\mathscr{A}_F^\pm$, $\mathscr{A}_F$, the flat connections in $\mathscr{A}^\pm$, $\mathscr{A}$. Let

$$\begin{aligned} \mathscr{M}_0^\pm &= \mathscr{A}_F^\pm / \mathscr{G}_0^\pm \quad & \mathscr{M}^\pm &= \mathscr{A}_F / \mathscr{G}^\pm \\ \mathscr{M}_0 &= \mathscr{A}_F / \mathscr{G}_0 & \mathscr{M} &= \mathscr{A}_F / \mathscr{G} \end{aligned} \tag{2.2}$$

be the quotient spaces. $\mathscr{M}^\pm$ and $\mathscr{M}$ are called the *moduli spaces* of flat connections on P, $P^\pm$, and $\mathscr{M}_0^\pm$, $\mathscr{M}_0$ are called the *based moduli spaces* of flat connections. The residual groups $\mathscr{G}/\mathscr{G}_0 \simeq \mathbf{G}$ and $\mathscr{G}^\pm/\mathscr{G}_0^\pm \simeq \mathbf{G}$ act on $\mathscr{M}_0$ and $\mathscr{M}_0^\pm$, and $\mathscr{M} = \mathscr{M}_0/\mathbf{G}$, $\mathscr{M}^\pm = \mathscr{M}_0^\pm/\mathbf{G}$. For the following proposition we refer to [D-W1, Subsect. 2.1] and [Ak-M].

(2.3) **Proposition.** *$\mathcal{M}_0^{\pm}$ is a smooth manifold diffeomorphic to $\mathbf{G}^{2g^{\pm}}$, and the holonomy map $q_{\pm}: \mathcal{M}_0^{\pm} \to \mathbf{G}$ which measures the holonomy around C^+, $(C^-)^{-1}$, respectively, is a smooth map. Moreover, $q_{\pm}$ has the following properties:*

(i) *$q_{\pm}$ is surjective;*
(ii) *$\{$critical points of $q_{\pm}\} = \{$reducible connections$\}$;*
(iii) *$\{$reducible connections$\} \subset q_{\pm}^{-1}(I)$;*
(iv) *$q_{\pm}^{-1}(pt.)\backslash\{$reducible connections$\}$ is a C^{∞} manifold of dimension $6g_{\pm} - 3$.*
(v) *$q_{\pm}: \mathcal{M}_0^{\pm}\backslash q_{\pm}^{-1}(I) \to \mathbf{G}\backslash\{I\}$ is a trivial fiber bundle.*

Let $q: \mathcal{M}_0 \to \mathbf{G}$ denote the map measuring the holonomy around C. Let $\mathcal{M}_0^s \subset \mathcal{M}_0$ be the subspace of irreducible connections, and let $\mathcal{M}^s \subset \mathcal{M}$ be the image of $\mathcal{M}_0^s \subset \mathcal{M}_0$ under the obvious quotient map. For the next two propositions we refer to [Ak-M].

(2.4) **Proposition.** *$\mathcal{M}_0$ is a real manifold of dimension $6g - 3$ and $\mathcal{M}_0\backslash\mathcal{M}_0^s \subset q^{-1}(I)$. Furthermore, the natural quotient map $\mathcal{M}_0^s \to \mathcal{M}^s$ is a principal $SO(3)$ bundle.*

(2.5) **Proposition.** *$q: \mathcal{M}_0\backslash q^{-1}(I) \to \mathbf{G}\backslash\{I\}$ is the fiber product of $q_+: \mathcal{M}_0^+\backslash q_+^{-1}(I) \to \mathbf{G}\backslash\{I\}$ and $q_-: \mathcal{M}_0^-\backslash q_-^{-1}(I) \to \mathbf{G}\backslash\{I\}$, and therefore q is also a trivial fiber bundle. Moreover, if $\mathbf{T}_a \subset \mathbf{G}$ is a maximal torus, a $\mathbf{T}_a$-equivariant deformation retract of $\mathbf{G}\backslash\{I\}$ induces $\mathbf{T}_a$-equivariant trivializations of q, $q_{\pm}$ compatible with the fiber product.*

Below is a diagram of this fiber product:

$$\begin{array}{ccccc} & & \mathcal{M}_0\backslash q^{-1}(I) & & \\ & {}^{p_+}\swarrow & & \searrow{}^{p_-} & \\ \mathcal{M}_0^+\backslash q_+^{-1}(I) & & \downarrow q & & \mathcal{M}_0^-\backslash q_-^{-1}(I) \\ & {}_{q_+}\searrow & & \swarrow{}_{q_-} & \\ & & \mathbf{G}\backslash\{I\}\,. & & \end{array}$$

Given $a \in \mathbf{G}\backslash\{I\}$ we let $\mathcal{F}_a = q^{-1}(a)$, $\mathcal{F}_a^{\pm} = q_{\pm}^{-1}(a)$ denote the fibers of q, $q_{\pm}$. Moreover, let $\mathbf{G}\cdot a$ denote the adjoint orbit in $\mathbf{G}$ through a, and let $\mathcal{M}_{0,a} = q^{-1}(\mathbf{G}\cdot a)$, $\mathcal{M}_{0,a}^{\pm} = q_{\pm}^{-1}(\mathbf{G}\cdot a)$. We let $\mathcal{M}_a$, $\mathcal{M}_a^{\pm}$ denote the images of $\mathcal{M}_{0,a}$, $\mathcal{M}_{0,a}^{\pm}$ under the natural quotient maps.

(2.6) **Corollary.** *If $a \neq I$, $\mathcal{M}_{0,a}$, $\mathcal{M}_{0,a}^{\pm}$ are smooth submanifolds of $\mathcal{M}_0$, $\mathcal{M}_0^{\pm}$ and the restrictions of q, $q_{\pm}$ are trivial fiber bundles with fibers $\mathcal{F}_a$, $\mathcal{F}_a^{\pm}$ respectively. Furthermore, the restrictions of q, $q_{\pm}$ to $\mathcal{M}_{0,a}$, $\mathcal{M}_{0,a}^{\pm}$ form a trivial fiber product*

$$\begin{array}{ccccc} & & \mathcal{M}_{0,a} & & \\ & {}^{p_+}\swarrow & & \searrow{}^{p_-} & \\ \mathcal{M}_{0,a}^+ & & \downarrow q & & \mathcal{M}_{0,a}^- \\ & {}_{q_+}\searrow & & \swarrow{}_{q_-} & \\ & & \mathbf{G}\cdot a\,. & & \end{array}$$

(2.7) **Proposition.** (i) *If $a \neq I$, $\mathcal{M}_a$, $\mathcal{M}_a^{\pm}$ are smooth submanifolds of $\mathcal{M}$, $\mathcal{M}^{\pm}$ and the natural quotient maps*

$$\mathcal{M}_{0,a} \to \mathcal{M}_a, \quad \mathcal{M}_{0,a}^{\pm} \to \mathcal{M}_a^{\pm}$$

are principal SO(3) *bundles.* (ii) *If* $a \neq \pm I$, *then*

$$\mathscr{F}_a \to \mathscr{M}_a, \quad \mathscr{F}_a^\pm \to \mathscr{M}_a^\pm$$

are principal $\mathbf{T}_a/\mathbb{Z}_2$ *bundles, where* $\mathbf{T}_a \subset \mathrm{SU}(2)$ *denotes the maximal torus through* a.

Proof. The action of $SO(3)$ on $\mathscr{M}_{0,a}$, $\mathscr{M}_{0,a}^\pm$ is free and admits local slices. Similarly for the action of $\mathbf{T}_a/\mathbb{Z}_2$ on $\mathscr{F}_a, \mathscr{F}_a^\pm$. The Proposition follows from a general theorem on group actions (cf. [V, Lemma 2.9.11]).

2.2 Description via the Neumann conditions

We now turn to a different description of the spaces $\mathscr{M}_0^\pm$, $\mathscr{M}^\pm$ and the fiber product (2.6) in terms of the Neumann boundary value problem. We continue our notation $\Sigma = \Sigma_t$, $\Sigma^\pm = \Sigma_t^\pm$. Again, as we are not interested in complex structure in this section there is no ambiguity in identifying a punctured Riemann surface with a surface with boundary.

We first recall the following standard slice theorem on the space $\mathscr{A}$ (cf. [Fr-U, Theorem 3.2]).

(2.8) **Theorem.** *Let* $\nabla \in \mathscr{A}_F$ *be an irreducible connection and let* $[\nabla]$ *be the resulting orbit in* $\mathscr{M}$.

(i) *A neighborhood of* $[\nabla]$ *in* $\mathscr{M}$ *is diffeomorphic to*

$$K_\nabla = \{\beta \in C^\infty(T^*\Sigma \otimes \mathfrak{g}_E) : \nabla\beta = 0,\ \nabla{*}\beta = 0\}\,.$$

(ii) *A neighborhood of* $[\nabla]$ *in* $\mathscr{M}_0$ *is* SO(3)*-equivariant diffeomorphic to* $K_\nabla \times SO(3)$.

(iii) *Under the identifications*

$$T_{[\nabla]}\mathscr{M} = K_\nabla\,,$$
$$T_{[\nabla]}\mathscr{M}_0 = K_\nabla \oplus \nabla\mathfrak{g}$$

as subspaces of $C^\infty(T^*\Sigma \otimes \mathfrak{g}_E)$, *the derivative of the natural quotient map* $\mathscr{M}_0 \to \mathscr{M}$ *corresponds to projection onto the first factor.*

The above theorem generalizes to the case of manifolds with boundary as follows (cf. [T1, Proposition 2.1 and 2.2]):

(2.9) **Theorem.** *Let* $\nabla^\pm \in \mathscr{A}_F^\pm$ *be an irreducible connection, and let* $[\nabla^\pm]$ *denote the resulting orbit in* $\mathscr{M}^\pm$.

(i) *A neighborhood of* $[\nabla^\pm]$ *in* $\mathscr{M}^\pm$ *is diffeomorphic to*

$$K_{\nabla^\pm}^{\pm,\mathrm{neu}} = \{\beta \in C^\infty(T^*\Sigma^\pm \otimes \mathfrak{g}_{E^\pm}) : \nabla^\pm\beta = 0,\ \nabla^\pm{*}\beta = 0,\ \text{and}\ {*}\beta|_C = 0\}\,.$$

where ${*}\beta|_C$ *denotes the restriction of* ${*}\beta$ *to* C.

(ii) *A neighborhood of* $[\nabla^\pm]$ *in* $\mathscr{M}_0^\pm$ *is* SO(3)*-equivariant diffeomorphic to* $K_{\nabla^\pm}^{\pm,\mathrm{neu}} \times SO(3)$.

(iii) *Under the identification* $T_{[\nabla^\pm]}\mathscr{M}^\pm = K_{\nabla^\pm}^{\pm,\mathrm{neu}}$, $T_{[\nabla^\pm]}\mathscr{M}_0^\pm = K_{\nabla^\pm}^{\pm,\mathrm{neu}} \oplus \nabla^\pm\mathfrak{g}$ *as subspaces of* $C^\infty(T^*\Sigma^\pm \otimes \mathfrak{g}_{E^\pm})$, *the derivative of the natural quotient map* $\mathscr{M}_0^\pm \to \mathscr{M}^\pm$ *corresponds to projection onto the first factor.*

Given a closed curve C in Σ or $\Sigma^\pm$, fix an identification of the space of adjoint orbits $\mathbf{G}/\mathbf{G}_{\mathrm{adj}}$ of $\mathbf{G}$ with the interval [0, 1/2], and let

$$h_C : \mathscr{A}_F(\text{resp. } \mathscr{A}_F^\pm) \to [0, 1/2] \tag{2.10}$$

denote the holonomy map. It is easy to verify that h_C is continuous and is smooth away from 0 and 1/2.

We can now state the main corollary of Theorem 2.9.

(2.11) **Corollary.** *Let $\nabla \in \mathscr{A}_F$ be of holonomy $a \neq \pm I$, and let $\nabla^{\pm}$ be the restriction to $A_F^{\pm}$. Then*

(i) *A neighborhood of $[\nabla]$, $[\nabla^{\pm}]$ in $\mathscr{M}_a$, $\mathscr{M}_a^{\pm}$ is diffeomorphic to*

$$K_\nabla^a = \{\beta \in K_\nabla : (h_C)_* \beta = 0\}$$
$$K_{\nabla^\pm}^{a,\pm,\mathrm{neu}} = \{\beta \in K_{\nabla^\pm}^{\pm,\mathrm{neu}} : (h_C)_* \beta = 0\}$$

respectively;

(ii) *A neighborhood of $[\nabla]$, $[\nabla^{\pm}]$ in $\mathscr{M}_{0,a}$, $\mathscr{M}_{0,a}^{\pm}$ is $SO(3)$-equivariant diffeomorphic to $K_\nabla^a \times SO(3)$, $K_{\nabla^\pm}^{a,\pm,\mathrm{neu}} \times SO(3)$, respectively;*

(iii) *A neighborhood of $[\nabla]$, $[\nabla^{\pm}]$ in $\mathscr{F}_a$, $\mathscr{F}_a^{\pm}$ is $\mathbf{T}_a/\mathbb{Z}_2$-equivariant diffeomorphic to $K_\nabla^a \times \mathbf{T}_a/\mathbb{Z}_2$, $K_{\nabla^\pm}^{a,\pm,\mathrm{neu}} \times \mathbf{T}_a/\mathbb{Z}_2$, respectively;*

(iv) *Let $\mathfrak{t}_a = \mathrm{Lie}\,\mathbf{T}_a$. Under the identifications*

$$T_{[\nabla]}\mathscr{M}_a = K_\nabla^a\,, \qquad T_{[\nabla]}\mathscr{M}_{0,a} = K_\nabla^a \times \nabla\mathfrak{g}$$
$$T_{[\nabla^\pm]}\mathscr{M}_a^{\pm} = K_{\nabla^\pm}^{a,\pm,\mathrm{neu}}\,, \qquad T_{[\nabla^\pm]}\mathscr{M}_{0,a}^{\pm} = K_{\nabla^\pm}^{a,\pm,\mathrm{neu}} \times \nabla^{\pm}\mathfrak{g}\,,$$
$$T_{[\nabla]}\mathscr{F}_a = K_\nabla^a \times \nabla\mathfrak{t}_a\,, \qquad T_{[\nabla^\pm]}\mathscr{F}_a^{\pm} = K_{\nabla^\pm}^{a,\pm,\mathrm{neu}} \times \nabla^{\pm}\mathfrak{t}_a$$

the derivative of the natural quotient maps

$$\mathscr{M}_{0,a} \to \mathscr{M}_a\,, \qquad \mathscr{F}_a \to \mathscr{M}_a$$
$$\mathscr{M}_{0,a}^{\pm} \to \mathscr{M}_a^{\pm}\,, \qquad \mathscr{F}_a^{\pm} \to \mathscr{M}_a^{\pm}$$

is just projection onto the first factor.

The next corollary gives an infinitesimal description of the fibration (2.6) in terms of the Neumann spaces above. This will be used in later sections.

(2.12) **Corollary.** *Let ∇, $\nabla^{\pm}$ be as in the previous corollary. Under the identifications between $T\mathscr{M}_{0,a}$, $T\mathscr{M}_{0,a}^{\pm}$ and harmonic 1-forms described in Corollary 2.11 we have the following description for $(p_\pm)_*$: Let*

$$\beta \in T\mathscr{M}_{0,a} \simeq K_\nabla \oplus \nabla\mathfrak{g} \subset C^\infty(T^*\Sigma \otimes \mathfrak{g}_E)\,,$$

$\beta_\pm$ the restriction to $C^\infty(T^*\Sigma^{\pm} \otimes \mathfrak{g}_{E^\pm})$, and $\hat{\beta}_\pm$ the Neumann representative of $\beta_\pm$ in $K_{\nabla_\pm}^{a,\pm}$. Then for some $g \in \mathfrak{g}$,

$$(p_\pm)_*(\beta) = \hat{\beta}_\pm + \nabla_\pm g \in K_{\nabla_\pm}^{a,\pm,\mathrm{neu}} \oplus \nabla_\pm \mathfrak{g} = T\mathscr{M}_{0,a}^{\pm}\,.$$

We end this subsection by identifying the kernel of the map q_* of (2.6).

(2.13) **Lemma.** *Under the identification in Theorem 2.8(iii),*

$$\ker q_{*,[\nabla]} = K_\nabla^a \oplus \nabla(\mathfrak{t}_a)\,.$$

Proof. For dimensional reasons it suffices to show that $\nabla(\mathfrak{t}_a) \subset \ker q_{*,[\nabla]}$ and $K_\nabla^a \subset \ker q_{*,[\nabla]}$. To show the first inclusion choose $\varphi \in \mathfrak{t}_a$ and a curve $\gamma(s)$ in $\mathbf{T}_a$ such that $\gamma(0) = \mathrm{id}$ and $\left.\frac{d}{ds}\right|_{s=0} \gamma(s) = \varphi$. Since

$$\nabla\varphi = \left.\frac{d}{ds}\right|_{s=0} \nabla^{\gamma(s)} = \left.\frac{d}{ds}\right|_{s=0} (\gamma(s)\,\nabla\gamma(s)^{-1})\,,$$

we obtain

$$q_{*,[\nabla]}(\nabla\varphi) = \frac{d}{ds}\bigg|_{s=0} (\gamma(s)q(\nabla)\gamma(s)^{-1}) = \frac{d}{ds}\bigg|_{s=0} (\gamma(s)\cdot a\cdot\gamma(s)^{-1}) = 0\ ,$$

since $\gamma(s)$ commutes with a. Hence $\nabla(\mathrm{t}_a) \subset \ker q_{*,[\nabla]}$. To show the second inclusion, observe that under the identification between K_∇ and ∇-harmonic 1-forms, $\beta \in K^a_\nabla$ if and only if $\nabla\beta = \nabla{*}\beta = 0$, and $(h_C)_*\beta = 0$. The desired inclusion then follows by the commutativity of the diagram

$$\begin{array}{ccc} K^a_\nabla & \xrightarrow{q_*} & \mathbb{R} \\ \downarrow & \nearrow (h_C)_* & \\ T_\nabla \mathscr{A}_F\ . & & \end{array}$$

2.3 The universal connection

We continue our notation as in Subsect. 2.2. Let $\mathscr{M}^\pm(2,0)$ (resp. $\mathscr{M}^\pm(2,-1)$) denote the moduli spaces of rank two semi-stable vector bundles on $\bar{\Sigma}^\pm$ of degree 0 (resp. -1) and fixed determinant. These may be identified with the moduli spaces of flat SU(2) (resp. $SO(3)$) connections via the Narasimhan-Seshadri Theorem (cf. [A-B]). We would like to define connections on the principal bundles

$$\mathbf{T}_a/\mathbb{Z}_2 \to \mathscr{F}^\pm_a \xrightarrow{q^\pm_a} \mathscr{M}^\pm_a \quad (a \neq \pm I)\ .$$

This can be done either by identifying the spaces with flat connections on a surface with boundary and using the Neumann boundary conditions, or by identifying the spaces with flat connections on the punctured surface and using the universal connection of Atiyah and Singer. We shall follow the second approach.

Let $(\mathbb{U}^\pm, \Gamma^\pm)$ therefore denote the universal $SO(3)$ bundle over $\bar{\Sigma}^\pm \times \mathscr{M}^\pm(2,-1)$ with universal connection (cf. [A-S]). By pulling back $(\mathbb{U}^\pm, \Gamma^\pm)$ via the maps

$$\mathscr{M}^\pm(2,-1) \simeq \{x^\pm_0\} \times \mathscr{M}^\pm(2,-1) \hookrightarrow \bar{\Sigma}^\pm \times \mathscr{M}^\pm(2,-1)\ ,$$

we obtain a connection on the principal bundle

$$SO(3) \to \mathscr{F}^\pm_{-I} \xrightarrow{q^\pm_{-I}} \mathscr{M}^\pm(2,-1)\ .$$

This connection is easy to describe: Given $[\nabla^\pm] \in \mathscr{M}^\pm(2,-1)$, fix representative connections $\nabla^\pm$ on bundles $\bar{E}^\pm$ of degree -1 on $\bar{\Sigma}^\pm$. According to the standard slice theorem for $\mathscr{M}^\pm(2,-1)$ (see also Theorem 2.8), the tangent space $T_{[\nabla^\pm]}\mathscr{M}^\pm(2,-1)$ may be identified with a finite dimensional subspace of

$$C^\infty(T^*\bar{\Sigma}^\pm \otimes \mathfrak{g}_{\bar{E}^\pm}) \subset L^2(T^*\bar{\Sigma}^\pm \otimes \mathfrak{g}_{\bar{E}^\pm})\ .$$

Since this identification can be chosen to vary smoothly in a neighborhood of $\nabla^\pm$, the L^2-metric defines a Riemannian metric on $\mathscr{F}^\pm_{-I}$. We can now define a horizontal subspace in $T_{[\nabla^\pm]}\mathscr{F}^\pm_{-I}$ to be the perpendicular subspace to the fiber of $q^\pm_{-I}$. It is easy to see, by simple integration by parts, that under the identification of $T_{[\nabla^\pm]}\mathscr{F}^\pm_{-I}$ with $\mathfrak{g}_{\bar{E}^\pm}$-valued 1-forms as described above, the horizontal subspace corresponds to the space $K^\pm_{\nabla^\pm}$ of harmonic $\nabla^\pm$ 1-forms in $C^\infty(T^*\bar{\Sigma}^\pm \otimes \mathfrak{g}_{\bar{E}^\pm})$. It also follows (cf. [A-S]) that $K^\pm$ is the desired pullback of the connections $(\mathbb{U}^\pm, \Gamma^\pm)$ described above.

By dividing $\mathscr{F}^{\pm}_{-I}$ by the maximal torus $\mathbf{T}_a/\mathbb{Z}_2 \subset SO(3)$ we obtain an $SO(3)$-invariant connection on

$$\mathbf{T}_a/\mathbb{Z}_2 \to \mathscr{F}^{\pm}_{-I} \to \mathscr{F}^{\pm}_{-I}/\mathbf{T}_a .$$

We define $K^{a,\pm}$ to be the pullback of the above connection on q_a via the diffeomorphism $\mathscr{M}^{\pm}_a \simeq \mathscr{F}^{\pm}_{-I}/\mathbf{T}_a$ induced by the natural $\mathbf{T}_a$-equivariant trivialization of the fibration $q_{\pm}$ (see Proposition 2.5).

Next we briefly review how a complex structure is defined on $\mathscr{M}^{\pm}_a$. Let $V^{\pm}$ represent points in $\mathscr{M}^{\pm}_a$. Twisting the usual $\bar{\partial}$-operator by the connections, we have the operator

$$\bar{\partial}^*_{V^{\pm}} : L^2((T^*\Sigma^{\pm})^{0,1} \otimes \mathfrak{g}_{E^{\pm}}) \to L^2(\mathfrak{g}^{\mathbb{C}}_{E^{\pm}}) ,$$

and in [D-W1, Subsect. 3.1 and 3.2] it was shown that $T_{[V^{\pm}]}\mathscr{M}^{\pm}_a$ may be identified with $\ker \bar{\partial}^*_{V^{\pm}}$. This immediately defines an almost complex structure on $\mathscr{M}^{\pm}_a$ which, by Theorem 3.13 of [D-W1], is equivalent to the complex structure on $\mathscr{M}^{\pm}_a$ defined by the theorem of Mehta and Seshadri [Me-Se]. The latter identify $\mathscr{M}^{\pm}_a$ with the moduli spaces of bundles on $\bar{\Sigma}^{\pm}$ with parabolic structure.

An important aspect of rank two parabolic bundles with one parabolic point is that they form correspondence varieties; that is, there are holomorphic surjections $\pi^{\pm}_0, \pi^{\pm}_1$ linking $\mathscr{M}^{\pm}(2,0)$ and $\mathscr{M}^{\pm}(2,-1)$ via the diagram

(2.14)
$$\begin{array}{ccc} & \mathscr{M}^{\pm}_a & \\ \pi^{\pm}_0 \swarrow & & \searrow \pi^{\pm}_1 \\ \mathscr{M}^{\pm}(2,0) & & \mathscr{M}^{\pm}(2,-1) . \end{array}$$

The maps $\pi^{\pm}_1$ are defined by simply forgetting the parabolic structure. Alternatively, given a degree -1 stable bundle and a quasi-parabolic structure at a point p one can take the kernel of the sheaf map from the bundle to the flag at p, and thus obtain a semi-stable bundle of degree 0; this defines the maps $\pi^{\pm}_0$. The key point is that $\pi^{\pm}_0$ and $\pi^{\pm}_1$ are holomorphic $\mathbb{P}^1$ bundles (off a set of codimension 3 for $\pi^{\pm}_0$). Since the details behind (2.14) (known as the *Hecke correspondence*) are presented in [Me-Se, Be, D-W1], we shall not elaborate further.

At this point we should make a remark concerning orientations. Let Σ^-_{opp} denote the surface Σ^- with the opposite orientation. As we have defined it, $\mathscr{F}^-_a$ denotes the based equivalence classes of flat connections on Σ^-_{opp} with holonomy a, whereas the complex structure on the quotient $\mathscr{M}^-_a$ was chosen to be induced from Σ^-. Now $\mathscr{F}^-_a$ inherits a natural orientation from this complex structure, and this is opposite the usual one coming from the complex structure on $\mathscr{M}^-_a$ that is induced from Σ^-_{opp}.

With this understood, we are now prepared to make a statement about the indefiniteness of the curvature of $K^{a,\pm}$ which we shall need later on. Recall from [D-W1] and [D-W2] that for $a \neq \pm I$,

$$\pi_1(\mathscr{M}^{\pm}_a) = 0, \quad \text{and} \quad H_2(\mathscr{M}^{\pm}_a, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} .$$

In the references above we explicitly defined generators $\beta_{\pm}, \gamma_{\pm}$ of $H_2(\mathscr{M}^{\pm}_a, \mathbb{Z})$ by the requirements that $\beta_{\pm}$ correspond to the generators of $H_2(\mathscr{F}^{\pm}_a, \mathbb{Z})$ and $\gamma_{\pm}$ to those coming from the fiber $\mathbf{T}_a/\mathbb{Z}_2 \simeq S^1$. Moreover, we demanded that $\beta_{\pm}$ and $\gamma_{\pm}$

correspond to holomorphic cycles (see [D-W2, Sect. 2]). This fixes the signs of $\beta_\pm, \gamma_\pm$. In particular, if $\iota_0^\pm, \iota_1^\pm$ denote the inclusion maps of the fibers in the Hecke correspondence (2.14), then

(2.15)
$$(\iota_0^\pm)_*[\mathbb{P}^1] = 2\gamma_\pm$$
$$(\iota_1^\pm)_*[\mathbb{P}^1] = \beta_\pm - 2\gamma_\pm \,,$$

as elements of $H_2(\mathscr{M}_a^\pm, \mathbb{Z})$ (see [D-W2, Lemmas 2.2 and 3.3]). We now prove

(2.16) **Proposition.** *$\frac{\sqrt{-1}}{2\pi}\mathrm{Curv}(K^{a,\pm})$ represents the dual classes $\pm\gamma_\pm^*$ in $H^2(\mathscr{M}_a^\pm, \mathbb{Z})$.*

Proof. First consider $K^{a,+}$. Then $\frac{\sqrt{-1}}{2\pi}\mathrm{Curv}(K^{a,+})$ represents the Euler class e of the fibration q_a^+. Let ψ be a 1-form on $\mathscr{F}_a^+$ with the properties that its restriction to the fiber $\mathbf{T}_a/\mathbb{Z}_2$ of q_a^+ equals the fundamental class of the fiber and $d\psi$ is co-homologous to $-(q_a^+)^*(e)$ in $H^2(\mathscr{F}_a^+, \mathbb{Z})$. Let $\hat{\beta}_+$ be the generator of $H_2(\mathscr{F}_a^+, \mathbb{Z})$ as described in [D-W2, Sect. 2]. Since $\hat{\beta}_+$ is a cycle,

$$0 = \int_{\partial\hat{\beta}_+} \psi = \int_{\hat{\beta}_+} d\psi = -\int_{\hat{\beta}_+} (q_a^+)^*(e) = -\int_{(q_a^+)_*\hat{\beta}_+} e = -\int_{\beta_+} e = -\langle e, \beta_+\rangle \,.$$

On the other hand, it is easy to see that the pullback of the fibration q_a^+ via the map ι_1^+ corresponds to the homogeneous bundle

$$\mathbf{T}_a/\mathbb{Z}_2 \to SO(3) \to \mathbb{P}^1$$

of Chern class -2. Thus

$$-2 = \langle(\iota_1^+)^*e, \mathbb{P}^1\rangle = \langle e, (\iota_1^+)_*\mathbb{P}^1\rangle = \langle e, \beta_+ - 2\gamma_+\rangle = -2\langle e, \gamma_+\rangle \,.$$

Therefore $\langle e, \beta_+\rangle = 0$, and $\langle e, \gamma_+\rangle = +1$, so e represents γ_+^* in $H^2(\mathscr{M}_a^+, \mathbb{Z})$. This proves the result for $\mathrm{Curv}(K^{a,+})$. For $\mathrm{Curv}(K^{a,-})$, the same argument works except that we have chosen the opposite orientation on $\mathscr{F}_a^-$ (see the remarks above), hence the minus sign.

The next Corollary is also a direct consequence of the main result of [D-W2].

(2.17) **Corollary.** *Off a set of codimension* 3 *in $\mathscr{M}_a^\pm$, $\frac{\sqrt{-1}}{2\pi}\mathrm{Curv}(K^{a,\pm})$ has at least one positive and one negative eigenvalue. Moreover, the eigenvalues are uniformly bounded away from zero.*

Proof. Let $\iota_0^\pm, \iota_1^\pm$ be as above. Then by (2.15),

(2.18)
$$\langle(\iota_0^\pm)^*\gamma_\pm^*, \mathbb{P}^1\rangle = \langle\gamma_\pm^*, (\iota_0^\pm)_*\mathbb{P}^1\rangle = +2$$
$$\langle(\iota_1^\pm)^*\gamma_\pm^*, \mathbb{P}^1\rangle = \langle\gamma_\pm^*, (\iota_1^\pm)_*\mathbb{P}^1\rangle = -2 \,.$$

On the other hand, since $\pi_0^\pm$ and $\pi_1^\pm$ are $SO(3)$ invariant, it follows that the inclusions $\iota_0^\pm$ and $\iota_1^\pm$ of $\mathbb{P}^1$ into $\mathscr{M}_a$ are $SO(3)$ equivariant. Hence, the pullbacks of the $SO(3)$ invariant $K^{a,\pm}$ via $\iota_0^{\pm}$, $\iota_1^{\pm}$ are the homogeneous connections on $\mathbb{P}^1$, hence the cohomological statements (2.18) and Proposition 2.16 actually imply

$$\pm(\iota_0^{\pm})^*\left(\frac{\sqrt{-1}}{2\pi}\mathrm{Curv}(K^{a,\pm})\right) > 0$$

$$\pm(\iota_1^{\pm})^*\left(\frac{\sqrt{-1}}{2\pi}\mathrm{Curv}(K^{a,\pm})\right) < 0\,,$$

with uniform bounds.

3 Local degeneration of the moduli space

In this section we show how to locally degenerate the moduli space around a connection ∇ and prove Theorem 1.4 of the Introduction. Our method is very similar to techniques used in "conic degeneration", and much of this section is a straightforward generalization of [S-S] (see also [B-S] and [S]). Perhaps the only novelty is the somewhat curious choice of weighted Sobolev spaces which, as we shall see, is more or less dictated by the requirement of a smooth limit as the surface degenerates.

There are really two distinct problems to consider: The first is to degenerate the operator acting on sections over the closed glued surface. The latter may be regarded as sections over the surfaces with boundary satisfying some "matching conditions" on the boundary. This we treat in Subsect. 3.2. The second is to degenerate the Neumann problem defined on the surfaces with boundary, and this we do in Subsect. 3.3. An important point is that the Neumann and matching conditions degenerate to the same conditions in the limit once restrictions coming from the holonomy map are taken into account.

3.1 Preliminaries

We first set up some notation. Throughout this section, nothing special to SU(2) is used, so unless otherwise specified we allow for arbitrary rank. Let Σ denote the punctured surface Σ^+ or Σ^-. Let $E \to \Sigma$ be the trivial $SU(n)$ bundle, and suppose $[\nabla]$ is an equivalence class of flat unitary connections on E of holonomy $a \in U(n)$ around p. Then according to [D-W1, Lemma 2.7], we may choose a unitary frame $e_1, \ldots, e_n$ over the punctured disk $\tilde{D}^*$ with respect to which $\nabla = d + i\hat{\alpha}d\theta$, where $\hat{\alpha}$ is the matrix $\mathrm{diag}(\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ and

$$0 \leqq \hat{\alpha}_1 \leqq \ldots \leqq \hat{\alpha}_n < 1\,.$$

Clearly, for holonomy a^{-1} we can find a frame with respect to which the connection has the form $d - i\hat{\alpha}d\theta$. Let $\bar{\partial}_\nabla$ be the corresponding $\bar{\partial}$-operator. If we define a new basis $\{s_i\}$ by $s_i = |z|^{\hat{\alpha}_i}e_i$, then the s_i are holomorphic with respect to $\bar{\partial}_\nabla$.

The bundle E and operator $\bar{\partial}_\nabla$ may be extended to a holomorphic bundle $\bar{E} \to \bar{\Sigma}$ by gluing in a disk via the transition function $z^{\hat{\alpha}}$ on the universal cover of Σ (cf. [M-S]). Note that in the case of rank two we may also normalize the weights $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$. Twisting by the transition function z^{α} extends E to a holomorphic bundle $\bar{F}$ of degree zero. The maps $E \to \bar{E}$ and $E \to \bar{F}$ are the two projections π_1, π_0, respectively, in the Hecke correspondence (2.14). We shall denote the adjoint bundles associated to E and $\bar{E}$ by $\mathfrak{g}_E$ and $\mathfrak{g}_{\bar{E}}$, respectively.

As in Sect. 2, let Σ_t be a glued Riemann surface degenerating as $t \to 0$ to $\Sigma_0 = \Sigma^+ \cup \Sigma^-$, and let $\varepsilon = |t|^{1/2}$. We shall work locally in the pinching region $\varepsilon < |z_\pm| < 1$, and we denote by $\Sigma_t^\pm$ the surfaces $\Sigma^\pm \setminus \{|z_\pm| < \varepsilon\}$. Furthermore, we write $z_\pm = r_\pm e^{i\theta_\pm}$ and $t = \varepsilon^2 e^{i\beta}$. The cylindrical metrics on Σ^+ and Σ^-, $ds_\pm^2 = d\tau_\pm d\theta_\pm$, where $\tau_\pm = -\log r_\pm$, naturally glue together to form a smooth metric on Σ_t.

Given flat connections $\nabla^\pm$ on $E^\pm \to \Sigma^\pm$ of holonomy a, a^{-1}, respectively, we assume the connections have the form expressed above, namely

$$\nabla^+ = d + i\hat{\alpha} d\theta_+$$

$$\nabla^- = d - i\hat{\alpha} d\theta_-$$

with respect to frames $\{e_i^\pm\}$. Then by identifying $e_i = e_i^+ = e_i^-$, we obtain a bundle $E_t \to \Sigma_t$ with (flat) connection ∇_t. We will be interested in relating the L^2 sections of bundles associated to E_t with the *weighted* L^2_δ *sections* of bundles associated with $E \to \Sigma_0$. These are defined as follows: Let $\nabla^\pm$ be connections on $E^\pm \to \Sigma^\pm$ as above. For $\delta \in \mathbb{R}$ we define the weighted L^p spaces of sections of $E^\pm$, denoted $L^p_\delta(E^\pm)$, to be the completion of the space of compactly supported sections $C_0^\infty(E^\pm)$ with respect to the norm

$$\|u^\pm\|_{L^p_\delta} = \left\{ \int_{\Sigma^\pm} e^{\tau_\pm \delta} |u^\pm|^p \right\}^{1/p} .$$

Similarly, for a positive integer k we define the weighted Sobolev spaces $L^p_{k,\delta}(E^\pm)$ as the completion of $C_0^\infty(E^\pm)$ in the norm

$$\|u^\pm\|_{L^p_{k,\delta}} = \left\{ \int_{\Sigma^\pm} e^{\tau_\pm \delta} (|(\nabla^\pm)^{(k)} u^\pm|^p + \ldots + |\nabla^\pm u^\pm|^p + |u^\pm|^p) \right\}^{1/p} .$$

For more details, we refer to [T2] in general and [D-W1, Sect. 3], for our situation.

3.2 Boundary parametrix for the closed problem

For $t \neq 0$, we have a continuous family of Fredholm operators:

$$\bar{\partial}^*_{\nabla_t} : L^2_{1,\delta}((T^*\Sigma_t)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}) \to L^2_\delta(\mathfrak{g}_E^{\mathbb{C}}) . \tag{3.1}$$

The L^2 spaces with the weighting factor are equivalent to the usual L^2 spaces, and the kernel of (3.1) may be identified with the tangent space to $\mathcal{M}$ at $[\nabla_t]$. On Σ_0 we have the same operator, now acting on $L^2_{1,\delta}((T^*\Sigma_0)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$, but for purely dimensional reasons we know that this operator cannot continuously fill in the point $t = 0$ (cf. [D-W1, Subsect. 3.2]). We therefore extend $\bar{\partial}^*_\nabla$ to a larger domain in $L^2_{-\delta}((T^*\Sigma_0)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$ (we could equally well have chosen $-\delta'$ for any sufficiently small δ'; the same δ is chosen for convenience). More precisely, let $\mathbb{D}_{\max}$ be the maximal subspace of $L^2_{-\delta}$ mapped to L^2_δ by the operator in (3.1). Denote by D_t the operator in (3.1) and by D_0 the same operator defined on $\mathbb{D}_{\max}$. As in [B-S, S, S-S], we will cut the domain to a subspace $\mathbb{D}_0$ by imposing matching conditions. For the moment, however, let us consider the local problem on Σ_t^+ (we temporarily drop the $+$).

In the local unitary frame $\{e_i\}$ we have $\nabla = d + i\hat{\alpha} d\theta$, hence

$$\nabla^{0,1} e_i = -\frac{1}{2}\hat{\alpha}_i \frac{d\bar{z}}{\bar{z}} \otimes e_i \,.$$

Let u be a section of $(T^*\Sigma)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}$. Then we may write $u = u_{ij}\dfrac{d\bar{z}}{\bar{z}} \otimes e_i \otimes e_j^*$, so

$$\nabla^{0,1} * u = i\left\{\frac{\partial \bar{u}_{ji}}{\partial \bar{z}}\frac{dz}{z} \wedge d\bar{z} \otimes e_i \otimes e_j^* - \frac{1}{2}(\hat{\alpha}_i - \hat{\alpha}_j)\bar{u}_{ji}\frac{dz \wedge d\bar{z}}{|z|^2} \otimes e_i \otimes e_j^*\right\}.$$

Expanding locally,

$$u_{ij}(z, \bar{z}) = \sum_{k \in \mathbf{Z}} u_{ij,k}(r)\, e^{ik\theta} \,,$$

and using

$$\bar{z}\frac{\partial}{\partial \bar{z}} = \frac{r}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right),$$

we have

$$*\nabla^{0,1} * u = \sum_{k \in \mathbf{Z}} r\left\{\dot{u}_{ij,k} + \frac{\lambda_{ij,k}}{r} u_{ij,k}\right\} e^{ik\theta} e_i \otimes e_j^* \,,$$

where the dot means $\partial/\partial r$ and $\lambda_{ij,k} = k + \hat{\alpha}_i - \hat{\alpha}_j$. Note that the $\lambda_{ij,k}$ are the eigenvalues of the boundary operator $\dfrac{d}{d\theta} + i\hat{\alpha}$. We now fix δ to be less than the first non-zero eigenvalue of the boundary operator (cf. [D-W1, Sect. 3]). Hence,

$$0 < \delta < |\lambda_{ij,k}| \quad \text{for } \lambda_{ij,k} \neq 0 \,. \tag{3.2}$$

Note further that since the $\hat{\alpha}_i$'s are normalized between 0 and 1, $\lambda_{ij,k} = 0$ only if $k = \hat{\alpha}_i - \hat{\alpha}_j = 0$.

We want to solve $*\nabla^{0,1} * u = f$, where $f \in L^2_\delta(\mathfrak{g}_E^{\mathbb{C}})$. Write $f(z, \bar{z}) = f_{ij}(z, \bar{z}) e_i \otimes e_j^*$, and expand

$$f_{ij} = \sum_{k \in \mathbf{Z}} f_{ij,k}(r)\, e^{ik\theta} \,.$$

We must solve

$$\dot{u}_{ij,k} + \frac{\lambda_{ij,k}}{r} u_{ij,k} = \frac{1}{r} f_{ij,k} \,,$$

The general solution is

$$u_{ij,k}(r) = r^{-\lambda_{ij,k}}\left\{\int_?^r dx\, x^{\lambda_{ij,k}-1} f_{ij,k}(x) + \gamma_{ij,k}(t)\right\},$$

where the lower endpoint will be fixed by requiring convergence as $t \to 0$.

Since the $f_{ij,k}$ are assumed in L^2_δ, the integral is convergent for arbitrary lower end-point if and only if

$$2\lambda_{ij,k} - 1 + \delta > -1$$

which by (3.2) is equivalent to $\lambda_{ij,k} \geqq 0$. We therefore define (restoring the $+$)

$$(3.3) \quad u^+_{ij,k}(r) = \begin{cases} r^{-\lambda_{ij,k}} \int_1^r dx x^{\lambda_{ij,k}-1} f^+_{ij,k}(x) & \text{if } \lambda_{ij,k} < 0 \\ \int_\varepsilon^r dx x^{-1} f^+_{ij,0}(x) & \text{if } k = \hat{\alpha}_i - \hat{\alpha}_j = 0 \\ r^{-\lambda_{ij,k}} \left\{ \int_\varepsilon^r dx x^{\lambda_{ij,k}-1} f^+_{ij,k}(x) + \gamma^+_{ij,k}(t) \right\} & \text{if } \lambda_{ij,k} > 0 \,. \end{cases}$$

Setting $\gamma^+_{ij,k} \equiv 0$ for $\lambda_{ij,k} = 0$ is not strictly necessary, but this will satisfy the matching conditions, which we now discuss.

On Σ_t we have locally $z_+ z_- = t$, so

$$\frac{dz_+}{z_+} = -\frac{dz_-}{z_-} \,.$$

Therefore $\nabla^{0,1} e_i = \frac{1}{2} \hat{\alpha}_i \frac{d\bar{z}_-}{\bar{z}_-} \otimes e_i$, and Eqs. (3.3) solve $* \nabla^{0,1} * u^- = f^-$, provided we substitute $\hat{\alpha}_i \mapsto -\hat{\alpha}_i$. Hence

$$(3.4) \quad u^-_{ij,k}(r) = \begin{cases} r^{-\lambda_{ji,k}} \int_1^r dx x^{\lambda_{ji,k}-1} f^-_{ij,k}(x) & \text{if } \lambda_{ji,k} < 0 \\ \int_\varepsilon^r dx x^{-1} f^-_{ij,0}(x) & \text{if } k = \hat{\alpha}_j - \hat{\alpha}_i = 0 \\ r^{-\lambda_{ji,k}} \left\{ \int_\varepsilon^r dx x^{\lambda_{ji,k}-1} f^-_{ij,k}(x) + \gamma^-_{ij,k}(t) \right\} & \text{if } \lambda_{ji,k} > 0 \,. \end{cases}$$

On the overlap,

$$u^+_{ij}(z_+, \bar{z}_+) \frac{dz_+}{z_+} \otimes e_i \otimes e^*_j = u^-_{ij}(z_-, \bar{z}_-) \frac{dz_-}{z_-} \otimes e_i \otimes e^*_j$$

$$\Rightarrow u^+_{ij}(\varepsilon e^{i\theta_+}, \varepsilon e^{-i\theta_+}) = -u^-_{ij}(\varepsilon e^{i\beta - i\theta_+}, \varepsilon e^{-i\beta + i\theta_+}) \,.$$

In terms of the Fourier coefficients

$$u^\pm_{ij}(z_\pm, \bar{z}_\pm) = \sum_{k \in \mathbf{Z}} u^\pm_{ij,k}(r_\pm) e^{ik\theta_\pm}$$

we have the matching conditions

$$(3.5) \qquad u^\pm_{ij,k}(\varepsilon) = -e^{-ik\beta} u^\mp_{ij,-k}(\varepsilon) \,.$$

Noting that $\lambda_{ij,k} = -\lambda_{ji,-k}$, the matching conditions require

$$(3.6) \quad \begin{cases} \gamma^+_{ij,k}(t) = -e^{-ik\beta} \varepsilon^{2\lambda_{ij,k}} \int_\varepsilon^1 dy y^{-(\lambda_{ij,k}+1)} f^-_{ij,-k}(y); \; \lambda_{ij,k} > 0 \\ \gamma^-_{ij,k}(t) = -e^{-ik\beta} \varepsilon^{2\lambda_{ji,k}} \int_\varepsilon^1 dx x^{-(\lambda_{ji,k}+1)} f^+_{ij,-k}(x); \; \lambda_{ji,k} > 0 \,. \end{cases}$$

(3.7). **Proposition.** *Regarded as a map from* $L^2_\delta \to L^2_{-\delta}$, *Eqs.* (3.3) *and* (3.4) *with constants* (3.6) *define a family of compact operators* Q_t *which is also defined at* $t = 0$. *Moreover,* $Q_t \to Q_0$ *in norm.*

Proof. Ignoring for the moment the constants γ, we want to estimate the integral operator defined by the kernel $K_\varepsilon(r, x) = r^{-\lambda_{ij,k}} x^{\lambda_{ij,k}-1}$. To show this is compact on each eigenspace, it suffices to show that the modified kernel

$$\tilde{K}_\varepsilon(r, x) = r^{-\frac{1}{2}(1-\delta)} K_\varepsilon(r, x) x^{\frac{1}{2}(1+\delta)}$$

is Hilbert-Schmidt as a map from $L^2 \to L^2$. Setting $\mu = \frac{1}{2}(1 - \delta)$, we have

$$\tilde{K}_\varepsilon(r, x) = \begin{cases} r^{-(\lambda_{ij,k}+\mu)} x^{\lambda_{ij,k}-\mu} & \text{if} \begin{cases} \varepsilon < r \leqq x < 1; & \lambda_{ij,k} < 0\,; \\ \varepsilon < x \leqq r < 1; & \lambda_{ij,k} \geqq 0\,; \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

By direct computation, this operator is Hilbert-Schmidt on each eigenspace. Explicitly,

$$\int\int dx dr \tilde{K}_\varepsilon(r, x)^2 = \frac{1}{2|\lambda_{ij,k}| - \delta} \left\{ \frac{1}{2\delta}(1 - \varepsilon^{2\delta}) + \frac{\varepsilon^{2\delta}}{2|\lambda_{ij,k}| + \delta}(\varepsilon^{2|\lambda_{ij,k}|+\delta} - 1) \right\}$$

for all $\lambda_{ij,k}$. The above is uniformly bounded as $\varepsilon \to 0$.

To show that the direct sum over the eigenspaces converges, and hence converges to a compact operator, we use the Schur test (cf. [H-S]). For $\lambda_{ij,k} \ll 0$, we estimate

$$\begin{aligned} \sup_{\varepsilon<r<1} \int_r^1 dx |\tilde{K}_\varepsilon(r, x)| &= \sup_{\varepsilon<r<1} \left| \frac{r^{-(\lambda_{ij,k}+\mu)}}{\lambda_{ij,k} - \mu + 1}(1 - r^{\lambda_{ij,k}-\mu+1}) \right| \\ &= \sup_{\varepsilon<r<1} \left| \frac{1}{\lambda_{ij,k} - \mu + 1}(r^\delta - r^{-(\lambda_{ij,k}+\mu)}) \right| \\ &= O((1 + k)^{-1})\,, \end{aligned}$$

uniformly as $\varepsilon \to 0$ and k sufficiently large. Similarly,

$$\begin{aligned} \sup_{\varepsilon<x<1} \int_\varepsilon^x dr |\tilde{K}_\varepsilon(r, x)| &\leqq \sup_{\varepsilon<x<1} \left| \frac{1}{-\lambda_{ij,k} - \mu + 1}\left(x^\delta - \left(\frac{\varepsilon}{x}\right)^{-\lambda_{ij,k}} \varepsilon^{1-\mu} x^{-\mu}\right) \right| \\ &= O((1 + k)^{-1}) \end{aligned}$$

uniformly as $\varepsilon \to 0$ and k sufficiently large. A similar result holds for $0 \ll \lambda_{ij,k}$. Therefore the direct sum over all the eigenspaces converges uniformly as $\varepsilon \to 0$.

Finally, we estimate the constants $\gamma^\pm_{ij,k}$. For $\lambda_{ij,k} > 0$, i.e. $\gamma^+_{ij,k}$ we compute

$$\begin{aligned} \left| \varepsilon^{2\lambda_{ij,k}} \int_1^\varepsilon dy y^{-(\lambda_{ij,k}+1)} f^-_{ij,-k}(y) \right| &\leqq \varepsilon^{2\lambda_{ij,k}} \left\{ \int_1^\varepsilon dy y^{-2\lambda_{ij,k}-1+\delta} \right\}^{1/2} \|f^-_{ij,-k}\|_{L^2_\delta} \\ &= \varepsilon^{\delta/2+\lambda_{ij,k}} \left\{ \frac{1 - \varepsilon^{2\lambda_{ij,k}-\delta}}{2\lambda_{ij,k} - \delta} \right\}^{1/2} \|f^-_{ij,-k}\|_{L^2_\delta}\,. \end{aligned}$$

Also,

$$\left| \int_{\varepsilon}^{1} dr r^{-2\lambda_{ij,k}-1+\delta} \right|^{1/2} = \varepsilon^{\delta/2-\lambda_{ij,k}} \left\{ \frac{1-\varepsilon^{2\lambda_{ij,k}-\delta}}{2\lambda_{ij,k}-\delta} \right\}^{1/2},$$

so the $L^2_{-\delta}$ norm of $r^{-\lambda_{ij,k}}\gamma^+_{ij,k}(t)$ is

$$\leqq C(2\lambda_{ij,k}-\delta)^{-1}\varepsilon^{\delta/2}(1-\varepsilon^{2\lambda_{ij,k}-\delta})\,\|f^-_{ij,-k}\|_{L^2_\delta}.$$

By the choice of δ, this bound is uniform as $\varepsilon \to 0$. The constants $\gamma^-_{ij,k}$ have similar bounds. This completes the proof of the proposition.

Now let us consider the formal adjoint D'_t of D_t. It is easy to verify that locally

$$D'_t f = e^{\tau\delta}\,\nabla^{0,1}(e^{\tau\delta} f).$$

We define $\mathbb{D}'_{\max} \subset L^2_\delta(\mathfrak{g}^{\mathbb{C}}_E)$ to be the maximal subspace mapped into $L^2_{-\delta}$ by D'_t. A parametrix for D'_t is constructed in the same way as for D_t. We must solve

$$r^{-\delta}\left(\frac{r}{2}\frac{\partial}{\partial r}(r^{-\delta}f_{ij,k}) - \frac{\lambda_{ij,k}}{2}r^{-\delta}f_{ij,k}\right) = u_{ij,k}.$$

The general solution is

$$f_{ij,k}(r) = 2r^{\lambda_{ij,k}+\delta}\left\{\int_{?}^{r} dx x^{-\lambda_{ij,k}-(1-\delta)}u_{ij,k}(x) + \mu_{ij,k}(t)\right\}.$$

Since the $u_{ij,k}$ are assumed to be in $L^2_{-\delta}$, the integral is convergent for arbitrary lower endpoint only if

$$-2\lambda_{ij,k} + \delta > 0,$$

and we therefore obtain the criterion $\lambda_{ij,k} \leqq 0$. Define

$$(3.8)\qquad f^+_{ij,k}(r) = \begin{cases} 2r^{\lambda_{ij,k}+\delta}\int_1^r dx x^{-\lambda_{ij,k}-(1-\delta)}u^+_{ij,k}(x) & \text{if } \lambda_{ij,k} > 0 \\ 2r^{\delta}\int_\varepsilon^r dx x^{-(1-\delta)}u^+_{ij,0}(x) & \text{if } k = \hat{\alpha}_i - \hat{\alpha}_j = 0 \\ 2r^{\lambda_{ij,k}+\delta}\left\{\int_\varepsilon^r dx x^{-\lambda_{ij,k}-(1-\delta)}u^+_{ij,k}(x) + \mu^+_{ij,k}(t)\right\} & \text{if } \lambda_{ij,k} < 0. \end{cases}$$

A similar solution holds for $f^-_{ij,k}$ with $\lambda_{ij,k} \to \lambda_{ji,k}$. The matching conditions are clearly

$$f^+_{ij,k}(\varepsilon) = f^-_{ij,-k}(\varepsilon)e^{-ik\beta},$$

and as in (3.6) the μ's may be chosen to satisfy these conditions. Take

$$(3.9)\qquad \begin{cases} \mu^+_{ij,k}(t) = e^{-ik\beta}\varepsilon^{-2\lambda_{ij,k}}\int_1^\varepsilon dy y^{\lambda_{ij,k}-(1-\delta)}u^-_{ij,-k}(y);\ \lambda_{ij,k} < 0 \\ \mu^-_{ij,k}(t) = e^{-ik\beta}\varepsilon^{-2\lambda_{ij,k}}\int_1^\varepsilon dx x^{\lambda_{ji,k}-(1-\delta)}u^+_{ij,-k}(x);\ \lambda_{ji,k} < 0. \end{cases}$$

The above, together with Eqs. (3.8) define a local parametrix Q'_t for D'_t which is also well-defined at $t = 0$. As in Proposition 3.7, one verifies that $Q'_t \to Q'_0$ in norm.

To describe the domain $\mathbb{D}_0$, we must check what happens to the matching conditions as $t \to 0$ (see Lemma 2 of [S-S]).

(3.10). **Lemma.** *Suppose $u \in L^2_{-\delta}$ and $D_0 u \in L^2_\delta$ in the region $0 < |z| < 1$. Then for $\lambda_{ij,k} \neq 0$, the Fourier coefficients $u_{ij,k}(r)$ are $O(r^{\delta/2})$. For $\lambda_{ij,k} = 0$, the coefficients $u_{ij,0}(r)$, $\hat{\alpha}_i = \hat{\alpha}_j$, have limits as $r \to 0$. Suppose also that $f \in L^2_\delta$ and $D_0' f \in L^2_{-\delta}$. Then all the Fourier coefficients $f_{ij,k}(r)$ are $O(r^\delta)$.*

Proof. We write $D_0 u = f$. Then as above,

$$u_{ij,k}(r) = r^{-\lambda_{ij,k}} \left\{ \int_1^r dx x^{\lambda_{ij,k}-1} f_{ij,k}(x) + \gamma_{ij,k} \right\},$$

for some constants $\gamma_{ij,k}$. For $\lambda_{ij,k} = 0$,

$$\begin{aligned} u_{ij,k}(r) &= \left\{ \int_1^r dx x^{-1} f_{ij,k}(x) + \gamma_{ij,k} \right\} \\ &\to \left\{ -\int_0^1 dx x^{-1} f_{ij,k}(x) + \gamma_{ij,k} \right\}, \end{aligned}$$

which exists by the assumption that $f \in L^2_\delta$. For $\lambda_{ij,k} > 0$,

$$\lim_{r \to 0} \int_1^r dx x^{\lambda_{ij,k}-1} f_{ij,k}(x) = -\int_0^1 dx x^{\lambda_{ij,k}-1} f_{ij,k}(x)$$

exists by the same assumption, but since $r^{-\lambda_{ij,k}} \notin L^2_{-\delta}$, we must have

$$\gamma_{ij,k} = \int_0^1 dx x^{\lambda_{ij,k}-1} f_{ij,k}(x)\,.$$

Then

$$u_{ij,k}(r) = r^{-\lambda_{ij,k}} \int_0^r dx x^{\lambda_{ij,k}-1} f_{ij,k}(x)$$

$$|u_{ij,k}(r)| \leqq C \cdot r^{\delta/2} \|f_{ij,k}\|_{L^2_\delta}\,.$$

For $\lambda_{ij,k} < 0$, we have by the same computation,

$$\left| \int_0^r dx x^{\lambda_{ij,k}-1} f_{ij,k}(x) \right| \leqq C \cdot r^{\lambda_{ij,k}+\delta/2} \|f_{ij,k}\|_{L^2_\delta}\,,$$

and the first part of the lemma is proven. For the second part, we write $D_0' f = u$. Then

$$f_{ij,k}(r) = 2r^{\lambda_{ij,k}+\delta} \left\{ \int_1^r dx x^{-(\lambda_{ij,k}+1-\delta)} u_{ij,k}(x) + \mu_{ij,k} \right\},$$

and we argue as above.

Lemma 3.10 determines the domain $\mathbb{D}_0$. More precisely, Eq. (3.5) reduces to

(3.11) $$u^+_{ij,0}(0) = -u^-_{ij,0}(0); \quad \text{when } \hat{\alpha}_i = \hat{\alpha}_j\,.$$

By the lemma, the other coefficients $u^\pm_{ij,k}$ vanish at 0, so (3.5) is satisfied automatically for these. Note that the matching conditions vanish for the adjoint, so $\mathbb{D}_0' = \mathbb{D}'_{\max}$.

Let us complete this subsection by identifying the kernel of the operator D_0 in $\mathbb{D}_0$. Recall from Subsect. 3.1 that the connection ∇ defines bundles $\bar{E}^\pm$ on $\bar{\Sigma}^\pm$ with an identification of the fibers $\bar{E}^+_{p_0^+} \simeq \bar{E}^-_{p_0^-}$. Let $\bar{E}$ denote the bundle on the disjoint union $\bar{\Sigma}^+ \cup \bar{\Sigma}^-$ and $\mathrm{End}_0 \bar{E}$ the sheaf of germs of traceless endomorphisms of $\bar{E}$. Note that we have a natural identification of the fibers

$$(\mathrm{End}_0 \bar{E})_{p_0^+} = (\mathrm{End}_0 \bar{E})_{p_0^-} = \mathfrak{g}^{\mathbb{C}} .$$

(3.12) **Proposition.** *Hermitian conjugation gives a complex anti-linear isomorphism identifying* $\ker D_0$ *with meromorphic sections* ω *of* $(T^* \bar{\Sigma}_0)^{1,0} \otimes \mathrm{End}_0 \bar{E}$ *having at most simple poles at* $p_0^\pm$ *with residue lower (resp. upper) triangular and satisfying*

$$\mathrm{proj}_{\mathfrak{p}_a^{\mathbb{C}}} \mathrm{res}(\omega; p_0^+) = -\mathrm{proj}_{\mathfrak{p}_a^{\mathbb{C}}} \mathrm{res}(\omega; p_0^-) ,$$

where the projection is onto the Lie algebra $\mathfrak{p}_a^{\mathbb{C}}$ *of the stabilizer of* a *in* $\mathbf{G}^{\mathbb{C}}$.

Proof. The hermitian conjugate of an element of $\ker D_0$ is a holomorphic $(1, 0)$ form ω on $\Sigma^+ \cup \Sigma^-$ with values in $\mathrm{End}_0 E$. We must show ω extends as a meromorphic form with the residue as stated above. Near p_0^+ we write

$$\omega(z_+) = \omega_{ij}(z_+) dz_+ \otimes s_i^+ \otimes (s_j^+)^* ,$$

where ω_{ij} are holomorphic functions on the punctured disk $\tilde{D}^*$ and $\{s_i^+\}$ is the local holomorphic basis introduced in Subsect. 3.1. Writing in terms of the orthonormal basis $\{e_i\}$,

$$\omega(z_+) = |z_+|^{\hat{\alpha}_i - \hat{\alpha}_j} \omega_{ij}(z_+) dz_+ \otimes e_i \otimes e_j^* .$$

Locally near p_0^-, we have a similar expression with $\hat{\alpha}_i \mapsto -\hat{\alpha}_i$. Using the relation $e^{-r_\pm \delta} = |z_\pm|^\delta$, we have

$$\|\omega\|_{L^2_{-\delta}} \sim \sum_{i,j} \int |dz_+|^2 |\omega_{ij}|^2 |z_+|^{2(\hat{\alpha}_i - \hat{\alpha}_j) + \delta}$$
$$+ \int |dz_-|^2 |\omega_{ij}|^{2(\hat{\alpha}_j - \hat{\alpha}_i) + \delta} < +\infty .$$

Now for $\hat{\alpha}_i - \hat{\alpha}_j \geqq 0$, we have chosen δ to satisfy

$$0 < \delta < 1 - (\hat{\alpha}_i - \hat{\alpha}_j) \Rightarrow 2(\hat{\alpha}_i - \hat{\alpha}_j) + \delta < 1 + (\hat{\alpha}_i - \hat{\alpha}_j) < 2 .$$

Therefore, ω_{ij} may have at most simple poles at $p_0^\pm$. If $\hat{\alpha}_i - \hat{\alpha}_j \geqq 0$, the residue at p_0^+ may be arbitrary. If $\hat{\alpha}_i - \hat{\alpha}_j < 0$, then

$$-2 < 2(\hat{\alpha}_i - \hat{\alpha}_j) + \delta < 0 ,$$

and the residue must vanish at p_0^+. Similarly for p_0^-. Finally, note that when $\hat{\alpha}_i = \hat{\alpha}_j$,

$$\mathrm{res}(\omega_{ij}; p_0^+) = \lim_{z_+ \to 0} z_+ \omega_{ij}(z_+) = \lim_{z_+ \to 0} u_{ij}^+ (z_+) = u_{ij,0}^+(0) .$$

The last part of the proposition is therefore a consequence of the matching conditions (3.11).

3.3 Boundary parametrix for the Neumann problem

We now carry out a similar construction of a boundary parametrix for the Neumann problem on the surfaces $\Sigma_t^\pm$ with boundary. Our limiting operator will again be defined on the weighted Sobolev spaces associated to $\Sigma_0^\pm$ – the Neumann conditions will degenerate to play the role of the matching conditions of the last section.

Let Σ_t denote either Σ_t^+ or Σ_t^-. As in Subsect. 2.2, we have the operator

$$\delta_t : L^2_{-\delta}(T^*\Sigma_t \otimes \mathfrak{g}_E) \supset \mathbb{D}^{\text{neu}} \to L^2_\delta\left(\bigwedge^2 T^*\Sigma_t \otimes \mathfrak{g}_E\right) \oplus L^2_\delta(\mathfrak{g}_E),$$

where $\mathbb{D}^{\text{neu}}$ denotes the subspace of forms β mapped into L^2_δ by $\delta_t = (\nabla, \nabla^*)$ and satisfying the Neumann boundary conditions $*\beta|_{\partial\Sigma_t} = 0$. As before, we wish to find an operator δ_0 which is a continuous limit of δ_t. Let $\mathbb{D}_{\max} \subset L^2_{-\delta}(T^*\Sigma \otimes \mathfrak{g}_E)$ be the maximal subspace mapped into $L^2_\delta(\wedge^2 T^*\Sigma \otimes \mathfrak{g}_E) \oplus L^2_\delta(\mathfrak{g}_E)$ by the operator (∇, ∇^*). By the isomorphisms

$$T^*\Sigma_t \otimes \mathfrak{g}_E \simeq (T^*\Sigma_t)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}, \quad \bigwedge^2 T^*\Sigma_t \otimes \mathfrak{g}_E \oplus \mathfrak{g}_E \simeq \mathfrak{g}_E^{\mathbb{C}},$$

we define a parametrix Q_t^{neu} for δ_t by Eqs. (3.3). The constants $\gamma_{ij,k}$ are now determined by the Neumann conditions: If $\beta \in L^2_\delta(T^*\Sigma_t \otimes \mathfrak{g}_E)$ is written $\beta = u - *u$ where $u \in L^2_\delta((T^*\Sigma_t)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$, and if we write $u = u_{ij}\dfrac{d\bar{z}}{\bar{z}} \otimes e_i \otimes e_j^*$, then the condition $*\beta|_{\partial\Sigma_t} = 0$ implies

$$\begin{aligned} 0 &= \left(u_{ij}\frac{dz}{z} - i\bar{u}_{ji}\frac{d\bar{z}}{\bar{z}}\right)\Bigg|_{|z|=\varepsilon} \\ &= \left(u_{ij}\left(\frac{dr}{r} + id\theta\right) - i\bar{u}_{ji}\left(\frac{dr}{r} - id\theta\right)\right)\Bigg|_{|z|=\varepsilon} \\ &= -(u_{ij} + i\bar{u}_{ji})|_{|z|=\varepsilon}\, id\theta\,. \end{aligned}$$

In terms of the Fourier coefficients $u_{ij,k}$, we have

$$u_{ij,k}(\varepsilon) = -i\bar{u}_{ji,-k}(\varepsilon)\,. \tag{3.13}$$

We therefore take the constants γ to be

$$\gamma_{ij,k}(\varepsilon) = -i\varepsilon^{2\lambda_{ij,k}}\int_1^\varepsilon dx x^{-(\lambda_{ij,k}+1)}\bar{f}_{ji,-k}(x);\ \lambda_{ij,k} > 0\,. \tag{3.14}$$

By Lemma 3.10 and Eq. (3.13), we see that the limiting conditions are

$$u_{ij,0}(0) = -i\bar{u}_{ji,0}(0), \quad \hat{\alpha}_i = \hat{\alpha}_j\,, \tag{3.15}$$

with all other Fourier coefficients vanishing, and we define $\mathbb{D}_0^{\text{neu}} \subset \mathbb{D}_{\max}$ to be the subspace satisfying (3.15). One easily verifies the following version of Proposition 3.7.

(3.16) **Proposition.** *Regarded as maps from $L^2_\delta \to L^2_{-\delta}$, the Q_t^{neu} form a family of compact operators with continuous limit $Q_t^{\text{neu}} \to Q_0^{\text{neu}}$.*

Let δ_t' denote the formal adjoint of δ_t acting on sections of $\wedge^2 T^*\Sigma_t \otimes \mathfrak{g}_E \oplus \mathfrak{g}_E$. The condition

$$\langle \delta_t\beta, (\psi, \varphi)\rangle = \langle \beta, \delta_t'(\psi, \varphi)\rangle$$

requires $\int_{\partial\Sigma_\varepsilon} \mathrm{Tr}(\beta * \psi) = 0$. Writing $\psi = *\mathrm{Im} f$ where $f \in L^2(\mathfrak{g}_E^{\mathbb{C}})$ and $\beta = u - *u$, we have

$$0 = \int_0^{2\pi} \sum_{i,j} \frac{1}{2}(u_{ij} - i\bar{u}_{ji})(f_{ij} - \bar{f}_{ji})|_{|z|=\varepsilon} d\theta$$
$$= \int_0^{2\pi} \sum_{i,j} u_{ij}(f_{ij} - \bar{f}_{ji})|_{|z|=\varepsilon} d\theta$$

by the Neumann conditions (3.13). Then

$$0 = \int_0^{2\pi} \sum_{i,j,k,l} u_{ij,\,l}(\varepsilon) e^{il\theta}(f_{ij,\,k}(\varepsilon) - \bar{f}_{ji,\,-k}(\varepsilon)) e^{ik\theta} d\theta$$
$$= 2\pi \sum_{i,j,k} u_{ij,\,-k}(\varepsilon)(f_{ij,\,k}(\varepsilon) - \bar{f}_{ji,\,-k}(\varepsilon)) .$$

Since u is arbitrary the conditions on the domain of the adjoint are

$$f_{ij,\,k}(\varepsilon) = \bar{f}_{ji,\,-k}(\varepsilon) .$$

Therefore, the parametrix $(Q_t^{\mathrm{neu}})'$ may be defined by Eqs. (3.8) with constants

$$\mu_{ij,\,k}(\varepsilon) = \varepsilon^{-2\lambda_{ij,k}} \int_1^{\varepsilon} dx\, x^{(\lambda_{ij,k} - (1-\delta))} \bar{u}_{ji,\,-k}(x);\ \lambda_{ij,\,k} < 0 .$$

The analogous statement to Proposition 3.16 for $(Q_t^{\mathrm{neu}})'$ is then straightforward to prove.

Recall from Corollary 2.11 (iv) that the tangent space to $\mathcal{M}_a^{\pm}$ may be described via the Neumann conditions with the additional requirement that the vectors be in the kernel of the holonomy map. The following discussion shows that on the degenerate surface $\Sigma_0 = \Sigma^+ \cup \Sigma^-$ this condition corresponds to the other half of the matching condition (3.15).

(3.17) **Proposition.** *Let $\beta \in \ker \delta_0$ satisfy $(h_{C_1})_* \beta = 0$. Then in terms of the representation $\beta = u - *u$ where u is a section of $(T^* \Sigma_0)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}$ with Fourier coefficients $u_{ij,\,k}$, we have*

$$u_{ij,\,0}(0) = i\bar{u}_{ji,\,0}(0), \quad for\ \hat{\alpha}_i = \hat{\alpha}_j .$$

Proof. Consider the map

$$\tilde{h}_{C_\varepsilon} : \mathcal{A}_F \to \mathbf{G}$$

measuring the holonomy around the circle (with basepoint) C_ε. Let c_0 be the matrix

$$(c_0)_{ij} = \begin{cases} -(iu_{ij}(0) + \bar{u}_{ji}(0)) & \text{if } \hat{\alpha}_i = \hat{\alpha}_j ; \\ 0 & \text{otherwise} . \end{cases}$$

Then we have the following

(3.18) **Lemma.** *Let V, β be as above. Then*

$$\lim_{\varepsilon \to 0} (\tilde{h}_{C_\varepsilon})_* \beta = \exp(2\pi i \hat{\alpha}) \cdot c_0 .$$

Assuming the lemma, the proposition follows from the fact the $h_{C_\varepsilon} = h_{C_1}$ for all $\varepsilon > 0$.

Proof of Lemma 3.18. Choose a family of flat connections $\nabla_s = d + A_s$ such that over $\tilde{D}^*$,

$$A_0 = i\hat{\alpha}d\theta, \quad \frac{d}{ds}\bigg|_{s=0} A_s = \beta .$$

With respect to the unitary frame $\{e_i\}$, we write

$$\beta = u - *u = \left(r^{\hat{\alpha}_i - \hat{\alpha}_j} u_{ij}(z, \bar{z}) \frac{d\bar{z}}{\bar{z}} + i r^{\hat{\alpha}_j - \hat{\alpha}_i} \bar{u}_{ji}(z, \bar{z}) \frac{dz}{z} \right) \otimes e_i \otimes e_j^* .$$

If β_θ denotes the $d\theta$ component of β, then

$$\beta_\theta = (-i r^{\hat{\alpha}_i - \hat{\alpha}_j} u_{ij}(0) - r^{\hat{\alpha}_j - \hat{\alpha}_i} \bar{u}_{ji}(0) + O(r^\delta)) e_i \otimes e_j^* .$$

The term $O(r^\delta)$ follows by our choice of δ in (3.2). By (3.11), $u_{ij}(0) = 0$ whenever $\hat{\alpha}_i - \hat{\alpha}_j < 0$, so the terms divide into terms with $\hat{\alpha}_i = \hat{\alpha}_j$ and those with $\hat{\alpha}_i - \hat{\alpha}_j > 0$. In terms of the matrix c_0 we may write

$$\beta_\theta = ((c_0)_{ij} + O(r^\delta)) e_i \otimes e_j^* .$$

the δ-term again coming from our choice (3.2). Moreover, the matrices $i\hat{\alpha}$ and c_0 clearly commute. Next, write

$$A_s = a_s dr + b_s d\theta ,$$

where b_s has a Taylor expansion

$$b_s(r, \theta) = i\hat{\alpha} + s c_0 + s r^\delta \tilde{c}_s(r, \theta) + s^2 \tilde{b}_s(r\theta) ,$$

and $\tilde{c}_s$, $\tilde{b}_s$ are smooth as $s \to 0$. In order to compute holonomy around C_ε, it suffices to find a global parallel frame on C_ε with given initial conditions. By writing $\psi(\theta) = \psi(\varepsilon, \theta)$ for a section over C_ε, we have to solve the linear system of ODE's

$$(3.19) \qquad \begin{cases} \nabla_{s, \partial/\partial\theta} \psi = \psi'(\theta) + b_s(\varepsilon, \theta)\psi(\theta) = 0 \\ \psi(0) = \psi^0 . \end{cases}$$

We solve (3.19) by Picard iteration. Define the operator

$$P_{\varepsilon, s}\psi(\theta) = \psi^0 - \int_0^\theta d\tilde{\theta}\, b_s(\varepsilon, \tilde{\theta})\psi(\tilde{\theta})$$

acting on the space of C^1 matrix valued functions with initial condition ψ^0. Define also the operators

$$R_s\psi(\theta) = \psi^0 - \int_0^\theta d\tilde{\theta}(i\hat{\alpha} + s c_0)\psi(\tilde{\theta})$$

$$R_{\varepsilon, s}\psi(\theta) = -s\varepsilon^\delta \int_0^\theta d\tilde{\theta}\tilde{c}_s(\varepsilon, \tilde{\theta})\psi(\tilde{\theta}) - s^2 \int_0^\theta d\tilde{\theta}\tilde{b}_s(\varepsilon, \tilde{\theta})\psi(\tilde{\theta}) .$$

Observe that $P_{\varepsilon, s} = R_s + R_{\varepsilon, s}$, and assume that all are contraction mappings for $\theta \leq \theta_0$. Let ψ_s be the fixed point of R_s. It is easy to see that

$$\psi_s(\theta) = \exp(\theta(i\hat{\alpha} + s c_0)) \cdot \psi^0 .$$

Let $\psi_{\varepsilon,s}$ be the fixed point of $P_{\varepsilon,s}$. We then have the following estimate (using sup norms):

$$\begin{aligned}\|\psi_{\varepsilon,s}-\psi_s\|_{\theta\le\theta_0} &= \|P_{\varepsilon,s}\psi_{\varepsilon,s}-R_s\psi_s\|_{\theta\le\theta_0}\\ &= \|R_s\psi_{\varepsilon,s}+R_{\varepsilon,s}\psi_{\varepsilon,s}-R_s\psi_s\|_{\theta\le\theta_0}\\ &\leqq \|R_s\psi_{\varepsilon,s}-R_s\psi_s\|_{\theta\le\theta_0}+\|R_{\varepsilon,s}\psi_{\varepsilon,s}\|_{\theta\le\theta_0}\,.\end{aligned}$$

Since R_s is a contraction mapping,

$$\begin{aligned}\text{(3.20)}\qquad \|\psi_{\varepsilon,s}-\psi_s\| &\leqq C\|R_{\varepsilon,s}\psi_{\varepsilon,s}\|_{\theta\le\theta_0}\\ &\leqq C(O(s\varepsilon^\delta)+O(s^2))\|\psi_{\varepsilon,s}\|_{\theta\le\theta_0}.\end{aligned}$$

Patching together the solutions $\psi_{\varepsilon,s}$ to (3.19), we can extend to the interval $[0,2\pi]$ maintaining the estimate (3.20). Writing

$$\psi_{\varepsilon,s}=h_{C_\varepsilon}(A_s)\cdot\psi^0\,,$$

we have

$$\|h_{C_\varepsilon}(A_s)\cdot\psi^0-\exp(2\pi(i\hat{\alpha}+sc_0))\cdot\psi^0\|\leqq(O(s\varepsilon^\delta)+O(s^2))\|\psi_{\varepsilon,s}\|_{0\le\theta\le2\pi}\,.$$

Since ψ^0 was arbitrary, and $\psi_{\varepsilon,s}$ is continuous with respect to variation of parameters, we have in matrix norm

$$\|h_{C_\varepsilon}(A_s)-\exp(2\pi(i\hat{\alpha}+sc_0))\|\leqq O(s\varepsilon^\delta)+O(s^2)\,.$$

for sufficiently small s and ε. Now since $h_{C_\varepsilon}(A_0)=\exp(2\pi i\hat{\alpha})$ and $i\hat{\alpha}$ commutes with c_0, we have

$$\|(h_{C_\varepsilon})_*\beta-\exp(2\pi i\hat{\alpha})\cdot c_0\|\leqq O(\varepsilon^\delta)\,,$$

which, upon taking $\varepsilon\to0$, completes the proof of the lemma.

Now consider the disjoint union of the two Neumann problems on $\Sigma_0=\Sigma^+\cup\Sigma^-$. We have the operator δ_0 with domain $\mathbb{D}_0^{\text{neu}}$. The following is immediate from the condition (3.15) and Propositions 3.12 and 3.17.

(3.21) **Corollary.** *The isomorphism* $T^*\Sigma_0\otimes\mathfrak{g}_E\simeq(T^*\Sigma_0)^{0,1}\otimes\mathfrak{g}_E^{\mathbb{C}}$ *identifies*

$$\ker\delta_0\cap\ker(h_{C_1^+})_*\cap\ker(h_{C_1^-})_*\subset\mathbb{D}_0^{\text{neu}}$$

with the subspace in $\ker D_0$ *satisfying the condition that the residues at* $p_0^\pm$ *be strictly lower (resp. upper) triangular. In the notation of Proposition* 3.12,

$$\operatorname{proj}_{\mathfrak{p}_{\hat{\alpha}}^{\mathbb{C}}}\operatorname{res}(\omega;p_0^+)=-\operatorname{proj}_{\mathfrak{p}_{\hat{\alpha}}^{\mathbb{C}}}\operatorname{res}(\omega;p_0^-)=0\,.$$

3.4. Global parametrices

In this subsection we patch the boundary parametrices constructed in Subsect 3.2 and Subsect 3.3 together with interior parametrices and show that the operators D_t and δ_t converge continuously in the graph norm to D_0 and δ_0, respectively. Graph continuity, along with the assumption of irreducibility of the corresponding connection, are enough to show the bundles $\ker D_t$ and $\ker\delta_t$ extend continuously past $t=0$. We shall closely follow Sect. 4 of [S-S] and Sect. 3 of [S].

We focus on the operator D_t. The formulation for δ_t is completely analogous. As reviewed in Subsect 3.1 the connection ∇ defines a $\bar{\partial}$-operator on End $\bar{E}$, where $\bar{E}$ is the bundle on the disjoint union $\bar{\Sigma}^+ \cup \bar{\Sigma}^-$. Let Q^{int} denote the usual pseudo-differential parametrix for this $\bar{\partial}$-operator. Thus

$$\bar{\partial}_\nabla^* Q^{\text{int}} = I + R^{\text{int}} ,$$

where R^{int} is smoothing.

Choose C^∞ cut-off functions on $\Sigma_0 = \Sigma^+ \cup \Sigma^-$, φ^{int}, φ^{cpt}, ψ^{int}, ψ^{cpt}, satisfying

$$\varphi^{\text{int}} = \begin{cases} 1 & \text{if } |z_\pm| > 1/2 , \\ 0 & \text{if } |z_\pm| < 1/4 , \end{cases}$$

$$\varphi^{\text{cpt}} = \begin{cases} 1 & \text{if } |z_\pm| < 3/4 , \\ 0 & \text{if } |z_\pm| > 1 , \end{cases}$$

$$\psi^{\text{int}} = \begin{cases} 1 & \text{if } |z_\pm| > 3/4 , \\ 0 & \text{if } |z_\pm| < 1/2 , \end{cases}$$

and $\psi^{\text{cpt}} = 1 - \psi^{\text{int}}$. Let Q_t be the operator defined by Eqs. (3.3), (3.4), and (3.6), and define (for small $|t|$)

$$Q_t^{\text{tot}} = \varphi^{\text{int}} Q^{\text{int}} \psi^{\text{int}} + \varphi^{\text{cpt}} Q_t \psi^{\text{cpt}} .$$

Then

$$\begin{aligned} D_t Q_t^{\text{tot}} &= *(\bar{\partial}\varphi^{\text{int}} \wedge * Q^{\text{int}} \psi^{\text{int}}) + \varphi^{\text{int}} D_t Q^{\text{int}} \psi^{\text{int}} \\ &\quad + *(\bar{\partial}\varphi^{\text{cpt}} \wedge * Q_t \psi^{\text{cpt}}) + \varphi^{\text{cpt}} D_t Q_t \psi^{\text{cpt}} \\ &= \varphi^{\text{int}} (I + R^{\text{int}}) \psi^{\text{int}} + \varphi^{\text{cpt}} I \psi^{\text{cpt}} \\ &\quad + *(\bar{\partial}\varphi^{\text{int}} \wedge * Q^{\text{int}} \psi^{\text{int}}) + *(\bar{\partial}\varphi^{\text{cpt}} \wedge * Q_t \psi^{\text{cpt}}) . \end{aligned}$$

By the choice of cut-off functions, $\varphi^{\text{int}} \psi^{\text{int}} = \psi^{\text{int}}$ and $\varphi^{\text{cpt}} \psi^{\text{cpt}} = \psi^{\text{cpt}}$, so $\varphi^{\text{int}} \psi^{\text{int}} + \varphi^{\text{cpt}} \psi^{\text{cpt}} = I$. Hence,

$$D_t Q_t^{\text{tot}} = I + R_t^{\text{tot}} ,$$

where

$$R_t^{\text{tot}} = \varphi^{\text{int}} R^{\text{int}} \psi^{\text{int}} + *(\bar{\partial}\varphi^{\text{int}} \wedge * Q^{\text{int}} \psi^{\text{int}}) + *(\bar{\partial}\varphi^{\text{cpt}} \wedge * Q_t \psi^{\text{cpt}}) .$$

By Proposition 3.7, R_t^{tot} is a continuous family of compact operators. Analogously, one constructs a total parametrix for $-D_t^*$ which combined with the one for D_t gives $\tilde{Q}_t^{\text{tot}}$ and $\tilde{R}_t^{\text{tot}}$ satisfying

$$\tilde{D}_t \tilde{Q}_t^{\text{tot}} = I + \tilde{R}_t^{\text{tot}} ,$$

where

$$\tilde{D}_t = \begin{pmatrix} I & -D_t^* \\ D_t & I \end{pmatrix} .$$

Again by Proposition 3.7 and the same result for Q_t', $\tilde{R}_t^{\text{tot}}$ is a continuous family of compact operators.

Let χ_t denote the diffeomorphism $\Sigma_0 \to \Sigma_t \backslash C_\varepsilon$ (see (2.1)). Then χ_t defines an operator

$$U_t : L^2_{-\delta}((T^*\Sigma_0)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}) \supset \mathbb{D}_{\max} \to \mathbb{D}_{\max}$$

as follows: First express $u \in \mathbb{D}_{\max}$ locally as $u_{ij} \frac{d\bar{z}}{\bar{z}} \otimes e_i \otimes e_j^*$. Then the functions $u_{ij}^{\pm} \circ (\chi_t^{\pm})^{-1}$ are clearly in L^2 in light of Lemma 3.10. U_t is defined to be this map, followed by the obvious inclusion

$$L^2((T^*\Sigma_t)^{0,1} \otimes \mathfrak{g}_{E_t}^{\mathbb{C}}) \subsetneq \mathbb{D}_{\max} \subset L^2_{-\delta}((T^*\Sigma_0)^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}}) \tag{3.22}$$

obtained by extending by zero. Similarly, define a map U_t^* by first restricting to the image $\Sigma_t \subsetneq \Sigma_0$ and then pulling back by χ_t. Using U_t and U_t^*, we may pull the operators D_t and $\tilde{D}_t$ back to act on the fixed Hilbert space $L^2_{-\delta}$. We denote these by $U_t^* D_t U_t$ and $U_t^* \tilde{D}_t U_t$. It now follows exactly the same way as in [S-S] and [S] that $U_t^* \tilde{D}_t U_t$ possesses a continuous inverse. The existence of such proves the following

(3.23) **Proposition.** *The operators $U_t^* D_t U_t$ form a continuous family of (unbounded) Fredholm operators with respect to the graph norm, and its limit as $t \to 0$ is D_0. The same is true for $(U_t^{\pm})^* \delta_t^{\pm} U_t^{\pm} \to \delta_0^{\pm}$.*

Our interest in Proposition 3.23 is the following

(3.24) **Corollary.** *Suppose ∇ is an irreducible connection. Then given a basis $\{\beta_i\}_{i=1}^N$ for* ker D_0, *we may find a family $\{\beta_i(t)\}_{i=1}^N$ such that*
(i) *For each i, $\beta_i(t) \to \beta_i$ is a continuous family as $t \to 0$;*
(ii) *for $|t|$ sufficiently small, $\{\beta_i(t)\}_{i=1}^N$ is a basis for* ker $U_t^* D_t U_t$. *Moreover, if we set $\psi_i(t) = U_t \beta_i(t)$, then $\psi_i(t) \to \beta_i$ is a continuous family, and $\{\psi_i(t)\}_{i=1}^N$ is a basis for the image of* ker D_t *by the map* (3.22).

Proof. (cf. [S-S, A]). Let Gr_t denote the graph of $U_t^* D_t U_t$ in $L^2_{-\delta} \oplus L^2_\delta$. By Proposition 3.23 we have a continuous family of isomorphisms

$$G_t : Gr_0 \to Gr_t, \quad G_0 = I .$$

Let π_1, π_2 denote the orthogonal projections from $L^2_{-\delta} \oplus L^2_\delta$ to the first and second factors, respectively. Then

$$\pi_2 \circ G_t : Gr_0 \to L^2_\delta$$

is a continuous family of bounded Fredholm operators. Notice that $\ker \pi_2 \circ G_0 = \ker D_0$, and since ∇ is assumed to be irreducible, $\pi_2 \circ G_0$ has no cokernel. The operators

$$F_t : G_0 \to \ker D_0 \oplus L^2_\delta$$

defined by $\beta \mapsto (\pi_1 \beta, \pi_2 \circ G_t(\beta))$ form a continuous family as well, and by the irreducibility of ∇, F_0 is an isomorphism. The continuity allows us to conclude that F_t is still an isomorphism for $|t|$ sufficiently small. Let $\tilde{\beta}_i(t) = F_t^{-1}(\beta_i, 0)$. Then $\{\tilde{\beta}_i(t)\}_{i=1}^N$ form a continuous family and are linearly independent. Furthermore, $\pi_2 \circ G_t(\tilde{\beta}_i(t)) = 0$ implies $\beta_i(t) = \pi_1 \circ G_t(\tilde{\beta}_i(t))$ is in the kernel of $U_t^* D_t U_t$, and it is easy to see that $\{\beta_i(t)\}_{i=1}^N$ are linearly independent. Parts (i) and (ii) of the corollary now follow from the continuity of the index (cf. [S-S, p. 125]) which implies, since $\operatorname{coker} D_0 = \{0\}$, that $\dim \ker U_t^* D_t U_t = \dim \ker D_0$. Since $U_t \to I$ continuously, to prove the last statement we need only show that $\{\psi_i(t)\}$ is a basis for $\ker D_t$. Let ψ_t

be the image of an element of $\ker D_t$ by (3.22). Then clearly $U_t U_t^* \psi_t = \psi_t$. This implies, since $D_t \psi_t = 0$, that $U_t^* \psi_t \in \ker U_t^* D_t U_t$, and by part (ii) we may therefore find constants $c_i(t)$ such that

$$U_t^* \psi_t = \sum_{i=1}^{N} c_i(t) \beta_i(t) .$$

But then

$$\psi_t = U_t U_t^* \psi_t = \sum_{i=1}^{N} c_i(t) \psi_i(t) .$$

This completes the proof of the corollary.

We conclude this section with a corollary which will be used in Subsect. 4.4. The convergence described above is not uniform with respect to the underlying connection. Consider the case of rank two, $0 < \alpha < 1/2$, and $\hat{\alpha}_1 = \alpha$, $\hat{\alpha}_2 = 1 - \alpha$. Then as $\alpha \to 1/2$, the first positive and negative non-zero eigenvalues of the boundary operator $\lambda_{21,0}, \lambda_{12,0} \to 0$, and it is evident from the estimates that this destroys the uniform convergence of the parametrices Q_t and Q_t'. However, if we project out these eigenspaces from the domain of our operators, then the restricted operators converge uniformly for α near 1/2. The restriction of $\ker D_0 \cap \ker(h_C)_*$ then consists of holomorphic forms, i.e. forms of vanishing residue, and this is identified via (2.14) with the tangent space to $\mathcal{M}(2, -1)^+ \times \mathcal{M}(2, -1)^-$. Note also that the entire construction goes through if we normalize the weights $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$ (see Subsect. 3.1). Then as $\alpha \to 0$ the first non-zero eigenvalues again converge to zero. As before, if we project out these eigenspaces the kernel of the restricted operators converges uniformly to holomorphic forms. This subbundle is identified via (2.14) with the tangent space to $\mathcal{M}(2,0)^+ \times \mathcal{M}(2,0)^-$. We have thus shown

(3.25) **Corollary.** *Let V be as in Corollary* 3.24. *Let $\{\beta_i'\}_{i=1}^{N'}$ be a linearly independent set of* holomorphic *one forms in* $\ker D_0$. *Then we can choose $\{\psi_i'(t)\}_{i=1}^{N'}$ a linearly independent set in $\ker D_t$ such that $\psi_i'(t) \to \beta_i'$ is a continuous family as $t \to 0$. Furthermore, the family above can be chosen to vary uniformly for V in a neighborhood of a connection of holonomy $-I$. A similar result holds in a neighborhood of a connection of holonomy I.*

4 CR-submanifolds of the moduli space

The purpose of this section is to study certain real hypersurfaces of the moduli space of stable bundles. We start by showing that the moduli $\mathcal{M}_a(0)$ of flat connections with holonomy conjugate to a on the degenerate surface naturally admits the structure of a CR-*manifold*. Moreover, $\mathcal{M}_a(0)$ is the limit as $t \to 0$ of a differentiable family $\{\mathcal{M}_a(t)\}_{t \in D^*}$ of CR-manifolds, where $\mathcal{M}_a(t)$ is the space of flat connections on Σ_t of holonomy a with the CR-structure induced from $\mathcal{M}(t)$ (see Theorem 4.11 and Theorem 1.5 of the Introduction). We use this fact to show that for sufficiently small choice of the degeneration parameter t, the *Levi form* of $\mathcal{M}_a(t)$ has everywhere at least two positive and two negative eigenvalues (see Theorem 4.19, Corollary 4.20, and Theorem 1.6 of the Introduction). This implies that for the same t any CR-section of the determinant line bundle on $\mathcal{M}_a$ extends to a holomorphic section of the determinant line bundle on the whole of $\mathcal{M}$ (see Theorem 4.25 and Theorem 1.7 of the Introduction).

4.1. Preliminaries

Let Σ^+ and Σ^- be the punctured surfaces as in Subsect. 2.1 and $\Sigma = \Sigma_t$ the compact Riemann surface obtained by gluing Σ^+ and Σ^- with the complex parameter $t \in D^*$. All the spaces in Sect. 2 carry a parameter t. In particular the moduli spaces $\mathcal{M}_0 = \mathcal{M}_0(t)$, $\mathcal{M} = \mathcal{M}(t)$, $\mathcal{M}_0^\pm = \mathcal{M}_0^\pm(t)$, and $\mathcal{M}^\pm = \mathcal{M}^\pm(t)$. Let

$$q : \mathcal{M}_0^s \to \mathbf{G}$$

be the holonomy map around C as defined in Sect. 2. Since q is $\mathbf{G}$-equivariant for the usual action of $\mathbf{G}$ on $\mathcal{M}_0^s$ and the adjoint action of $\mathbf{G}$ on itself, q induces a continuous map

$$r : \mathcal{M}^s \to [0, 1/2] \simeq \mathbf{G}/\mathbf{G}_{\mathrm{adj}} . \tag{4.1}$$

which is smooth on the pre-image of the open interval (0, 1/2).

Let $\mathcal{M}_a = \mathcal{M}_a(t)$ denote the real hypersurface in $\mathcal{M}(t)$ defined by the equation $r = a$, where $a \in (0, 1/2)$ denotes also the conjugacy class of $a \in \mathbf{G}_R = \mathbf{G}\backslash\{\pm I\}$. The complex structure on $\mathcal{M}$ induces a CR-structure on $\mathcal{M}_a$. This is defined to be the maximal complex invariant subbundle of $T\mathcal{M}_a$. We shall explain this in greater detail presently, but first it will be useful to introduce the abstract notion of a CR-structure.

(4.2) **Definition.** Let $\mathcal{M}_a$ be a smooth manifold of (real) dimension $2n + 1$. A CR-structure is a C^∞ rank n subbundle $\mathcal{S}^a$ of the complexified tangent bundle $T\mathcal{M}_a \otimes \mathbb{C}$ of $\mathcal{M}_a$ satisfying the properties

(i) $\mathcal{S}^a \cap \overline{\mathcal{S}^a} = \{0\}$, and
(ii) $\mathcal{S}^a$ is closed under Lie brackets.

We say that the pair $(\mathcal{M}_a, \mathcal{S}^a)$ gives $\mathcal{M}_a$ the structure of a CR-*manifold*. We shall sometimes omit the subbundle $\mathcal{S}^a$ when it is understood that $\mathcal{M}_a$ carries a CR-structure. The examples of CR-manifolds we have in mind are the hypersurfaces $\mathcal{M}_a = \mathcal{M}_a(t)$ in $\mathcal{M}^s = \mathcal{M}^s(t)$.

Recall that the complex structure on $\mathcal{M}$ is induced by the $*$-operator on Σ. More precisely, for $[V] \in \mathcal{M}$ and under the usual identification

$$T_{[V]}\mathcal{M} \simeq T_{[V]}^{1,0}\mathcal{M} , \tag{4.3}$$

the $*$-operator acting on $\mathfrak{g}_E$-valued 1-forms corresponds to multiplication by $\sqrt{-1}$ on $T_{[V]}^{1,0}\mathcal{M}$. Let

$$\mathcal{S}_{[V]}^a = T_{[V]}\mathcal{M}_a \cap *T_{[V]}\mathcal{M}_a . \tag{4.4}$$

As $\mathcal{M}_a \subset \mathcal{M}$ is a real hypersurface, the dimension of $\mathcal{S}_{[V]}^a$ is constant equal to $6g - 8$ for all $[V] \in \mathcal{M}_a$, and hence it defines a subbundle $\mathcal{S}^a \subset T\mathcal{M}_a$. Under the identification (4.3), $\mathcal{S}^a$ defines a CR-structure on $\mathcal{M}_a$. This is the CR-structure *induced* from the complex structure on $\mathcal{M}$.

(4.5) **Definition.** Let $(\mathcal{M}_a, \mathcal{S}^a)$ be a CR-manifold. An *hermitian* CR-*vector bundle* on $\mathcal{M}_a$ is an hermitian complex vector bundle V on $\mathcal{M}_a$ with an hermitian connection ∇ whose curvature Ω satisfies the property $\Omega(X, Y) = 0$ for all X, $Y \in \mathcal{S}^a$. A *local* CR-*section* of V is a local smooth section s satisfying $\nabla_{\bar{X}} s = 0$ for all local vector fields $X \in \mathcal{S}^a$.

We denote by $\mathcal{O}_{\mathrm{CR}}(V)$ the sheaf of germs of local CR-sections of V, and let $H^0_{\mathrm{CR}}(\mathcal{M}_a, V)$ denote its space of global sections. For example, if we set $\mathcal{M}_a = \mathcal{M}_a(t)$ with the induced CR-structure from $\mathcal{M}^s(t)$, we can take V to be the restriction of the k-th power of the determinant bundle Δ^k on $\mathcal{M}_a$. The line bundle has a natural hermitian structure and hermitian connection which we shall describe in Subsect. 6.1. The space $H^0_{\mathrm{CR}}(\mathcal{M}_a, V^k)$ is of fundamental interest in this paper.

As in the case of holomorphic bundles on complex manifolds, the sheaf $\mathcal{O}_{\mathrm{CR}}(V)$ only depends on the values of the connection in the directions $\overline{\mathcal{S}^a}$. More precisely, the connection ∇ induces an operator $\bar{\partial}_b$ mapping smooth sections of V to smooth sections of $(\overline{\mathcal{S}^a})^* \otimes V$. Two connections which induce the same $\bar{\partial}_b$-operator give rise to the same sheaf $\mathcal{O}_{\mathrm{CR}}(V)$.

We next review the definition of the Levi form [Lv] associated to a CR-manifold.

(4.6) **Definition.** Given a CR-manifold $(\mathcal{M}_a, \mathcal{S}^a)$, we define the *Levi form* $\mathbb{L}$ to be the map

$$\mathbb{L}: \mathcal{S}^a \to T\mathcal{M}_a \otimes \mathbb{C}/\mathcal{S}^a \oplus \overline{\mathcal{S}^a} : X \mapsto \operatorname{proj}[X, \bar{X}],$$

where

$$\operatorname{proj}: T\mathcal{M}_a \otimes \mathbb{C} \to T\mathcal{M}_a \otimes \mathbb{C}/\mathcal{S}^a \oplus \overline{\mathcal{S}^a}$$

is the natural quotient map.

The Levi form can be expressed in terms of a local basis of vector fields as follows: Set $\dim \mathcal{M}_a = 2n + 1$ and let $\{L_1, \ldots, L_n\}$ be a set of independent local vector fields lying in $\mathcal{S}^a$. Let N be a local vector field on $\mathcal{M}_a$ such that the set $\{L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n, N\}$ span $T\mathcal{M}_a \otimes \mathbb{C}$ locally. Without loss of generality we may also assume that N is purely imaginary. Write

$$[L_i, \bar{L}_j] = \sum_{k=1}^{n} a^k_{ij} L_k + \sum_{l=1}^{n} b^l_{ij} \bar{L}_l + c_{ij} N. \tag{4.7}$$

Then the coefficients c_{ij} represent the Levi form $\mathbb{L}$ expressed in terms of the local basis $\{L_i, \bar{L}_j, N\}$. It follows from the invariant Definition 4.6 of the Levi form that the number of non-zero eigenvalues and the absolute value of the signature of c_{ij} at each point are independent of the choice of basis. Therefore, conditions such as the statement that the Levi form has at least two pairs of non-zero eigenvalues of opposite sign are well-defined (see Definition 5.1).

In the case where $\mathcal{M}_a$ is the boundary of a complex manifold $\mathcal{M}$ defined via the equation $r = 0$ for a smooth function r such that $r < 0$ on the interior of $\mathcal{M}$ and $r > 0$ on the exterior of $\mathcal{M}$, there is yet another equivalent definition of the Levi form (cf. [F-K, p. 56]). In this case for $L \in \mathcal{S}^a$ we have

$$\mathbb{L}(L) = 2\partial\bar{\partial} r(L \wedge \bar{L}). \tag{4.8}$$

We shall use the description (4.8) in Subsect. 4.4.

4.2 The limiting CR-*structure*

The notion of a differentiable family of complex manifolds (cf. [Kod]) extends naturally to the appropriate notion of a differentiable family of CR-manifolds. This is nothing but a differentiable family of complex subbundles $\mathscr{S}^a \subset T\mathscr{M}_a$ each of which satisfies the conditions of Definition 4.2. In our case, $\mathscr{M}_a(t)$ defines a differentiable family $\{\mathscr{M}_a(t)\}_{t\in D^*}$ of CR-manifolds parameterized by the punctured disk D^*. We wish to extend this to a differentiable family parameterized by D. The first step is to define a limiting CR-manifold $\mathscr{M}_a(0)$.

As before (see Subsect. 2.1), let $\mathscr{F}_a^\pm$ denote the space of equivalence classes of flat connections on $E^\pm$ over $\Sigma^\pm$ of holonomy $a\in \mathbf{G}_R = \mathbf{G}\backslash\{\pm I\}$. Then $\mathscr{F}_a^\pm$ is a smooth manifold of dimension $6g_\pm - 3$ with a smooth $\mathbf{T}_a$-action. Moreover,

$$T_{[\nabla^\pm]}\mathscr{F}_a^\pm = \ker \delta^0_{\nabla^\pm} \oplus \nabla_\pm \mathfrak{t}_a , \tag{4.9}$$

where $\delta^0_{\nabla^\pm}$ is the operator $(\nabla^\pm, (\nabla^\pm)^*)$ of [D-W1, Proposition 3.5]. Of course, under the usual identification between real 1-forms and complex (0, 1)-forms (see Subsect. 3.3), $\ker \delta^a_{\nabla^\pm}$ corresponds to the L^2-ker $\bar\partial^*_{\nabla^\pm}$. Let

$$\mathscr{M}_a(0) = \mathscr{F}_a^+ \times_{\mathbf{T}_a} \mathscr{F}_a^- .$$

Given $[\nabla_0] = [(\nabla^+, \nabla^-)] \in \mathscr{M}_a(0)$, let

$$\mathscr{S}^a_{[\nabla_0]}(0) = \ker \delta^0_{\nabla^+} \times \ker \delta^0_{\nabla^-} . \tag{4.10}$$

Under the identification (4.9), $\mathscr{S}^a(0)$ defines a subbundle of $T\mathscr{M}_a(0)$ satisfying properties (i) and (ii) of Definition 4.2, hence it defines a CR-structure on $\mathscr{M}_a(0)$. We refer to $\mathscr{S}^a(0)$ as the *limiting* CR-*structure* on $\mathscr{M}_a$. We now prove

(4.11) **Theorem.** *Let* $\mathfrak{M}_a = \bigcup_{t\in D}\mathscr{M}_a(t)$. *Then* $\mathfrak{M}_a$ *is a differentiable family of* CR-*manifolds parameterized by the disk D.*

The rest of this subsection is devoted to the proof of Theorem 4.11.

Choose $\nabla_0 = (\nabla^+, \nabla^-)$, a flat connection on $E_0 \to \Sigma_0 = \Sigma^+ \cup \Sigma^-$ such that in a neighborhood of the punctures

$$\nabla^\pm = d \pm i\hat\alpha d\theta_\pm \tag{4.12}$$

with $\exp(2\pi i\hat\alpha) = a$. Without loss of generality we may assume that this neighborhood contains the pinching region $\{0 < |z_\pm| < 1\}$. Let ∇_t be the connection on $E_t \to \Sigma_t$ constructed by gluing ∇^+ and ∇^- as in Subsect. 3.1. Then ∇_t has the form (4.12) in the pinching region. Let $T_{[\nabla_t]}\mathscr{M}(t) \simeq K_{\nabla_t}(t)$ and $T_{[\nabla_t^\pm]}\mathscr{M}^\pm(t) \simeq K_{\nabla_t^\pm}(t)$ be the tangent spaces of $\mathscr{M}(t)$ and $\mathscr{M}^\pm(t)$, respectively, viewed as subspaces of

$$L^2(T^*\Sigma_t \otimes \mathfrak{g}_{E_t}) = L^2(T^*\Sigma_t^+ \otimes \mathfrak{g}_{E_t^+}) \oplus L^2(T^*\Sigma_t^- \otimes \mathfrak{g}_{E_t^-})$$

the usual way (see Subsect. 2.2). On the other hand we can apply the map (3.22) as in Corollary 3.24(ii) to view $K_{\nabla_t}(t)$ and $K_{\nabla_t^\pm}(t)$ as subspaces of $L^2_{-\delta}(T^*\Sigma_0 \otimes \mathfrak{g}_E)$. With this understood, the next proposition is immediate from Corollary 3.24.

(4.13) **Proposition.** *The following are smooth families of vector spaces in* $L^2_{-\delta}(T^*\Sigma_0 \otimes \mathfrak{g}_E)$:

(i) $K_{\nabla_t}(t) \to \ker D_0$, *where* D_0 *is the operator in Subsect.* 3.2;

(ii) $K^{\pm, \mathrm{neu}}_{\nabla_t^\pm}(t) \to \ker \delta_0^\pm$, *where* $\delta_0^\pm$ *is the operator in Subsect.* 3.3.

Next we are going to restrict to a fixed $a \in \mathbf{G}_R$. Let $K^a_{\tilde V_t}(t)$, $K^{a,\pm,\mathrm{neu}}_{\tilde V^\pm_t}$ be the restrictions of $K_{\tilde V_t}(t)$, $K^{a,\pm,\mathrm{neu}}_{\tilde V^\pm_t}$ to $\mathscr{M}_{0,a}$, $\mathscr{M}^{\pm}_{0,a}$. Of course, we have the identifications

$$(4.14) \qquad T_{[V_t]}\mathscr{M}_a(t) = K^a_{\tilde V_t}(t)$$
$$T_{[V^\pm_t]}\mathscr{M}^\pm_a(t) = K^{a,\pm,\mathrm{neu}}_{\tilde V^\pm_t} .$$

We start with the following obvious

(4.15) **Lemma.** *Let* $h_C: \mathscr{A}_F$ *(resp.* $\mathscr{A}^\pm_F$*)* $\to [0, 1/2]$ *be the holonomy map* (2.10) *measuring holonomy around the circle* $C = C_1$ *(resp.* $C^\pm_1$*). Then*
(i) $K^a_{\tilde V_t}(t) = K_{\tilde V_t}(t) \cap \ker(h_C)_*$;
(ii) $K^{a,\pm,\mathrm{neu}}_{\tilde V^\pm_t}(t) = K^{\pm,\mathrm{neu}}_{\tilde V^\pm_t}(t) \cap \ker(h_C)_*$.

(4.16) **Corollary.** *Let*

$$K^a_{\tilde V}(0) = \{\beta \in \ker D_0 : (h_C)_*\beta = 0\} ,$$
$$K^{a,\pm}_{\tilde V^\pm}(0) = \{\beta \in \ker \delta^\pm_0 : (h_C)_*\beta = 0\} ,$$

Then the following families are smooth in t.
(i) $K^a_{\tilde V_t}(t) \to K^a_{\tilde V_0}(0)$;
(ii) $K^{a,\pm,\mathrm{neu}}_{\tilde V^\pm_t}(t) \to K^{a,\pm}_{\tilde V^\pm}(0)$.

Proof. Immediate from Proposition 4.13 and Lemma 4.15.

Recall that we are interested in the limiting behavior of $\mathscr{S}^a_{[V_t]}(t)$ as $t \to 0$. Under the identification (4.14), $\mathscr{S}^a_{[V_t]}(t)$ corresponds to the maximal complex subspace $\mathscr{H}^a_{\tilde V_t}(t)$ of $K^a_{\tilde V_t}(t)$. Also, let $\mathscr{H}^a_{\tilde V_0}(0)$ be the maximal complex subspace of $K^a_{\tilde V_0}(0)$. We first show

(4.17) **Corollary.** $\mathscr{H}^a_{\tilde V_t}(t) \to \mathscr{H}^a_{\tilde V_0}(0)$ *is a smooth family of complex vector spaces as* $t \to 0$.

Proof. By Proposition 4.13, $K_{\tilde V_t} \to \ker D_0$ is a smooth family of complex vector spaces as $t \to 0$. Then if J_0 denotes the complex structure on the vector space $\ker D_0$, $*_t \to J_0$ is smooth family of vectors in the smooth family of vector spaces

$$\operatorname{Hom}(K_{\tilde V_t}(t), K_{\tilde V_t}(t)) \to \operatorname{Hom}(\ker D_0, \ker D_0) .$$

The result now follows from Corollary 4.16.

In the next proposition we identify the various spaces in terms of the corresponding $\bar\partial$-operator.

(4.18) **Proposition.** *Under the obvious identification between real* 1*-forms and complex* (0, 1) *forms,*
(i) $K^{a,\pm}_{\tilde V^\pm}(0) = L^2\text{-}\ker \bar\partial^*_{\tilde V^\pm}$;
(ii) $\mathscr{H}^a_{\tilde V_0}(0) = L^2\text{-}\ker \bar\partial^*_{\tilde V^+} \oplus L^2\text{-}\ker \bar\partial^*_{\tilde V^-}$;
(iii) $K^{a,\pm}_{\tilde V^\pm}$ *can naturally be identified with* $K^{a,\pm}_{\tilde V^\pm}(0) = L^2\text{-}\ker \bar\partial^*_{\tilde V^\pm}$. *In particular, the* CR*-manifolds* $\mathscr{M}_a(0)$ *with the limiting* CR*-structure are mutually* CR*-isomorphic for all* $a \in \mathbf{G}_R = \mathbf{G}\backslash\{\pm I\}$.

Proof. (i) $\beta \in K^a_{\tilde V^\pm}(0) \Leftrightarrow \beta = u - *u$, where u is an anti-holomorphic $\mathfrak{g}^{\mathbf{C}}_E$-valued one-form with residue upper (resp. lower) triangular at $p^\pm_0$ and satisfying the Neumann matching condition (3.15) *and* the condition $(h_C)_*\beta = 0$. By Corollary 3.21, $\beta = u - *u$, where u is an anti-meromorphic one-form for which diagonal part of the residue vanishes, hence $u \in L^2\text{-}\ker \bar\partial^*_{\tilde V_0}$ by [D-W1, Subsect. 3.3]. (ii) $\beta \in \mathscr{H}^a_{\tilde V_0}(0) \Leftrightarrow \beta = u - *u$, where u is anti-holomorphic $\mathfrak{g}^{\mathbf{C}}_E$ with residues as above. The conditions $(h_C)_*\beta = (h_C)_*(*\beta) = 0$ are equivalent by Proposition 3.17 to the vanishing of the diagonal part of the residue. Therefore the result follows again by [D-W1, Subsect. 3.3]. (iii) The first statement in (iii) follows immediately from the definition of $K^{a,\pm}_{V^\pm}$ given in Subsect. 2.3. The second statement follows by (4.9), (4.10), the identification $\ker \delta^0_{V^\pm} = L^2\text{-}\ker \bar\partial^*_{V^\pm}$, and the fact that the derivative of the natural identification between $\mathscr{F}^\pm_a$ and $\mathscr{F}^\pm_{-I}$ induces the identification $K^{a,\pm}_{V^\pm} = K^{a,\pm}_{V^\pm}(0)$.

Proof of Theorem 4.11. Let $\Psi^a_t: \mathscr{M}_a(t) \to \mathscr{M}_a(0)$ be defined as follows: Let $[V_t] \in \mathscr{M}(t)$. Choose representatives V_t such that in the pinching region V_t has the form (4.12). Extend constantly to Σ_0 and take $\Psi^a_t[V_t]$ to be the equivalence class of the extension in $\mathscr{M}_0$. Clearly, Ψ^a_t defines a diffeomorphism between $\mathscr{M}_a(t)$ and $\mathscr{M}_a(0)$. Moreover, $\Psi^a = \{\Psi^a_t\}_{t\in D^*}$ defines a bijection between $\bigcup_{t\in D^*}\mathscr{M}_a(t)$ and the trivial family $\mathscr{M}_a(0)\times D^*$. By declaring Ψ^a to be a diffeomorphism, this defines a C^∞-structure on $\bigcup_{t\in D^*}\mathscr{M}_a(t)$, and the family $\{\mathscr{M}_a(t)\}_{t\in D^*}$ extends via this diffeomorphism to the trivial family $\mathscr{M}_a(0)\times D$. In order to analyze the family of CR-structures, we have first to understand the derivative $(\Psi^a_t)_*$ of Ψ^a_t. It is not difficult to see that under our identification

$$T_{[V_t]}\mathscr{M}_a(t) = K^a_{V_t}(t) \subset L^2_{-\delta}(T^*\Sigma_t \otimes \mathfrak{g}_E)$$
$$T_{[V_0]}\mathscr{M}_a(0) = K^a_{V_0}(0) \subset L^2_{-\delta}(T^*\Sigma_0 \otimes \mathfrak{g}_E)\,,$$

$(\Psi^a_t)_*$ corresponds to the inclusion of $K^a_{V_t} \subset L^2_{-\delta}(T^*\Sigma_0 \otimes \mathfrak{g}_E)$ followed by harmonic projection onto $K^a_{V_0}(0)$. Corollary 4.17, together with the fact that harmonic projection equals the identity map on $K^a_{V_0}(0)$, imply that

$$[(\Psi^a_t)^{-1}]^*\mathscr{S}^a_{[V_t]} \to \mathscr{S}^a_{[V_t]}(0)$$

is smooth as $t\to 0$. Therefore $\{\mathscr{S}^a(t)\}_{t\in D}$ is a smooth family of complex vector bundles, proving the theorem.

We conclude this subsection with two remarks: First observe that the normal bundle $\mathscr{S}^{a,\perp}(t)$ to $\mathscr{S}^a(t)$ in $\mathscr{M}_a(t)$ with respect to the induced L^2-metric on $\mathscr{M}(t)$ defines also a differentiable family of real line bundles. Moreover, the limiting bundle $\mathscr{S}^{a,\perp}(0)$ can be used to define a distinguished transverse direction to the diagonal action of the gauge group on $\mathscr{F}^+_a \times \mathscr{F}^-_a$. This will be further explained in Lemma 4.26. The S^1 action is CR on $\mathscr{M}_a(0)$, i.e. it preserves the bundle $\mathscr{S}^a(0)$, but it does not do so on $\mathscr{M}_a(t)$ for $t \neq 0$ (see Subsect. 5.2).

The second remark concerns the dependence of the families $\mathscr{M}_a$ upon the parameter a. It is not hard to see that the families $\mathfrak{M}_a$ glue together for all $a \in \mathbf{G}_R/\mathbf{G}_{\mathrm{adj}} \simeq (0, 1/2)$ to define a differentiable family of complex manifolds parameterized by the disk. The limiting complex structure on $\bigcup_{a\in(0,1/2)}\mathscr{M}_a(0)$ will be explicitly constructed in Sect. 5 (see Proposition 5.10).

Since we will not use these facts in the rest of the paper, we shall not elaborate further.

4.3 Indefiniteness of the Levi form

The purpose of this subsection is to prove the following properties of the Levi form for the CR-manifolds $\mathscr{M}_a(t)$:

(4.19) Theorem. *The Levi form $\mathbb{L}_a(0)$ of $\mathscr{M}_a(0)$ has everywhere at least two positive and two negative eigenvalues.*

(4.20) **Corollary.** *Given $a \in \mathbf{G}_R$, there is an $\varepsilon_0 > 0$ such that for all t satisfying $0 < |t| < \varepsilon_0$, the Levi form $\mathbb{L}_a(t)$ of $\mathcal{M}_a(t)$ has everywhere at least two positive and two negative eigenvalues.*

In the language of Folland and Kohn (cf. [F-K, p. 94], and Definition 5.1 below), the statement that the Levi form has everywhere two pairs of non-zero eigenvalues of opposite sign means that $\mathcal{M}_a(t)$ *satisfies condition* $Y(1)$. As we shall see in Sect. 5, CR-manifolds of this type have nice properties.

Proof of Theorem 4.19 and Corollary 4.20. First observe that Theorems 4.11 and 4.19 together immediately imply Corollary 4.20, so we shall prove Theorem 4.19. Start with a basis $\{X_i^\pm\}_{i=1}^{3g_\pm-2}$ of local vector fields on $\mathcal{M}_a^\pm(0)$ of type (1, 0) (recall that $\mathcal{M}_a^\pm(0)$ has a complex structure via the theorem of Mehta and Seshadri (see [D-W1, Sects. 2 and 3]). Use the universal connection $K^{a,\pm}$ to define horizontal lifts $\tilde{X}_i^\pm$ of $X_i^\pm$ to $\mathcal{F}_a^\pm$ (see Subsect. 2.3). Set $\tilde{X}_i$, $1 \leqq i \leqq 3g-4$ to be $\tilde{X}_i = \tilde{X}_i^+$ for $1 \leqq i \leqq 3g_+ - 2$, and $\tilde{X}_i = \tilde{X}_{i-3g_++2}^-$ for $3g_+ - 2 < i \leqq 3g-4$. Let

$$p : \mathcal{F}_a^+ \times \mathcal{F}_a^- \to \mathcal{M}_a(0) = \mathcal{F}_a^+ \times \mathcal{F}_a^- / \mathbf{T}_a \tag{4.21}$$

be the natural quotient map, and let

$$L_i = p_* \tilde{X}_i\,. \tag{4.22}$$

By (4.10) and Lemma 4.18(iii), $\{L_i\}$ is a local basis of vector fields of $\mathcal{S}^a(0) \subset T\mathcal{M}_a(0)$. Let N be a local, purely imaginary vector field of $\mathcal{M}_a$ such that $\{L_i, \bar{L}_j, N\}$ span $T\mathcal{M}_a \otimes \mathbb{C}$, and let Y be a local vector field of $\mathcal{F}_a^+ \times \mathcal{F}_a^-$ such that

$$p_* Y = N\,. \tag{4.23}$$

In terms of a local basis of vertical vector fields $\mathbf{t}_+$ and $\mathbf{t}_-$ for the fibrations $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$, we can take without loss of generality, $Y = \pi\sqrt{-1}(\mathbf{t}_+ - \mathbf{t}_-)$. By setting $\Omega^\pm = \operatorname{Curv}(K^{a,\pm})$, we obtain

$$[\tilde{X}_i^\pm, \bar{\tilde{X}}_j^\pm] = \Omega^\pm(X_i^\pm, \bar{X}_j^\pm)\mathbf{t}_\pm \quad \text{mod horizontal } (K^{a,\pm})\,.$$

Let $\mathbf{t}$ be the local vector field $\mathbf{t} = (\mathbf{t}_+, \mathbf{t}_-)$ on $\mathcal{F}_a^+ \times \mathcal{F}_a^-$. Then for $1 \leqq i, j \leqq 3g_+ - 2$,

$$[\tilde{X}_i, \bar{\tilde{X}}_j] = \frac{1}{2\pi\sqrt{-1}}\Omega^+(X_i^+, \bar{X}_j^+)Y \quad \operatorname{mod}(K^{a,+} \times K^{a,-} \oplus \mathbb{C}\cdot\mathbf{t})\,,$$

and for $3g_+ - 2 < i, j < 3g - 4$,

$$[\tilde{X}_i, \bar{\tilde{X}}_j] = \frac{-1}{2\pi\sqrt{-1}}\Omega^-(X_i^-, \bar{X}_j^-)Y \quad \operatorname{mod}(K^{a,+} \times K^{a,-} \oplus \mathbb{C}\cdot\mathbf{t})\,,$$

whereas for all other values of i, j, $[\tilde{X}_i, \bar{\tilde{X}}_j] = 0$. Therefore, since $p_*(\mathbf{t}) = 0$ in $\mathcal{M}_a(0)$, we obtain from (4.10), Proposition 4.18(iii), (4.22), and (4.23) that with respect to the local basis of vector fields $L_i, \bar{L}_j, N$, the Levi form $\mathbb{L}(0)$ splits as a block diagonal matrix

$$(c_{ij}) = \frac{1}{2\pi\sqrt{-1}}\begin{pmatrix} \Omega^+(X_i^+, \bar{X}_j^+) & 0 \\ 0 & -\Omega^-(X_i^-, \bar{X}_j^-) \end{pmatrix}. \tag{4.24}$$

Theorem 4.19 then follows from Corollary 2.17.

4.4 Extension of CR*-sections*

Let Δ^k denote the k-th power of the determinant line bundle on $\mathcal{M}$, and let $H^0(\mathcal{M}, \Delta^k)$ denote its space of global holomorphic sections. Strictly speaking, Δ^k is only defined on the non-singular part $\mathcal{M}^s$ of $\mathcal{M}$, but it extends uniquely as an invertible sheaf to the whole of $\mathcal{M}$ (cf. [Dr-N]). Fix $a \in \mathbf{G}_R = \mathbf{G} \setminus \{\pm I\}$ and let $\mathcal{M}_a \subset \mathcal{M}$ be the hypersurface defined as in the previous sections. Let $H^0_{\mathrm{CR}}(\mathcal{M}_a, \Delta^k)$ denote the space of global CR-sections of the restriction of Δ^k to $\mathcal{M}_a$ (cf. Definition 4.5). Since $\mathcal{M}_a$ has *real* codimension 1 in $\mathcal{M}^s$, the restriction map

$$\rho : H^0(\mathcal{M}, \Delta^k) \to H^0_{\mathrm{CR}}(\mathcal{M}_a, \Delta^k)$$

is injective. The main goal of this section is to prove

(4.25) **Theorem.** *For sufficiently small choice of the degeneration parameter t, the restriction map ρ is an isomorphism.*

The key to the proof of Theorem 4.25 is the following technical generalization of Corollary 4.20.

(4.26) **Lemma.** *Given $a_0 \in (0, 1/2)$ there exists an $\varepsilon > 0$ and a positive smooth function $\tilde{r}$ on $r^{-1}(0, 1/2)$ such that*

$$r^{-1}(0, a_0) \subset \{\tilde{r} < 1\},\ r^{-1}(a_0, 1/2) \subset \{\tilde{r} > 1\}\ , \tag{4.27}$$

and such that for all $0 < |t| < \varepsilon$, the complex hessian $\partial\bar{\partial}\tilde{r}$ has at least two positive eigenvalues everywhere in $r^{-1}(0, a_0)$ and at least two negative eigenvalues in $r^{-1}(a_0, 1/2)$.

Proof of Theorem 4.25, assuming Lemma 4.26. It suffices to show that for sufficiently small $|t|$, the map ρ is surjective. Let $a_0 \in (0, 1/2)$ and fix $0 < |t| < \varepsilon$ as in Lemma 4.26. By the Lewy Extension Theorem (cf. [W]), any CR-section φ of Δ^k on $\mathcal{M}_{a_0}$ extends to a local holomorphic section of Δ^k defined in a neighborhood $\mathcal{U}$ of $\mathcal{M}_{a_0}$. We may without loss of generality assume $\mathcal{U} = r^{-1}(a_0 - \eta, a_0 + \eta)$. It follows by Lemma 4.26 and the Andreotti-Grauert version of Hartog's Theorem (cf. [He-Le, Theorem 15.11]) that the section can in fact be extended all the way from $\mathcal{U}$ to $r^{-1}(0, 1/2)$. The idea is as follows: Let

$$\alpha_- = \inf\{\alpha > 0 : \varphi \text{ extends holomorphically to } r^{-1}(\alpha, \alpha_0)\}$$

$$\alpha_+ = \sup\{\alpha < 1/2 : \varphi \text{ extends holomorphically to } r^{-1}(\alpha_0, \alpha)\}\ .$$

Then in particular φ extends holomorphically to $r^{-1}(\alpha_-, \alpha_+)$. If $\alpha_- \neq 0$ or $\alpha_+ \neq 1/2$, we again apply the Lewy Extension Theorem to extend φ to a larger interval, contradicting the choice of $\alpha_\pm$. Finally, since $r^{-1}\{0, 1/2\}$ has real codimension 3 in $\mathcal{M}^s$, the section extends to the whole of $\mathcal{M}^s$ by a theorem of Shiffman (cf. [Sh]). This completes the proof of the surjectivity of the map ρ, and therefore also of Theorem 4.25.

Proof of Lemma 4.26. By setting $\tilde{r} = (1 - \cos 2\pi r)/(1 - \cos 2\pi a_0)$ it is easy to verify that $\tilde{r}$ satisfies (4.27). We are going to show that $\partial\bar{\partial}\tilde{r}$ has at least two positive eigenvalues in $\{\tilde{r} < 1\}$ and at least two negative eigenvalues in $\{\tilde{r} > 1\}$. In view of (4.8) and Corollary 4.20 we relate $\partial\bar{\partial}\tilde{r}$ to the Levi form of $\mathcal{M}_a$, $0 < \alpha < 1/2$. However, in Subsect. 4.1 the Levi form was only defined up to a scalar, so we have to fix signs. In particular, we must choose the vector N carefully. According to

Wells [W, p. 150], the complex hessian $\partial\bar{\partial}\tilde{r}$ is equal to the Levi form c_{ij} (up to an overall positive multiple) provided the choice of N satisfies

$$J(\sqrt{-1}N) = \frac{\partial}{\partial \tilde{r}} \mod T\mathscr{M}_a ,$$

(up to an overall positive multiple) where J denotes the complex structure on $\mathscr{M}$. Since $\partial/\partial r$ and $\partial/\partial\tilde{r}$ are also related by a positive scalar, it suffices to take

$$J(\sqrt{-1}N) = \frac{\partial}{\partial r} \mod T\mathscr{M}_a . \tag{4.28}$$

Let us verify that with respect to the natural orientation on $\mathscr{F}_a$ and the moduli space $\mathscr{M}$, the choice of N (and Y, see (4.23)) used in the proof of Theorem 4.19 satisfies (4.28). Let $\mathscr{M}_0$ be as in Subsect. 2.1 and $\mathbf{t}_\pm$ as in Subsect. 4.3. Let $X = \mathscr{M}_a^+ \times \mathscr{M}_a^-$, and let $\mathscr{F}_a = \mathscr{F}_a^+ \times \mathscr{F}_a^-$ have the product orientation. In addition, we decompose the tangent space to $\mathbf{G}$ at a as $\mathfrak{g} = \mathbb{C}\cdot\partial/\partial r \oplus \mathfrak{h}_a$. Then on the one hand,

$$\begin{aligned} T\mathscr{M}_0 &= T\mathscr{F}_a \oplus \mathfrak{g} \\ &= T\mathscr{F}_a \oplus \mathbb{C}\cdot\frac{\partial}{\partial r} \oplus \mathfrak{h}_a \\ &= TX \oplus \mathbb{C}\cdot(\mathbf{t}_+ + \mathbf{t}_-) \oplus \mathbb{C}\cdot(-\mathbf{t}_+ + \mathbf{t}_-) \oplus \mathbb{C}\cdot\frac{\partial}{\partial r} \oplus \mathfrak{h}_a . \end{aligned}$$

On the other hand,

$$T\mathscr{M}_0 = T\mathscr{M} \oplus \mathfrak{g} = T\mathscr{M} \oplus \mathbb{C}\cdot(\mathbf{t}_+ + \mathbf{t}_-) \oplus \mathfrak{h}_a .$$

This implies that $J(-\mathbf{t}_+ + \mathbf{t}_-)$ has a component in the $\partial/\partial r$ direction, and since $\{(-\mathbf{t}_+ + \mathbf{t}_-), \partial/\partial r\}$ is an oriented pair this multiple must be positive. Thus, in order to make the signs in the Levi form agree with those in the complex hessian we should set

$$\sqrt{-1}N = \text{positive multiple of } (-\mathbf{t}_+ + \mathbf{t}_-) ,$$

and indeed this is in accord with the choice of N in Subsect. 4.3 (see 4.23).

Now consider the case $\alpha \to 1/2$. In a small neighborhood $\mathscr{V}$ of a point in $r^{-1}(1/2)$, we can choose local vector fields $N(t)$ and a *subset* of the $L_i(t)$, $\bar{L}_j(t)$ converging as $t \to 0$ uniformly in $\mathscr{V}\backslash r^{-1}(1/2)$ (see Corollary 3.25). The limit $N(0)$ may be chosen to be the vector N as above, and the $L_i(0)$, $\bar{L}_j(0)$ may be chosen to span

$$\{T\mathscr{M}^+(2, -1) \oplus T\mathscr{M}^-(2, -1)\} \otimes \mathbb{C}$$

when projected by the maps $\pi_1^\pm$ of the Hecke correspondence (2.14). By (4.24) and Corollary 2.17 we see that the positive eigenvalues of the Levi form have been lost by restricting to these L_i's, but the negative eigenvalues remain. By the uniformity of the convergence and the compactness of the fibers $r^{-1}(\alpha)$, we can find a sufficiently small $|t|$ such that the Levi form has negative eigenvalues everywhere near $r^{-1}(1/2)$. A similar argument applies for the case $\alpha \to 0$. This completes the proof of the lemma.

5 Deformation of CR-sections

In order to identify the space of CR-sections we shall have to degenerate the surface completely, i.e. let $t \to 0$. To prove that the dimension of the space of CR-sections does not jump, we associate a complex to the $\bar{\partial}_b$ operator acting on smooth sections of the line bundle and study the cohomology groups of this complex. This *Kohn-Rossi cohomology* is different from the ordinary Dolbeault cohomology – in particular, finite dimensionality is not at all evident. Nevertheless, under certain assumptions of convexity which, by the results of Sect. 4 hold in the case at hand, many of the usual properties are still valid. In Subsect. 5.1, we define the Kohn-Rossi cohomology groups and review the results concerning the $\bar{\partial}_b$-operator we shall need. We also state a semi-continuity theorem analogous to that of ordinary Dolbeault cohomology. Since its proof is so similar, we relegate it to the Appendix. Then in Subsect. 5.2 we prove the vanishing of the (0, 1) Kohn-Rossi cohomology associated to the boundary complex of the limiting CR-structure on $\mathcal{M}_a$ (Theorem 1.8 of the Introduction). This combined with semi-continuity will show that the dimension of the (0, 0) cohomology, i.e. the CR-sections, does not jump.

5.1 The $\bar{\partial}$-Neumann complex

The basic references for this subsection are [F-K] and [K-R]. Let $D(M)$ be a complex manifold of dimension n with smooth boundary M. We assume that $D(M)$ is imbedded in a slightly larger open manifold $D(M)'$, that $\overline{D(M)} = D(M) \cup M$ is compact, and that M is defined by the equation $r = 0$ where r is a real C^∞ function satisfying $r < 0$ inside $D(M)$, $r > 0$ outside $\overline{D(M)}$, and $|dr| = 1$ on M.

(5.1) **Definition.** We say that M (or $D(M)$) *satisfies condition* $Z(q)$ if the Levi form of M has at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues at each point of M. We say that M (or $D(M)$) *satisfies condition* $Y(q)$ if it satisfies both conditions $Z(q)$ and $Z(n - q - 1)$.

For example, according to Theorem 4.19, $\mathcal{M}_a(0)$ satisfies conditions $Y(0)$ and $Y(1)$.

Let V be a holomorphic vector bundle on $D(M)'$. Let $\tilde{\mathscr{A}}^{p,q} = \tilde{\mathscr{A}}^{p,q}(\overline{D(M)}, V)$ denote the sheaf of germs of $C^\infty(p, q)$-forms on $\overline{D(M)}$ with values in V. Let $\tilde{\mathscr{C}}^{p,q} = \tilde{\mathscr{C}}^{p,q}(D(M), V)$ denote the subsheaf of $\tilde{\mathscr{A}}^{p,q}$ consisting of germs of V-valued forms such that $\varphi \wedge \bar{\partial} r = 0$ (where $\bar{\partial}$ denotes the $\bar{\partial}$-operator on $D(M)'$ – we shall use the same notation for the operator associated to V). Let $\tilde{\mathscr{B}}^{p,q} = \tilde{\mathscr{A}}^{p,q}/\tilde{\mathscr{C}}^{p,q}$ be the quotient sheaf. Then $\tilde{\mathscr{B}}^{p,q} = \tilde{\mathscr{B}}^{p,q}(M, V)$ is a locally free sheaf supported on M. It is easy to see that the $\bar{\partial}$-operator satisfies $\bar{\partial}(\tilde{\mathscr{C}}^{p,q}) \subset \tilde{\mathscr{C}}^{p,q+1}$, and thus by the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \tilde{\mathscr{C}}^{p,q} & \longrightarrow & \tilde{\mathscr{A}}^{p,q} & \longrightarrow & \tilde{\mathscr{B}}^{p,q} & \longrightarrow 0 \\ & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & & \\ 0 \longrightarrow & \tilde{\mathscr{C}}^{p,q+1} & \longrightarrow & \tilde{\mathscr{A}}^{p,q+1} & \longrightarrow & \tilde{\mathscr{B}}^{p,q+1} & \longrightarrow 0 \end{array}$$

it induces a map

$$\bar{\partial}_b : \tilde{\mathscr{B}}^{p,q} \to \tilde{\mathscr{B}}^{p,q+1}\,, \tag{5.2}$$

satisfying $\bar{\partial}_b \circ \bar{\partial}_b = 0$.

Let $\mathscr{A}^{p,q}$, $\mathscr{C}^{p,q}$, and $\mathscr{B}^{p,q}$ denote the space of global sections of the sheaves $\tilde{\mathscr{A}}^{p,q}$, $\tilde{\mathscr{C}}^{p,q}$, and $\tilde{\mathscr{B}}^{p,q}$. Then we obtain complexes $\mathscr{A} = (\mathscr{A}^{p,\cdot}, \bar{\partial})$, $\mathscr{C} = (\mathscr{C}^{p,\cdot}, \bar{\partial})$, and $\mathscr{B} = (\mathscr{B}^{p,\cdot}, \bar{\partial})$, and we will denote by $H_a^{p,q} = H_a^{p,q}(D(M), V)$, $H_c^{p,q} = H_c^{p,q}(D(M), V)$, and $H_b^{p,q} = H_b^{p,q}(M, V)$ the cohomology groups associated to the complexes $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{B}$. These groups form what is known as the *absolute, Dirichlet, and boundary Kohn-Rossi cohomologies of* $D(M)$.

By the long exact sequence in cohomology associated to the short exact sequence of complexes

$$0 \to \mathscr{C}^{p,q} \to \mathscr{A}^{p,q} \to \mathscr{B}^{p,q} \to 0$$

we obtain

$$\ldots \to H_c^{p,q} \to H_a^{p,q} \to H_b^{p,q} \to H_c^{p,q+1} \to \ldots . \tag{5.3}$$

Under condition $Z(q)$ the cohomology groups above have an interpretation via harmonic theory which we shall now explain. Let $\mathscr{D}^{p,q}$ denote the elements $\varphi \in \mathscr{A}^{p,q}$ such that on M, $\iota_{\partial r}(\varphi) = 0$, $\iota_{\partial r}$ denotes interior multiplication by ∂r. The importance of $\mathscr{D}^{p,q}$ lies in the fact that on $\mathscr{D}^{p,q}$, the formal adjoint $\mathfrak{d}$ of $\bar{\partial}$ equals its Hilbert space adjoint $\bar{\partial}^*$. We therefore set Dom $\bar{\partial}^* = \mathscr{D}^{p,q}$. Let $\square = \bar{\partial}\mathfrak{d} + \mathfrak{d}\bar{\partial}$ be the associated Laplacian defined on

$$\text{Dom}\,\square = \{\varphi \in \mathscr{D}^{p,q} : \bar{\partial}\varphi \in \mathscr{D}^{p,q+1}\}\,,$$

and let

$$\mathbf{H}^{p,q} = \mathbf{H}^{p,q}(D(M), V) = \ker \square\ . \tag{5.4}$$

In general, one cannot expect the spaces $\mathbf{H}^{p,q}$ to be finite dimensional. However, *provided that condition $Z(q)$ hold, $\mathbf{H}^{p,q}$ is finite dimensional and isomorphic to $H_a^{p,q}$.* This result, due to Kohn, follows from the analogues of the coercive estimates for elliptic operators which we state precisely in the Appendix (see Theorems A.1 and A.2).

We next turn to a version of the Dolbeault isomorphism due to Andreotti and Hill (cf. [Hör, Theorem 3.4.8; F-K, Theorem 4.3.1; An-H, Theorem 5]). Let $H^q(D(M), \Omega^p(V))$ denote the q-th cohomology of the sheaf of germs of holomorphic $(p, 0)$-forms with values in V on the interior of $D(M)$. Then we have

(5.5) **Theorem** [An-H, Theorem 5]. *If the Levi form of M has at least p positive and q negative eigenvalues, then*

$$\mathbf{H}^{r,s}(D(M), V) \simeq H^s(D(M), \Omega^r(V))\,,$$

for $s < q$ *and* $s > n - p - 1$.

Consider now the Dirichlet cohomology groups $H_c^{p,q}$. As in ordinary de Rham cohomology, the Dirichlet cohomology groups are related to the absolute cohomology groups via Lefschetz duality ($*$-operator). The same holds in our case as well by the following duality theorem of Kohn and Rossi (cf. [K-R, Proposition 6.8], and [F-K, Proposition 5.1.7]):

(5.6) **Theorem.** *If M satisfies condition $Z(q)$, then*

$$H_c^{n-p,n-q}(D(M), V) \simeq [H_a^{p,q}(D(M), V^*)]^*\,.$$

Finally, we turn to the groups $H_b^{p,q}$ which are our primary interest. Let $\vartheta_b : \mathscr{B}^{p,q} \to \mathscr{B}^{p,q-1}$ be the formal adjoint of $\bar{\partial}_b$ (hence also the Hilbert space adjoint of $\bar{\partial}_b$, since M has no boundary). Let $\Box_b = \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b$ be the associated Laplacian, and let

$$(5.7) \qquad \mathbf{H}_b^{p,q} = \mathbf{H}_b^{p,q}(M, V) = \ker \Box_b .$$

Thus $\mathbf{H}_b^{0,0} = H_{\mathrm{CR}}^0(M, V)$, the space of global CR-sections of V. The operator $\Box_b$ is not elliptic. However, *under the condition* $Y(q)$, $\mathbf{H}_b^{p,q}$ *is finite dimensional and is isomorphic to* $H_b^{p,q}$. For statements of the precise estimates for $\Box_b$, we refer to the Appendix (see Theorems A.3 and A.4).

The finite dimensional $H_b^{p,q}$ also satisfy a semi-continuity result. Recall from Sect. 4 the definitions of a differentiable family of CR-structures and CR-line bundles.

(5.8) **Theorem.** *Let $M(t)$ be a differentiable family of compact* CR*-manifolds, $L(t)$ a differentiable family of* CR*-hermitian line bundles, and suppose that for all $t \in D$, $M(t)$ satisfies the conditions $Y(0)$ and $Y(1)$. Then*

$$\dim H_{\mathrm{CR}}^0(M(t), L(t)), \dim H_b^{0,1}(M(t), L(t))$$

are upper semi-continuous functions of $t \in D$. Moreover, if $\dim H_b^{0,1}(M(t), L(t))$ is constant for t in some neighborhood of the origin, then the same is true for $\dim H_{\mathrm{CR}}^0(M(t), L(t))$.

The proof of Theorem 5.8 is not difficult; it consists of verifying that all the necessary ingredients for the proof of the ordinary semi-continuity theorem for Dolbeault cohomology hold in the case of CR-manifolds with positive and negative eigenvalues for the Levi form. For the details, we refer to the Appendix.

5.2 A vanishing theorem

The goal of this section is to provide a proof of Theorem 1.8 of the Introduction. We begin by realizing $\mathcal{M}_a = \mathcal{M}_a(0)$ (with the limiting CR-structure) as the boundary of a complex manifold.

(5.9) **Lemma.** *Let $X = \mathcal{M}_a^+ \times \mathcal{M}_a^-$ with the complex structure induced by the theorem of Mehta-Seshadri. Then $\mathcal{M}_a$ is a principal S^1-bundle over X such that*
(i) *the S^1 action on $\mathcal{M}_a$ is* CR;
(ii) *the projection $\pi : \mathcal{M}_a \to X$ is* CR;
(iii) *the map $d\pi$ restricted to the complex subbundle $\mathscr{S}$ is surjective.*

Proof. The first statement follows from the identification of $\mathcal{M}_a$ with $\mathscr{F}_a^+ \times \mathscr{F}_a^- / \mathbf{T}_a$ where $\mathbf{T}_a$ acts diagonally (see Corollary 2.6 and Proposition 2.7). The map π is then just projection onto the first and second factors. The rest of the lemma follows from the definition (4.10), Proposition 4.18(i), and the fact that $\mathrm{K}^{a,\pm}$ define connections on $\mathscr{F}_a^\pm$ (see Subsect. 2.3).

By crossing $\mathcal{M}_a$ with the positive real axis and coupling $\mathscr{S}^a$ with the complex structure on $S^1 \times \mathbb{R}^+$, it follows exactly as in Rossi [R, Proposition 2.3], that $\mathbb{C}^*(\mathcal{M}_a) = \mathcal{M}_a \times \mathbb{R}^+$ has the structure of a holomorphic $\mathbb{C}^*$-bundle over X. The embedding

$$\mathcal{M}_a \hookrightarrow \mathbb{C}^*(\mathcal{M}_a) : x \mapsto (x, 1)$$

is furthermore a CR-embedding. If we restrict to the interval (0, 1], then we have a punctured disk bundle over X which can, by the Riemann Extension Theorem, be completed to a holomorphic disk bundle. Thus we have

(5.10) **Proposition.** *There exists a holomorphic disk bundle $D(\mathcal{M}_a) \xrightarrow{\pi} X$ where $\mathcal{M}_a$ is* CR-*isomorphic to $\partial D(\mathcal{M}_a)$. The restriction of the map π to the boundary is isomorphic to the S^1-bundle in Lemma* 5.9.

Associated to $\mathcal{M}_a \to X$ we have a holomorphic line bundle $\varepsilon \to X$ obtained by taking the standard representation of the S^1 fiber, and it is apparent from Proposition 2.16 that in terms of the generators $\beta_\pm^*, \gamma_\pm^*$ of $H^2(X, \mathbb{Z})$, $c_1(\varepsilon) = \gamma_+^* + \gamma_-^*$. Given a holomorphic line bundle $L \to X$ and regarding $D(\mathcal{M}_a)$ as a complex manifold with boundary $\mathcal{M}_a$, we can form the complexes $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ with values in $\pi^* L$ as in Subsect. 5.1. Since the Levi form is intrinsic, Theorem 4.19 implies that $D(\mathcal{M}_a)$ satisfies conditions $Y(0)$ and $Y(1)$. We are now prepared to prove the main result of this section.

(5.11) **Theorem.** *Let L be a holomorphic line bundle on X, and assume that for all integers λ, $H^1(X, L \otimes \varepsilon^\lambda) = 0$. Then $H_b^{0,1}(\mathcal{M}_a, \pi^* L) = 0$.*

Proof. From the exact sequence (5.3) it suffices to show

$$H_a^{0,1}(D(\mathcal{M}_a), \pi^* L) = H_c^{0,2}(D(\mathcal{M}_a), \pi^* L) = 0 .$$

By Definition 5.1, the condition $Y(1)$ implies $Z(1)$ and $Z(n-2)$, where $n = \dim_{\mathbb{C}} D(\mathcal{M}_a)$. Hence by duality, Theorem 5.6,

$$H_c^{0,2}(D(\mathcal{M}_a), \pi^* L) = [H_a^{n,n-2}(D(\mathcal{M}_a), \pi^* L^*)]^* .$$

Applying Theorem 5.5, it suffices to prove that

$$H^1(D(\mathcal{M}_a), \pi^* L) = H^{n-2}(D(\mathcal{M}_a), K_{D(\mathcal{M}_a)} \otimes \pi^* L^*) = 0 .$$

We shall apply a degenerate case of the Leray spectral sequence (cf. [Gd, II, 4.17.1]). Since $D(\mathcal{M}_a)$ is a holomorphic disk bundle and $H^q(D, \mathcal{O}) = 0$ for $q > 0$, the higher direct images $R^q \pi_* \pi^* L$, $q > 0$, vanish. The image $\pi_* \mathcal{O}_{D(\mathcal{M}_a)}$ of the structure sheaf of $D(\mathcal{M}_a)$ has infinite rank, and it is easy to see that it is the direct limit as $m \to \infty$ of the finite rank bundles

$$\bigoplus_{0 \leq \lambda \leq m} \varepsilon^\lambda .$$

Since cohomology commutes with direct limits (cf. [H, III, Proposition 2.9]), the Leray spectral sequence implies that $H^1(D(\mathcal{M}_a), \pi^* L)$ is isomorphic to the direct limit of

$$\bigoplus_{0 \leq \lambda \leq m} H^1(X, L \otimes \varepsilon^\lambda) ,$$

and this vanishes by assumption. For $K_{D(\mathcal{M}_a)} \otimes \pi^* L^*$, note that $K_{D(\mathcal{M}_a)} = \pi^* (K_X \otimes \varepsilon)$. Then by the same argument as above $H^{n-2}(D(\mathcal{M}_a), K_{D(\mathcal{M}_a)} \otimes \pi^* L^*)$ is isomorphic to the direct limit of

$$\bigoplus_{0 \leq \lambda \leq m} H^{N-1}(X, K_X \otimes L^* \otimes \varepsilon^{\lambda+1}) ,$$

where $N = n - 1 = \dim_{\mathbb{C}} X$. If we apply Serre duality to each term, we find that

$$H^{N-1}(X, K_X \otimes L^* \otimes \varepsilon^{\lambda+1}) = [H^1(X, L \otimes \varepsilon^{-(\lambda+1)})]^* ,$$

and this again vanishes by assumption. This completes the proof of the theorem.

It is interesting to compare the disk bundle constructed in this section with the Hopf bundles over projective spaces. For computations of the Kohn-Rossi cohomology in that case, we refer to Folland [F].

6. Factorization of theta functions

In this final section we apply the results of Sects. 4 and 5 to prove the Main Theorem

6.1. Factorization of line bundles with connections

Here we briefly review the cocycle construction of the determinant line bundles we are interested in (cf. [R-S-W]). Let $E_t \to \Sigma_t$ be the trivial rank two bundle on the glued surface Σ_t, and let $\mathscr{A}_F, \mathscr{G}$ be the C^∞ flat connections and gauge transformations as in Subsect. 2.1. Let $\tilde{\mathscr{G}}$ denote the space of smooth paths $\tilde{g}: [0, 1] \to \mathscr{G}$ with $\tilde{g}(0) = I$. The evaluation map $e_1: \tilde{\mathscr{G}} \to \mathscr{G}$ defined by $e_1(\tilde{g}) = g(1)$ is smooth and surjective. We then define a map

(6.1)
$$\begin{gathered} c: \mathscr{A}_F \times \tilde{\mathscr{G}} \to \mathrm{U}(1) \\ c(A, \tilde{g}) = \exp\left\{\frac{i}{4\pi} \int_{\Sigma_t} \mathrm{Tr}(A dg g^{-1}) - \frac{i}{12\pi} \int_{\Sigma_t \times [0,1]} \mathrm{Tr}(\tilde{d}\tilde{g}\tilde{g}^{-1})^3\right\} \end{gathered}$$

where $g = e_1(\tilde{g})$ and $\tilde{d} = d + \frac{d}{dt}$. One can show that $c(A, \tilde{g})$ actually only depends on $e_1(\tilde{g})$, and therefore descends to a map $c: \mathscr{A}_F \times \mathscr{G} \to \mathrm{U}(1)$ which satisfies the cocycle condition $c(A, gh) = c(A^g, h)\, c(A, g)$. We define a line bundle $L(t)$ on $\mathscr{A}_F^s/\mathscr{G} = \mathscr{M}^s$ whose smooth sections are complex valued functions s on $\mathscr{A}_F^s$ satisfying $s(A^g) = c(A, g)s(A)$. In [D-W1] it was shown how the very same cocycle (6.1) can be used to define line bundles $L_\pm \to \mathscr{F}_a^\pm$. Since the cocycle is the identity on constant gauge transformations, the equation

(6.2)
$$t \cdot s_\pm(A_\pm) = s_\pm(A_\pm^t)$$

for $s_\pm$ a section of $L_\pm$ and $t \in \mathbf{T}_a$, defines a smooth action of T_a on sections of $L_\pm$. Hence the line bundles descend to $\mathscr{M}_a^\pm$ by taking as the sheaf of smooth sections those smooth sections of $L_\pm$ which are invariant under the action (6.2). Similarly, by taking the diagonal action of $\mathbf{T}_a$ we obtain a line bundle

$$L(0) \to \mathscr{M}_a = \mathscr{F}_a^+ \times \mathscr{F}_a^- / \mathbf{T}_a .$$

(6.3) Lemma. *Let $\Psi_t^a: \mathscr{M}_a(t) \to \mathscr{M}_a(0)$ denote the diffeomorphism in the proof of Theorem* 4.11, *$L(t) \to \mathscr{M}_a(t)$ the restriction of the line bundle defined by* (6.1), *and $L(0) \to \mathscr{M}_a(0)$ as above. Then $[(\Psi_t^a)^{-1}]^* L(t) \simeq L(0)$.*

Proof. The proof is clear, since on representatives V_t having the form (4.7) in the pinching region the cocycle (6.1) is the product of cocycles defined on Σ^+ and Σ^-. We omit the details.

We may define connections $\omega_t, \omega^\pm$ on the line bundles $L(t), L_\pm$ by pushing down the trivial connection induced by the form

$$\omega_A(\beta) = \frac{1}{4\pi} \int_{\Sigma_t} \operatorname{Tr} A \wedge \beta .$$

For $L_\pm$, we need to fix a base connection A_0 in the form (4.7). Then

$$\omega^\pm_{A_0+A}(\beta) = \frac{1}{4\pi} \int_{\Sigma^\pm} \operatorname{Tr}(A_0 + A) \wedge \beta .$$

These define connections $L_\pm \to \mathscr{F}^\pm_a$ which are, however, not invariant under the action of $\mathbf{T}_a$. Indeed, a computation shows that for $\xi \in \operatorname{Lie} \mathbf{T}_a$ and $\xi^\#$ the induced vector field on $\mathscr{F}^\pm_a$,

$$-i\omega^\pm_{A_0+A}(\xi^\#) = -\operatorname{Tr}(\xi \cdot \alpha) , \tag{6.4}$$

(see [D-W1, Sect. 5]). Nevertheless, for the product connection on $L_+ \otimes L_- \to \mathscr{F}^+_a \times \mathscr{F}^-_a$, the terms (6.4) cancel, and we therefore have connection ω_0 on $L(0)$. Again using the diffeomorphism Ψ^a_t we have the following simple

(6.5) **Lemma.** *Let* $[(\Psi^a_t)^{-1}]^* \omega_t$ *denote the pullback connection on* $L(0)$. *Then* $[(\Psi^a_t)^{-1}]^* \omega_t \to \omega_0$ *smoothly.*

The curvature of the connections ω_t is given by the symplectic form

$$\Omega_V(\beta_1, \beta_2) = \frac{1}{2\pi} \int_{\Sigma_t} \operatorname{Tr} \beta_1 \wedge \beta_2 ,$$

with a similar expression for the curvature of ω_0. The form Ω satisfies the condition in Definition 4.5 to give $L(t)$ and $L(0)$ the structure of CR-line bundles. Indeed, by Theorem 2 of [R-S-W], the bundle $(L(t), \omega_t)$ is CR-isomorphic to the restriction of the determinant bundle Δ to the hypersurface $\mathscr{M}_a(t)$. We therefore have for all integers k,

$$H^0_{\mathrm{CR}}(\mathscr{M}_a(t), \Delta^k) \simeq H^0_{\mathrm{CR}}(\mathscr{M}_a(t), L(t)^k) .$$

Combining Lemmas 6.3 and 6.5 with the remarks above, we have

(6.6) **Theorem.** *The bundles* $\bigcup_{t\in D}(L(t), \omega_t)$ *give a differentiable family of* CR-*line bundles on the family* $\mathfrak{M}_a$.

As mentioned before, the connections $\omega^\pm$ are not invariant under the action of $\mathbf{T}_a$. Using the universal connections $K^{a,\pm}$ on the bundle $\mathscr{F}^\pm_a \to \mathscr{M}^\pm_a$ we may put a connection on $L_\pm \to \mathscr{M}^\pm_a$ as follows: Given sections $s_\pm$ and vector fields $Z_\pm$ on $\mathscr{M}^\pm_a$, realize $s_\pm$ as $\mathbf{T}_a$-invariant sections $\tilde{s}_\pm$ of $L_\pm \to \mathscr{F}^\pm_a$ and lift $Z_\pm$ to horizontal vector fields $\tilde{Z}_\pm$ on $\mathscr{F}^\pm_a$. Then define

$$d_{\omega^\pm} s_\pm(Z_\pm) = d_{\omega^\pm} \tilde{s}_\pm(\tilde{Z}_\pm) .$$

We must check that the right-hand-side is invariant under the action of $\mathbf{T}_a$. It suffices to show that it vanishes in the infinitesimal direction $\xi^\#$ for $\xi \in \operatorname{Lie} \mathbf{T}_a$. By (6.4)

$$\begin{aligned} \xi^\#(d_{\omega^\pm} \tilde{s}_\pm(\tilde{Z}_\pm)) &= (d_{\omega^\pm, \xi^\#} + \operatorname{Tr}(\xi \cdot \alpha))(d_{\omega^\pm} \tilde{s}_\pm(\tilde{Z}_\pm)) \\ &= d_{\omega^\pm, \tilde{Z}_\pm}(\xi^\# \cdot \tilde{s}_\pm) - i\Omega_\pm(\xi^\#, \tilde{Z}_\pm) \cdot \tilde{s}_\pm . \end{aligned}$$

The first term vanishes, since $\tilde{s}_\pm$ is assumed to be $\mathbf{T}_a$-invariant, and the second term vanishes because the symplectic forms $\Omega_\pm$ are invariant in the $\mathbf{T}_a$-directions (cf. [D-W1, Sect. 5]).

The curvature of the connection is clearly of type (1, 1) on $\mathscr{M}_a^\pm$, and so the product connection gives a holomorphic structure to the product bundle $L \to X = \mathscr{M}_a^+ \times \mathscr{M}_a^-$. In terms of the generators $\beta_\pm^*$ and $\gamma_\pm^*$ of

$$H^2(X, \mathbb{Z}) \simeq H^2(\mathscr{M}_a^+, \mathbb{Z}) \oplus H^2(\mathscr{M}_a^-, \mathbb{Z})$$

(see Subsect. 2.3), it follows from Lemma 2.13. and the fact that $\Omega_\pm$ on $\mathscr{M}_a^\pm$ represents $\beta_\pm^* + k\alpha\gamma_\pm^*$ (see [D-W1, Sect. 5]) that

$$c_1(L) = \beta_+^* + \beta_-^* \,. \tag{6.7}$$

By Lemma 5.9, $\mathscr{M}_a(0) \xrightarrow{\pi} X$ is a principal S^1-bundle. The connection on π^*L is obviously trivial in the fiber direction, but by (6.4) the connection ω_0 on $L(0) \to \mathscr{M}_a(0)$ is not. Nevertheless, we have

(6.8) **Proportion.** *$L(0)$ is* CR-*isomorphic to π^*L.*

Proof. The smooth sections of L are by definition the S^1-invariant sections of $L(0)$, so $\pi^*L = L(0)$ as smooth bundles. To prove the proposition, we must show that the $\bar{\partial}_b$-operators associated to the connections $L(0)$ and π^*L coincide (see the discussion following Definition 4.5). But this is clear, since by Lemma 5.9 the CR-structure on $\mathscr{M}_a(0)$ is induced by the horizontal lifts of the (0, 1) vector fields on X, and by construction the connections on $L(0)$ and π^*L agree in the horizontal directions.

6.2. *Proof of the Main Theorem*

We continue with the notation of the previous section. Recall that in terms of the generators $\beta_\pm^*$ and $\gamma_\pm^*$ of $H^2(\mathscr{M}_a^\pm, \mathbb{Z})$, $c_1(L_\pm^k \otimes \varepsilon^\lambda) = k\beta_\pm^* + \lambda\gamma_\pm^*$, We set

$$\mathscr{L}_\pm(k, \lambda) = L_\pm^k \otimes \varepsilon^\lambda \,.$$

(6.9) **Theorem.** *For any integer λ, $H^q(\mathscr{M}_a^\pm, \mathscr{L}_\pm(k, \lambda))$ vanishes either for $q = 0$ or for $q = 1$. More precisely, we have*
(i) *If $0 \leqq \lambda \leqq k/2$, then $H^q(\mathscr{M}_a^\pm, \mathscr{L}_\pm(k, \lambda)) = 0$ for all $q > 0$.*
(ii) *If $\lambda < 0$ or $\lambda > k/2$, then $H^0(\mathscr{M}_a^\pm, \mathscr{L}_\pm(k, \lambda)) = 0$.*

Proof. We begin by noting that the natural map

$$\mathrm{Pic}(\mathscr{M}_a^\pm) \to H^2(\mathscr{M}_a^\pm, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

is an isomorphism. Indeed, from the exponential sequence the map is seen to be injective because $\mathscr{M}_a^\pm$ is simply connected (cf. [D-W1, Theorem 4.1]), and from the Hecke correspondence we conclude that $\mathrm{Pic}(\mathscr{M}_a^\pm)$ must have rank at least two. The canonical bundle therefore has the form $K_{\mathscr{M}_a^\pm} = m\beta_\pm^* + l\gamma_\pm^*$. By the adjunction formula (cf. [G-H, p. 147]) and using the fact that the normal bundle to a fiber is trivial, we have

$$K_{\mathscr{M}_a^\pm}|_{(\iota_0^\pm)_*\mathbb{P}^1} = K_{\mathbb{P}^1} = \mathcal{O}(-2)$$

$$K_{\mathscr{M}_a^\pm}|_{(\iota_1^\pm)_*\mathbb{P}^1} = K_{\mathbb{P}^1} = \mathcal{O}(-2) \,.$$

In the above formulas, $\iota_0^\pm$ and $\iota_1^\pm$ denote the inclusion maps of the fibers of the Hecke correspondence (2.14) into $\mathscr{M}_a^\pm$. The integers m and l may be determined from (2.15); this yields $K_{\mathscr{M}_a^\pm} = -4\beta_\pm^* - \gamma_\pm^*$. Note that the formulas (2.15) for $(\iota_0^\pm)_*\mathbb{P}^1$ and $(\iota_1^\pm)_*\mathbb{P}^1$ immediately give part (ii) of the theorem – for if $\lambda < 0$, then $\mathscr{L}_\pm(k,\lambda)$ restricted to $(\iota_0^\pm)_*\mathbb{P}^1$ is negative, and $\mathscr{L}_\pm(k,\lambda)$ restricted to $(\iota_1^\pm)_*\mathbb{P}^1$ has Chern class $k - 2\lambda$ which is negative if $\lambda > k/2$. Since the Hecke correspondence is a fiber bundle in both directions over open dense sets, we conclude that $\mathscr{L}_\pm(k,\lambda)$ cannot have non-zero global holomorphic sections in either of these two cases. For part (i) it suffices, by Serre duality and the Kodaira–Nakano Vanishing Theorem (cf. G-H, pp. 153–4]) to show that the bundle $K_{\mathscr{M}_a^\pm} \otimes \mathscr{L}_\pm(k,\lambda)^*$ is negative for $k \geqq 0$ and $0 \leqq \lambda \leqq k/2$. In terms of the ample generators $\beta_{0,\pm}^*, \beta_{1,\pm}^*$ of $\mathrm{Pic}(\mathscr{M}^\pm(2,0))$ and $\mathrm{Pic}(\mathscr{M}^\pm(2,-1))$ we have by the results of [D-W2, Sect. 3], and the computation of the canonical bundle above,

$$K_{\mathscr{M}_a^\pm} \otimes \mathscr{L}_\pm(k,\lambda)^* = -(k - 2\lambda + 2)p_0^*\beta_{0,\pm}^* - (\lambda + 1)p_1^*\beta_{1,\pm}^* .$$

The negativity follows, since both coefficients are strictly negative, and $p_0^*\beta_{0,\pm}^*$ and $p_1^*\beta_{1,\pm}^*$ are semi-positive, vanishing only in transverse directions. This completes the proof of Theorem 6.9.

The rest of this section, including the proof of the Main Theorem, consists of corollaries of Theorem 6.9. We begin with

(6.10) **Corollary.** *Let $\mathscr{V}_\lambda^\pm$ be as in* (1.1). *Then for* $0 \leqq \lambda \leqq k/2$,

$$\dim \mathscr{V}_\lambda^\pm = \chi(\mathscr{L}_\pm(k,\lambda)) ,$$

where χ denotes the Euler characteristic.

Proof. According to Theorem C of [D-W1],

$$\mathscr{V}_\lambda^\pm \simeq H^0(\mathscr{M}_a^\pm, \mathscr{L}_\pm(k,\lambda)) \tag{6.11}$$

in the particular case $a = \exp(2\pi i\lambda/k)$. But for all $a \neq \pm I$, the $\mathscr{M}_a^\pm$ are mutually biholomorphic (see [D-W1, Subsect. 2.3]). Thus (6.11) holds for any $a \neq \pm I$. The corollary now follows from part (i) of Theorem 6.9.

(6.12). **Corollary.** *Let $L^k \to X$ be as in Subsect.* 6.1. *Then*
(i) *For all integers λ, $H^1(X, L^k \otimes \varepsilon^\lambda) = 0$.*
(ii) *If $\lambda < 0$ or $\lambda > k/2$, $H^0(X, L^k \otimes \varepsilon^\lambda) = 0$*

Proof. Since

$$L^k \otimes \varepsilon^\lambda = \mathrm{pr}_+^*(\mathscr{L}_+(k,\lambda)) \otimes \mathrm{pr}_-^*(\mathscr{L}_-(k,\lambda)) ,$$

where $\mathrm{pr}_\pm : X \to \mathscr{M}_a^\pm$ denote the projection maps, we have by the Künneth formula (cf. [EGA, p. 29])

$$H^1(X, L^k \otimes \varepsilon^\lambda) \simeq \{H^0(\mathscr{M}_a^+, \mathscr{L}_+(k,\lambda)) \otimes H^1(\mathscr{M}_a^-, \mathscr{L}_-(k,\lambda))\}$$
$$\oplus \{H^1(\mathscr{M}_a^+, \mathscr{L}_+(k,\lambda)) \otimes H^0(\mathscr{M}_a^-, \mathscr{L}_-(k,\lambda))\} ,$$

and the right-hand side vanishes for all λ by Theorem 6.9. This proves part (i). The Künneth formula also yields

$$H^0(X, L^k \otimes \varepsilon^\lambda) \simeq H^0(\mathscr{M}_a^+, \mathscr{L}_+(k,\lambda)) \otimes H^0(\mathscr{M}_a^-, \mathscr{L}_-(k,\lambda)) , \tag{6.13}$$

and so part (ii) of the corollary follows from part (ii) of Theorem 6.9.

(6.14) **Corollary.** *Recall the notation $H_b^{p,q}$ from Subsect.* 5.1. *Then* $H_b^{0,1}(\mathcal{M}_a(0), \pi^*L^k) = 0$.

Proof. Immediate from part (i) of Corollary 6.12 and Theorem 5.11.

Proof of the Main Theorem. By the result Theorem 4.25 on the extension of CR-sections, we have

$$H^0(\mathcal{M}(t), \Delta^k(t)) \simeq H^0_{CR}(\mathcal{M}_a(t), L^k(t)), \tag{6.15}$$

for sufficiently small $|t|$. The isomorphism above is given by restriction. The pair $(\mathcal{M}_a(t), L^k(t))$ forms a differentiable family of CR-manifolds and CR-bundles. By Theorem 4.11, the CR-structures converge to the limiting one on $\mathcal{M}_a$, and by Theorem 6.6 and Proposition 6.8, $L^k(t) \to \pi^*L^k$ as a CR-bundle. By semi-continuity Theorem 5.8 and Corollary 6.14, $H_b^{0,1}(\mathcal{M}_a(t), L^k(t)) = 0$ for $|t|$ sufficiently small. In particular, the dimension of $H_b^{0,1}$ is constant. Again applying Theorem 5.8 we obtain

$$H^0_{CR}(\mathcal{M}_a(t), L^k(t)) \simeq H^0_{CR}(\mathcal{M}_a(0), \pi^*L^k), \tag{6.16}$$

where the isomorphism is given by degeneration. By Lemma 5.9(i), the S^1-action on $\mathcal{M}_a(0)$ is CR, and therefore lifts to an action on $H^0_{CR}(\mathcal{M}_a(0), \pi^*L^k)$. Decomposing in terms of irreducible representations and using Corollary 6.12(ii), we have

$$H^0_{CR}(\mathcal{M}_a(0), \pi^*L^k) \simeq \bigoplus_{0 \leq \lambda \leq k/2} H^0(X, L^k \otimes \varepsilon^\lambda). \tag{6.17}$$

The Main Theorem now follows from (6.13) and the identification (6.11). The equality (1.3) follows from the Main Theorem and Corollary 6.10.

Appendix

In this Appendix we state explicitly the estimates alluded to in Subsect. 5.1 and provide a proof of the semi-continuity for Kohn-Rossi cohomology, Theorem 5.8.

Recall that the Laplacian $\square$ acts on Dom $\square \subset \mathcal{D}^{p,q} \subset \mathcal{A}^{p,q}$ (see Subsect. 5.1). The condition $\varphi \in \mathcal{D}^{p,q}$ and $\bar\partial\varphi \in \mathcal{D}^{p,q}$ are called the *$\bar\partial$-Neumann conditions* and the *$\bar\partial$-Neumann problem* consists of solving the equation $\square\varphi = a$ subject to the $\bar\partial$-Neumann conditions. The $\bar\partial$-Neumann problem is *non-coercive* (cf. [F-K]). However, provided that condition $Z(q)$ hold, one proves the following estimate which is the analogue of the coercive estimate for elliptic operators (cf. [F-K, 2.1.7 and 3.1.1]):

(A.1) **Theorem** (Main Estimate for $\square$). *Suppose condition $Z(q)$ holds. Given $a \in L^2(\mathcal{A}^{p,q})$, let φ be the unique solution of*

$$(\square + I)\varphi = a.$$

Let $U \subset \overline{D(M)}$ be an open set and f, f_1 C^∞ real functions with $\operatorname{supp} f \subset \operatorname{supp} f_1 \subset U$ *and $f_1 = 1$ on* $\operatorname{supp} f$. *Let $k = 1$ or 2, according to whether $U \cap M \neq \emptyset$ or $U \cap M = \emptyset$. If $a|_U \in L^2_s(\mathcal{A}^{p,q})$, then $f \cdot \varphi \in L^2_{k+s}(\mathcal{A}^{p,q})$, and there exists a constant c_s such that*

$$\|f\varphi\|^2_{k+s} \leqq c_s(\|f_1 a\|^2_s + \|a\|^2_0).$$

From the Main Estimate A.1 one proves the analogue of the Hodge Theorem the usual way (cf. [F-K, Proposition 3.1.12 and 3.1.14]). Recall (5.4) that $\mathbf{H}^{p,q}$ denotes the kernel of $\square$.

(A.2) **Theorem.** *Assume condition $Z(q)$ holds. Then*

(1) $\mathbf{H}^{p,q} = \mathbf{H}^{p,q}(D(M), V)$ *is finite dimensional and* $\mathbf{H}^{p,q} \simeq H_a^{p,q}$. *Moreover, the following decompositions holds:*

$$L^2(\mathscr{A}^{p,q}) = \mathbf{H}^{p,q} \oplus \bar{\partial}\mathfrak{d}\,\mathrm{Dom}\,\square \oplus \mathfrak{d}\bar{\partial}\,\mathrm{Dom}\,\square\ ;$$

(2) *Let* $H: L^2(\mathscr{A}^{p,q}) \to \mathbf{H}^{p,q}$ *denote the harmonic projection. Then there is a* compact *operator*

$$N: L^2(\mathscr{A}^{p,q}) \to \mathrm{Dom}(\square + I)$$

satisfying

(i) *For any* $a \in L^2(\mathscr{A}^{p,q})$,

$$a = \bar{\partial}\mathfrak{d}Na + \mathfrak{d}\bar{\partial}Na + Ha\ ;$$

(ii) $NH = HN = 0$, $N\square = \square N = I - H$ *on* $\mathrm{Dom}(\square + I)$, *and if N is also defined on* $L^2(\mathscr{A}^{p,q})$ *(resp.* $L^2(\mathscr{A}^{p,q-1})$*), then* $N\bar{\partial} = \bar{\partial}N$ *on* $\mathrm{Dom}\ \bar{\partial}$ *(resp.* $N\mathfrak{d} = \mathfrak{d}N$ *on* $\mathrm{Dom}\ \bar{\partial}^*$*).*

(iii) $N(\mathscr{A}^{p,q}) \subset A^{p,q}$, *and for all s there is a constant c_s such that*

$$\| Na \|_{s+1} \leqq c_s \| a \|_s\ ,$$

for all $a \in \mathscr{A}^{p,q}$.

We have a similar result for the Laplacian $\square_b$ acting on the boundary complex $\mathscr{B}^{p,q}$. The operator $\square_b$ is *not elliptic.* However, under the condition $Y(q)$, $\square_b$ satisfies the analogue of the Main Estimate A.1 (cf. [F-K, Proposition 5.4.10]).

(A.3) **Theorem** (Main Estimate for $\square_b$). *Suppose M satisfies $Y(q)$. If* $U \subset \bar{U} \subset W \subset M$, *and f_1 is a smooth function supported in W, then for each smooth f supported in U,* $\varphi \in \mathscr{B}^{p,q}$, *and each positive integer s there exists a constant c_s such that*

$$\| f\varphi \|_{s+1}^2 \leqq c_s(\| f_1(\square_b + I)\varphi \|_s^2 + \|(\square_b + I)\varphi\|_0^2)\ .$$

Again as for the $\bar{\partial}$-Neumann problem, Theorem A.3 implies a Hodge decomposition for $\square_b$ (cf. [F-K, Theorem 5.4.12]). Recall (5.7) that $\mathbf{H}_b^{p,q}$ denotes the kernel of $\square_b$.

(A.4) **Theorem.** *Assume condition $Y(q)$ holds. Then*

(1) $\mathbf{H}_b^{p,q} = \mathbf{H}_b^{p,q}(M, V)$ *is finite dimensional and* $\mathbf{H}_b^{p,q} \simeq H_b^{p,q}$. *Moreover, the following decomposition holds:*

$$L^2(\mathscr{B}^{p,q}) = \mathbf{H}_b^{p,q} \oplus \bar{\partial}_b\mathfrak{d}_b\ \mathrm{Dom}\ \square_b \oplus \mathfrak{d}_b\bar{\partial}_b\ \mathrm{Dom}\ \square_b\ ;$$

(2) *Let* $H_b: L^2(\mathscr{B}^{p,q}) \to \mathbf{H}_b^{p,q}$ *denote the harmonic projection. Then there is a* compact *operator*

$$G_b: (\mathbf{H}_b^{p,q})^{\perp} \to L^2(\mathscr{B}^{p,q})$$

defined by $G_b = \square_b^{-1}$ *such that*

(i) *For any* $a \in L^2(\mathscr{B}^{p,q})$,

$$a = \bar{\partial}_b \mathfrak{d}_b G_b a + \mathfrak{d}_b \bar{\partial}_b G_b a + H_b a ;$$

(ii) $G_b H_b = H_b G_b = 0$, $G_b \Box_b = \Box_b G_b = I - H_b$ *on* Dom $\Box_b$, *and if* G_b *is also defined on* $L^2(\mathscr{B}^{p,q+1})$ *(resp.* $L^2(\mathscr{B}^{p,q-1})$*), then* $G_b \bar{\partial}_b = \bar{\partial}_b G_b$ *on* Dom $\bar{\partial}_b$ *(resp.* $G_b \mathfrak{d}_b = \mathfrak{d}_b G_b$ *on* Dom $\mathfrak{d}_b$*).*

(iii) $G_b(\mathscr{B}^{p,q}) \subset \mathscr{B}^{p,q}$, *and for all s there is a constant* c_s *such that*

$$\| G_b a \|_{s+1} \leqq c_s \| a \|_s ,$$

for all $a \in \mathscr{A}^{p,q}$.

With these theorems in hand, we turn now to the proof of the semi-continuity, Theorem 5.8. We model our arguments on those of Kodaira [Kod, Chap. 7]. Since nearly the entire proof of the ordinary semi-continuity theorem for Dolbeault cohomology goes through for this situation as well, mutatis mutandis, we shall be brief.

In what follows, we denote by $\Box_b(t)$ the $\bar{\partial}_b$ Laplacian associated to the line bundles $L(t)$. According to Theorem A.4, the conditions $Y(0)$ and $Y(1)$ guarantee the existence of Green's operators, discrete spectrum, finite dimensionality of the (0, 0) and (0, 1) cohomology groups, and the usual harmonic decomposition. The following lemma is essential (cf. [Kod, p. 338]). Its proof makes use of the optimal regularity results for $\Box_b$ which hold when $Y(q)$ is satisfied.

(A.5) **Lemma.** *Given* $\xi_0 \in \mathbb{C}$, $\xi_0 \notin \operatorname{spec}(\Box_b(t))$, *then for small* δ *there exists a constant* $C > 0$ *such that for all* $|t| < \delta$, $|\xi - \xi_0| < \delta$ *and smooth sections* ψ,

$$\| \Box_b(t, \xi) \psi \|_0 \geqq C \| \psi \|_0 ,$$

where $\Box_b(t, \xi) = \Box_b(t) + \xi$.

Proof. Suppose not. Then we can find sequences t_j, ξ_j, and ψ_j satisfying

$$|t_j| < 1/j, |\xi_j - \xi_0| < 1/j, \| \Box_b(t_j, \xi_j) \psi_j \|_0 < 1/j ,$$

and $\| \psi_j \|_0 = 1$. Now

(A.6)
$$\| \Box_b(t_j, \xi_j) \psi_j - \Box_b(0, \xi_0) \psi_j \|_0 \leqq \| \Box_b(t_j) \psi_j - \Box_b(0) \psi_j \|_0 + \| \psi_j \|_0 |\xi_j - \xi_0| .$$

Writing out $\Box_b(t_j)\psi_j$ in local coordinates, the coefficients are smooth by assumption, and the derivatives of the second order are in the *complex directions* only, i.e. correspond to vector fields in the subbundle $\mathscr{S}$. Following Rothschild and Stein [R-S], we introduce the spaces $S_k^2(L_t)$ which are the usual L^2 completions of smooth sections of $L(t)$, but where only the k-th derivatives in the complex directions are assumed to be in L^2. Then by [R-S, Theorem 19], we have the estimate

$$\| \psi_j \|_{S_2^2}^2 \leqq c(\| \Box_b(t_j, \xi_j) \psi_j \|_0^2 + \| \psi_j \|_0^2) \leqq 2c ,$$

and it is clear that C may be chosen uniformly in t. For the derivatives of order one or less, we apply the Main Estimate A.3

$$\| \psi_j \|_1^2 \leqq c_1(\| \Box_b(t, \xi) \psi_j \|_0^2 + \| \psi_j \|_0^2) \leqq 2c_1 .$$

Since the coefficients are smooth, we conclude that

$$\| \Box_b(t_j)\psi_j - \Box_b(0)\psi_j \|_0 \to 0$$

as $t \to 0$, and the fact that $|\xi_j - \xi_0| \to 0$ combined with (A.6) means that

$$\| \Box_b(t_j, \xi_j)\psi_j - \Box_b(0, \xi_0)\psi_j \|_0 \to 0 \, .$$

The assumption $\| \Box_b(t_j, \xi_j)\psi_j \|_0 \to 0$ implies $\| \Box_b(0, \xi_0)\psi_j \|_0 \to 0$ by the above. But since ξ_0 is assumed to be outside the spectrum of $\Box_b(0)$, we can find some bound $\mu > 0$ depending on ξ_0 such that

$$\| \Box_b(0, \xi_0)\psi \|_0 \geqq \mu \| \psi \|_0 \, ,$$

for all smooth sections ψ. Applied to ψ_j, the result above forces $\| \psi_j \|_0 \to 0$, which is a contradiction. This proves the lemma.

(A.7) **Lemma.** *Suppose $\xi_0 \notin \mathrm{spec}(\Box_b(0))$. Then for $|\xi - \xi_0|$ and $|t|$ sufficiently small, the Green's function $G_b(t, \xi) = \Box_b(t, \xi)^{-1}$ is C^∞ in (t, ξ).*

Proof. The ξ variable is a zero order term, so it suffices to show differentiability in t. By Lemma A.5 we have

$$\| \psi \|_0 \leqq \frac{1}{C} \| \Box_b(t, \xi)\psi \|_0 \leqq \frac{1}{C} \| \Box_b(t, \xi)\psi \|_k \, .$$

Combining the above with the Main Estimate A.3 we have

$$\| \psi \|_{k+1} \leqq \tilde{c}_k \| \Box_b(t, \xi)\psi \|_k \, .$$

By Sobolev's inequality, we have for $k + 1 - l > n/2$

$$|D_x^l \psi(x)| \leqq c_{k+1-l,l} \| \psi \|_{k+1} \, ,$$

hence

$$|D_x^l \psi(x)| \leqq c'_{k,l} \| \Box_b(t, \xi)\psi \|_k \, , \tag{A.8}$$

for $k > l - 1 + n/2$. Note that $c'_{k,l}$ is independent of t. The proof now proceeds by induction: assume $\varphi(t) = \Box_b(t, \xi)\psi(t)$ is continuous. We must show that for any l, $D_x^l \psi(x, t)$ is continuous in (x, t). Continuity in x for fixed t is clear. We must show that for any s, l,

$$D_x^l \psi(x, t) \to D_x^l \psi(x, s)$$

uniformly. For a given l choose $k > l - 1 + n/2$, $c = c'_{k,l}$. By (A.8) we have

$$\begin{aligned} |D_x^l \psi(x, t) - D_x^l \psi(x, s)| &\leqq c \| \Box_b(t, \xi)(\psi(t) - \psi(s)) \|_k \\ &\leqq c \| \Box_b(t, \xi)\psi(t) - \Box_b(s, \xi)\psi(s) \|_k \\ &\quad + c \| (\Box_b(t, \xi) - \Box_b(s, \xi))\psi(s) \|_k \\ &= c \| \varphi(t) - \varphi(s) \|_k + c \| (\Box_b(t, \xi) - \Box_b(s, \xi))\psi(s) \|_k \, . \end{aligned}$$

By hypothesis $\| \varphi(t) - \varphi(s) \|_k \to 0$, and $\Box_b(t, \xi)\psi(s)$ is a C^∞ function of (x, t), so

$$\| \Box_b(t, \xi)\psi(s) - \Box_b(s, \xi)\psi(s) \|_k \to 0 \, .$$

This completes the proof that $\psi(x, t)$ is continuous. The proof of the continuity of higher derivatives is in Kodaira, [Kod, p. 332].

Given Lemma A.7, it is clear that for a Jordan curve $C \subset \mathbb{C}$ disjoint from spec($\Box_b(0)$) and t sufficiently close to the origin, the operator

$$F_t(C) = \frac{1}{2\pi i} \int_C d\xi \, \Box_b(t, \xi)^{-1}$$

is C^∞ in t. The image is finite dimensional and consists of the eigenspaces of $\Box_b(t)$ corresponding to eigenvalues inside the contour C (cf. [Kod, p. 340]). From this it is not difficult to show (cf. [Kod, Theorem 7.2])

(A.9) **Proposition.** *Let* $\{\lambda_n^q\}_{n=1}^\infty$ *denote the eigenvalues of* $\Box_b^{0,q}(t) = \Box_b(t)$ *acting on* $(0, q)$ *forms. Then for each* n, $\lambda_n^q(t)$ *is continuous in* $t \in D$.

The semi-continuity of dim $H_b^{0,q}(M(t), L(t))$, $q = 0, 1$ is immediate from Proposition A.9. To prove the second statement in Theorem 5.8 we use the harmonic decomposition, Theorem A.4(1):

$$L^2(\mathscr{B}^{0,0}) = \mathbf{H}^{0,0}(M(t), L(t)) \oplus \mathfrak{d}_b(t)\bar{\partial}_b(t) \operatorname{Dom} \Box_b^{0,0}$$

$$L^2(\mathscr{B}^{0,1}) = \mathbf{H}^{0,1}(M(t), L(t)) \oplus \mathfrak{d}_b(t)\bar{\partial}_b(t) \operatorname{Dom} \Box_b^{0,1} \oplus \bar{\partial}_b(t)\mathfrak{d}_b(t) \operatorname{Dom} \Box_b^{0,1} .$$

Choose an orthonormal basis $\{\psi_n^1(t)\}$ of $\{\bar{\partial}_b(t)\mathfrak{d}_b(t) \operatorname{Dom} \Box_b^{0,1}\}$ with eigenvalues

$$\Box_b(t)\psi_j^1(t) = \mu_n^1(t)\psi_n^1(t), \quad \mu_n^1(t) > 0 .$$

If we define

$$\varphi_n(t) = \frac{1}{\sqrt{\mu_n^1(t)}} \mathfrak{d}_b(t)\psi_n^1(t) ,$$

then $\{\varphi_n(t)\}$ form a basis for $\{\mathfrak{d}_b(t)\bar{\partial}_b(t) \operatorname{Dom} \Box_b^{0,0}\}$, and

$$\Box_b(t)\varphi_n(t) = \frac{1}{\sqrt{\mu_n^1(t)}} \Box_b(t)\mathfrak{d}_b(t)\psi_n^1(t) = \frac{1}{\sqrt{\mu_n^1(t)}} \mathfrak{d}_b(t)\Box_b(t)\psi_n^1(t)$$

$$= \mu_n^1(t) \frac{1}{\sqrt{\mu_n^1(t)}} \mathfrak{d}_b(t)\psi_n^1(t) = \mu_n^1(t)\varphi_n(t) .$$

Hence $\{\varphi_n(t)\}$ are eigenfunctions with eigenvalues $\{\mu_n^1(t)\}$. But if dim $H_b^{0,1}(M(t), L(t))$ is constant, then by continuity there must be an $\varepsilon > 0$ such that $\mu_n^1(t) \geqq \varepsilon$ for t sufficiently close to the origin. Then again by continuity, dim $H_{CR}^0(M(t), L(t))$ must be constant. This completes the proof of Theorem 5.8.

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