The Algebraic and Analytic Compactifications of the Hitchin Moduli Space

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Abstract

Following the work of Mazzeo-Swoboda-Weiß-Witt [MSWW16] and Mochizuki [Moc16], there is a map $\overline{\Xi}$ between the algebraic compactification of the Dolbeault moduli space of $\mathrm{SL}(2,\mathbb{C})$ Higgs bundles on a smooth projective curve coming from the $\mathbb{C}^*$ action, and the analytic compactification of Hitchin's moduli space of solutions to the $\mathsf{SU}(2)$ self-duality equations on a Riemann surface obtained by adding solutions to the decoupled equations, known as "limiting configurations". This map extends the classical Kobayashi-Hitchin correspondence. The main result of this paper is that $\overline{\Xi}$ fails to be continuous at the boundary over a certain subset of the discriminant locus of the Hitchin fibration. This suggests the possibility of a third, refined compactification which dominates both.

1. Introduction

Let Σ be a closed Riemann surface of genus $g \geqslant 2$. The coarse Dolbeault moduli space of $\mathrm{SL}(2,\mathbb{C})$ semistable Higgs bundles on Σ, denoted by $\mathcal{M}_{\mathrm{Dol}}$, and Hitchin's moduli space of solutions to the $\mathsf{SU}(2)$ self-duality equations on Σ, denoted by $\mathcal{M}_{\mathrm{Hit}}$, have been extensively studied since their introduction over 35 years ago. The Kobayashi-Hitchin correspondence, proved in [Hit87a], gives a homeomorphism between these two moduli spaces:

$$\Xi : \mathcal{M}_{\mathrm{Dol}} \xrightarrow{\sim} \mathcal{M}_{\mathrm{Hit}} \ . \tag{1}$$

Both spaces are noncompact: $\mathcal{M}_{\mathrm{Dol}}$ is naturally a quasiprojective variety [Nit91, Sim94], and like monopole moduli spaces, $\mathcal{M}_{\mathrm{Hit}}$ admits Higgs fields of arbitrarily large norm. Nevertheless, the map Ξ is proper. Recently, there has been interest from several directions on natural compactifications of these two spaces. A key feature on the Dolbeault side is the existence of a $\mathbb{C}^*$ action with the Białynicki-Birula property, and this may be used to define a completion of $\mathcal{M}_{\mathrm{Dol}}$ as a projective variety [Hau98, dC21, Fan22a]. The ideal points are identified with the $\mathbb{C}^*$ orbits in the complement of the nilpotent cone of $\mathcal{M}_{\mathrm{Dol}}$. The Hitchin moduli space also admits a more recently introduced compactification, $\overline{\mathcal{M}}_{\mathrm{Hit}}$, based on the work of several authors (see [MSWW16, Moc16, Tau13b]). The boundary of $\overline{\mathcal{M}}_{\mathrm{Hit}}$ is given by gauge equivalence classes of *limiting configurations*. This compactification is relevant to many aspects of Hitchin's moduli space. For more details, we refer the reader to [DN19, MSWW14, Fre20, FMSW22, OSWW, KNPS15, CL22], and the

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references therein.

By the work of [MSWW16, Moc16], there is a natural extension

$$\overline{\Xi} : \overline{\mathcal{M}}_{\mathrm{Dol}} \longrightarrow \overline{\mathcal{M}}_{\mathrm{Hit}} \tag{2}$$

of the Kobayashi-Hitchin correspondence to the two compactifications described above, and it is of interest to study the geometry of this map. This involves another key feature of Hitchin's moduli space; namely, spectral curves. Spectral curves and spectral data [Hit92] realize the Dolbeault moduli space as an algebraically complete integrable system $\mathcal{H} : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{B}$. In the case of $\mathrm{SL}(2, \mathbb{C})$, the base $\mathcal{B}$ is the space of holomorphic quadratic differentials on Σ. Given $q \in H^0(K^2)$, one obtains a (scheme theoretic) spectral curve S_q. This curve is reduced if $q \neq 0$, irreducible if q is not the square of an abelian differential, and smooth if q has simple zeros. Let $\mathcal{B}^{\mathrm{reg}} \subset \mathcal{B}$ denote the open cone of quadratic differentials with simple zeros.

The ideal points of both compactifications $\overline{\mathcal{M}}_{\mathrm{Dol}}$ and $\overline{\mathcal{M}}_{\mathrm{Hit}}$ have associated nonzero quadratic differentials, and therefore spectral curves. We write $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}}$ for the elements in $\overline{\mathcal{M}}_{\mathrm{Dol}}$ with smooth spectral curves, and $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} = \overline{\mathcal{M}}_{\mathrm{Dol}} \setminus \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}}$ for those with singular spectral curves; similarly for $\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$ and $\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}}$. We then have the following result.

Theorem 1.1. *The restriction of the compactified Kobayashi-Hitchin map $\overline{\Xi} : \overline{\mathcal{M}}_{\mathrm{Dol}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}$ to the locus with smooth associated spectral curves defines a homeomorphism $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} \simeq \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$. On the singular spectral curve locus, however, $\overline{\Xi}^{\mathrm{sing}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}}$ is neither surjective nor injective.*

It will be convenient to analyze the behavior along rays in $\mathcal{B}$, where the spectral curve is simply rescaled. For $q \neq 0$ a quadratic differential, we set $\overline{\mathcal{M}}_{\mathrm{Dol},q^+}$ (resp. $\overline{\mathcal{M}}_{\mathrm{Hit},q^+}$) to be the points in $\overline{\mathcal{M}}_{\mathrm{Dol}}$ (resp. $\overline{\mathcal{M}}_{\mathrm{Hit}}$) with spectral curves S_{tq}, $t \in \mathbb{R}^+$. The restriction of $\overline{\Xi}$ gives a map $\overline{\Xi}_{q^+} : \overline{\mathcal{M}}_{\mathrm{Dol},q^+} \to \overline{\mathcal{M}}_{\mathrm{Hit},q^+}$. We shall study the continuous behavior of $\overline{\Xi}_{q^+}$ for points in the fiber $\mathcal{H}^{-1}(tq)$ as $t \to +\infty$. For convenience, we set $\mathcal{M}_{q^+} := \overline{\mathcal{M}}_{\mathrm{Dol},q^+} \cap \mathcal{M}_{\mathrm{Dol}}$. When q is irreducible, i.e. not a square, all elements in $\mathcal{M}_q$ are stable. Via the Hitchin [Hit87b] and Beauville-Narasimhan-Ramanan (BNR) correspondence [BNR89], this reduces the description of the fiber $\mathcal{M}_q := \mathcal{H}^{-1}(q)$ to the characterization of rank 1 torsion free sheaves on the integral curve S_q.

In [Reg80], parameter spaces for rank 1 torsion free sheaves on algebraic curves with Gorenstein singularities were studied in the context of compactified Jacobians, and the crucial notion of a parabolic module was introduced. This was extensively investigated by Cook in [Coo93, Coo98], partially following ideas of Bhosle [Bho92]. For simple plane curve singularities of the type appearing in spectral curves, one makes use of the local classification of torsion free modules of Greuel-Knörrer [GK85]. These methods were applied to study the Hitchin fibration by Gothen-Oliveira in [GO13] (see also [KSZ22] for recent study). In parallel, Horn [Hor22a] defines a stratification of $\mathcal{M}_q = \bigcup_D \mathcal{M}_{q,D}$ by certain effective divisors contained in the divisor of q (see Section 5.5, and also [HN] for the more general situation).

Using the results from these references, we reinterpret the work of Mochizuki [Moc16] and Mochizuki-Szabó [MS23]. We first prove that the restriction of the compactified Kobayashi-Hitchin map to the boundary is discontinuous in general. Following that, by utilizing the exponential decay results from Mochizuki-Szabó [MS23], which play an essential role, we demonstrate that the entirety of $\overline{\Xi}_{q^+}$ is discontinuous.

Theorem 1.2. *Let $q \neq 0$ be an irreducible quadratic differential.*

(i) *If q has only zeros of odd order, then $\overline{\Xi}_{q^+}$ is continuous.*

(ii) *If q has at least one zero of even order, then for each $D \neq 0$ there exists an even integer $n_D \geqslant 1$ so that for any Higgs bundle $(\mathcal{F}, \psi) \in \mathcal{M}_{q,D}$, there exist $2n_D$ sequences of Higgs bundles $(\mathcal{E}_i^k, \varphi_i^k)$, $k = 1, \ldots, 2n_D$ satisfying*
   - $\lim_{i\to\infty}(\mathcal{E}_i^k, \varphi_i^k) = (\mathcal{F}, \psi)$ *for* $k = 1, \ldots, 2n_D$,
   - *and if we write*

$$\eta^k := \lim_{i\to\infty} \overline{\Xi}_{q^+}(\mathcal{E}_i^k, \varphi_i^k) \quad , \quad \xi := \lim_{i\to\infty} \overline{\Xi}_{q^+}(\mathcal{F}, t_i\psi) \ ,$$

   for some sequence $t_i \in \mathbb{R}^+$, $t_i \to +\infty$,
     * *if $(\mathcal{F}, \psi)$ doesn't lie in the real locus, then $\xi, \eta^1, \ldots, \eta^{2n_D}$ are $2n_D + 1$ different limiting configurations,*
     * *if $(\mathcal{F}, \psi)$ lies in the real locus, then $\eta^i \cong \eta^{n_D+i}$ for $i = 1, \cdots, n$ and we obtain n_D+1 different limiting configurations.*

When q is reducible, the description of Higgs bundles in the fiber over q becomes more complicated because of, among other things, the existence of strictly semistable objects. To understand this, we use the local descriptions of Gothen-Oliveira and Mochizuki (see [GO13, Moc16]). By contrast to the irreducible case, the analogous exponential decay result to that of Mochizuki-Szabó [Moc16] is unfortunately currently not available. This results in a weaker statement for the reducible fiber. Recall that we have defined $\overline{\Xi}_{q^+} : \overline{\mathcal{M}}_{\mathrm{Dol},q^+} \to \overline{\mathcal{M}}_{\mathrm{Hit},q^+}$ as the compactified Kobayashi-Hitchin map, and $\partial\overline{\Xi}_{q^+} : \partial\overline{\mathcal{M}}_{\mathrm{Dol},q^+} \to \partial\overline{\mathcal{M}}_{\mathrm{Hit},q^+}$ as its restriction to the compactified boundary. With this notation, the following holds:

Theorem 1.3. *Suppose $q \neq 0$ is reducible, if $g \geqslant 3$, then the boundary map $\partial\overline{\Xi}_{q^+}|_{\partial\overline{\mathcal{M}}^{\mathrm{st}}_{\mathrm{Dol},q^+}}$ is discontinuous. However, if $g = 2$, the boundary map $\partial\overline{\Xi}_{q^+}|_{\partial\overline{\mathcal{M}}^{\mathrm{st}}_{\mathrm{Dol},q^+}}$ is continuous.*

This paper is organized as follows: in Section 2, we provide a brief overview of Higgs bundles and the BNR correspondence. In Section 3, we introduce the concepts of filtered bundles and their compactness properties. Section 4 defines the algebraic and analytic compactifications. Section 5 introduces parabolic modules and examines their connection to spectral curves. The main results for Hitchin fibers with irreducible singular spectral curves are established in Section 6. In Section 7, the results for the reducible case are proven. Finally, in Section 8, we construct the compactified Kobayashi-Hitchin map and prove the main results. The Appendix, based on the work of Greuel-Knörrer, calculates some invariants of rank 1 torsion free sheaves on the spectral curves we consider.

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2. Background on Higgs bundles

This section gives a very brief overview of the Dolbeault and Hitchin moduli spaces, spectral curve descriptions, and the nonabelian Hodge correspondence. For more details on these topics, see [Hit87a, Hit87b, Sim92].

2.1 Higgs bundles

As in the Introduction, throughout this paper Σ will denote a closed Riemann surface of genus $g \geqslant 2$ with structure sheaf $\mathcal{O} = \mathcal{O}_\Sigma$ and canonical bundle $K = K_\Sigma$. Let $E \to \Sigma$ be a complex vector bundle. A Higgs bundle consists of a pair $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a holomorphic bundle and $\varphi \in H^0(\mathrm{End}(\mathcal{E}) \otimes K)$. If $\mathrm{rank}(\mathcal{E}) = 1$, then a Higgs field is just an abelian differential ω. The pair $(\mathcal{E}, \varphi)$ is called an $\mathrm{SL}(2, \mathbb{C})$ Higgs bundle if $\mathrm{rank}(E) = 2$, $\det(\mathcal{E})$ has a fixed isomorphism with the trivial bundle, and $\mathrm{Tr}(\varphi) = 0$. In this paper we will focus mainly on $\mathrm{SL}(2, \mathbb{C})$ Higgs bundles, but the rank 1 case will also be important.

Let $(\mathcal{E}, \varphi)$ be an $\mathrm{SL}(2, \mathbb{C})$ Higgs bundle. A (proper) Higgs subbundle of $(\mathcal{E}, \varphi)$ is a holomorphic line bundle $\mathcal{L} \subset \mathcal{E}$ that is φ-invariant, i.e. $\varphi : \mathcal{L} \to \mathcal{L} \otimes K$. In this case, the restriction $\varphi_{\mathcal{L}} := \varphi\big|_{\mathcal{L}}$, makes $(\mathcal{L}, \varphi_{\mathcal{L}})$ a rank 1 Higgs bundle. Moreover, φ induces a Higgs bundle structure on the quotient $\mathcal{E}/\mathcal{L}$. We say $(\mathcal{E}, \varphi)$ is stable (resp. semistable) if for all Higgs subbundles $\mathcal{L}$, $\deg \mathcal{L} < 0$ (resp. $\deg \mathcal{L} \leqslant 0$). We say $(\mathcal{E}, \varphi)$ is polystable if $(\mathcal{E}, \varphi) \simeq (\mathcal{L}, \omega) \oplus (\mathcal{L}^{-1}, -\omega)$, where $\mathcal{L}$ is a degree zero holomorphic line bundle and $\omega \in H^0(K)$.

If $(\mathcal{E}, \varphi)$ is strictly semistable, i.e. semistable but not polystable, the Seshadri filtration [Ses67] gives a unique Higgs subbundle $0 \subset (\mathcal{L}, \omega) \subset (\mathcal{E}, \varphi)$ with $\deg(\mathcal{L}) = \frac{1}{2}\deg(\mathcal{E}) = 0$. Write $(\mathcal{L}', \omega') := (\mathcal{E}, \varphi)/(\mathcal{L}, \omega)$, then we have $\omega' = -\omega$ and $\mathcal{L}' = \mathcal{L}^{-1}$. The associated graded bundle $\mathrm{Gr}(\mathcal{E}, \varphi) = (\mathcal{L}, \omega) \oplus (\mathcal{L}^{-1}, -\omega)$ of this filtration is a polystable $\mathrm{SL}(2, \mathbb{C})$ Higgs bundle. We say that $(\mathcal{E}, \varphi)$ is S-equivalent to $\mathrm{Gr}(\mathcal{E}, \varphi)$.

Holomorphic bundles $\mathcal{E}$ with underlying C^∞ bundle E are in 1-1 correspondence with $\bar\partial$-operators $\bar\partial_E : \Omega^0(E) \to \Omega^{0,1}(E)$. We use the notation $\mathcal{E} := (E, \bar\partial_E)$. Let $\mathcal{C}$ denote the space of pairs $(\bar\partial_E, \varphi)$, $\bar\partial_E \varphi = 0$. Let $\mathcal{C}^s$ and $\mathcal{C}^{ss}$ denote the subspaces of $\mathcal{C}$ where the Higgs bundles are stable (resp. semistable). The complex gauge transformation group $\mathcal{G}_{\mathbb{C}} := \mathrm{Aut}(E)$ has a right action on $\mathcal{C}$ by defining for $g \in \mathcal{G}_{\mathbb{C}}$, $(\bar\partial_E, \varphi)g := (g^{-1} \circ \bar\partial \circ g, g^{-1} \circ \varphi \circ g)$.

There is a quasiprojective scheme $\mathcal{M}_{\mathrm{Dol}}$ whose closed points are in 1-1 correspondence with isomorphism classes of polystable $\mathrm{SL}(2, \mathbb{C})$ Higgs bundles constructed via (finite dimensional) Geometric Invariant Theory (see [Nit91, Sim94]). In [Fan22b] it was shown that the infinite dimensional quotient $\mathcal{C}^{\mathrm{ss}} /\!\!/ \mathcal{G}_{\mathbb{C}}$, where the double slash indicates that S-equivalent orbits are identified, admits the structure of a complex analytic space that is biholomorphic to the analytification $\mathcal{M}^{\mathrm{an}}_{\mathrm{Dol}}$ of $\mathcal{M}_{\mathrm{Dol}}$. Henceforth, we shall work in the complex analytic category, identify the algebro-geometric and gauge theoretic moduli spaces as complex analytic spaces, and simply denote them both by $\mathcal{M}_{\mathrm{Dol}}$. We note that the set of stable Higgs bundles modulo gauge transformations, $\mathcal{M}^s_{\mathrm{Dol}} := \mathcal{C}^{\mathrm{s}}/\mathcal{G}_{\mathbb{C}}$, is a geometric quotient and an open subset of $\mathcal{M}_{\mathrm{Dol}}$.

Finally, notice that the pair $(\mathcal{E}, \varphi)$ is stable (resp. semistable) if and only if the same is true for $(\mathcal{E}, \lambda\varphi)$, $\lambda \in \mathbb{C}^*$. Hence, $\mathcal{M}_{\mathrm{Dol}}$ admits an action of $\mathbb{C}^*$ that preserves $\mathcal{M}^s_{\mathrm{Dol}}$. Though $\mathcal{M}_{\mathrm{Dol}}$ is only quasiprojective, the $\mathbb{C}^*$ action satisfies the Białynicki-Birula property:

Theorem 2.1 ([Hit87a, Sim92]). *For any* $[(\mathcal{E}, \varphi)] \in \mathcal{M}_{\mathrm{Dol}}$,

$$\lim_{\lambda \to 0} \lambda \cdot [(\mathcal{E}, \varphi)] := \lim_{\lambda \to 0} [(\mathcal{E}, \lambda\varphi)]$$

exists in $\mathcal{M}_{\mathrm{Dol}}$.

2.2 Spectral curves and the Hitchin fibration

The Hitchin map is defined as

$$\mathcal{H} : \mathcal{M}_{\mathrm{Dol}} \longrightarrow H^0(K^2)\ ,\ [(\mathcal{E}, \varphi)] \mapsto \det(\varphi)\ ,$$

where $H^0(K^2) =: \mathcal{B}$ is known as the Hitchin base. Hitchin [Hit87a, Hit87b] showed that $\mathcal{H}$ is a proper map and a fibration by abelian varieties over the open cone $\mathcal{B}^{\mathrm{reg}} \subset \mathcal{B}$ consisting of nonzero quadratic differentials with only simple zeros. The discriminant locus $\mathcal{B}^{\mathrm{sing}} := \mathcal{B} \setminus \mathcal{B}^{\mathrm{reg}}$ consists of quadratic differentials that are either identically zero or have at least one zero with multiplicity. For $q \in \mathcal{B}$, let $\mathcal{M}_q := \mathcal{H}^{-1}(q)$. The "most singular fiber" $\mathcal{M}_0$ is called the *nilpotent cone.*

Consider the total space $\mathrm{Tot}(K)$ of K, along with its projection $\pi : \mathrm{Tot}(K) \to \Sigma$. The pullback bundle π^*K has a tautological section, which we denote by $\lambda \in H^0(\mathrm{Tot}(K), \pi^*K)$. Given any $q \neq 0 \in H^0(K^2)$, the *spectral curve* S_q associated with q is the zero scheme of the section $\lambda^2 - \pi^*q \in H^0(\mathrm{Tot}(K), \pi^*K)$. This is a reduced, but possibly reducible, projective algebraic curve. The restriction of π to S_q, also denoted by $\pi : S_q \to \Sigma$, is a double covering branched along the zeros of q.

The spectral curve S_q is smooth if and only if q has only simple zeros. It is reducible if and only if $q = -\omega \otimes \omega$ for some $\omega \in H^0(K)$. In the latter case, we call such quadratic differentials *reducible*, and otherwise we refer to them as *irreducible*. There is a noteworthy observation regarding irreducible spectral curves.

Proposition 2.2 (cf. [Hit87b]). *Let $(\mathcal{E}, \varphi)$ be a Higgs bundle with $q = \det(\varphi)$, and suppose q is irreducible. Then $(\mathcal{E}, \varphi)$ has no proper invariant subbundles. In particular, $(\mathcal{E}, \varphi)$ is stable.*

Proof. Suppose $\mathcal{L} \subset \mathcal{E}$ is φ-invariant, and let $\varphi_{\mathcal{L}}$ be the restriction. Then

$$\det \varphi = -\frac{1}{2}\mathrm{Tr}(\varphi^2) = -(\varphi_{\mathcal{L}})^2\ ,$$

contradicting the assumption. □

Let us emphasize that being reducible is not the same as having only even zeros. To see this, suppose that $\mathrm{Div}(q) = 2D$. Then $K \simeq \mathcal{O}(D) \otimes \mathcal{I}$, where $\mathcal{I}$ is a 2-torsion point in the Jacobian. The spectral curve S_q is reducible if and only if $\mathcal{I}$ is trivial.

2.3 Rank 1 torsion free sheaves and the BNR correspondence

In this subsection, we provide some background on rank 1 torsion free sheaf theory over spectral curves in the context of the Hitchin and BNR correspondence, as developed in [Hit87b, BNR89].

Let S be a reduced and irreducible complex projective curve and $\mathcal{O}_S$ its structure sheaf. The moduli space of invertible sheaves on S is denoted by $\mathrm{Pic}(S)$, and $\mathrm{Pic}^d(S) \subset \mathrm{Pic}(S)$ is the degree d component. If $\mathcal{F}$ is a coherent analytic sheaf on S, we can define its cohomology groups $H^i(S, \mathcal{F})$. Since $\dim S = 1$, $H^i(S, \mathcal{F}) = 0$ for $i \geqslant 2$. The Euler characteristic is defined as $\chi(\mathcal{F}) = \dim H^0(S, \mathcal{F}) - \dim H^1(S, \mathcal{F})$. The degree of a torsion free sheaf $\mathcal{F}$ is given by $\deg(\mathcal{F}) = \chi(\mathcal{F}) - \mathrm{rank}(\mathcal{F})\chi(\mathcal{O}_S)$. If $\mathcal{F}$ is locally free, then $\deg(\mathcal{F})$ coincides with the degree of the invertible sheaf $\det(\mathcal{F})$.

Let $\overline{\mathrm{Pic}}^d(S)$ be the moduli space of degree d rank 1 torsion free sheaves on S, and $\overline{\mathrm{Pic}}(S) = \prod_{d \in \mathbb{Z}} \overline{\mathrm{Pic}}^d(S)$ [D'S79]. Then $\overline{\mathrm{Pic}}^d(S)$ is an irreducible projective scheme containing $\mathrm{Pic}^d(S)$ as an open subscheme. When S is smooth, we have $\overline{\mathrm{Pic}}^d(S) = \mathrm{Pic}^d(S)$. The relationship to Higgs bundles is given by the following.

Theorem 2.3 ([Hit87b, BNR89]). *Let $q \in H^0(K^2)$ be an irreducible quadratic differential with spectral curve S_q. There is a bijective correspondence between points in $\overline{\mathrm{Pic}}(S_q)$ and isomorphism classes of rank 2 Higgs pairs $(\mathcal{E}, \varphi)$ with $\mathrm{Tr}(\varphi) = 0$ and $\det(\varphi) = q$. Explicitly: if $\mathcal{L} \in \overline{\mathrm{Pic}}(S_q)$, then $\mathcal{E} := \pi_*(\mathcal{L})$ is a rank 2 vector bundle, and the homomorphism $\pi_*\mathcal{L} \to \pi_*\mathcal{L} \otimes K \cong \pi_*(\mathcal{L} \otimes \pi^*K)$ given by multiplication by the canonical section λ defines the Higgs field φ.*

This correspondence gives the very useful exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{I} \to \pi^*\mathcal{E} \xrightarrow{\pi^*\varphi - \lambda} \pi^*\mathcal{E} \otimes \pi^*K \to \mathcal{L} \otimes \pi^*K \to 0 \ , \tag{3}$$

for some ideal sheaf $\mathcal{I}$. In case S is smooth, then $\mathcal{I} = \mathcal{O}_S(-\Delta)$, where Δ is the ramification divisor. The sequence (3) will be used below in Section 6.

Let q be a quadratic differential with only simple zeros, and to simplify notation write $S = S_q$. Let $\Lambda := \mathrm{Div}(\lambda)$ be the ramification divisor of the map $\pi : S \to \Sigma$. By the Riemann-Hurwitz formula, the genus of S is $g(S) = 4g - 3$, where g is the genus of Σ. Furthermore, for any $\mathcal{L} \in \mathrm{Pic}(S)$, Riemann-Roch gives $\deg(\pi_*\mathcal{L}) = \deg(\mathcal{L}) - (2g - 2)$. The $\mathrm{SL}(2, \mathbb{C})$ Higgs bundles are characterized by

$$\mathcal{T} := \{\mathcal{L} \in \mathrm{Pic}^{2g-2}(S) \mid \det(\pi_*\mathcal{L}) = \mathcal{O}_\Sigma\}. \tag{4}$$

By the Hitchin-BNR correspondence (Theorem 2.3), the map $\chi_{BNR} : \mathcal{T} \to \mathcal{M}_q$ is a bijection.

The branched double cover $\pi : S \to \Sigma$ is given by an involution $\sigma : S \to S$. We have the norm map $\mathrm{Nm}_{S/\Sigma} : \mathrm{Jac}(S) \to \mathrm{Jac}(\Sigma)$, where $\mathrm{Jac}(S)$ is the connected component of the trivial line bundle in $\mathrm{Pic}(S)$ and $\mathrm{Nm}_{S/\Sigma}(\mathcal{O}_S(D)) := \mathcal{O}_\Sigma(\pi(D))$. The Prym variety is defined as

$$\mathrm{Prym}(S/\Sigma) := \ker(\mathrm{Nm}_{S/\Sigma}) = \{\mathcal{L} \in \mathrm{Pic}(S) \mid \mathcal{L} \otimes \sigma^*\mathcal{L} = \mathcal{O}_S\} \ .$$

Also, we have $\det(\pi_*\mathcal{L}) \cong \mathrm{Nm}_{S/\Sigma}(\mathcal{L}) \otimes K^{-1}$. Thus, $\mathcal{T}$ can be expressed as

$$\mathcal{T} = \{\mathcal{L} \in \mathrm{Pic}^{2g-2}(S) \mid \mathrm{Nm}_{S/\Sigma}(\mathcal{L}) \cong K\} \ .$$

Hence, $\mathcal{T}$ is a torsor over $\mathrm{Prym}(S/\Sigma)$. Explicitly, by choosing $\mathcal{L}_0 \in \mathcal{T}$, we obtain an isomorphism $\mathcal{T} \xrightarrow{\sim} \mathrm{Prym}(S/\Sigma)$ given by $\mathcal{L} \to \mathcal{L} \otimes \mathcal{L}_0^{-1}$.

To summarize, we have the following:

Proposition 2.4. *Let q be a quadratic differential with simple zeros. Then $\mathcal{M}_q \cong \mathcal{T} \cong \mathrm{Prym}(S/\Sigma)$.*

If $q \neq 0$ is irreducible but nongeneric, the spectral curve S is singular and irreducible. We may still define the set $\overline{\mathcal{T}} \subset \overline{\mathrm{Pic}}^{2g-2}(S)$ as follows:

$$\overline{\mathcal{T}} := \{\mathcal{L} \in \overline{\mathrm{Pic}}^{2g-2}(S) \mid \det(\pi_*\mathcal{L}) \cong \mathcal{O}_\Sigma\} \ ,$$

We also set $\mathcal{T} := \overline{\mathcal{T}} \cap \mathrm{Pic}^{2g-2}$.Then $\overline{\mathcal{T}}$ is the natural compactification of $\mathcal{T}$ induced by the inclusion $\mathrm{Pic}^{2g-2}(S) \subset \overline{\mathrm{Pic}}(S)$. The BNR correspondence, as stated in Theorem 2.3, implies that $\chi_{\mathrm{BNR}} : \overline{\mathcal{T}} \to \mathcal{M}_q$ is an isomorphism.

2.4 The Hitchin moduli space and the nonabelian Hodge correspondence

We now recall the well-known nonabelian Hodge correspondence (NAH), which relates the space of flat $\mathrm{SL}(2, \mathbb{C})$ connections, Higgs bundles, and solutions to the Hitchin equations. This result was developed in the work of Hitchin [Hit87a], Simpson [Sim88], Corlette [Cor88], and Donaldson [Don87].

As above, let E be a trivial, smooth, rank 2 vector bundle over the Riemann surface Σ, and let H_0 be a fixed Hermitian metric on E. We denote by $\mathfrak{sl}(E)$ (resp. $\mathfrak{su}(E)$) the bundle of traceless (resp. traceless skew-hermitian) endomorphisms of E. Let A be a unitary (with respect to H_0) connection on E that induces the trivial connection on $\det E$, and let $\phi \in \Omega^1(i\mathfrak{su}(E))$. We will sometimes also refer to ϕ as a Higgs field. The Hitchin equations for the pair (A, ϕ) are given by:

$$F_A + \phi \wedge \phi = 0 \ , \ d_A\phi = d_A^*\phi = 0 \ . \tag{5}$$

If we split the Higgs field into type: $\phi = \varphi + \varphi^\dagger$, with $\varphi \in \Omega^{1,0}(\mathfrak{sl}(E))$, then (5) is equivalent to:

$$F_A + [\varphi, \varphi^\dagger] = 0 \ , \ \bar\partial_A\varphi = 0 \ . \tag{6}$$

Notice that $(\bar\partial_E, \varphi)$ then defines an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle. The Hitchin moduli space, denoted by $\mathcal{M}_{\mathrm{Hit}}$, is the moduli space of solutions to the Hitchin equation, given by

$$\mathcal{M}_{\mathrm{Hit}} := \{(A,\phi) \mid (A,\phi) \text{ satisfies } (5)\}/\mathcal{G},$$

where $\mathcal{G}$ is the gauge group of unitary automorphisms of E. Recall that a flat connection $\mathcal{D}$ is called completely reducible if and only if it is a direct sum of irreducible flat connections. The NAH can be summarized as follows:

Theorem 2.5 ([Hit87a, Sim90, Cor88, Don87]). *A Higgs bundle $(\mathcal{E}, \varphi)$ is polystable if and only if there exists a Hermitian metric H such that the corresponding Chern connection A and Higgs field $\phi = \varphi + \varphi^\dagger$ solve the Hitchin equations (5). Moreover, the connection $\mathcal{D}$ defined by $\mathcal{D} = \nabla_A + \phi$ is a completely reducible flat connection, and it is irreducible if and only if $(\mathcal{E}, \varphi)$ is stable.*

Conversely, a flat connection $\mathcal{D}$ is completely reducible if and only if there exists a Hermitian metric H on E such that when we express $\mathcal{D} = \nabla_A + \varphi + \varphi^\dagger$, we have $\bar\partial_{\mathcal{E}}\varphi = 0$. Moreover, the corresponding Higgs bundle $(\mathcal{E}, \varphi)$ is polystable, and it is stable if and only if $\mathcal{D}$ is irreducible.

The nonabelian Hodge correspondence gives the Kobayashi-Hitchin homeomorphism (1), which when restricted to the stable locus is a diffeomorphism onto irreducible solutions of (5).

Finally, we note that there is an action of S^1 on $\mathcal{M}_{\mathrm{Hit}}$ defined by $(A,\phi) \to (A, e^{i\theta}\cdot\phi)$, where $e^{i\theta}\cdot\phi = e^{i\theta}\varphi + e^{-i\theta}\varphi^\dagger$. With respect to this and the $S^1 \subset \mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Dol}}$, the map Ξ is S^1-equivariant.

3. Filtered bundles and compactness

Filtered (or parabolic) bundles are described, for example, in [Sim90]. They play a key role in the analytic compactification. This section provides a brief overview of filtered line bundles and demonstrates a compactness result.

3.1 Filtered line bundles and nonabelian Hodge

Let Z be a finite collection of distinct points on a closed Riemann surface Σ, and let $\Sigma' = \Sigma \backslash Z$. Viewing Σ as a projective algebraic curve, an algebraic line bundle L over the affine curve Σ' is a line bundle defined by regular transition functions on Zariski open sets over Σ'. The sheaf of sections of L can be extended in infinitely many different ways over Z to obtain coherent (invertible) sheaves on Σ. The sections of L are then realized as meromorphic sections of any such extension that are regular on Σ'.

A *filtered line bundle* $\mathcal{F}_*(L)$ is an algebraic line bundle $L \to \Sigma'$, along with a collection $\{L_\alpha\}_{\alpha\in\mathbb{R}}$ of coherent extensions across the punctures Z such that $L_\alpha \subset L_\beta$ for $\alpha \geqslant \beta$, for a fixed sufficiently small ϵ, $L_{\alpha-\epsilon} = L_\alpha$, and $L_\alpha = L_{\alpha+1} \otimes \mathcal{O}_\Sigma(Z)$. Let $\mathrm{Gr}_\alpha = L_\alpha/L_{\alpha+\epsilon}$ denote the quotient (torsion) sheaf. A value α where $\mathrm{Gr}_\alpha \neq 0$ is called a jump. Since we are considering line bundles, for each p in the support of Gr_{α_p}, there is exactly one jump α_p in the interval $[0, 1)$. The collection of jumps α_p, $p \in Z$, fully determines the filtered bundle structure. If we denote by $\mathcal{L} := L_0$, the degree of a filtered line bundle is defined as

$$\deg(\mathcal{F}_*(L)) := \deg(\mathcal{L}) + \sum_{p\in Z} \alpha_p \ .$$

Alternatively, a *weighted line bundle* is a pair $(\mathcal{L}, \chi)$ where $\mathcal{L} \to \Sigma$ is a holomorphic line bundle and $\chi : Z \to \mathbb{R}$ is a weight function. The degree of a weighted bundle is defined as

$$\deg(\mathcal{L}, \chi) := \deg(\mathcal{L}) + \sum_{p\in Z} \chi_p \ .$$

The notions of filtered and weighted line bundles are nearly equivalent. Namely, given a filtered line bundle $\mathcal{F}_*(L)$, we define $\mathcal{L} := L_0$ and $\chi_p = \alpha_p$. Conversely, given a weighted line bundle $(\mathcal{L}, \chi)$, let $\alpha_p = \chi_p + n_p$, where $n_p \in \mathbb{Z}$ is the unique integer so that $0 \leqslant \chi_p + n_p < 1$. A filtered bundle $\mathcal{F}_*(L)$, $L := \mathcal{L}\big|_{\Sigma'}$, is then determined by setting $L_0 = \mathcal{L}(-\sum_{p\in Z} n_p p)$ with jumps α_p. Clearly, $\deg(\mathcal{F}(L)) = \deg(\mathcal{L}, \chi)$. We shall use the notation $\mathcal{F}_*(\mathcal{L}, \chi)$ for the filtered bundle associated to a weighted bundle $(\mathcal{L}, \chi)$ in this way.

Different weighted bundles can give rise to the same filtered bundle. The following is a fact that will be frequently used in this paper. If $D = \sum_{x\in Z} d_x x$ is a divisor supported on Z, let

$$\chi_D(x) := \begin{cases} d_x \ , & x \in Z \ ; \\ 0 \ , & x \in \Sigma \setminus Z \ . \end{cases}$$

Then for any weighted bundle $(\mathcal{L}, \chi)$ we have $\mathcal{F}_*(\mathcal{L}(D), \chi - \chi_D) = \mathcal{F}_*(\mathcal{L}, \chi)$.

Let $(\mathcal{L}_1, \chi_1)$ and $(\mathcal{L}_2, \chi_2)$ be two weighted lines bundles. We define the tensor product

$$(\mathcal{L}_1, \chi_1) \otimes (\mathcal{L}_2, \chi_2) := (\mathcal{L}_1 \otimes \mathcal{L}_2, \chi_1 + \chi_2) \ .$$

Then the degree is additive on tensor products. For filtered bundles, we *define*

$$\mathcal{F}_*(\mathcal{L}_1, \chi_1) \otimes \mathcal{F}_*(\mathcal{L}_2, \chi_2) := \mathcal{F}_*(\mathcal{L}_1 \otimes \mathcal{L}_2, \chi_1 + \chi_2) \ . \tag{7}$$

The degree is again additive for the tensor product of filtered bundles. This agrees with the usual definition of tensor product for parabolic bundles.

3.2 Harmonic metrics for weighted line bundles

Proposition 3.1. *Let $(\mathcal{L}, \chi)$ be a degree 0 weighted bundle. Then there exists a Hermitian metric h on $\mathcal{L}_{\Sigma'}$, which is unique up to a multiplication by a nonzero constant, and such that:*

(i) the Chern connection A_h of $(\mathcal{L}, h)$ is flat: $F_{A_h} = 0$;

(ii) for $p \in Z$, and (U_p, z) a holomorphic coordinate centered at p, $|z|^{-2\chi_p} h$ extends to a $\mathcal{C}^\infty$ Hermitian metric on $\mathcal{L}|_{U_p}$.

Proof. We first choose a background Hermitian metric h_0 such that $|z|^{-2\chi_p} h_0$ defines a $\mathcal{C}^\infty$ Hermitian metric defined on U_p. Let A_{h_0} be the Chern connection, and F_{A_0} the curvature. Note that F_{A_0} is smooth on Σ. By the Poincaré-Lelong formula, we have $\frac{\sqrt{-1}}{2\pi}\int_\Sigma F_{A_0} = \deg(\mathcal{L},\chi) = 0$. Therefore, there exists a $\mathcal{C}^\infty$ function ρ such that $\Delta\rho + \frac{\sqrt{-1}}{2\pi}\Lambda F_{A_0} = 0$. We define $h = h_0 e^\rho$. For the corresponding Chern connection A_h, we have $F_{A_h} = 0$, which implies (i). (ii) follows from the property for h_0, since ρ is a smooth function on Σ. As ρ is well-defined up to a constant, h is well-defined up to a constant, which implies the uniqueness of h up to a constant. □

The metric obtained above is called the *harmonic metric*. For a weighted bundle $(\mathcal{L},\chi)$, the holomorphic bundle $\mathcal{L}$ and the harmonic metric h define a filtration as follows. Given α, there exists $\epsilon > 0$ sufficiently small(probably depends on α) such that we could define

$$L_\alpha := \{s \in \mathcal{L}(*Z) \mid |s|_h \leqslant Cr^{\alpha-\epsilon} \text{ for some } C\} .$$

Here, r denotes the distance to Z in any smooth conformal metric on Σ. It is straightforward to check that this defines a filtered bundle that matches $\mathcal{F}_*(\mathcal{L},\chi)$ under the correspondence given in the previous section.

Even though the harmonic metric is only well-defined up to a constant, the Chern connection $A = (\mathcal{L}, h)$ is independent of this choice. The $(1,0)$ part of A, denoted ∇_h, then defines logarithmic connections $\nabla_h : L_\alpha \to L_\alpha \otimes K(Z)$.

3.3 Convergence of weighted line bundles

In this subsection, we consider the convergence of weighted line bundles. The main result we prove here is a consequence of [MS23, Theorem 1.8]. For the reader's convenience, we present a short proof in our situation.

Let (Σ_0, g_0) be a metrized Riemann surface (i.e. a Riemann surface Σ_0 with conformal metric g_0). We view Σ_0 as given by an underlying surface C with almost complex structure J_0. Consider a neighborhood U_1 of J_0 in the moduli space of holomorphic structures and a neighborhood U_2 of g_0 in the space of smooth metrics. We denote the product of these neighborhoods by $U = U_1 \times U_2$. We can define the fiber bundle $\mathrm{Pic}_U \to U$, where each fiber is the Picard group defined by the holomorphic structure. Let $(\Sigma_t = (C, J_t), g_t)$ be a family of metrized Riemann surfaces that converge smoothly to (Σ_0, g_0) as $t \to 0$. Let $Z_t \subset \Sigma_t$ be a collection of a finite number of points that converge to Z_0 in suitable symmetric products of C. For each $p \in Z_0$, we can write $Z_t = \cup_{p\in Z_0} Z_{t,p}$ such that all points in $Z_{t,p}$ converge to p. We define the convergence of weighted line bundles as follows.

Definition 3.2. *A family of weighted line bundles $(\mathcal{L}_t, \chi_t)$ over Σ_t with weights $\chi_t : Z_t \to \mathbb{R}$ converges to $(\mathcal{L}_0, \chi_0)$ if*

(i) $\mathcal{L}_t$ converges to $\mathcal{L}_0$ in Pic_U,

(ii) for all $p \in Z_0$ and t sufficiently small, $\sum_{q\in Z_{t,p}} \chi_t(q) = \chi_0(p)$.

A sequence of filtered bundles $\mathcal{F}_*(\mathcal{L}_t)$ converges to $\mathcal{F}_*(\mathcal{L}_0)$ if the corresponding weighted bundles converge. The following theorem provides insight into the compactness of a sequence of weighted line bundles.

Theorem 3.3. *Consider a one parameter family of weighted line bundles $(\mathcal{L}_t, \chi_t)$ defined over $(\Sigma_t \setminus Z_t)$, and with $\deg(\mathcal{L}_t, \chi_t) = 0$. Let h_t be the corresponding harmonic metrics from Proposition 3.1. If Z_t converges to Z_0, we write $Z_t = \cup_{p\in Z_0} Z_{t,p}$. Then there exists a weighted line bundle $(\mathcal{L}_0, \chi_0)$ over Z_0 with a harmonic metric h_0 such that:*

(i) After rescaling by $c_t > 0$, $c_t h_t$ converges to h_0 over $\Sigma_0 \setminus Z_0$ in the $\mathcal{C}^\infty_{\mathrm{loc}}$ sense.

(ii) If A_{h_t} is the Chern connection of $(\mathcal{L}_t, h_t)$, then on $\Sigma_0 \setminus Z_0$, $\lim_{t\to 0} \nabla_t = \nabla_0$ in $\mathcal{C}^\infty_{\mathrm{loc}}$.

Proof. By the assumptions on weights, $\deg(\mathcal{L}_t)$ is a fixed, t-independent constant. Let $\gamma_t = (J_t, g_t)$ be a path in U. Then $\mathrm{Pic}_U|_{\gamma_t}$ is compact, and there exists an $\mathcal{L}_0 \in \mathrm{Pic}(\Sigma_0)$ such that $\mathcal{L}_t$ converges to $\mathcal{L}_0$. For $p \in Z_0$, define $\chi_0(p) = \sum_{q\in Z_{t,p}} \chi_t(q)$, and thus obtain a weighted line bundle $(\mathcal{L}_0, \chi_0)$. Choose a family of approximate harmonic metrics h_t^{app}, such that $|z|^{-2\chi_p} h_t^{\mathrm{app}}$ extends to a smooth metric in a neighborhood of p and h_t^{app} converges to h_0^{app} in $\mathcal{C}^\infty_{\mathrm{loc}}(\Sigma_0 \setminus Z_0)$. Moreover, write $h_t = h_t^{\mathrm{app}} e^{s_t}$. After a suitable rescaling of h_t, we can assume $\|s_t\|_{L^2} = 1$. Let $\rho_t := \Delta_t h_t^{\mathrm{app}}$ be the curvature defined by the metric h_t^{app}. Then s_t satisfies the equation $\Delta_t s_t = \rho_t$ over Σ. As ρ_t converges to $\rho_0 \in \mathcal{C}^\infty_{\mathrm{loc}}(\Sigma \setminus Z_0)$, and g_t is a family with bounded geometry, we obtain the estimate

$$\|s_t\|_{\mathcal{C}^{k+2,\alpha}(\Sigma)} \leqslant C_{k,\alpha}(\|\rho_t\|_{\mathcal{C}^{k,\alpha}(\Sigma)} + 1) ,$$

where $C_{k,\alpha}$ is a t-independent constant. Therefore, passing to a subsequence, s_t converges to s_0 in $\mathcal{C}^\infty(\Sigma)$, which implies (i). The assertion (ii) follows from (i). $\square$

4. The algebraic and analytic compactifications

4.1 The algebraic compactification of the Dolbeault moduli space

In this subsection, we present the algebraic method for compactifying the Dolbeault moduli space. This technique is based on the $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Dol}}$, and was introduced in [Sim97, Sch98, Hau98, dC21, KNPS15]. The gauge theoretic approach can be found in [Fan22a].

Theorem 4.1 ([Sim97, Thm. 11.2],[dC21]). *Let V be an algebraic variety with $\mathbb{C}^*$ action. Suppose*

(i) the fixed point set of the $\mathbb{C}^*$ action is proper,

(ii) for every $t \in \mathbb{C}^*$, $v \in V$, the limit $\lim\limits_{t\to 0} t\cdot v$ exists.

Then the space $U := \{v \in V \mid \lim\limits_{t\to\infty} t\cdot v \text{ does not exist}\}$ is open in V, and the quotient $U/\mathbb{C}^*$ is separated and proper.

We apply this to the Dolbeault moduli space. The first step is to note that the possible isotropy subgroups are limited.

Lemma 4.2 ([Hau98, Thm. 6.2]). *Let $\xi = [(\mathcal{E}, \varphi)]$ be a $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle equivalence class with $\mathcal{H}(\xi) \neq 0$. Then the stabilizer Γ_ξ of ξ for the $\mathbb{C}^*$ action is either trivial or $\mathbb{Z}/2$. The latter case holds if and only if $(\mathcal{E}, \varphi)$ and $(\mathcal{E}, -\varphi)$ are complex gauge equivalent.*

Proof. For $t \in \Gamma_\xi$, $\mathcal{H}(t\cdot\xi) = t^2\mathcal{H}(\xi)$, hence $t^2 = 1$ if $\mathcal{H}(\xi) \neq 0$. $\square$

By this Lemma, the space $(\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*$ has an orbifold structure. In passing, we note that the fixed points of the $\mathbb{Z}/2$ action correspond to real representations under the nonabelian Hodge correspondence [Hit87a, Sec. 10].

By the properness of the Hitchin map $\mathcal{H}$ (see Theorem 2.1), it follows that $\lim\limits_{t\to\infty} t\cdot\xi$ exists if and only if $\mathcal{H}(\xi) = 0$. Now define

$$\overline{\mathcal{M}}_{\mathrm{Dol}} = \Big\{(\mathcal{M}_{\mathrm{Dol}} \times \mathbb{C}^*) \coprod (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))\Big\} / \mathbb{C}^* \ . \tag{8}$$

The analytic topology on the disjoint union is generated by open sets $U \times W_1$ and $V \times (W_2 \cap \mathbb{C}^*) \amalg V \cap (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))$, where $U, V \subset \mathcal{M}_{\mathrm{Dol}}$, $W_1, W_2 \subset \mathbb{C}$ are open, and $0 \notin W_1$, $0 \in W_2$.

The topology on $\overline{\mathcal{M}}_{\mathrm{Dol}}$ is then the quotient topology, and it is straightforward to see that with this topology, it is compact.

Since $(\mathcal{M}_{\mathrm{Dol}} \times \mathbb{C}^*)/\mathbb{C}^* = \mathcal{M}_{\mathrm{Dol}}$, there is a natural inclusion

$$\iota : \mathcal{M}_{\mathrm{Dol}} \to \overline{\mathcal{M}}_{\mathrm{Dol}},\ \iota(\xi) = [(\xi, 1)]\ ,$$

where brackets denote the equivalence class under the $\mathbb{C}^*$ action. The boundary of $\overline{\mathcal{M}}_{\mathrm{Dol}}$ is

$$\partial\overline{\mathcal{M}}_{\mathrm{Dol}} = \overline{\mathcal{M}}_{\mathrm{Dol}} \setminus \iota(\mathcal{M}_{\mathrm{Dol}}) = (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*\ .$$

There is a *boundary map*

$$\iota_\partial : \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0) \longrightarrow \partial\overline{\mathcal{M}}_{\mathrm{Dol}},\ \xi \mapsto [(\xi, 0)]\ ,$$

which is invariant under the $\mathbb{C}^*$ action, i.e., $\iota_\partial(\lambda\xi) = \iota_\partial(\xi)$ for $\lambda \in \mathbb{C}^*$.

The $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Dol}}$ covers the square of the action on $\mathcal{B}$. Hence, it is natural to compactify $\mathcal{B}$ by projectivizing:

$$\overline{\mathcal{B}} := \mathbb{P}(H^0(K^2) \oplus \mathbb{C})\ .$$

The inclusion is given, as usual, by

$$\iota_0 : \mathcal{B} \to \overline{\mathcal{B}},\ \iota(q) = [q \times \{1\}]\ ,$$

where $q \times \{1\} \in H^0(K^2) \oplus \mathbb{C}$. We also define $\partial\overline{\mathcal{B}} = \overline{\mathcal{B}} \setminus \iota_0(\mathcal{B}) \simeq \mathbb{P}(H^0(K^2))$, with boundary projection map

$$\iota_{0,\partial} : \mathcal{B} \setminus \{0\} \to \partial\overline{\mathcal{B}}\ ,\ \iota_{0,\partial}(q) = [q \times \{0\}]\ .$$

The Hitchin map $\mathcal{H} : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{B}$ extends to $\overline{\mathcal{H}} : \overline{\mathcal{M}}_{\mathrm{Dol}} \to \overline{\mathcal{B}}$, where $\overline{\mathcal{H}}|_{\mathcal{M}_{\mathrm{Dol}}} := \iota_0 \circ \mathcal{H}$, and for every $[(\mathcal{E}, \varphi)]/\mathbb{C}^* \in \partial\overline{\mathcal{M}}_{\mathrm{Dol}}$,

$$\overline{\mathcal{H}}([(\mathcal{E}, \varphi)]/\mathbb{C}^*) := [(\mathcal{H}(\varphi), 0)] \subset \overline{\mathcal{B}}\ .$$

This is well defined, since $\det(\varphi) \neq 0$ if $[(\mathcal{E}, \varphi)]/\mathbb{C}^* \in \partial\overline{\mathcal{M}}_{\mathrm{Dol}}$. Moreover,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Dol}} \arrow[r, "\iota"] \arrow[d, "\mathcal{H}"] & \overline{\mathcal{M}}_{\mathrm{Dol}} \arrow[d, "\overline{\mathcal{H}}"] \\ \mathcal{B} \arrow[r, "\iota_0"] & \overline{\mathcal{B}} \end{tikzcd}$$

commutes.

There is a good algebraic structure on this compactification.

Theorem 4.3 ([Sim97, Sch98, Hau98, dC21, Fan22a]). *The compactified space $\overline{\mathcal{M}}_{\mathrm{Dol}}$ is a normal projective variety, and $\partial\overline{\mathcal{M}}_{\mathrm{Dol}}$ is a Cartier divisor of $\overline{\mathcal{M}}_{\mathrm{Dol}}$.*

The following characterization of sequential convergence is useful. As $H^0(\Sigma)$ is a finite dimensional space, the L^2 norm on $q \in H^0(\Sigma)$ could be chosen arbitrarily and we fix one such choice.

Proposition 4.4. *Let $[(\mathcal{E}_i, \varphi_i)] \in \mathcal{M}_{\mathrm{Dol}}$ be a sequence of Higgs bundles, and write $q_i = \det(\varphi_i)$ and $r_i = \|q_i\|_{L^2}^{\frac{1}{2}}$. Suppose $\limsup r_i \to \infty$. Then up to subsequence:*

(i) there exists a Higgs bundle $[(\widehat{\mathcal{E}}_\infty, \hat{\varphi}_\infty)]$ with $\tilde{q}_\infty = \det(\tilde{\varphi}_\infty)$ and $\|\hat{q}_\infty\|_{L^2} = 1$ such that $\lim_{i\to\infty}[(\mathcal{E}_i, r_i^{-1}\varphi_i)] = [(\widehat{\mathcal{E}}_\infty, \hat{\varphi}_\infty)]$ in $\mathcal{M}_{\mathrm{Dol}}$ and $\lim_{i\to\infty} r_i^{-1} q_i = \hat{q}_\infty$ in $H^0(K^2)$;

(ii)

$$\lim_{i\to\infty} \iota[(\mathcal{E}_i,\varphi_i)] = \iota_{\partial}[(\widehat{\mathcal{E}}_\infty,\hat{\varphi}_\infty)] \ , \ \text{on } \overline{\mathcal{M}}_{\mathrm{Dol}} \ ,$$
$$\lim_{i\to\infty} \iota_0(q_i) = \iota_{0,\partial}(\hat{q}_\infty) \ , \ \text{on } \overline{\mathcal{B}} \ .$$

Proof. The first point follows since the Hitchin map $\mathcal{H}$ is proper and $\mathcal{H}(r_i^{-1}\varphi_i)$ is bounded. The second follows directly from the definition. □

4.2 The analytic compactification of the Hitchin moduli space

We next describe the compactification of the Hitchin moduli space, as developed in [MSWW14, Moc16, Tau13a].

4.2.1 *Decoupled Hitchin equations.* We begin by defining the decoupled Hitchin equations. Recall the notation from Section 2.4, let E be a trivial, smooth, rank 2 vector bundle over a Riemann surface Σ, and let H_0 be a background Hermitian metric on E. Let Z be a finite set of distinct points in Σ. For a smooth unitary connection A on $E|_{\Sigma\setminus Z}$ and smooth $\phi = \varphi + \varphi^\dagger \in \Omega^1(i\mathfrak{su}(E))|_{\Sigma\setminus Z}$, the *decoupled Hitchin equations* on $\Sigma \setminus Z$ are:

$$F_A = 0 \ , \ [\varphi,\varphi^\dagger] = 0 \ , \ \bar{\partial}_A\varphi = 0 \ . \tag{9}$$

Solutions to (9) may be quite singular near Z, so we make the following restriction:

Definition 4.5. *A solution (A,ϕ) to the decoupled Hitchin equations over $\Sigma \setminus Z$ is called* admissible *if $\phi \neq 0$, and $|\phi|$ extends to a continuous function on Σ with $|\phi|^{-1}(0) = Z$.*

By a *limiting configuration* we always mean an admissible solution to the decoupled Hitchin equations. Clearly, Z is determined by (A,ϕ). Admissibility guarantees that $\det(\varphi)$ extends to a holomorphic quadratic differential $q = \det(\varphi)$ on Σ, with $Z = q^{-1}(0)$ the zero locus. Hence, the spectral curve S_q is well-defined. We emphasize that Z may vary for different admissible solutions, but one always has that $\#Z \leqslant 4g-4$.

The equivalence relation on limiting configurations is that $(A_1,\phi_1) \sim (A_2,\phi_2)$ if $Z_1 = Z_2$ and $(A_1,\phi_1)g = (A_2,\phi_2)$ for a smooth unitary gauge transformation g on $\Sigma \setminus Z_1$. The moduli space of decoupled Hitchin equations is then

$$\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}} = \{\text{admissible solutions to } (9)\}/\sim \ .$$

We denote by $\mathcal{M}_{\mathrm{Hit},q}^{\mathrm{Lim}}$ the elements in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ with the determinant of the Higgs field equal to a quadratic differential q. In this case, the equivalence relation is induced by the action of the unitary gauge group over $\Sigma \setminus Z$, $Z = q^{-1}(0)$.

There is a natural $\mathbb{C}^*$ action on the moduli space $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$: given $(A,\phi = \varphi+\varphi^\dagger) \in \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ and $t \in \mathbb{C}^*$, we set $t\cdot[(A,\phi)] = [(A, t\varphi + \bar{t}\varphi^\dagger)]$, which is also a solution to (9).

4.2.2 *Compactification of the Hitchin moduli space.* The following compactness result is due to Taubes [Tau13b] and Mochizuki [Moc16] (see also [He20]).

Proposition 4.6. *Let (A_i,φ_i) be a sequence of solutions to (5), with $q_i = \det(\varphi_i) \in H^0(K^2)$. Then*

(i) if $\limsup \|q_i\|_{L^2(\Sigma)} < \infty$, then there is a subsequence (also denoted $\{i\}$), a smooth solution (A_∞,ϕ_∞) to (5), and a sequence g_i of smooth unitary gauge transformations on Σ, such that $(A_i,\phi_i)g_i$ converges smoothly to (A_∞,ϕ_∞) on Σ;

(ii) if $\lim \|q_i\|_{L^2(\Sigma)} = \infty$*, then there is a subsequence (also denoted* $\{i\}$*), and* $q_\infty \in H^0(K^2)$ *so that*

$$\frac{q_i}{\|q_i\|_{L^2}} \longrightarrow q_\infty$$

over Σ*, and an admissible solution* $(A_\infty, \phi_\infty = \varphi_\infty + \varphi_\infty^\dagger)$ *to* (9)*, with* $Z_\infty := q_\infty^{-1}(0)$*, and smooth unitary gauge transformations* g_i *on* $\Sigma \setminus Z_\infty$*, such that over any open set* $\Omega \Subset \Sigma \setminus Z_\infty$*,* $(A_i)g_i \to A_\infty$*, and*

$$\frac{g_i^{-1}\phi_i g_i}{\|\phi\|_{L^2}} \longrightarrow \phi_\infty$$

smoothly on Ω*.*

There is also a compactness result for sequences of solutions in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$.

Proposition 4.7. *Let* $[(A_i, \phi_i = \varphi_i + \varphi_i^\dagger)] \in \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ *be a sequence of admissible solutions to* (9)*, and let* $q_i = \det(\varphi_i)$ *be the corresponding quadratic differentials. Then after passing to a subsequence, there are* $t_i \in \mathbb{C}^*$*, a limiting configuration* $(A_\infty, \phi_\infty = \varphi_\infty + \varphi_\infty^\dagger)$ *with quadratic differential* $q_\infty = \det(\varphi_\infty) \neq 0$*, and a sequence* g_i *of smooth gauge transformations on* $\Sigma \setminus Z_\infty$*, such that:*

(i) $t_i^2 q_i$ *converges smoothly to* q_∞*,*

(ii) over any open set $\Omega \Subset X \setminus Z_\infty$*,* $(A_i, t_i \cdot \phi_i)g_i$ *converges smoothly to* (A_∞, ϕ_∞)*.*

Proof. Write $q_i = \det(\varphi_i) \in H^0(K^2)$. Adjusting by t_i if necessary, we may assume q_i converges to q_∞ over Σ. Also, since $F_{A_i} = 0$ over $\Sigma \setminus Z_i$ and Z_i converges to Z_∞, we can apply both Uhlenbeck compactness and the classical bootstrapping method to obtain A_∞ such that up to gauge A_i converges smoothly to A_∞ over $\Sigma \setminus Z_\infty$. Finally, the convergence of φ_i follows by the bound on q_i's. □

4.2.3 *The topology on the compactified space.* We now carefully define the topology on the space $\mathcal{M}_{\mathrm{Hit}} \coprod \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. Choose a metric in the conformal class on Σ. Let $W^{k,2}$ denote the Sobolev spaces on Σ of distributional sections with at least k derivatives in L^2. For a finite set of points $Z \subset \Sigma$ (or indeed any closed subset),

$$W^{k,2}_{\mathrm{loc}}(\Sigma \setminus Z) := \{f \mid f \in W^{k,2}(K),\ K \subset \Sigma \setminus Z,\ K \text{ compact}\}.$$

These definitions extend easily to the space of connections and $\Omega^1(i\mathfrak{su}(E))$ for a Hermitian vector bundle (E, H_0) over Σ with a fixed smooth background connection.

Let ω_n be a nested collection of open sets with $\omega_n \subset \overline{\omega_n} \subset \omega_{n+1}$, with $\bigcup_n \omega_n = \Sigma \setminus Z$. We then define the seminorms $\|f\|_n := \|f\|_{W^{k,2}(\omega_n)}$; in terms of these, $W^{k,2}_{\mathrm{loc}}(\Sigma \setminus Z)$ a Fréchet space.

For any $q \in H^0(K^2) \setminus \{0\}$, set $Z_q := q^{-1}(0)$, and consider the moduli space

$$\mathbb{M}_q = \{[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q^*} \cap W^{k,2}(\Sigma)\} \cup \{[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q}^{\mathrm{Lim}} \cap W^{k,2}_{\mathrm{loc}}(\Sigma \setminus Z_q)\}/\mathbb{C}^*.$$

Here we give more precise explanation of the above notation. The space $\mathcal{M}_{\mathrm{Hit},q^*}$ consists of solutions $(A, \phi = \varphi + \varphi^\dagger)$ to the Hitchin equations such that $\det(\varphi) = tq$ for some complex number t. Moreover, the notation $[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q^*} \cap W^{k,2}(\Sigma)$ means the equivalence class of $(A,\phi) \in W^{k,2}(\Sigma)$ modulo unitary gauge transformations in $W^{k+1,2}(\Sigma)$. Similarly, $[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q}^{\mathrm{Lim}} \cap W^{\mathrm{loc}}$ consists of the equivalence class of $(A,\phi) \in W^{k,2}_{\mathrm{loc}}(\Sigma \setminus Z_q)$ modulo unitary gauge transformations in $W^{k+1,2}_{\mathrm{loc}}(\Sigma \setminus Z_q)$. By classical bootstrapping of the gauge-theoretic elliptic equations, $\mathbb{M}_q$ is independent of $k \geqslant 2$.

Next define $\mathbb{M} := \mathcal{M}_0 \cup \bigcup_{q \in H^0(K^2)\setminus\{0\}} \mathbb{M}_q$ and based on definition, we have $\mathbb{M} = \mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}^{\mathrm{Lim}}_{\mathrm{Hit}}/\mathbb{C}^*$. Its topology is generated by two types of open sets. For interior points $\xi = [(A,\phi)] \in \mathcal{M}_{\mathrm{Hit}} \subset \mathbb{M}$ we use the open sets

$$V_{\xi,\epsilon} := \{[(A',\phi')] \in \mathcal{M}_{\mathrm{Hit}} \mid \|A'-A\|_{W^{k,2}(\Sigma)} + \|\phi'-\phi\|_{W^{k,2}(\Sigma)} < \epsilon\}$$

from the topology of $\mathcal{M}_{\mathrm{Hit}}$. For any boundary point $\xi_0 \in \mathcal{M}^{\mathrm{Lim}}_{\mathrm{Hit}}/\mathbb{C}^*$, choose a representative (A_0,ϕ_0) with $\|\phi_0\|_{L^2} = 1$. Let $q = \det(\phi_0)$, and fix any open set $\omega \Subset \Sigma \setminus Z_q$. Then, setting $\mathcal{M}^*_{\mathrm{Hit}} = \mathcal{M}_{\mathrm{Hit}} \setminus \mathcal{H}^{-1}(0)$,

$$\begin{aligned} U_{\xi_0,\omega,\epsilon} :=& \{(A,\phi) \in \mathcal{M}^*_{\mathrm{Hit}} \mid \|A-A_0\|_{W^{k,2}(\omega)} + \inf_{\theta \in S^1} \| \|\phi\|_{L^2}^{-\frac{1}{2}}\phi - e^{i\theta}\phi_0\|_{W^{k,2}(\omega)} < \epsilon,\ \|\phi\|_{L^2} > \frac{1}{\epsilon}\} \\ & \bigcup \{(A,\phi) \in \mathcal{M}^{\mathrm{Lim}}_{\mathrm{Hit}} | \|A-A_0\|_{W^{k,2}(\omega)} + \|\phi-\phi_0\|_{W^{k,2}(\omega)} < \epsilon\} \end{aligned}$$

defines an open set around ξ_0. The sets $U_{\xi_0,\omega,\epsilon}$ and $V_{\xi,\epsilon}$ generate the topology on $\mathbb{M}$.

Theorem 4.8. *The space $\mathbb{M}$ is a Hausdorff and compact.*

Proof. The Hausdorff property follows from the definition of the topology. By Propositions 4.6 and 4.7, $\mathbb{M}$ is sequentially compact. Moreover, using this explicit base for the topology $\mathbb{M}$ is first countable, and hence compact. □

We may now define the compactification of the Hitchin moduli space as the closure $\overline{\mathcal{M}}_{\mathrm{Hit}} \subset \mathbb{M}$; we write $\partial\overline{\mathcal{M}}_{\mathrm{Hit}}$ for the boundary of the closure, and $\overline{\mathcal{M}}_{\mathrm{Hit},q^*} := \overline{\mathcal{M}}_{\mathrm{Hit}} \cap \mathbb{M}_q$ for the subset of elements with a fixed quadratic differential.

The following result is described in [MSWW16, OSWW, MSWW19].

Theorem 4.9 ([MSWW19, Prop. 3.3]). *If q has only simple zeros, then $\overline{\mathcal{M}}_{\mathrm{Hit},q^*} = \mathbb{M}_q$.*

In other words, the compactification of any slice where q does not lie in the discriminant locus is "the obvious one".

5. Parabolic modules and stratification of BNR data

In this section, we review the notion of a parabolic module, as described in [Reg80, Coo93, Coo98, GO13]. This concept leads to a partial normalization of the generalized Jacobian and Prym varieties of the spectral curve.

5.1 Normalization of the spectral curve

Let $q \neq 0$ be a quadratic differential with an irreducible, singular spectral curve $S = S_q$. The zeros of q define a divisor $\mathrm{Div}(q) = \sum_{i=1}^{r_1} m_i p_i + \sum_{j=1}^{r_2} n_j p'_j$, where the m_i and n_j are even and odd integers, respectively, and hence r_1 and r_2 are the numbers of even and odd zeros, respectively, counted without multiplicity. Write $Z_{\mathrm{even}} = \{p_1,\dots,p_{r_1}\}$, $Z_{\mathrm{odd}} = \{p'_1,\dots,p'_{r_2}\}$, and $Z = Z_{\mathrm{even}} \cup Z_{\mathrm{odd}}$, so $\#Z = r = r_1 + r_2$.

The map $\pi : S \to \Sigma$ is a double covering branched along Z; hence, we may view p_i and p'_i as points in S. For $x \in S$, let $\mathcal{O}_x$ be the algebraic local ring, $\mathcal{O}^*_x$ its group of units, and R_x the completion. We say that S has an A_n singularity at x if $R_x \cong \mathbb{C}[[r,s]]/(r^2 - s^{n+1})$, where $n \geqslant 1$.

If S has an A_1 singularity at x, we call it a *nodal* singularity, and if S has an A_2 singularity at x, we call it a *cusp* singularity.

Let $p : \widetilde{S} \to S$ be the normalization of S, and let $\tilde{\pi} := \pi \circ p$:

$$\begin{tikzcd} \widetilde{S} \arrow[r, "p"] \arrow[dr, "\tilde{\pi}"'] & S \arrow[d, "\pi"] \\ & \Sigma \end{tikzcd} \tag{10}$$

For even zeros p_i we write $p^{-1}(p_i) = \{\tilde{p}_i^+, \tilde{p}_i^-\}$, and for odd zeros p_i' we write $p^{-1}(p_i') = \tilde{p}_i'$. Since $\pi : S \to \Sigma$ is a branched double cover, the involution σ on S extends to an involution of $\widetilde{S}$ which we also denote by σ. Note that $\sigma(\tilde{p}_i') = \tilde{p}_i'$ while $\sigma(\tilde{p}_i^{\pm}) = \tilde{p}_i^{\mp}$.

The ramification divisor $\Lambda' = \frac{1}{2}\sum_{i=1}^{r_1} m_i p_i + \frac{1}{2}\sum_{j=1}^{r_2}(n_j - 1)p_j'$, is a (Weil) divisor on S, and there is an exact sequence:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow p_*\mathcal{O}_{\widetilde{S}} \longrightarrow \sum_{x \in \mathrm{Supp}(\Lambda')} \widetilde{\mathcal{O}}_x/\mathcal{O}_x \longrightarrow 0\ . \tag{11}$$

The genus of $\widetilde{S}$ is $g(\widetilde{S}) = 4g - 3 - \deg(\Lambda') = 2g - 1 + r_2/2$.

5.2 Jacobian under the pull-back to the normalization

We now recall some facts about the Jacobian under the pull-back to the normalization (cf. [GO13]). Let $x \in Z \subset S$ be a singular point, i.e. either $x \in Z_{\text{even}}$ or $x = p_j'$ with $n_j \geqslant 3$. Let $\widetilde{\mathcal{O}}_x$ be the integral closure of $\mathcal{O}_x$. Set $V := \prod_{x \in Z} \widetilde{\mathcal{O}}_x^*/\mathcal{O}_x^*$. Then we have the following well-known short exact sequence.

$$0 \longrightarrow V \longrightarrow \mathrm{Jac}(S) \xrightarrow{p^*} \mathrm{Jac}(\widetilde{S}) \longrightarrow 0\ . \tag{12}$$

This will play an important role later on.

5.2.1 *Hitchin fiber.* We examine the locally free part $\mathcal{T}$ of the Hitchin fiber under the pull-back. Here, $\mathcal{T}$ is defined to be the set of $L \in \mathrm{Pic}^{2g-2}(S)$ such that $\det(\pi_* L) = \mathcal{O}_\Sigma$ (see 4). Though Λ' is a divisor S, it could also be considered as a divisor on Σ by the identification of p_i, p_j' and $\pi(p_i), \pi(p_j')$. To save notation, we write $\mathcal{O}_\Sigma(\Lambda')$ for the corresponding line bundle on Σ. For any $L \in \mathrm{Pic}(S)$, from (11) we see that $\det(\tilde{\pi}_* p^* L) \cong \det(\pi_* L) \otimes \mathcal{O}_\Sigma(\Lambda')$. We define a new set, $\widetilde{\mathcal{T}}$, as follows:

$$\widetilde{\mathcal{T}} := \{\widetilde{L} \in \mathrm{Pic}^{2g-2}(\widetilde{S}) \mid \det(\tilde{\pi}_* L) \cong \mathcal{O}(\Lambda')\}.$$

Then p^* maps $\mathcal{T}$ to $\widetilde{\mathcal{T}}$. Furthermore, if $L_1, L_2 \in \mathrm{Pic}(S)$ satisfy $p^* L_1 \cong p^* L_2$, then we have $\pi_* L_1 \cong \pi_* L_2$. This means that the fiber of $p^* : \mathrm{Jac}(S) \to \mathrm{Jac}(\widetilde{S})$ is the same as that of $p^* : \mathcal{T} \to \widetilde{\mathcal{T}}$, resulting in the following fibration:

$$V \longrightarrow \mathcal{T} \xrightarrow{p^*} \widetilde{\mathcal{T}}\ . \tag{13}$$

5.3 Torsion free sheaves

Now we present Cook's parametrization of rank 1 torsion free sheaves on curves with Gorenstein singularities (see [Coo98, p. 40] and also [Coo93, Reg80]). An explicit computation of the invariants used in this paper is provided in Appendix A. Let $x \in Z$ be a singular point of S, and let $L \to S$ be a rank 1 torsion free sheaf. We again let $\mathcal{O}_x$ denote the local ring at x, $\widetilde{\mathcal{O}}_x$ its integral closure, and $\delta_x = \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/\mathcal{O}_x)$. According to [GP93, Lemma 1.1], there exists a fractional ideal I_x that is isomorphic to L_x, uniquely defined up to multiplication by a unit of $\widetilde{\mathcal{O}}_x$, such that $\mathcal{O}_x \subset I_x \subset \widetilde{\mathcal{O}}_x$. We define $\ell_x := \dim_{\mathbb{C}}(I_x/\mathcal{O}_x)$ and $b_x := \dim_{\mathbb{C}}(T(I_x \otimes_{\mathcal{O}_x} \widetilde{\mathcal{O}}_x))$, where T means the torsion subsheaf. Then, ℓ_x and b_x are invariants of L.

Let $\mathcal{K}_x$ be the field of fractions of $\mathcal{O}_x$. The *conductor* of $I_x \subset \widetilde{\mathcal{O}}_x$ is defined to be

$$C(I_x) = \{u \in \mathcal{K}_x \mid u \cdot \widetilde{\mathcal{O}}_x \subset I_x\} \ .$$

The singularity is characterized by the following dimensions:

$$\underbrace{C(\mathcal{O}_x) \subset \overbrace{C(I_x) \subset \overbrace{\mathcal{O}_x \subset \underbrace{I_x \subset \widetilde{\mathcal{O}}_x}_{\delta_x-\ell_x}}^{\delta_x}}^{2\delta_x-2\ell_x}}_{2\delta_x} \ . \tag{14}$$

For $x = p_i \in Z_{\text{even}}$, we have $\delta_{p_i} = m_i/2$, and there are two maximal ideals $\mathfrak{m}_\pm$ in $\widetilde{\mathcal{O}}_x$ corresponding to the points $\tilde{p}_i^\pm$. We let $(\widetilde{\mathcal{O}}_{p_i}/C(I_{p_i}))_{\mathfrak{m}_\pm}$ be the localization by the ideals $\mathfrak{m}_\pm$, and define $a_{\tilde{p}_i^\pm} := \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i}/C(I_{p_i}))_{\mathfrak{m}_\pm}$. Moreover, we have $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i}/C(\mathcal{O}_{p_i}))_{\mathfrak{m}_\pm} = m_i/2 = \delta_{p_i}$. By Appendix A, $a_{\tilde{p}_i^\pm} = (m_i/2) - \ell_{p_i}$, and therefore $a_{\tilde{p}_i^+} + a_{\tilde{p}_i^-} = 2\delta_{p_i} - 2\ell_{p_i}$, and also $b_{p_i} = \ell_{p_i}$. Define

$$V(L_{p_i}) = \{(c_i^+, c_i^-) \mid c_i^\pm \in \mathbb{Z}_{\geqslant 0} \ , \ c_i^+ + c_i^- = \ell_{p_i}\} \ .$$

For $x = p_i' \in Z_{\text{odd}}$, we have $\delta_{p_i'} = (n_i - 1)/2$, and the maximal ideal $\mathfrak{m}$ of $\widetilde{\mathcal{O}}_x$ is unique. Define $a_{\tilde{p}_i'} := \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i'}/C(I_{p_i'}))_{\mathfrak{m}}$. By Appendix A, we have $a_{\tilde{p}_i'} = 2\delta_{p_i'} - 2\ell_{p_i'}$ and $b_{p_i'} = \ell_{p_i'}$. Moreover, $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i'}/C(\mathcal{O}_{p_i'}))_{\mathfrak{m}} = n_i - 1 = 2\delta_{p_i'}$. In this case we set $V(L_{p_i'}) = \{\ell_{p_i'}\}$.

Let $\eta : \widetilde{\mathcal{O}}_x \to \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x)$ be the quotient map. Define

$$S(L_x) := \{\mathcal{O}_x\text{-submodules } F_x \subset \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x) \mid \dim_{\mathbb{C}}(F_x) = \delta_x \ , \ \eta^{-1}(F_x) \cong L_x\} \ .$$

Hence, if $J_x = \eta^{-1}(F_x)$ with $F_x \in S(L_x)$, there exists an ideal $\mathfrak{t}_x$ in $\widetilde{\mathcal{O}}_x$ such that $J_x = \mathfrak{t}_x \cdot L_x$. For $x = p_i \in Z_{\text{even}}$, we obtain two integers $c_i^\pm = \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/(\mathfrak{t}_x \cdot \widetilde{\mathcal{O}}_x))_{\mathfrak{m}_\pm}$. By [Coo98, Lemma 6], $(c_i^+, c_i^-) \in V(L_{p_i})$, for $x = p_i' \in Z_{\text{odd}}$, $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/(\mathfrak{t}_x \cdot \widetilde{\mathcal{O}}_x)) = \ell_{p_i'} \in V(L_{p_i'})$, and these only depend on F_x. Hence, there is a well-defined map:

$$\kappa_x : S(L_x) \longrightarrow V(L_x) \ : \ \begin{cases} F_x \to (c_i^+, c_i^-) & \text{when } x = p_i, \\ F_x \to \ell_{p_i'} & \text{when } x = p_i' \ . \end{cases}$$

Lemma 5.1 ([Coo98, Lemma 6]). *For $x \in Z$, the connected components of $S(L_x)$ are parameterized by elements in $V(L_x)$.*

Set $V(L) := \prod_{x \in Z} V(L_x)$ and $S(L) := \prod_{x \in Z} S(L_x)$. Write $N(L) := |V(L)|$ for the number of points in $V(L)$. There is a map

$$\kappa := \prod_{x \in Z} \kappa_x : S(L) \longrightarrow V(L) \ .$$

For any $\mathbf{c} \in V(L)$, write $\mathbf{c} = (c_1^\pm, \ldots, c_{r_1}^\pm, \ell_{p_1'}, \ldots, \ell_{p_{r_2}'})$. Associate to this the divisor

$$D_{\mathbf{c}} = \sum_{i=1}^{r_1} (c_i^+ \tilde{p}_i^+ + c_i^- \tilde{p}_i^-) + \sum_{i=1}^{r_2} \ell_{p_i'} \tilde{p}_i'$$

on $\widetilde{S}$. Composing κ with the map above, we define

$$\varkappa : S(L) \longrightarrow \mathrm{Div}(\widetilde{S}) \ : \ \prod_{x\in Z} F_x \mapsto \mathbf{c} \mapsto D_{\mathbf{c}} \ . \tag{15}$$

The following result is straightforward but important:

Proposition 5.2. *L is locally free if and only if $\varkappa = 0$ on $S(L)$.*

Proof. L is locally free if and only if $\ell_x = 0$ for $x \in Z$. The claim then follows directly from the definition of $D_{\mathbf{c}}$. □

5.4 Parabolic modules

In this subsection, we define the notion of a parabolic module, following [Reg80, Coo93, Coo98].

First note that $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/C(\mathcal{O}_x)) = 2\delta_x$ (cf. (14)). Let $\mathrm{Gr}(\delta_x, \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x))$ be the Grassmannian of δ_x dimensional subspaces of the vector space $\widetilde{\mathcal{O}}_x/C(\mathcal{O}_x)$. Then $\widetilde{\mathcal{O}}_x^*$ acts on $\mathrm{Gr}(\delta_x, \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x))$ by multiplication, with fixed points corresponding to δ_x-dimensional $\mathcal{O}_x$ submodules of $\widetilde{\mathcal{O}}_x/C(\mathcal{O}_x)$. We write $\mathcal{P}(x)$ for the (reduced) variety of fixed points. This is a closed subvariety of $\mathrm{Gr}(\delta_x, \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x))$.

Suppose x is an A_n singularity. For notational convenience, we write $\mathcal{P}(A_n) := \mathcal{P}(x)$. We have the following:

Proposition 5.3 ([Coo98, Prop. 2]). *The following holds:*

(i) *$\mathcal{P}(A_n)$ is connected and depends only on δ_x. Also, $\dim_{\mathbb{C}} \mathcal{P}(A_{2n}) = n$, and we have isomorphisms $\mathcal{P}(A_{2n-1}) \cong \mathcal{P}(A_{2n})$.*

(ii) *If $P(A_0)$ is defined to be a point, then the inclusions $\mathcal{P}(A_0) \subset \mathcal{P}(A_2) \subset \cdots \subset \mathcal{P}(A_{2n})$ give a cell decomposition of $\mathcal{P}(A_{2n})$.*

(iii) *The singular locus $\mathrm{Sing}(\mathcal{P}(A_{2n})) \cong \mathcal{P}(A_{2n-4})$. In particular, it has codimension $\geqslant 2$. Moreover, $\mathcal{P}(A_1) = \mathcal{P}(A_2) \cong \mathbb{C}P^1$, and $\mathcal{P}(A_4)$ is a quadric cone.*

Define $\mathscr{P}(S) = \prod_{x\in Z} \mathcal{P}(x)$. This only depends on the curve singularity of S. Let $J \in \mathrm{Pic}(\widetilde{S})$. As vector spaces,

$$J_{\tilde{p}_i^+}^{\oplus \frac{m_i}{2}} \oplus J_{\tilde{p}_i^-}^{\oplus \frac{m_i}{2}} \cong \widetilde{\mathcal{O}}_{p_i}/C(\mathcal{O}_{p_i}) \ , \ J_{\tilde{p}_i'}^{\oplus (n_i-1)} \cong \widetilde{\mathcal{O}}_{p_i'}/C(\mathcal{O}_{p_i'}) \ .$$

Definition 5.4. *A parabolic module $\mathrm{PMod}(\widetilde{S})$ consists of pairs (J, v), where $J \in \mathrm{Jac}(\widetilde{S})$ and $v = \prod_{x\in Z} v_x$, with $v_x \in \mathcal{P}(x)$.*

By [Coo98, p. 41], $\mathrm{PMod}(\widetilde{S})$ has a natural algebraic structure. Let $\mathrm{pr} : \mathrm{PMod}(\widetilde{S}) \to \mathrm{Jac}(\widetilde{S})$ be the projection to the first component. Then pr defines a fibration of $\mathrm{PMod}(\widetilde{S})$ with fiber $\mathscr{P}(S)$. Moreover, there is a finite morphism $\tau : \mathrm{PMod}(\widetilde{S}) \to \overline{\mathrm{Jac}}(S)$ defined by sending $(J, v) \to L$, where L is given by:

$$0 \longrightarrow L \longrightarrow p_* J \longrightarrow (J \otimes \mathcal{O}_\Lambda)/v \longrightarrow 0 \ .$$

There is a diagram:

$$\begin{tikzcd} \mathscr{P}(S) \arrow[r] & \mathrm{PMod}(\widetilde{S}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\tau"] & \mathrm{Jac}(\widetilde{S}) \\ & \overline{\mathrm{Jac}}(S) & \end{tikzcd} \ .$$

The map τ may be regarded as the compactification of the pull-back normalization map p^* in (12).

Theorem 5.5 ([Coo98, Thm. 1]). *For the map* $\tau : \mathrm{PMod}(\widetilde{S}) \to \overline{\mathrm{Jac}}(S)$ *defined above,*

(i) τ *is a finite morphism, where the fiber over* L *consists of* $N(L)$ *points,*
(ii) The restriction $\tau : \tau^{-1}\mathrm{Jac}(S) \to \mathrm{Jac}(S)$ *is an isomorphism. Moreover, for* $L \in \mathrm{Jac}(S)$*, we have* $\mathrm{pr} \circ \tau^{-1}(L) = p^*(L)$.
(iii) Suppose $\tau(J, v) = L$*. For* $x \in Z$*, we have* $v \in S(L_x)$*. Let* $D_v = \varkappa(v)$ *be the divisor defined in (15). Then*

$$0 \longrightarrow p^*L/T(p^*L) \longrightarrow J \longrightarrow J \otimes \mathcal{O}_{D_v} \longrightarrow 0 \ .$$

In particular, $p^*L/T(p^*L) = J(-D_v)$.

Suppose all of the zeros of the quadratic differential q are odd. Then for $L \in \overline{\mathrm{Jac}}(S)$, $N(L) = 1$, and we can deduce the following.

Corollary 5.6. *If* $q^{-1}(0) = \{p_1', \ldots, p_r'\}$ *and all zeroes have odd multiplicity, then* $\tau : \mathrm{PMod}(\widetilde{S}) \to \overline{\mathrm{Jac}}(S)$ *is a bijection. Moreover, for* $L \in \overline{\mathrm{Jac}}(S)$ *with* $\tau(J, v) = L$*, we have*

$$p^*L/T(p^*L) = J(-\sum \ell_{p_i'}\tilde{p}_i') \ .$$

For convenience, we recall the canonical example of a parabolic module.

Example 5.7 ([Coo98, Ex. 2]). *Suppose* q *contains* $4g-2$ *simple zeros and one zero* x *of order* 2*. Then the spectral curve* S *has one nodal singularity at* x*. Denote* $p : \widetilde{S} \to S$ *the normalization, with* $p^{-1}(x) = \{\tilde{x}_+, \tilde{x}_-\}$*. Then* $\mathscr{P}(S) = \mathbb{C}P^1$*, and we obtain a fibration* $\mathbb{C}P^1 \to \mathrm{PMod}(\widetilde{S}) \to \mathrm{Jac}(\widetilde{S})$*. Let* $L \in \overline{\mathrm{Jac}}(S) \setminus \mathrm{Jac}(S)$*. If we write* $\widetilde{L} := p^*L/T(p^*L)$*, then*

$$\tau^{-1}(L) = \{(\widetilde{L} \otimes \mathcal{O}(\tilde{x}_+), v_+), (\widetilde{L} \otimes \mathcal{O}(\tilde{x}_-), v_-)\} \ .$$

We can define two sections:

$$s_\pm : \mathrm{Jac}(\widetilde{S}) \longrightarrow \mathrm{PMod}(\widetilde{S}) : J \mapsto (J, v_\pm) \ ,$$

where $v_+ = [1, 0]$*,* $v_- = [0, 1]$*. Then* $\overline{\mathrm{Jac}}(S)$ *is the quotient of* $\mathrm{PMod}(\widetilde{S})$ *given by the identification*

$$\overline{\mathrm{Jac}}(S) \cong \mathrm{PMod}(\widetilde{S})/(s_+ \sim \mathcal{O}(\tilde{x}_- - \tilde{x}_+)s_-) \ .$$

In particular, $\mathrm{PMod}(\widetilde{S})$ *is not a fibration over* $\overline{\mathrm{Jac}}(S)$.

Proposition 5.8. *The singular set of* $\mathrm{PMod}(\widetilde{S})$ *has codimension at least* 2*. Moreover, if the spectral curve* S *contains only cusp or nodal singularities, then* $\mathrm{PMod}(\widetilde{S})$ *is smooth.*

Proof. As the singularities of $\mathrm{PMod}(\widetilde{S})$ come from the space $\mathscr{P}(S)$, the claim follows from Proposition 5.3. □

Since we focus on $\mathrm{SL}(2, \mathbb{C})$ Higgs bundles, we must consider the parabolic module compactification of the fibration

$$0 \longrightarrow V \longrightarrow \mathcal{P} \xrightarrow{p^*} \mathrm{Prym}(\widetilde{S}/\Sigma) \longrightarrow 0 \ .$$

Setting, $\widehat{\mathrm{PMod}}(\widetilde{S}) := \tau^{-1}(\overline{\mathcal{P}})$, then there is a diagram from [GO13, p. 17]

$$\begin{tikzcd} \mathscr{P}(S) \arrow[r] & \widehat{\mathrm{PMod}}(\widetilde{S}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\tau"] & \mathrm{Prym}(\widetilde{S}/\Sigma) \\ & \overline{\mathcal{P}} & \end{tikzcd} \tag{16}$$

Theorem 5.5 proves that $\mathrm{pr}\circ\tau^{-1}|_{\mathcal{P}} = p^*$.

5.5 Stratifications of the BNR data

Recall that $\overline{\mathcal{T}}$ (resp. $\overline{\mathcal{P}}$) is the natural compactification of $\mathcal{T}$ (resp. $\mathcal{P}$) induced by the inclusion $\mathrm{Pic}(S)\subset\overline{\mathrm{Pic}}(S)$. Parabolic modules define a stratification of $\overline{\mathcal{P}}$ and $\overline{\mathcal{T}}$. In the following, $\pi: S\to\Sigma$ is a branched double cover, σ the associated involution on S, and by σ we also denote its extension to an involution on the normalization $\widetilde{S}$ of S.

For a rank 1 torsion free sheaf $L\in\overline{\mathrm{Pic}}(S)$, consider the map

$$p_{\mathrm{tf}}^{\star}:\overline{\mathrm{Pic}}(S)\longrightarrow \mathrm{Pic}(\widetilde{S})\ ,\ p_{\mathrm{tf}}^{\star}(L):=p^*L/T(p^*L)\ ,$$

i.e. the torsion free part of the pull-back to the normalization. By [Rab79], $p_{\mathrm{tf}}^{\star}(L)=p^*L$ at $x\in\widetilde{S}$ if and only if L is locally free at $p(x)\in S$.

Using the previous conventions, recall that we have the divisor

$$\Lambda=\sum_{i=1}^{r_1}\frac{m_i}{2}(\tilde{p}_i^{+}+\tilde{p}_i^{-})+\sum_{j=1}^{r_2}n_j\tilde{p}_j'$$

on $\widetilde{S}$.

Definition 5.9 ([Hor22a]). *An effective divisor $D\in\mathrm{Div}(\widetilde{S})$ is called a σ-divisor if*

(i) $D\leqslant\Lambda$ and $\sigma^*D=D$;

(ii) and for any $x\in\mathrm{Fix}(\sigma)$, $D|_x=d_x x$, where $d_x\equiv 0 \mod 2$.

The σ-divisors play an important role in describing the singular Hitchin fibers.

Proposition 5.10 ([Hor22a, Moc16]). *Let $L\in\overline{\mathcal{P}}$ and write $\widetilde{L}:=p_{\mathrm{tf}}^{\star}L$. Then we have $\widetilde{L}\otimes\sigma^*\widetilde{L}=\mathcal{O}(-D)$ for D a σ-divisor.*

For a σ-divisor D, define

$$\begin{aligned}\widetilde{\mathcal{T}}_D&=\{J\in\mathrm{Pic}(\widetilde{S})\mid J\otimes\sigma^*J=\mathcal{O}(\Lambda-D)\}\ ;\\ \widetilde{\mathcal{P}}_D&=\{J\in\mathrm{Pic}(\widetilde{S})\mid J\otimes\sigma^*J=\mathcal{O}(-D)\}\ .\end{aligned}\tag{17}$$

Then by [Hor22a, Prop. 5.6], $\widetilde{\mathcal{T}}_D$ and $\widetilde{\mathcal{P}}_D$ are abelian torsors over $\mathrm{Prym}(\widetilde{S}/\Sigma)$ with dimension $g(\widetilde{S})-g=g-1+\frac{1}{2}r_2$. In addition, we define

$$\begin{aligned}\overline{\mathcal{T}}_D&=\{L\in\overline{\mathcal{T}}\mid p_{\mathrm{tf}}^{\star}L\in\widetilde{\mathcal{T}}_D\}\ ;\\ \overline{\mathcal{P}}_D&=\{L\in\overline{\mathcal{P}}\mid p_{\mathrm{tf}}^{\star}L\in\widetilde{\mathcal{P}}_D\}\ .\end{aligned}\tag{18}$$

Then the partial order on divisors defines a stratification of $\overline{\mathcal{T}}$ (resp. $\overline{\mathcal{P}}$) by: $\cup_{D'\leqslant D}\overline{\mathcal{T}}_{D'}$ (resp. $\cup_{D'\leqslant D}\overline{\mathcal{P}}_{D'}$). The top strata are $\overline{\mathcal{T}}_{D=0}$ (resp. $\overline{\mathcal{P}}_{D=0}$), and these consist of the locally free sheaves. From the definition, $\mathcal{T}=\overline{\mathcal{T}}_{D=0}$ and $\mathcal{P}=\overline{\mathcal{P}}_{D=0}$.

Theorem 5.11 ([Hor22a, Thm. 6.2]). *(i) Suppose q contains at least one zero of odd order. For each stratum indexed by a σ-divisor D, if we let n_{ss} be the number of p such that $D|_p=\Lambda|_p$, then there are holomorphic fiber bundles*

$$\begin{aligned}(\mathbb{C}^*)^{k_1}\times\mathbb{C}^{k_2}&\longrightarrow\overline{\mathcal{T}}_D\xrightarrow{p_{\mathrm{tf}}^{\star}}\widetilde{\mathcal{T}}_D\ ;\\ (\mathbb{C}^*)^{k_1}\times\mathbb{C}^{k_2}&\longrightarrow\overline{\mathcal{P}}_D\xrightarrow{p_{\mathrm{tf}}^{\star}}\widetilde{\mathcal{P}}_D\ ,\end{aligned}\tag{19}$$

where $k_1 = r_1 - n_{ss}$, $k_2 = 2g-2-\frac{1}{2}\deg(D) - r_1 + n_{ss} - \frac{r_2}{2}$, *and* r_1, r_2 *are the number of even and odd zeros.*[1]

(ii) Suppose q *is irreducible but all zeros are of even order. Then there exists an analytic space* $\overline{\mathcal{T}}'_D$ *and a double branched covering* $p : \overline{\mathcal{T}}_D \to \overline{\mathcal{T}}'_D$*, with* $\overline{\mathcal{T}}'_D$ *a holomorphic fibration*

$$(\mathbb{C}^*)^{k_1} \times \mathbb{C}^{k_2} \longrightarrow \overline{\mathcal{T}}'_D \xrightarrow{p^\star_{\mathrm{tf}}} \widetilde{\mathcal{T}}_D \ .$$

In particular, $\dim(\overline{\mathcal{P}}_D) = \dim(\overline{\mathcal{T}}_D) = 3g-3-\frac{1}{2}\deg(D)$.

As explained in [Hor22a], via the BNR correspondence the stratification above translates into a stratification of the Hitchin fiber. Let $\chi_{\mathrm{BNR}} : \overline{\mathcal{T}} \xrightarrow{\sim} \mathcal{M}_q$ be the bijection in Theorem 2.3. Let D be a σ-divisor. Define $\mathcal{M}_{q,D} := \chi_{\mathrm{BNR}}(\overline{\mathcal{T}}_D)$. Then the stratification of $\overline{\mathcal{T}}$ induces a stratification on $\mathcal{M}_q = \bigcup_D \mathcal{M}_{q,D}$.

For each σ-divisor D, since $\sigma^* D = D$ and for any $x \in \mathrm{Fix}(\sigma)$, $D|_x = d_x x$, where $d_x \equiv 0 \mod 2$. We can write $D' := \frac{1}{2}\tilde{\pi}(D)$. Then D' is an effective divisor with supp $D' \subset Z$. Moreover, for $x \in q^{-1}(0)$, $D'_x \leqslant \frac{1}{2}\lfloor \mathrm{ord}_x(q) \rfloor$. Therefore, $\mathcal{M}_q$ may be regarded as also being stratified by divisors D' defined over Σ.

5.6 The structure of the parabolic module projection

We now explain the relationship between the divisor D_v in Theorem 5.5 and the σ-divisor. Given $L \in \overline{\mathcal{P}}$, define

$$\begin{aligned} \mathscr{N}_L &:= \{(J,v) \in \widehat{\mathrm{PMod}}(\widetilde{S}) \mid \tau(J,v) = L\} \ ; \\ \mathscr{D}_L &:= \{D_v \mid (J,v) \in \mathscr{N}_L\} \ , \end{aligned} \tag{20}$$

where $\mathscr{N}_L$ is $\tau^{-1}(L)$, and $\mathscr{D}_L$ is the collection of divisors D_v such that $J(-D_v) = p^\star_{\mathrm{tf}}(L)$. Moreover, it follows straight forward from the definition that the cardinalities have $|\mathscr{N}_L| = |\mathscr{D}_L|$. Furthermore, we define $N_L := |\mathscr{N}_L| = |\mathscr{D}_L|$. If L is locally free, then $J = p^* L$, and $\mathscr{D}_L$ is the zero divisor. Moreover, if $\tau(J,v) = \tau(J',v)$, then $J' = J(D_{v'} - D_v)$.

The divisor D_v satisfies the following symmetry property.

Proposition 5.12. *Let* D *be a* σ*-divisor and* $L \in \overline{\mathcal{P}}_D$*. For any* $D_v \in \mathscr{D}_L$*, we have* $D_v + \sigma^* D_v = D$*.*

Proof. Let $\tau(J,v) = L$. Then by Theorem 5.5 we have $\widetilde{L} = J(-D_v)$, where $\widetilde{L} = p^\star_{\mathrm{tf}}(L)$. As $L \in \overline{\mathcal{P}}_D$ and $J \in \mathrm{Prym}(\widetilde{S}/\Sigma)$, we have $\widetilde{L} \otimes \sigma^* \widetilde{L} = \mathcal{O}(-D)$ and $J \otimes \sigma^* J = \mathcal{O}_{\widetilde{S}}$, which implies $D_v + \sigma^* D_v = D$. ☐

As a consequence, we have the following.

Corollary 5.13. *Suppose* q *has only zeroes of odd order. Then for* $L \in \overline{\mathcal{P}}_D$ *and* $D_v \in \mathscr{D}_L$*, we have* $\sigma^* D_v = D_v$ *and* $D_v = \frac{1}{2} D$*. In addition,* $\tau : \widehat{\mathrm{PMod}}(\widetilde{S}) \to \overline{\mathcal{P}}$ *is a bijection.*

Proof. Since each zero has odd order, supp $(D_v) \subset \mathrm{Fix}(\sigma)$, which implies $D_v = \sigma^* D_v$. By Proposition 5.12, we must have $D_v = \frac{1}{2} D$. ☐

There are relationships between the integers appearing in the construction of the parabolic module:

[1] It was pointed out by Horn to us that the formula in the paper [Hor22a, Theorem 6.2] needs to be modified by incorporating n_{ss}. The expressions for k_1 and k_2 are derived from [Hor22a, Proposition 5.12] and [Hor22a, Theorem 5.13]. Specifically, in [Hor22a, Proposition 5.12], it is stated that the local contribution of p is null when $D|_p = \Lambda|_p$, which leads to the expression $k_1 = r_1 - n_{ss}$.

LEMMA 5.14 ([GK85]). *Let $D = \sum_{i=1}^{r_1} d_i(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{i=1}^{r_2} d_i'\tilde{p}_i'$ be a σ-divisor, and let $L \in \overline{\mathcal{P}}_D$. Then we have*

(i) $\ell_{p_i} = d_i$ *and* $\ell_{p_i'} = d_i'/2$;

(ii) $a_{\tilde{p}_i^+} = a_{\tilde{p}_i^-} = (m_i/2) - d_i$ *and* $a_{\tilde{p}_i'} = n_i - 1 - d_i'$.

Proof. Since $L \in \overline{\mathcal{P}}_D$, we have $\dim T(p^*L_{p_i}) = d_i$ and $\dim T(p^*L_{p_i'}) = d_i'/2$. The claim then follows from Proposition A.1. □

PROPOSITION 5.15. *Let $D = \sum_{i=1}^{r_1} d_i(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{i=1}^{r_2} d_i'\tilde{p}_i'$ be a σ-divisor, and let $L \in \overline{\mathcal{P}}_D$. Then $N_L = \prod_{i=1}^{r_1}(d_i + 1)$. The number N_L depends only on the σ-divisor D.*

Proof. By Lemma 5.14, $V(L)$ can be rewritten as

$$V(L) = \{(c_1^\pm, \ldots, c_{r_1}^\pm, c_1' = l_{p_1'}, \ldots, c_{r_2}' = l_{p_{r_2}'}) \mid c_i^+ + c_i^- = d_i, c_i^\pm \in \mathbb{Z}_{\geqslant 0}\} ,$$

□

If we define n_L to be the number of $D_v \in \mathscr{D}_L$ such that $\sigma^*D_v \neq D_v$, then we have the following.

PROPOSITION 5.16. *(i) n_L is even;*

(ii) if $L \in \overline{\mathcal{P}}_D$ with

$$D = \sum_{i=1}^{r_1} d_i(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{i=1}^{r_2} d_i'\tilde{p}_i' ,$$

and if there exists $i_0 \in \{1, \ldots, r_1\}$ such that d_{i_0} is not even, then $n_L = N_L$; otherwise, $n_L = N_L - 1$.

Proof. To prove (i), note that if $\sigma^*D_v \neq D_v$, then $\sigma^*(\sigma^*D_v) \neq \sigma^*D_v$, which means that n_L is even. For (ii), by Proposition 5.15, $D_v = \sigma^*D_v$ for $D_v \in \mathscr{D}_L$ if and only if $c_i^+ = c_i^- = d_i/2$. Therefore, $n_L \neq N_L$ if and only if all d_i are even, which implies (ii). □

We should note that the integer n_L only depends on the Higgs divisor D and in the rest of paper, we define $n_D := \frac{n_L}{2}$.

6. Irreducible singular fibers and the Mochizuki map

In this section, we provide a reinterpretation of the limiting configuration construction of a Higgs bundle on an irreducible fiber, as introduced by Mochizuki in [Moc16] (see also [Hor22a]). We also investigate the relationship between limiting configurations and the stratification.

6.1 Abelianization of a Higgs bundle

Let q be a fixed irreducible quadratic differential with spectral curve S, with normalization $p : \widetilde{S} \to S$. We define $\widetilde{K} := \widetilde{\pi}^*K$ (but note that $\widetilde{K} \neq K_{\widetilde{S}}$) and $\widetilde{q} := \widetilde{\pi}^*q \in H^0(\widetilde{K}^2)$, where $\widetilde{\pi}$ is as in (10). Choose a square root $\omega \in H^0(\widetilde{K})$ such that $\widetilde{q} = -\omega \otimes \omega$ (i.e. $\omega = p^*\lambda$). Let $\Lambda := \mathrm{Div}(\omega)$ and $\widetilde{Z} := \mathrm{supp}(\Lambda)$. We can then write

$$\Lambda = \sum_{i=1}^{r_1} \frac{m_i}{2}(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{j=1}^{r_2} n_j\tilde{p}_j' . \tag{21}$$

If $\sigma : \widetilde{S} \to \widetilde{S}$ denotes the involution, then $\sigma^*\omega = -\omega$.

Let $(\mathcal{E}, \varphi)$ be a Higgs bundle on Σ with $\det\varphi = q$. Consider the pullback $(\widetilde{\mathcal{E}}, \tilde{\varphi}) := (\widetilde{\pi}^*\mathcal{E}, \widetilde{\pi}^*\varphi)$ to $\widetilde{S}$. We have $\tilde{\varphi} \in H^0(\mathrm{End}(\widetilde{\mathcal{E}}) \otimes \widetilde{K})$ and $\widetilde{q} = \det(\tilde{\varphi})$. Since $\widetilde{q} = -\omega \otimes \omega$, $\pm\omega$ are well-defined eigenvalues of $\tilde{\varphi}$ over $\widetilde{S}$. Let $\tilde{\lambda}$ be the canonical section of the pullback of $\widetilde{K}$ to the total space $\mathrm{Tot}(\widetilde{K})$. The spectral curve for $(\widetilde{\mathcal{E}}, \tilde{\varphi})$ is defined by the equation

$$\widetilde{S}' := \{\tilde{\lambda}^2 - \tilde{q} = 0\}.$$

The set $\widetilde{S}' = \mathrm{Im}(\omega) \cup \mathrm{Im}(-\omega) \subset \mathrm{Tot}(\widetilde{K})$ decomposes into two irreducible pieces.

Having fixed a choice of ω, the eigenvalues of $\tilde{\varphi}$ are globally well-defined, and we can define the line bundle $\widetilde{L}_+ \subset \widetilde{\mathcal{E}}$ as $\widetilde{L}_+ := \ker(\tilde{\varphi} - \omega)$. Since $\sigma^*\omega = -\omega$, $\widetilde{L}_- = \sigma^*\widetilde{L}_+ = \ker(\tilde{\varphi} + \omega)$, and there is an isomorphism $\widetilde{\mathcal{E}}|_{\widetilde{S}\setminus\widetilde{Z}} \cong \widetilde{L}_+ \oplus \widetilde{L}_-|_{\widetilde{S}\setminus\widetilde{Z}}$.

There is a local description of $(\widetilde{\mathcal{E}}, \tilde{\varphi})$.

Lemma 6.1 ([Hor22a, Lemma 5.1, Thm. 5.3],[Moc16, Lemma 4.2]). *Let $x \in \widetilde{Z}$ and write $\Lambda|_x = m_x x$. Let U be a holomorphic coordinate neighborhood of x. Then there exists a frame $\mathfrak{e} \in H^0(U, \widetilde{K})$ such that, under a suitable trivialization of $\mathcal{E}|_U \cong U \times \mathbb{C}^2$, we can write*

$$\tilde{\varphi} = z^{d_x}\begin{pmatrix} 0 & 1 \\ z^{2m_x - 2d_x} & 0 \end{pmatrix} \otimes \mathfrak{e}. \tag{22}$$

Moreover, if we define $D := \sum_{x\in\widetilde{Z}} d_x x$, then D is a σ-divisor.

Lemma 6.2 ([Moc16, Sec. 4.1]). *For the $\widetilde{L}_\pm$ defined above, we have $\widetilde{L}_+ \otimes \widetilde{L}_- = \mathcal{O}_{\widetilde{S}}(D - \Lambda)$. Moreover, if we denote $\widetilde{L}_0 := \widetilde{L}_+(\Lambda - D)$ and $\widetilde{L}_1 := \sigma^*\widetilde{L}_0$, then $\widetilde{L}_+ = \widetilde{\mathcal{E}} \cap \widetilde{L}_0$, $\widetilde{L}_- = \widetilde{\mathcal{E}} \cap \widetilde{L}_1$, and we have the exact sequences*

$$0 \longrightarrow \widetilde{L}_+ \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{L}_1 \longrightarrow 0\ ;$$
$$0 \longrightarrow \widetilde{L}_- \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{L}_0 \longrightarrow 0\ .$$

Proof. The inclusion of $\widetilde{L}_\pm \to \widetilde{\mathcal{E}}$ defines an exact sequence:

$$0 \longrightarrow \widetilde{L}_+ \oplus \widetilde{L}_- \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{T} \longrightarrow 0\ ,$$

where $\mathcal{T}$ is a torsion sheaf with $\mathrm{supp}\,\mathcal{T} \subset \widetilde{Z}$. From the local description in (22), in the same trivialization, $\widetilde{L}_\pm$ are spanned by the bases $s_\pm = \begin{pmatrix} 1 \\ \pm z^{m_x - d_x} \end{pmatrix}$. Therefore, as $\det(\mathcal{E}) = \mathcal{O}_\Sigma$, we obtain $\widetilde{L}_+ \otimes \widetilde{L}_- = \mathcal{O}_{\widetilde{S}}(D - \Lambda)$. Since s_+, s_- are linear independent away from z, $\widetilde{\mathcal{E}}/\widetilde{L}_+$ is locally generated by the section $z^{d_x - m_x}s_-$. Therefore, $\widetilde{\mathcal{E}}/\widetilde{L}_+ \cong \widetilde{L}_-(\Lambda - D) = \widetilde{L}_1$. Using the involution, we obtain the other exact sequence. □

Therefore, if $\widetilde{L} \otimes \sigma^*\widetilde{L} = \mathcal{O}_{\widetilde{S}}(D - \Lambda)$, we have $\widetilde{L}_0 = \widetilde{L}(\Lambda - D) \in \widetilde{\mathcal{T}}_D$. In summary, the construction above leads us to consider the composition of the following maps given by the composition

$$\delta : \mathcal{M}_q \to \widetilde{\mathcal{T}}_D,\ \ (\mathcal{E}, \varphi) \mapsto \widetilde{L}_+ \mapsto \widetilde{L}_+(\Lambda - D)\ ,$$

where the first map is obtained by taking the kernel of $(\tilde{\pi}^*\varphi - \omega)|_{\tilde{\pi}^*\mathcal{E}}$.

This construction is directly related to the torsion free pull-back. Recall that $\chi_{\mathrm{BNR}} : \overline{\mathcal{T}} \to \mathcal{M}_q$ is the BNR correspondence map, and $p_{\mathrm{tf}}^\star : \overline{\mathrm{Pic}}(S) \to \mathrm{Pic}(\widetilde{S})$ is the torsion free pull-back. Then we have

PROPOSITION 6.3. *$\delta \circ \chi_{\mathrm{BNR}} = p_{\mathrm{tf}}^{\star}$. In particular, if $J \in \overline{\mathcal{T}}_D$, then $\delta \circ \chi_{\mathrm{BNR}}(J) \in \widetilde{\mathcal{T}}_D$.*

Proof. Let $J \in \overline{\mathcal{T}}$, and write $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(J)$, $(\widetilde{\mathcal{E}}, \tilde{\varphi}) := \tilde{\pi}^*(\mathcal{E}, \varphi)$. Recall the BNR exact sequence on S (see (3)). As p^* is right exact, we obtain

$$\widetilde{\mathcal{E}} \xrightarrow{\tilde{\varphi}-\tilde{\lambda}} \widetilde{\mathcal{E}} \otimes \tilde{K} \longrightarrow p^*J \otimes \tilde{K} \longrightarrow 0 \ .$$

Since the spectral curve is $\widetilde{S}' = \mathrm{Im}(\omega) \cup \mathrm{Im}(-\omega)$, we can consider the restriction to the component $\mathrm{Im}(\omega)$ and write $\tilde{\lambda} = \omega$, $\widetilde{L}_{\pm} := \ker(\tilde{\varphi} \mp \omega)$. We obtain an exact sequence

$$0 \longrightarrow \widetilde{L}_{+} \longrightarrow \widetilde{\mathcal{E}} \xrightarrow{\tilde{\varphi}-\omega} \widetilde{\mathcal{E}} \otimes \tilde{K} \longrightarrow p^*J \otimes \tilde{K} \longrightarrow 0 \ ,$$

which breaks into short exact sequences

$$0 \longrightarrow \widetilde{L}_{+} \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathrm{Im}(\tilde{\varphi} - \omega) \longrightarrow 0 \ ;$$
$$0 \longrightarrow \mathrm{Im}(\tilde{\varphi} - \omega) \longrightarrow \widetilde{\mathcal{E}} \otimes \tilde{K} \longrightarrow p^*J \otimes \tilde{K} \longrightarrow 0 \ .$$

Using the local trivialization in Lemma 6.1, $\mathrm{Im}(\tilde{\varphi} - \omega)$ is locally spanned by $\begin{pmatrix} z^{d_x} \\ -z^{m_x} \end{pmatrix} \mathfrak{e}$. From Lemma 6.2, if we write $\widetilde{L}_0 := \widetilde{L}_{+}(\Lambda - D)$ and $\widetilde{L}_1 := \sigma^* \widetilde{L}_0$, then

$$\delta \circ \chi_{BNR}(J) = \widetilde{L}_{+}(\Lambda - D) \ .$$

Moreover, there is an isomorphism $\mathrm{Im}(\tilde{\varphi} - \omega) \cong \widetilde{L}_1$. Letting $\widetilde{L}_1'$ be the saturation of $\widetilde{L}_1$, then we obtain the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{L}_1 \arrow[r] \arrow[d, "i"] & \widetilde{\mathcal{E}} \otimes \tilde{K} \arrow[r] \arrow[d, "\cong"] & p^*J \otimes \tilde{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{L}_1' \arrow[r] & \widetilde{\mathcal{E}} \otimes \tilde{K} \arrow[r] & p_{\mathrm{tf}}^{\star} J \otimes \tilde{K} \arrow[r] & 0
\end{tikzcd}$$

where $i : \widetilde{L}_1 \to \widetilde{L}_1'$ is the natural inclusion. Moreover, in the same trivialization, $\widetilde{L}_1'$ is spanned by the section $\begin{pmatrix} 1 \\ -z^{m_x - d_x} \end{pmatrix} \mathfrak{e}$. Therefore, $\widetilde{L}_1' \cong \widetilde{L}_{-} \otimes \tilde{K}$ and from Lemma 6.2, $p_{\mathrm{tf}}^{\star} J = \delta \circ \chi_{\mathrm{BNR}}(J)$. $\square$

If $(\mathcal{E}, \varphi)$ is a Higgs bundle with $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(L)$, and $\widetilde{L}_0 = \delta \circ \chi_{BNR}(L)$, then by Proposition 6.3, $\widetilde{L}_0 = p_{\mathrm{tf}}^{\star}(L)$. We define a Higgs bundle $(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ as follows

$$\widetilde{\mathcal{E}}_0 = \widetilde{L}_0 \oplus \sigma^* \widetilde{L}_0, \ \tilde{\varphi}_0 = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} .$$

Moreover, $\widetilde{\mathcal{E}}$ is an $\mathcal{O}_{\widetilde{S}}$ submodule of $\widetilde{\mathcal{E}}_0$ with a natural inclusion $\iota : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}_0$ satisfying the following:

(i) the induced morphism $\widetilde{\mathcal{E}} \to \widetilde{L}_0$, $\widetilde{\mathcal{E}} \to \sigma^* \widetilde{L}_0$ is surjective,

(ii) the restriction of $\iota|_{\widetilde{S} \backslash \widetilde{Z}}$ is an isomorphism,

(iii) $\tilde{\varphi}_0 \circ \iota = \iota \circ \tilde{\varphi}$.

Following [Moc16, Sec. 4.1], we call $(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ the *abelianization of the Higgs bundle* $(\mathcal{E}, \varphi)$.

6.2 The construction of the algebraic Mochizuki map

In this subsection, we define the algebraic Mochizuki map, as introduced in [Moc16]. Recall that for any divisor $D=\sum_{x\in Z} d_x x$, there is a canonical weight function

$$\chi_D(x):=\begin{cases} d_x & x\in \text{supp } D \ ; \\ 0 & x\notin \text{supp } D \ . \end{cases}$$

We also have the stratification $\overline{\mathcal{T}}=\cup_D \overline{\mathcal{T}}_D$, for σ divisors D. Let $\mathscr{F}(\widetilde{S})$ be the space of all degree zero filtered line bundles over $\widetilde{S}$. The *algebraic Mochizuki map* Θ^{Moc} is defined as

$$\Theta^{\mathrm{Moc}}:\overline{\mathcal{T}}\longrightarrow \mathscr{F}(\widetilde{S}) \ , \ L\mapsto \mathcal{F}_*(p_{\mathrm{tf}}^\star(L),\tfrac{1}{2}\chi_{D-\Lambda}) \ .$$

Example 6.4. *When q has only simple zeroes, this construction generalizes that of [MSWW16] (see also [Fre18]). In the case of a quadratic differential with simple zeros, the spectral curve S is smooth, and every torsion free sheaf is locally free, so that $\mathcal{T}=\overline{\mathcal{T}}$. If $Z=\{p_1,\dots,p_{4g-4}\}$ are the branch points of S, and $\Lambda=\sum_{i=1}^{4g-4}p_i$, then the weight function $\frac{1}{2}\chi_{-\Lambda}$ assigns a weight of $-\frac{1}{2}$ at each p_i. For $L\in\mathcal{T}$, $\Theta^{\mathrm{Moc}}(L)=\mathcal{F}_*(L,\frac{1}{2}\chi_{-\Lambda})$.*

Below are some additional properties of Θ^{Moc}.

Proposition 6.5. *$\Theta^{\mathrm{Moc}}|_{\overline{\mathcal{T}}_D}$ is a continuous map.*

Proof. This follows directly from the definition of Θ^{Moc} and Theorem 3.3. □

From Theorem 5.11, we know that for a σ-divisor D, the preimage of the map $p_{\mathrm{tf}}^\star:\overline{\mathcal{T}}_D\to\widetilde{\mathcal{T}}_D$ has dimension $2g-2-\frac{1}{2}\deg(D)-r_2/2$, where r_2 is the number of odd zeros of q. Even for the top stratum $D=0$, $p_{\mathrm{tf}}^\star$ is not injective if the spectral curve is not smooth. Indeed, if $L_1,L_2\in\overline{\mathcal{T}}_D$ with $p_{\mathrm{tf}}^\star(L_1)=p_{\mathrm{tf}}^\star(L_2)$, then based on the construction we have $\Theta^{\mathrm{Moc}}(L_1)=\Theta^{\mathrm{Moc}}(L_2)$. In summary, we have the following result:

Proposition 6.6. *If $q\in H^0(K^2)$ is irreducible, then Θ^{Moc} is injective if and only if q has simple zeros.*

6.3 Convergence of subsequences

Fix a locally free $L_0\in\mathcal{T}$. Using the isomorphism $\psi_{L_0}:\overline{\mathcal{T}}\to\overline{\mathcal{P}}$ defined by $\psi_{L_0}(L)=LL_0^{-1}$, we can extend the Mochizuki map Θ^{Moc} to $\overline{\mathcal{P}}$. For $J\in\overline{\mathcal{P}}_D$, we write $\widetilde{J}:=p_{\mathrm{tf}}^\star(J)$ and choose the weight function $\frac{1}{2}\chi_D$. We then define:

$$\Theta_0^{\mathrm{Moc}}:\overline{\mathcal{P}}_D\longrightarrow\mathscr{F}(\widetilde{S}) \ , \ J\mapsto \mathcal{F}_*(\widetilde{J},\tfrac{1}{2}\chi_D) \ .$$

Proposition 6.7. *The map Θ_0^{Moc} satisfies the following properties:*

(i) Let $J\in\overline{\mathcal{P}}$ and $L:=L_0J$, then

$$\Theta_0^{\mathrm{Moc}}(J)=\Theta^{\mathrm{Moc}}(L)\otimes\Theta^{\mathrm{Moc}}(L_0)^{-1} \ ,$$

where $\otimes$ is the tensor product for filtered line bundles (7).

(ii) Suppose $L=\tau(I,v)$ with $(I,v)\in\widehat{\mathrm{PMod}}(\widetilde{S})$ and $L\in\overline{\mathcal{P}}_D$, then

$$\Theta_0^{\mathrm{Moc}}\circ\tau(I,v)=\mathcal{F}_*(I(-D_v),\tfrac{1}{2}\chi_{D_v+\sigma^*D_v}) \ ,$$

where D_v is the corresponding divisor defined in Theorem 5.5.

(iii) If $\sigma^*D_v=D_v$, then $\Theta_0^{\mathrm{Moc}}\circ\tau(I,v)=\mathcal{F}_*(I,0)$, where 0 means all parabolic weights are zero.

Proof. As L_0 is locally free, we have $p_{\rm tf}^\star J = (p^*L_0)^{-1} \otimes p_{\rm tf}^\star L$. By definition,

$$\Theta_0^{\rm Moc}(J) = \mathcal{F}_*(p_{\rm tf}^\star J, \tfrac{1}{2}\chi_D) \ , \ \Theta^{\rm Moc}(L) = \mathcal{F}_*(p_{\rm tf}^\star L, \tfrac{1}{2}\chi_{D-\Lambda}) \ , \ \Theta^{\rm Moc}(L_0) = \mathcal{F}_*(p_{\rm tf}^\star L_0, \tfrac{1}{2}\chi_{-\Lambda}) \ ,$$

which implies (i). For (ii), by Theorem 5.5, $p_{\rm tf}^\star L = I(-D_v)$, and from Proposition 5.12, we have $D = D_v + \sigma^* D_v$, which implies (ii). When $\sigma^* D_v = D_v$, we compute

$$\mathcal{F}_*(I(-D_v) \ , \ \tfrac{1}{2}\chi_{D_v+\sigma^*D_v}) = \mathcal{F}_*(I(-D_v) \ , \ \chi_{D_v}) = \mathcal{F}_*(I, 0) \ ,$$

which implies (iii). □

We now give a criterion for the continuity of the map $\Theta^{\rm Moc}$. By Proposition 6.7, it is sufficient to study the map $\Theta_0^{\rm Moc}$. Recall that for $L \in \overline{\mathcal{P}}$, we have

$$\mathscr{N}_L := \{(J,v) \in \widehat{\rm PMod}(\widetilde{S}) \mid \tau(J,v) = L\}, \quad \mathscr{D}_L := \{D_v \mid (J,v) \in \mathscr{N}_L\},$$

and the number n_L is defined to be the number of divisors $D_v \in \mathscr{D}_L$ such that $\sigma^* D_v \neq D_v$.

Proposition 6.8. *Let D be a σ-divisor, $L \in \overline{\mathcal{P}}_D$, and assume that $\Theta_0^{\rm Moc}$ is continuous at L. Then, for $(J,v) \in \mathscr{N}_L$ and $D_v \in \mathscr{D}_L$, we have $\sigma^* D_v = D_v$, i.e., $n_L = 0$.*

Proof. As the top stratum $\mathcal{P}$ is dense in $\overline{\mathcal{P}}$, there exists a family $L_i \in \mathcal{P}$ such that $\lim_{i\to\infty} L_i = L$. Let $(J_i, v_i) \in \widehat{\rm PMod}(\widetilde{S})$ be such that $\tau(J_i, v_i) = L_i$. Then, after passing to subsequences, $\lim_{i\to\infty}(J_i, v_i) = (J_\infty, v_\infty)$, and $\tau(J_\infty, v_\infty) = L$. As L_i is locally free, we have $D_{v_i} = 0$. Moreover, by Theorem 5.5, we have $p_{\rm tf}^\star L = J_\infty(-D_{v_\infty})$, and from Proposition 5.12, we have $D = D_{v_\infty} + \sigma^* D_{v_\infty}$. By Proposition 6.7, we have

$$\Theta_0^{\rm Moc}(L_i) = \Theta_0^{\rm Moc} \circ \tau(J_i, v_i) = \mathcal{F}_*(J_i, 0) \ ,$$

and we compute

$$\lim_{i\to\infty} \Theta_0^{\rm Moc}(L_i) = \mathcal{F}_*(J_\infty, 0) = \mathcal{F}_*(J_\infty(-D_{v_\infty}), \chi_{D_{v_\infty}}) \ .$$

Moreover, by Proposition 6.7, we have

$$\Theta_0^{\rm Moc}(L) = \mathcal{F}_*(J_\infty(-D_{v_\infty}), \frac{1}{2}(\chi_{D_{v_\infty}} + \chi_{\sigma^* D_{v_\infty}})).$$

Since $\Theta_0^{\rm Moc}$ is continuous on L, we have $\lim_{i\to\infty} \Theta_0^{\rm Moc}(L_i) = \Theta^{\rm Moc}(L)$, which implies that $\chi_{D_{v_\infty}} = \chi_{\sigma^* D_{v_\infty}}$. □

By Proposition 5.16, $n_L > 0$ if and only if q has at least one zero of even order. Hence, the following is immediate.

Corollary 6.9. *Suppose q is irreducible and has a zero of even order, then $\Theta_0^{\rm Moc}$ is not continuous.*

By contrast, we have the following.

Proposition 6.10. *If q is irreducible with all zeroes of odd order, then $\Theta_0^{\rm Moc}$ is continuous.*

Proof. Since all zeroes of q are odd, for any $L \in \overline{\mathcal{P}}$, we have $n_L = 0$. Let $L_\infty \in \overline{\mathcal{P}}$ be fixed and let $L_i \in \overline{\mathcal{P}}$ be any sequence such that $\lim_{i\to\infty} L_i = L_\infty$. Since $\tau : \widehat{\rm PMod}(\widetilde{S}) \to \overline{\mathcal{P}}$ is bijective, we take $(J_i, v_i) \in \widehat{\rm PMod}(\widetilde{S})$ with $\tau(J_i, v_i) = L_i$. Moreover, we assume $\lim_{i\to\infty}(J_i, v_i) = (J_\infty, v_\infty)$ with $\tau(J_\infty, v_\infty) = L_\infty$. Since q only contains odd zeros, it follows that supp $D_v \subset {\rm Fix}(\sigma)$. By Proposition 6.7, we have $\Theta_0^{\rm Moc}(L_i) = \mathcal{F}_*(J_i, 0)$. Therefore, we have:

$$\lim_{i\to\infty} \Theta_0^{\rm Moc}(L_i) = \lim_{i\to\infty} \mathcal{F}_*(J_i, 0) = \mathcal{F}_*(J_\infty, 0) = \Theta_0^{\rm Moc}(L_\infty).$$

This concludes the proof. □

Theorem 6.11. *Suppose q is irreducible. For the map $\Theta^{\mathrm{Moc}} : \mathcal{M}_q \to \mathscr{F}(\widetilde{S})$, we have:*

(i) Θ^{Moc} *is injective if and only if q only has only simple zeros;*

(ii) if q has only zeroes of odd order, Θ^{Moc} *is continuous;*

(iii) if q contains a zero of even order, Θ^{Moc} *is not continuous.*

Proof. (i) follows from Proposition 6.6. (ii) follows from Proposition 6.10. (iii) follows from Corollary 6.9. □

Proposition 6.12. *Suppose $n_L > 0$. Then for $k = 1, \dots, n_L$, there exist sequences $L_i^k \in \mathcal{P}$ with $\lim_{i\to\infty} L_i^k = L$ such that if we denote $\mathcal{F}_*^k := \lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i^k)$, $\mathcal{F}_*^0 := \Theta_0^{\mathrm{Moc}}(L)$, then $\mathcal{F}_*^{k_1} \neq \mathcal{F}_*^{k_2}$ for $k_1 \neq k_2$. Moreover, there exist $\{D_1, \dots, D_{n_L}\} \subset \mathscr{D}_L$ such that $\mathcal{F}_*^k = \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, \chi_{D_k})$.*

Proof. By the definition of n_L, we can find (J^k, v^k) with $\tau(J^k, v^k) = L$. If we define $D_k := D_{v^k}$, then $\sigma^* D_k \neq D_k$. Moreover, by Theorem 5.5, we have $p_{\mathrm{tf}}^{\star} L = J^k(-D_k)$. As $\tau^{-1}(\mathcal{P})$ is dense in $\widehat{\mathrm{PMod}}(\widetilde{S})$, for each (J^k, v^k), we can find a sequence $(J_i^k, v_i^k) \in \tau^{-1}(\mathcal{P})$ such that $\lim_{i\to\infty}(J_i^k, v_i^k) = (J^k, v^k)$ and we define $L_i^k := \tau(J_i^k, v_i^k)$. Since L_i^k is locally free, $D_{v_i^k} = 0$, and thus $\Theta_0^{\mathrm{Moc}}(L_i^k) = \mathcal{F}_*(J_i^k, 0)$. We compute

$$\lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i^k) = \mathcal{F}_*(J^k, 0) = \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, \chi_{D_k})$$

and $\Theta_0^{\mathrm{Moc}}(L) = \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, \frac{1}{2}\chi_D)$. Based on our assumptions, we have $D_{k_1} \neq D_{k_2}$ for $k_1 \neq k_2$ and $\sigma^* D_k \neq D_k$, which implies that $\chi_{D_{k_1}} \neq \chi_{D_{k_2}}$ for $k_1 \neq k_2$ and $\chi_{D_k} \neq \frac{1}{2}\chi_D$. □

We now present a computation for the case of a simple nodal curve.

Example 6.13. *Let q be a quadratic differential with $2g - 4$ simple zeros, and let x be an even zero of q of order two. Then S has a singular point, which we also denote by x. Let $p : \widetilde{S} \to S$ be the normalization map and $p^{-1}(x) = \{x_1, x_2\}$. Consider the σ-divisor $D = x_1 + x_2$, and let $L \in \overline{\mathcal{P}}_D$. Then $n_L = 2$, and we can write $\mathscr{N}_L = (J_1, v_1), (J_2, v_2)$, where $D_{v_1} = x_1$ and $D_{v_2} = x_2$. Moreover, we have $p_{\mathrm{tf}}^{\star} L = J_1 \otimes \mathcal{O}(-x_1) = J_2 \otimes \mathcal{O}(-x_2)$. Let (α, β) denote the parabolic weight that is equal to α at x_1, β at x_2, and $\frac{1}{2}$ at all other zeros. Then the filtered bundles obtained in Proposition 6.12 are*

$$\mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, (1, 0)) \ , \ \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, (0, 1)) \ , \ \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, (\tfrac{1}{2}, \tfrac{1}{2})) \ .$$

6.4 Mochizuki's convergence theorem for irreducible fibers

In this subsection, we recall Mochizuki's construction of the limiting configuration metric [Moc16, Section 4.2.1, 4.3.2] and the convergence theorem.

6.4.1 *Limiting configuration metric.* Let q be an irreducible quadratic differential and $(\mathcal{E}, \varphi) \in \mathcal{M}_q$ a Higgs bundle with $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(L)$. We write $\widetilde{L}_0 = p_{\mathrm{tf}}^{\star} L$ and $(\widetilde{\mathcal{E}}, \tilde{\varphi}) := p^*(\mathcal{E}, \varphi)$. Then the abelianization of $(\mathcal{E}, \varphi)$, which is a Higgs bundle over $\widetilde{S}$, can be written as $\widetilde{\mathcal{E}}_0 = \widetilde{L}_0 \oplus \sigma^* \widetilde{L}_0$, $\tilde{\varphi}_0 = \mathrm{diag}(\omega, \ -\omega)$. The natural inclusion $\iota : (\widetilde{\mathcal{E}}, \tilde{\varphi}) \to (\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ is an isomorphism over $\widetilde{S} \setminus \widetilde{Z}$. Moreover, we write D be the σ-divisor of $(\mathcal{E}, \varphi)$.

From the construction of $\Theta^{\mathrm{Moc}}(L)$ and Proposition 6.12, we have n_L different divisors D_k for $k = 1, \cdots, n_L$ with $\sigma^* D_k \neq D_k$ and $D_k + \sigma^* D_k = D$. Moreover, we can find $n_L + 1$ different filtered bundles with deg 0. Define

$$\mathcal{F}_{*,0} := \Theta^{\mathrm{Moc}}(L) = \mathcal{F}_*(\widetilde{L}_0, \chi_{\frac{1}{2}(D-\Lambda)}), \ \mathcal{F}_{*,k} := \mathcal{F}_*(\widetilde{L}_0, \chi_{(D_k - \frac{1}{2}\Lambda)}) \ ,$$

which are all degree zero filtered bundles with different level of filtrations.

Now, we will introduce the construction in [Moc16, Section 4.2.1, 4.3.2]. For $k = 0, \cdots, n_L$, we define $\tilde{h}_k$ to be the harmonic metric for the filtered bundle $\mathcal{F}_{*,k}$; this is well-defined up to a multiplicative constant. To fix this constant, assume that $\sigma^*\tilde{h}_k \otimes \tilde{h}_k = 1$. This gives a unique choice of $\tilde{h}_k$. We then define the metric $\widetilde{H}_k = \mathrm{diag}(\tilde{h}_k,\ \sigma^*\tilde{h}_k)$ on $\widetilde{\mathcal{E}}_0$, with $\det(\widetilde{H}_k) = 1$. For the resulting harmonic bundle $(\widetilde{\mathcal{E}}_0, \varphi_0, \widetilde{H}_k)$, we define $\tilde{\nabla}_k$ to be the unitary connection determined by $\widetilde{H}_k$. Since $\widetilde{H}_k$ is diagonal, over $\widetilde{S} \setminus \widetilde{Z}$, it follows that $F_{\tilde{\nabla}_k} = 0$, and we have $[\varphi_0, \varphi_0^{\dagger_{\widetilde{H}_k}}] = 0$. Furthermore, as ι is an isomorphism on $\widetilde{S} \setminus \widetilde{Z}$, the metric $\widetilde{H}_k$ also defines a metric on $(\widetilde{\mathcal{E}}, \tilde{\varphi})$ over $\widetilde{S} \setminus \widetilde{Z}$.

For any $\tilde{x} \in \widetilde{S} \setminus \widetilde{Z}$ with $x := p(\tilde{x})$, we have isomorphisms

$$(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)|_{\sigma(\tilde{x})} \cong (\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)|_{\tilde{x}} \cong (\widetilde{\mathcal{E}}, \tilde{\varphi})|_{\tilde{x}} \cong (\mathcal{E}, \varphi)|_x \ .$$

Therefore, $\widetilde{H}_k$ induces a metric H_k^{Lim} on $\Sigma \setminus Z$, and we may consider H_k^{Lim} as the push-forward of $\tilde{h}_k$. In [Hor22b, Theorem 5.2], the push-forward metric of $\Theta^{\mathrm{Moc}}(L)$ is explicitly written in local coordinates.

Recall the notation from Section 2.4, let E be a trivial, smooth, rank 2 vector bundle over a Riemann surface Σ, and let K be a background Hermitian metric on E. Over $\Sigma \setminus Z$, we write ∇_k^{Lim} for the Chern connection defined by H_k^{Lim}, which is unitary w.r.t. H_0 and $\phi_k^{\mathrm{Lim}} = \varphi_k^{\mathrm{Lim}} + \varphi_k^{\dagger_{\mathrm{Lim}}}$ be the corresponding Higgs field in the unitary gauge. They satisfy the decoupled Hitchin equations over $\Sigma \setminus Z$. Thus from any Higgs bundle $(\mathcal{E}, \varphi)$, we obtain $n_L + 1$ limiting configurations

$$(\nabla_k^{\mathrm{Lim}}, \phi_k^{\mathrm{Lim}} = \varphi + \varphi_k^{\dagger_{\mathrm{Lim}}}) \in \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}} \ .$$

The flat connection, which is defined over $\Sigma \setminus Z$, may be understood by using the nonabelian Hodge correspondence for filtered vector bundles [Sim90]. Given the filtered line bundles $\mathcal{F}_{*,k}$, define filtered vector bundles $\widetilde{\mathcal{E}}_{*,k} := \mathcal{F}_{*,k} \oplus \sigma^*\mathcal{F}_{*,k}$, which can be explicitly written as

$$\begin{aligned} \widetilde{\mathcal{E}}_{*,0} &:= \mathcal{F}_*(\widetilde{L}_0, \chi_{\frac{1}{2}(D-\Lambda)}) \oplus \mathcal{F}_*(\sigma^*\widetilde{L}_0, \chi_{\frac{1}{2}(D-\Lambda)}) \ ; \\ \widetilde{\mathcal{E}}_{*,k} &:= \mathcal{F}_*(\widetilde{L}_0, \chi_{D_k - \frac{1}{2}\Lambda}) \oplus \mathcal{F}_*(\sigma^*\widetilde{L}_0, \chi_{\sigma^* D_k - \frac{1}{2}\Lambda}) \ , \ k \neq 0 \ . \end{aligned} \tag{23}$$

These are polystable filtered vector bundles over $\widetilde{S} \setminus \widetilde{Z}$. As for each $k = 0, \cdots, n_L$, $\sigma^*\widetilde{\mathcal{E}}_{*,k} = \widetilde{\mathcal{E}}_{*,k}$, the filtered bundles $\widetilde{\mathcal{E}}_{*,k}$ induce a filtered vector bundles $\mathcal{E}_{*,k}$ over $\Sigma \setminus Z$. The flat connection ∇_k^{Lim} will be the unique harmonic unitary connection corresponding to the $\mathcal{E}_{*,k}$. Moreover, for $0 \leqslant k_1 \neq k_2 \leqslant n_L$, based on the definition of D_{k_1} and D_{k_2}, we can always find $\tilde{x} \in \widetilde{Z}_{\mathrm{even}}$, a preimage of an even zero x of q, such that $\widetilde{\mathcal{E}}_{*,k_1}$ and $\widetilde{\mathcal{E}}_{*,k_2}$ have different filtered structures near $\tilde{x}$. Since over even zeros, $\widetilde{S} \to \Sigma$ is not a branched covering, we conclude that near x, $\mathcal{E}_{*,k_1}$ and $\mathcal{E}_{*,k_2}$ are different filtered bundles. By [Sim90, Main theorem], the harmonic connections ∇_{k_1} and ∇_{k_2} are not gauge equivalent.

We therefore conclude the following:

Proposition 6.14. *For $0 \leqslant k_1 \neq k_2 \leqslant n_L$, $(\nabla_{k_1}^{\mathrm{Lim}}, \phi_{k_1}^{\mathrm{Lim}})$ and $(\nabla_{k_2}^{\mathrm{Lim}}, \phi_{k_2}^{\mathrm{Lim}})$ are not gauge equivalent in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$.*

Moreover, as with the algebraic compactification of the elements in the $\mathbb{C}^*$ orbit, we would like to compare with the limiting configurations in the space $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. Over the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$, there is a natural $\mathbb{Z}_2$ action given by $(\mathcal{E}, \varphi) \to (\mathcal{E}, -\varphi)$, and the fixed point of the $\mathbb{Z}_2$ action is defined to be the real locus of the Dolbeault moduli space, which we denoted by $\mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$.

It follows from [Hau98, Theorem 6.2] that the source of the orbitfold points of the algebraic compactification comes from the quotient of the real locus. Moreover, for $(\mathcal{E},\varphi) = \chi_{\mathrm{BNR}}(L)$, $(\mathcal{E},\varphi) \in \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$ if and only if $\sigma^* L = L$.

Given a Higgs bundle $(\mathcal{E},\varphi)$, under the previous convention, we write $(\widetilde{\mathcal{E}}_0 = \widetilde{L}_0 \oplus \sigma^*\widetilde{L}_0, \tilde{\varphi}_0 = \mathrm{diag}(\omega,-\omega))$ be the abelianzation of $(\mathcal{E},\varphi)$. Note that $\sigma^*(\widetilde{\mathcal{E}}_0,\tilde{\varphi}_0) = (\widetilde{\mathcal{E}}_0,\tilde{\varphi}_0)$ and $\widetilde{L}_0$ is the eigenline bundle for $\tilde{\varphi}_0$ with eigenvalue ω. Therefore, $(\widetilde{\mathcal{E}}_0,\tilde{\varphi}_0)$ is gauge equivalent to $(\widetilde{\mathcal{E}}_0,-\tilde{\varphi}_0)$ if and only if $\widetilde{L}_0 = \sigma^*\widetilde{L}_0$, which is equivalent to $(\mathcal{E},\varphi)$ lies in the real locus. For the collection of divisors $\mathscr{D} := \{D_1,\cdots,D_{n_L}\}$, for any $D_k \in \mathscr{D}$, we have $\sigma^* D_k \in \mathscr{D}$. Therefore, there exists a permutation of the index $\tau : \{1,\cdots,n_L\} \to \{1,\cdots,n_L\}$ such that $D_{\tau(k)} = \sigma^* D_k$ with $\tau^2 = \mathrm{id}$.

Suppose for the limiting configurations in (23), $(\widetilde{\mathcal{E}}_{*,k_1},\varphi_0)$ is gauge equivalent to $(\widetilde{\mathcal{E}}_{*,k_2},-\varphi_0)$. Then the eigenline bundle for eigenvalue ω will be gauge equivalent, which implies

$$\mathcal{F}_*(\widetilde{L}_0, \chi_{D_{k_1}-\frac{1}{2}\Lambda}) \cong \mathcal{F}_*(\sigma^*\widetilde{L}_0, \chi_{\sigma^* D_{k_2}-\frac{1}{2}\Lambda}).$$

The above equality holds if and only if $\widetilde{L}_0 = \sigma^*\widetilde{L}_0$ and $k_1 = \tau(k_2)$. In summary, we conclude the following:

Proposition 6.15. *Let $[(\nabla_k^{\mathrm{Lim}},\phi_k^{\mathrm{Lim}})]$ be the $\mathbb{C}^*$ equivalent class of $(\nabla_k^{\mathrm{Lim}},\phi_k^{\mathrm{Lim}})$ in the space $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. Then the following holds:*

(i) Suppose $(\mathcal{E},\varphi) \notin \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, then for any $0 \leqslant k_1 \neq k_2 \leqslant n_L$, $[(\nabla_{k_1}^{\mathrm{Lim}},\phi_{k_1}^{\mathrm{Lim}})] \neq [(\nabla_{k_2}^{\mathrm{Lim}},\phi_{k_2}^{\mathrm{Lim}})]$ in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$.

(ii) Suppose $(\mathcal{E},\varphi) \in \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, then $[(\nabla_{k_1}^{\mathrm{Lim}},\phi_{k_1}^{\mathrm{Lim}})] = [(\nabla_{k_2}^{\mathrm{Lim}},\phi_{k_2}^{\mathrm{Lim}})]$ in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$ if and only if $k_1 = k_2$ or $k_1 = \tau(k_2)$.

In particular, when $(\mathcal{E},\varphi) \notin \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, we obtain $1+2n_D$ different $\mathbb{C}^*$ equivalent classes of the limiting configurations in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$ and when $(\mathcal{E},\varphi) \in \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, we obtain $1+n_D$ different $\mathbb{C}^*$ equivalent classes limiting configurations in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$.

We define the *analytic Mochizuki map* Υ^{Moc} as

$$\Upsilon^{\mathrm{Moc}} : \mathcal{M}_q \longrightarrow \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}} \;:\; [(\mathcal{E},\varphi)] \mapsto [(\nabla_0^{\mathrm{Lim}},\phi_0^{\mathrm{Lim}})], \tag{24}$$

which we recall is the limiting configuration defined by $\Theta^{\mathrm{Moc}}(L)$.

6.4.2 *The continuity of the limiting configurations.* We now introduce the main result of Mochizuki [Moc16]. Fix $(\mathcal{E},\varphi) = \chi_{\mathrm{BNR}}(L) \in \mathcal{M}_q$. For any real parameter $t > 0$, $(\mathcal{E},t\varphi)$ is a stable Higgs bundle. By the Kobayashi-Hitchin correspondence, there exists a unique metric H_t solving the Hitchin equation. Denote by ∇_t the unitary connection defined by H_t and write $\mathcal{D}_t = \nabla_t + t\phi_t$ for the full $\mathrm{SL}(2,\mathbb{C})$ flat connection. We then have:

Theorem 6.16 ([Moc16], [MS23, Theorem 1.7]. *The family $(\mathcal{E},t\varphi)$ has a unique limiting configuration as limit for $t \to \infty$. Moreover, let D be the corresponding σ-divisor of $(\mathcal{E},\varphi)$, suppose $D = 0$, then for any compact set $K \subset \Sigma \setminus Z$, there exist t-independent positive constants $C_{l,K}$ and $C'_{l,K}$ such that*

$$|(\nabla_t,\phi_t) - \Upsilon^{\mathrm{Moc}}(\mathcal{E},\varphi)|_{\mathcal{C}^l} \leqslant C_{l,K} e^{-C'_{l,K} t}\,.$$

If write $(\mathcal{E},\varphi) = \chi_{\mathrm{BNR}}(L)$, and if $D = 0$, by (18), $L \in \mathcal{T} = \overline{\mathcal{T}}_{D=0}$, which is the top strata. In addition, $\mathcal{T}$ is dense in $\overline{\mathcal{T}}$.

As the map Υ^{Moc} is the composition of $\Theta^{\mathrm{Moc}} \circ \chi_{\mathrm{BNR}}^{-1}$ with the nonabelian Hodge correspondence, the behavior of Υ^{Moc} is the same as Θ^{Moc}. Recall the decomposition $\mathcal{M}_q = \bigcup \mathcal{M}_{q,D}$ from the end of Section 5. By Theorem 6.11, Proposition 6.12, Proposition 6.14 and Proposition 6.15, we obtain:

Theorem 6.17. *Let q be an irreducible quadratic differential. The map $\Upsilon^{\mathrm{Moc}} : \mathcal{M}_q \to \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ satisfies the following properties:*

(i) if all the zeros of q are odd, then Υ^{Moc} is continuous;

(ii) if at least one zero of q is even, then for each $(\mathcal{E}, \varphi) \in \mathcal{M}_{q,D}$, there exists an integer $2n_D$ that only depends on D, and $2n_D$ sequences $\{(\mathcal{E}_i^k, \varphi_i^k)\}$ for $k = 1, \ldots, 2n_D$, such that $\lim_{i\to\infty}(\mathcal{E}_i^k, \varphi_i^k) = (\mathcal{E}_\infty, \varphi_\infty)$, and

$$\lim_{i\to\infty} \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i^{k_1}, \varphi_i^{k_1}) \neq \lim_{i\to\infty} \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i^{k_2}, \varphi_i^{k_2}) \neq \Upsilon^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty) \ , \ \textit{for } k_1 \neq k_2 \ .$$

7. Reducible singular fiber and the Mochizuki map

We now investigate properties of the Hitchin fiber associated with a reducible quadratic differential, as discussed in [GO13]. Additionally, we provide an overview of Mochizuki's technique for constructing limiting configurations of Hitchin fibers for reducible quadratic differentials, as detailed in [Moc16]. We also analyze the continuity of the Mochizuki map.

7.1 Local description of a Higgs bundle

Write $q = -\omega\otimes\omega$ with $\omega \in H^0(K)$, $\Lambda = \mathrm{Div}(\omega)$, $Z = \mathrm{supp}\ (\Lambda)$, and $\mathcal{M}_q = \mathcal{H}^{-1}(q)$. Compared to the irreducible case, $\mathcal{M}_q$ contains strictly semistable Higgs bundles, so we let $\mathcal{M}_q^{\mathrm{st}}$ denote the stable locus. We point out that there is a sign ambiguity in the choice of ω, which actually plays an important role in the following.

7.1.1 *Local description.* Given a Higgs bundle $(\mathcal{E}, \varphi)$ with $\det(\varphi) = q$, define line bundles

$$L_\pm := \ker(\varphi \pm \omega) \ . \tag{25}$$

Then the inclusion maps $L_\pm \to \mathcal{E}$ are injective. Similarly, we may define an abelianization of $(\mathcal{E}, \varphi)$ by $(\mathcal{E}_0 = L_+ \oplus L_-, \varphi_0 = \mathrm{diag}(\omega, -\omega))$. We then have a natural inclusion $\iota : \mathcal{E}_0 \to \mathcal{E}$, which is is an isomorphism on $\Sigma \setminus Z$, and $\varphi \circ \iota = \iota \circ \varphi_0$.

It follows from [GO13, Prop. 7.10] that $L_\pm$ are the only φ-invariant subbundles of $\mathcal{E}$. If we write $d_\pm := \deg(L_\pm)$, then $(\mathcal{E}, \varphi)$ is stable (resp. semistable) if and only if $d_\pm < 0$ (resp. $\leqslant 0$). As $\det(\mathcal{E}) = \mathcal{O}$, the map $\det(\iota) : L_+ \otimes L_- \to \mathcal{O}$ defines a divisor $D = \mathrm{Div}(\det(\iota))$ such that $L_+ \otimes L_- = \mathcal{O}(-D)$. Therefore, we obtain

$$d_+ + d_- + \deg D = 0 \ ,$$

and $0 \leqslant D \leqslant \Lambda$. The Higgs bundle $(\mathcal{E}, \varphi)$ is semistable if and only if $-\deg D \leqslant d_+ \leqslant 0$ and stable if the equalities are strict. For the rest of this section, we always write $D = \sum_{p\in Z} \ell_p p$.

We define $\mathcal{M}_{q,D}$ as consisting of Higgs bundles $(\mathcal{E}, \varphi) \in \mathcal{M}_q$ for which the relation $L_+ \otimes L_- = \mathcal{O}(-D)$ holds. Consequently, we have $\mathcal{M}_q = \bigcup_{0\leqslant D\leqslant\Lambda} \mathcal{M}_{q,D}$.

7.1.2 *Semistable settings.* As the fiber $\mathcal{M}_q$ might contain strictly semistable Higgs bundles, we now explicitly enumerate all of the possible S-equivalence classes. When $D = 0$, then $L_- = L_+^{-1}$ and $\deg(L_+) = 0$. The corresponding Higgs bundle is polystable and can be explicitly written as

$$\left(L \oplus L^{-1}, \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\right) ,$$

where $L \in \mathrm{Jac}(\Sigma)$. When $D \neq 0$, suppose $\deg(L_+) = -\deg(D)$. Then $L_- = L_+^{-1}(-D)$ and $\deg(L_-) = 0$. Under S-equivalence, the polystable Higgs bundle is

$$\left(L_+(D) \oplus L_+^{-1}(-D), \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\right) ,$$

where $L_+ \in \mathrm{Pic}^{-\deg(D)}(\Sigma)$.

7.2 Reducible spectral curves

In this subsection, we introduce the algebraic data in [GO13] which describes the singular fiber with a reducible spectral curve. This plays a similar role to the parabolic modules. See [GO13, Sec. 7.1] for more details.

For any effective divisor D, and line bundle L, define the space

$$H^0(D, L) = \bigoplus_{p \in \mathrm{supp}\ D} \mathcal{O}(L)_p / \sim ,$$

where $s_1 \sim s_2$ if and only if $\mathrm{ord}_p([s_1] - [s_2]) \geqslant D_p$.

Let $L \in \mathrm{Pic}(\Sigma)$, define the following subspaces of $H^0(\Lambda, L^2K)$:

$$\begin{aligned} \mathcal{V}(D, L) &:= \{s \in H^0(\Lambda, L^2K) \mid \mathrm{ord}_p(s) = \Lambda_p - D_p,\ \text{if } D_p > 0;\ s|_p = 0,\ \text{if } D_p = 0\}, \\ \mathcal{W}(D, L) &= \{s \in H^0(\Lambda, L^2K) \mid s|_{\mathrm{supp}\ (\Lambda - D)} = 0\} . \end{aligned}$$

One checks that $\mathcal{W}(D, L) = \cup_{D' \leqslant D} \mathcal{V}(D', L)$. Moreover, the space $\mathcal{V}(D, L)$ is a linear subspace of $H^0(\Lambda, L^2K)$ with a hyperplane removed. In addition, $\mathbb{C}^*$ acts on $\mathcal{V}(D, L)$ by multiplication, and $\dim(\mathcal{V}(D, L)/\mathbb{C}^*) = \deg(D) - 1$.

We define the fibrations

$$p_m : \mathscr{V}(D, m) \longrightarrow \mathrm{Pic}^m(\Sigma),\ p_m : \mathscr{W}(D, m) \longrightarrow \mathrm{Pic}^m(\Sigma)$$

such that for $L \in \mathrm{Pic}^m(\Sigma)$, the fibers are $\mathcal{V}(D, L)$ and $\mathcal{W}(D, L)$.

7.2.1 *Algebraic data from the extension.* The Higgs bundle $(\mathcal{E}, \varphi)$ can be understood in terms of an extension of exact sequence. As $\det(\mathcal{E}) = \mathcal{O}$, we have the exact sequence

$$0 \longrightarrow L_+ \longrightarrow \mathcal{E} \longrightarrow L_+^{-1} \longrightarrow 0 .$$

For each $p \in Z$, with $U \subset \Sigma$ a neighborhood of p, $(\mathcal{E}, \varphi)$ can be written as the following splitting of $\mathcal{C}^\infty$ bundles

$$\mathcal{E} = L_+ \oplus_{\mathcal{C}^\infty} L_+^{-1} ,\ \bar{\partial}_{\mathcal{E}} = \begin{pmatrix} \bar{\partial}_{L_+} & b \\ 0 & \bar{\partial}_{L_+^{-1}} \end{pmatrix} ,\ \varphi = \begin{pmatrix} \omega & c \\ 0 & -\omega \end{pmatrix} .$$

We would like to consider the restriction of $\varphi + \omega \cdot \mathrm{id}$ to Λ. As $\omega|_\Lambda = 0$, $\varphi + \omega \cdot \mathrm{id}|_{L_+} = 0$ and the image of $\varphi + \omega \cdot \mathrm{id} \subset (0 \oplus L_+) \otimes K \subset \mathcal{E} \otimes K$. Therefore, the restriction of $\varphi + \omega \cdot \mathrm{id}$ to Λ defines a holomorphic map $s : L_+^{-1}|_\Lambda \to L_+K|_\Lambda$, or equivalently a section $s \in H^0(\Lambda, L_+^2 K)$. Moreover, by [GO13, Lemma 7.12], $\mathrm{Div}(s) = \Lambda - D$, where $\mathrm{Div}(s)$ is the divisor defined by zeros of s. Therefore, given any $(\mathcal{E}, \varphi) \in \mathcal{M}_q$, we obtain an $L \in \mathrm{Pic}^m(\Sigma)$ and an element in $\mathcal{V}(D, L)$. Moreover, the stability condition implies that $0 \leqslant D \leqslant \Lambda$, we have $-\deg D \leqslant \deg L \leqslant 0$.

7.2.2 *Inverse construction.* The inverse of the construction above also holds; for further details, see [GO13, Sec. 7] and [Hor22a, Sec. 5]. Given $L \in \mathrm{Pic}^m(\Sigma)$ and $q \in \mathcal{V}(D, L)$, we define a Higgs bundle via extensions as follows. From $q = \omega \otimes \omega, L$, we have a short exact sequence of complexes of sheaves:

$$\begin{tikzcd}
 & C_1^* & C_2^* & C_3^* & \\
0 \arrow[r] & L^2 \arrow[r, "="] \arrow[d, "\mathrm{id}"] & L^2 \arrow[r, "\mathrm{pr}"] \arrow[d, "c"] & 0 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & L^2 \arrow[r, "c"] & L^2K \arrow[r, "\mathrm{res}(\Lambda)"] & L^2K|_\Lambda \arrow[r] & 0
\end{tikzcd},$$

where, for a section $s \in \Gamma(L^2)$, $c(s) := \sqrt{-1}\omega s$, and $\mathrm{res}(\Lambda)$ is the restriction map to the divisor Λ. The long exact sequence in hypercohomology implies that $\mathrm{res}(\Lambda)$ induces an isomorphism

$$\mathrm{res}(\Lambda) : \mathbf{H}^1(C_2^*) \cong \mathbf{H}^1(C_3^*) = H^0(\Lambda, L^2K) \ .$$

Moreover, $\mathbf{H}^1(C_2^*)$ fits into an exact sequence

$$0 \longrightarrow W_1 \longrightarrow \mathbf{H}^1(C_2^*) \longrightarrow W_2 \longrightarrow 0 \ ,$$

where

$$W_1 = \mathrm{coker}\left(c : H^0(L^2) \longrightarrow H^0(L^2K)\right) \ ;$$
$$W_2 = \ker\left(c : H^1(L^2) \longrightarrow H^1(L^2K)\right) \ .$$

Now $H^1(\Sigma, L^2)$ parameterizes extensions

$$0 \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow L^{-1} \longrightarrow 0 \ .$$

Given $b \in W_2$, we can find $c' \in \Gamma(L^2K)$, $\bar\partial c' = 2b\omega$, and construct a Higgs bundle

$$E = L \oplus_{\mathcal{C}^\infty} L^{-1},\ \bar\partial_E = \begin{pmatrix} \bar\partial_L & b \\ 0 & \bar\partial_{L^{-1}} \end{pmatrix},\ \varphi = \begin{pmatrix} \omega & c' \\ 0 & -\omega \end{pmatrix}, \tag{26}$$

For $0 \leqslant D \leqslant \Lambda$ and $-\deg D \leqslant m \leqslant 0$, the construction above defines a map

$$\wp : \mathscr{V}(D, m) \longrightarrow \mathcal{M}_q \ ,\ \ s \in \mathcal{V}(D, L) \mapsto [(\mathcal{E}, \varphi)] \ ,$$

where $[(\mathcal{E}, \varphi)]$ is the S-equivalence class of the Higgs bundle constructed in (26) (note that the orbit of $(\mathcal{E}, \varphi)$ is closed in the semistable locus if and only if $\deg(L) \neq 0$). When $D = 0$, $\mathcal{V}(\Lambda, L) = \{0\}$ and the image of $\wp : \mathscr{V}(\Lambda, 0) \to \mathcal{M}_q$ consists of the polystable Higgs bundles $\mathcal{E} = L \oplus L^{-1}$, $\varphi = \mathrm{diag}(\omega, -\omega)$ such that $L^2 \cong \mathcal{O}_\Sigma$.

Theorem 7.1 ([GO13, Thm. 7.7]). *For $0 \leqslant D \leqslant \Lambda$ and $-\deg(D) \leqslant m_1 \leqslant 0$ and the map $\wp : \mathscr{V}(D, m_1) \to \mathcal{M}_q$, we have*

- *(i) for $m_2 = -\deg(D) - m_1$, we have $\wp(\mathscr{V}(D, m_1)) = \wp(\mathscr{V}(D, m_2))$,*
- *(ii) for the $\mathbb{C}^*$ action on $\mathscr{V}(D, m_1)$ by multiplication, for $\xi \in \mathscr{V}(D, m_1)$, $\wp(\mathbb{C}^*\xi) = \wp(\xi)$,*
- *(iii) when $m_1 \neq -\frac{1}{2}\deg(D), 0, -\deg(D)$, $\wp : \mathscr{V}(D, m_1)/\mathbb{C}^* \to \mathcal{M}_q$ is an isomorphism onto its image,*
- *(iv) when $m_1 = -\frac{1}{2}\deg(D)$, $\wp : \mathscr{V}(D, m_1)/\mathbb{C}^* \to \mathcal{M}_q$ is a double branched covering, which branched along line bundles $L \in \mathrm{Pic}^{m_1}(\Sigma)$ such that $L^2 \cong \mathcal{O}(-D)$,*

(v) *when* $D = 0$, *then* $\wp : \mathscr{V}(\Lambda, 0) \to \mathcal{M}_q$ *is a double branched covering, branched along* $L \in \mathrm{Pic}^0(\Sigma)$ *such that* $L^2 \cong \mathcal{O}$.

(vi) *when* $m_1 = 0, -\deg(D)$, $\wp : \mathscr{V}(\Lambda, 0) \to \mathcal{M}_q$ *is surjective but not injective. The image of* $\wp$ *are all polystable Higgs bundles.*

Remark. *Parts (iii) and (vi) of Theorem 7.1 are different from the statements in [GO13, Theorem 7.7]. Because of the S-equivalence, when* $m_1 = 0, -\deg(D)$, *the map* $\wp$ *will not be injective. We thank the authors of [GO13] for clarification of this point.*

Example 7.2. *When* $g = 2$, *for* $q = -\omega \otimes \omega$, *we can write* $\Lambda = p_1 + p_2$ *or* $\Lambda = 2p$. *In either case, the* $\mathcal{M}_q^{\mathrm{st}} = \wp(\mathscr{V}(D, m))$ *for* $-\deg(D) < m < 0$ *and* $0 \leqslant D \leqslant \Lambda$. *Therefore,* $m = -1$, $D = \Lambda$ *and* $\wp(\mathscr{V}(\Lambda, -1)) = \mathcal{M}_q^{\mathrm{st}}$. *Moreover, generically, the map* $\wp : (\mathscr{V}(\Lambda, -1))/\mathbb{C}^* \to \mathcal{M}_q^{\mathrm{st}}$ *is two-to-one.*

7.3 The stratification of the singular fiber

We now present two stratifications of $\mathcal{M}_q$. Recall that from any Higgs bundle $(\mathcal{E}, \varphi)$ we obtain two line bundles $L_\pm$ and a divisor D. There are two different stratifications: one given by the divisor D and the other by the degree of L_+.

7.3.1 *Divisor stratification.* We first discuss the stratification defined by the divisor. Indeed, using D, decompose into strata: $\mathcal{M}_q = \bigcup_{0 \leqslant D \leqslant \Lambda} \mathcal{M}_D$. As the definition of $L_\pm$ depends on the choice of the square root, there is no natural map from $\mathcal{M}_D$ to $\mathrm{Pic}(\Sigma)$. Consider the following space: $\mathbb{V}_D = \bigcup_{-\deg(D) \leqslant m \leqslant 0} \mathscr{V}(D, m)$. This forms a fibration

$$\tau : \mathbb{V}_D \longrightarrow \bigcup_{-\deg(D) \leqslant m \leqslant 0} \mathrm{Pic}^m(\Sigma) \ .$$

Moreover, for $L \in \mathrm{Pic}^m(\Sigma)$, we have $\tau^{-1}(L) = \mathscr{V}(D, L)$ and $\dim(\tau^{-1}(L)/\mathbb{C}^*) = \deg(D) - 1$. By Theorem 7.1, $\wp|_{\mathbb{V}_D} : \mathbb{V}_D \to \mathcal{M}_D$ is surjective. Moreover, since

$$\wp|_{\mathscr{V}(D,m)} = \wp|_{\mathscr{V}(D, -\deg(D) - m)}$$

generically, $\wp|_{\mathbb{V}_D}$ is a two-to-one map.

In summary, we obtain the following map which characterizes the singular fiber.

$$\wp : \mathbb{V} = \bigcup_{0 \leqslant D \leqslant \Lambda} \mathbb{V}_D \to \mathcal{M}_q = \bigcup_{0 \leqslant D \leqslant \Lambda} \mathcal{M}_{q,D}.$$

The top stratum is given by $D = \Lambda$.

7.3.2 *Degree stratification.* We next introduce the stratification defined by degrees; this encodes how different divisor stratifications are glued together. For $-(2g-2) \leqslant m \leqslant 0$ and $L \in \mathrm{Pic}^m(\Sigma)$, define $\mathbb{W}(L) := \bigcup_{\deg D \geqslant -m} \mathcal{V}(D, L)$. This set is connected, based on the definition and [GO13, Lemma 7.14]. Moreover, if we define

$$\mathbb{W}_m := \bigcup_{-m \leqslant \deg D,\ 0 \leqslant D \leqslant \Lambda} \mathscr{V}(D, m) \ , \ \mathbb{W} := \bigcup_{-(2g-2) \leqslant m \leqslant 0} \mathbb{W}_m \ ,$$

then we have $\wp(\mathbb{W}) = \wp(\mathbb{V})$. We should also note that though $\mathbb{W}_m \cap \mathbb{W}_n = \emptyset$ for any $m \neq n$, $\mathbb{W}$ is connected. As L_+, L_- are symmetric, by Theorem 7.1, we have

$$\wp(\mathscr{V}(D, m)) = \wp(\mathscr{V}(D, -\deg(D) - m)) \ ,$$

which implies that for any integer $-(2g - 2 + m) \leqslant n \leqslant 0$, $\wp\mathbb{W}_m \cap \wp\mathbb{W}_n \neq \emptyset$.

We now give an example of the degree stratification when $g = 2$.

Example 7.3. *Suppose ω has only one zero with order 2. Then $\Lambda = 2p$, and all possible divisors are $D_2 = 2p, D_1 = p, D_0 = 0$. The degree stratification is*

$$\mathbb{W}_{-2} = \mathscr{V}(D_2, -2) \ , \ \mathbb{W}_{-1} = \mathscr{V}(D_2, -1) \cup \mathscr{V}(D_1, -1)$$
$$\mathbb{W}_0 = \mathscr{V}(D_0, 0) \cup \mathscr{V}(D_1, 0) \cap \mathscr{V}(D_2, 0) \ .$$

The image of $\wp(\mathscr{V}(D_2, -1))$ is stable, $\wp(\mathscr{V}(D_0, 0))$ is poly-stable and $\wp(\mathscr{W} \setminus (\mathscr{V}(D_2, -1) \cup \mathscr{V}(D_0, 0)))$ is semistable.

Moreover, we have $\wp(\mathscr{V}(D_2, -2)) = \wp(\mathscr{V}(D_2, 0))$, $\wp(\mathscr{V}(D_1, -1)) = \wp(\mathscr{V}(D_1, 0))$ and $\wp|_{\mathscr{V}(D_2,-1)}$ is a branched covering. Moreover, we have $\wp(\mathscr{V}(D_2, -1)) \cap \wp(\mathscr{V}(D_1, 0)) \neq 0$ and $\wp(\mathscr{V}(D_2, -1)) \cap \wp(\mathscr{V}(D_0, 0)) = 0$.

7.4 Algebraic Mochizuki map

Based on the study of the local rescaling properties of Higgs bundles, Mochizuki introduced a weight for each $p \in Z$ in [Moc16, Sec. 3]. To be more specific, let c be a real number. For each $p \in Z$, the weight we consider is given by

$$\chi_p(c) = \min\{\ell_p, (m_p + 1)c + \ell_p/2\} \ ,$$

where $\mathrm{Div}(\omega) = \sum_p m_p p$ and ℓ_p is defined in Section 7.1.1.

By utilizing the global geometry of a Higgs bundle, we can uniquely determine the constant c. We aim to choose the sign of ω such that $d_+ \leqslant d_-$.

Lemma 7.4 ([Moc16, Lemma 4.3]). *If $(\mathcal{E}, \varphi)$ is stable, then there exists a unique constant $c \geqslant 0$ such that*

$$d_+ + \sum_{p \in Z} \chi_p(c) = 0 \ , \ d_- + \sum_{p \in Z} (\ell_p - \chi_p(c)) = 0 \ .$$

Proof. Since $(\mathcal{E}, \varphi)$ is stable, we have $-\sum \ell_p < d_\pm < 0$. We define the function

$$f(c) = d_+ + \sum_p \chi_p(c) \ , \tag{27}$$

which is strictly increasing. Moreover, for c sufficiently large, $\chi_p(c) = \ell_p$, and therefore $f(c) = d_+ + \sum_p \ell_p = -d_- > 0$. Additionally, $f(0) = d_+ + \sum_p (\ell_p/2)$. Since $d_+ \leqslant d_-$ and $d_+ + d_- + \sum_p \ell_p = 0$, we obtain $f(0) \leqslant 0$. The monotonicity of f implies the existence of c_0 such that $f(c_0) = 0$. □

From the construction, if $d_+ \leqslant d_-$, two weighted bundles $(L_+, \chi_p(c_0))$ and $(L_-, \ell_p - \chi_p(c_0))$ are obtained with weights $\chi_p(c_0)$ and $\ell_p - \chi_p(c_0)$ at each $p \in Z$, respectively. On the other hand, if $d_+ \geqslant d_-$, by symmetry, weighted bundles $(L_+, \ell_p - \chi_p(c_0))$ and $(L_-, \chi_p(c_0))$ are obtained. When $(\mathcal{E}, \varphi)$ is strictly semistable, S-equivalent to $(L, \omega) \oplus (L^{-1}, -\omega)$, then we would like to consider the weighted bundles $(L, 0) \oplus (L^{-1}, 0)$ with weight zero.

Next, we define the algebraic Mochizuki map. Let $\mathscr{F}_\pm(\Sigma)$ be the space of rank 1 degree zero filtered bundles on Σ, and let $\mathscr{F}_2(\Sigma) := \mathscr{F}_+(\Sigma) \times \mathscr{F}_-(\Sigma)$ be the direct product. Fix a choice of ω. Then from any Higgs bundle $(\mathcal{E}, \varphi)$, we obtain the subbundles $L_\pm$ with degree $d_\pm$ and define the algebraic Mochizuki map

$$\Theta^{\mathrm{Moc}}:\mathcal{M}_q\longrightarrow\mathscr{F}_2(\Sigma),$$
$$\Theta^{\mathrm{Moc}}(\mathcal{E},\varphi):=\begin{cases}\mathcal{F}_*(L_+,\chi_p(c_0))\oplus\mathcal{F}_*(L_-,\ell_p-\chi_p(c_0)), \text{ if } d_+\leqslant d_-\\ \mathcal{F}_*(L_+,\ell_p-\chi_p(c_0))\oplus\mathcal{F}_*(L_-,\chi_p(c_0)), \text{ if } d_-\leqslant d_+\end{cases},\ (\mathcal{E},\varphi)\text{ stable},$$
$$\Theta^{\mathrm{Moc}}(\mathcal{E},\varphi):=\mathcal{F}_*(L,0)\oplus\mathcal{F}_*(L^{-1},0),\ (\mathcal{E},\varphi)\text{ semistable}.$$

We list some properties of this map.

Proposition 7.5. *For Θ^{Moc}, we have:*

(i) for each $\mathscr{V}(D,m)$ with $0\leqslant D\leqslant\Lambda$, $-\deg(D)\leqslant m\leqslant 0$, $\Theta^{\mathrm{Moc}}|_{\wp(\mathscr{V}(D,m))}$ is continuous,

(ii) for i=1,2 and $s_i\in\mathbb{V}_D$ with $(\mathcal{E}_i,\varphi_i):=\wp(s_i)$, suppose $\tau(s_1)=\tau(s_2)$, then $\Theta^{\mathrm{Moc}}(\mathcal{E}_1,\varphi_1)=\Theta^{\mathrm{Moc}}(\mathcal{E}_2,\varphi_2)$. In particular, Θ^{Moc} is not injective.

Proof. The proof follows directly from the definition. □

A Higgs bundle $(\mathcal{E},\varphi)\in\mathcal{M}_q$ is called "exotic" if the constant c in Lemma 7.4 satisfies $c\neq 0$. This new behavior only appears in the Hitchin fiber with reducible spectral curve.

Proposition 7.6. *A Higgs bundle $(\mathcal{E},\varphi)$ is not exotic if and only if its corresponding degrees satisfies $d_+=d_-$.*

Proof. This is straight forward from the definition and Lemma 7.4. □

7.5 Discontinuous behavior

In this subsection, we study the discontinuous behavior of Θ^{Moc}. Consider a sequence of algebraic data $(L_i,q_i)\in\mathbb{W}_m$, where $L_i\in\mathrm{Pic}^m$ and $q_i\in\mathscr{V}(D,L_i)$. We assume that $\lim_{i\to\infty}L_i=L_\infty$ in Pic^m and $\lim_{i\to\infty}q_i=q_\infty\in\mathscr{V}(D_\infty,L)$, for $D_\infty\neq D$. As the space $\bigcup_{\deg D'\geqslant -m}\mathscr{V}(D',m)$ is connected, we can always find such a sequence.

Let $L_+^i:=L_i$ and $L_-^i:=L_i^{-1}\otimes\mathcal{O}(-D)$. By Lemma 7.4, the weight function, which we denote by $\chi_\pm$, is independent of i. In addition, we have

$$\lim_{i\to\infty}\Theta^{\mathrm{Moc}}\circ\wp(L_i,q_i)=\mathcal{F}_*(L_\infty,\chi_+)\oplus\mathcal{F}_*(L_\infty^{-1}(-D),\chi_-)\ .$$

For $(L_\infty,q_\infty\in\mathscr{V}(D_\infty,L))$, let $\chi_\pm^\infty$ be the corresponding weights. These depend on D_∞ and m. Then

$$\Theta^{\mathrm{Moc}}\circ\wp(L_\infty,q_\infty)=\mathcal{F}_*(L_\infty,\chi_+^\infty)\oplus\mathcal{F}_*(L_\infty^{-1}\otimes\mathcal{O}(-D_\infty),\chi_-^\infty)\ .$$

Therefore, we obtain

$$\begin{aligned}&\lim_{i\to\infty}\Theta^{\mathrm{Moc}}\circ\wp(L_i,q_i)\\ =&\Theta^{\mathrm{Moc}}\circ\wp(L_\infty,q_\infty)\otimes(\mathcal{F}_*(\mathcal{O},\chi_+-\chi_+^\infty)\oplus\mathcal{F}_*(\mathcal{O}(D_\infty-D),\chi_--\chi_-^\infty))\ .\end{aligned}\tag{28}$$

Proposition 7.7. *When $g\geqslant 3$, there exists a sequence $(\mathcal{E}_i,\varphi_i)\in\mathcal{M}_q$ of stable Higgs bundles with stable limit $(\mathcal{E}_\infty,\varphi_\infty)=\lim_{i\to\infty}(\mathcal{E}_i,\varphi_i)$ such that*

$$\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)\neq\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty)\ .$$

Proof. Choose $D=\Lambda$ and $d_+=-(g-1)$ with $L_i=L\in\mathrm{Pic}^{d_+}(\Sigma)$, and study the degenerate behavior for a family $q_i\in\mathcal{V}(\Lambda,L)$ which converges to $q_\infty\in\mathscr{V}(D_\infty,L)$. Here, D_∞ satisfies $D_\infty\leqslant D$ and $\deg(D_\infty)=\deg(D)-1$. As q_i lies in the top stratum, we can always find such a family. Take $(\mathcal{E}_i,\varphi_i)=\wp(L_i,q_i)$ and $(\mathcal{E}_\infty,\varphi_\infty)=\wp(L,q_\infty)$. When $g\geqslant 3$, we have $-\deg(D_\infty)<d_+\leqslant -\frac{1}{2}\deg(D_\infty)$, which implies $(\mathcal{E}_\infty,\varphi_\infty)$ is a stable Higgs bundle.

Write $D=\sum_p \ell_p$. As $(\mathcal{E}_i,\varphi_i)$ is nonexotic, the weights will be $\chi_+(p)=\chi_-(p)=\ell_p/2$. However, as $\deg(D_\infty)\neq 2d_+$, $(\mathcal{E}_\infty,\varphi_\infty)$ is exotic. By Proposition 7.6, if we write $\chi_\pm^\infty(p)$ for the weight functions with corresponding constant c, then $c>0$. Therefore, for $p\neq p_0$, we have $\chi_+^\infty(p)=(m_p+1)c+m_p/2>m_p/2=\chi_+(p)$. By (28), $\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)\neq\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty)$. □

When $g=2$, the stratification is simpler, and we have the following.

Proposition 7.8. *When $g=2$, the following holds:*

(i) Suppose $\Lambda=p_1+p_2$ for $p_1\neq p_2$, then $\Theta^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous. Moreover, there exists a sequence of stable Higgs bundles $(\mathcal{E}_i,\varphi_i)\in\mathcal{M}_q$ where the limit $(\mathcal{E}_\infty,\varphi_\infty)=\lim_{i\to\infty}(\mathcal{E}_i,\varphi_i)$ is semistable, and $\gamma(0)$ is also semistable and furthermore

$$\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)\neq\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty)\ .$$

(ii) Suppose $\Lambda=2p$. Then $\Theta^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous.

Proof. For (i), suppose $\Lambda=p_1+p_2$, then by Example 7.2, we have $\mathcal{M}_q^{\mathrm{st}}=\wp(\mathscr{V}(\Lambda,-1))$. By Proposition 7.5, $\Theta_q^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous. However, for semistable elements other strata must be taken into consideration. Take $L\in\mathrm{Pic}^{-1}(\Sigma)$ and $q_i\in\mathcal{V}(\Lambda,L)$ such that q_i convergence to $q_\infty\in\mathcal{V}(p_1,L)$. We define $(\mathcal{E}_i,\varphi_i)=\wp(L,q_i)$ and $(\mathcal{E}_\infty,\varphi_\infty)=\wp(L,q_\infty)$. For each i,

$$\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)=\mathcal{F}_*(L,(\tfrac{1}{2},\tfrac{1}{2}))\oplus\mathcal{F}_*(L^{-1}(-\Lambda),(\tfrac{1}{2},\tfrac{1}{2})).$$

Moreover, we have

$$\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty)=\mathcal{F}_*(L(D),(0,0))\oplus\mathcal{F}_*(L^{-1}(-D),(0,0))\neq\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i).$$

For (ii), by Example 7.3, $\wp(\mathscr{V}(D_2,-1))=\mathcal{M}_q^{\mathrm{st}}$ and by Proposition 7.5, $\Theta_q^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous. We now consider the behavior of the filtered bundle when crossing the divisors. □

7.6 The analytic Mochizuki map and limiting configurations

In this subsection, we construct the analytic Mochizuki map for the Hitchin fiber with a reducible spectral curve. We also introduce the convergence theorem of Mochizuki as stated in [Moc16] and examine the discontinuous behavior of the analytic Mochizuki map.

For $(\mathcal{E},\varphi)\in\mathcal{M}_q$, we can express the abelianization as $(\mathcal{E}_0,\varphi_0)=(L_+\oplus L_-,\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix})$, thus $\Theta^{\mathrm{Moc}}(\mathcal{E},\varphi)=\mathcal{F}_*(L_+,\chi_+)\oplus\mathcal{L}_-(L_-,\chi_-)\in\mathcal{F}_2(\Sigma)$. Via the nonabelian Hodge correspondence for filtered bundles, we obtain two Hermitian metrics $h_\pm^{\mathrm{Lim}}$ with corresponding Chern connections $A_{h_\pm^{\mathrm{Lim}}}$. These metrics satisfy the following proposition.

Proposition 7.9 ([Moc07, Lemma 4.4]). *The metrics $h_\pm^{\mathrm{Lim}}$ over $L_\pm$ satisfy*

i) $F_{A_{h_\pm^{\mathrm{Lim}}}}=0$ and $h_+^{\mathrm{Lim}}h_-^{\mathrm{Lim}}=1$,

ii) for every $p\in\Sigma$, there exists an open neighborhood (U,z) with $P=\{z=0\}$ such that $|z|^{-2\chi_p(c_0)}h_+^{\mathrm{Lim}}$ and $|z|^{2\chi_p(c_0)+2l_P}h_-^{\mathrm{Lim}}$ extends smoothly to $L_\pm|_U$.

Now, $H^{\mathrm{Lim}}:=h_+^{\mathrm{Lim}}\oplus h_-^{\mathrm{Lim}}$ is a metric on $\mathcal{E}_0$ which induces a metric on $(\mathcal{E},\varphi)|_{\Sigma\setminus Z}$ because $(\mathcal{E},\varphi)|_{\Sigma\setminus Z}\cong(\mathcal{E}_0,\varphi_0)|_{\Sigma\setminus Z}$. Let $(A^{\mathrm{Lim}},\phi^{\mathrm{Lim}})$ be the Chern connection defined by $(\mathcal{E},\varphi,H^{\mathrm{Lim}})$ over

$\Sigma\backslash Z$. Then $(A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}})$ is a limiting configuration that satisfies the decoupled Hitchin equations (9). The analytic Mochizuki map Υ^{Moc} is defined as:

$$\Upsilon^{\mathrm{Moc}} : \mathcal{M}_q \longrightarrow \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}, \quad \Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi) = (A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}}). \tag{29}$$

Note that H^{Lim} is not unique: for any constant c, the metric $ch_+^{\mathrm{Lim}} \oplus c^{-1}h_-^{\mathrm{Lim}}$ defines the same Chern connection as H^{Lim}. In any case, the map Υ^{Moc} is well-defined.

Suppose $(\mathcal{E}, \varphi)$ is an S-equivalence class of a semistable Higgs bundle. Let H_t be the harmonic metric for $(\mathcal{E}, t\varphi)$. For each constant $C > 0$, define μ_C to be the automorphism of $L_+ \oplus L_-$ given by $\mu_C = C\mathrm{id}_{L_+} \oplus C^{-1}\mathrm{id}_{L_-}$. As $\mathcal{E} \cong L_+ \oplus L_-$ on $\Sigma \setminus Z$, $\mu_C^* H_t$ can be regarded as a metric on $\mathcal{E}|_{\Sigma\setminus Z}$. Take any point $x \in \Sigma \setminus Z$ and a frame e_x of $L_+|x$, and define:

$$C(x,t) := \left(\frac{h_{L_+}^{\mathrm{Lim}}(e_x, e_x)}{H_t(e_x, e_x)}\right)^{1/2}.$$

Writing $\nabla_t + t\phi_t$ as the corresponding flat connection of $(\mathcal{E}, t\varphi)$ under the nonabelian Hodge correspondence, then

Theorem 7.10 ([Moc16]). *On any compact subset K of $\Sigma \setminus Z$, $\mu_{C(x,t)}^* H_t$ converges smoothly to H^{Lim}. In addition, we have $\lim_{t\to 0} |(\nabla_t, \phi_t) - \Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi)|_{\mathcal{C}^k(K)} = 0$.*

Comparing to the irreducible case Theorem 6.16, it is currently not known that the convergence of (∇_t, ϕ_t) to $\Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi)$ is uniform.

Propositions 7.7 and 7.8 now give

Theorem 7.11. *(Theorem 1.3) When $g \geqslant 3$, $\Upsilon^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is discontinuous, and when $g = 2$, $\Upsilon^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous.*

8. The Compactified Kobayashi-Hitchin map

In this section, we define a compactified version of the Kobayashi-Hitchin map and prove the main theorem of our paper. The Kobayashi-Hitchin map Ξ is a homeomorphism between the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$ and the Hitchin moduli space $\mathcal{M}_{\mathrm{Hit}}$. We wish to extend this to a map $\overline{\Xi}$ from the compactified Dolbeault moduli space $\overline{\mathcal{M}}_{\mathrm{Dol}}$ to the compactification $\overline{\mathcal{M}}_{\mathrm{Hit}} \subset \mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ of the Hitchin moduli space, and to study the properties of this extended map.

8.1 The compactified Kobayashi-Hitchin map

We first summarize the results obtained above. By the construction in Section 4, there is an identification $\partial\overline{\mathcal{M}}_{\mathrm{Dol}} \cong (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*$. Moreover, through (24) and (29), we have constructed the analytic Mochizuki map $\Upsilon^{\mathrm{Moc}} : \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0) \to \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$. Writing

$$(A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}} = \varphi + \varphi^{\dagger_{\mathrm{Lim}}}) = \Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi)\ ,$$

then for $w \in \mathbb{C}^*$, we have

$$\Upsilon^{\mathrm{Moc}}(\mathcal{E}, w\varphi) = (A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}} = w\varphi + \bar{w}\varphi^{\dagger_{\mathrm{Lim}}})\ .$$

Hence, Υ^{Moc} descends to a map $\partial\overline{\Xi}$ between $\mathbb{C}^*$ orbits:

$$\partial\overline{\Xi} : \partial\overline{\mathcal{M}}_{\mathrm{Dol}} = (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^* \longrightarrow \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*\ .$$

Together with the initial Kobayashi-Hitchin map $\Xi : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{M}_{\mathrm{Hit}}$, we obtain (2):

$$\overline{\Xi} : \overline{\mathcal{M}}_{\mathrm{Dol}} = \mathcal{M}_{\mathrm{Dol}} \cup \partial\overline{\mathcal{M}}_{\mathrm{Dol}} \longrightarrow \mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^* \ . \tag{30}$$

Theorems 6.16 and 7.10 show that for a Higgs bundle $(\mathcal{E},\varphi) \in \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0)$ and real t, $\lim_{t\to\infty} \Xi(\mathcal{E}, t\varphi) = \partial\overline{\Xi}[(\mathcal{E},\varphi)/\mathbb{C}^*]$. Thus the image of $\overline{\Xi}$ lies in $\overline{\mathcal{M}}_{\mathrm{Hit}}$, the closure of $\mathcal{M}_{\mathrm{Hit}}$ in $\mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0)$. There are natural extensions $\overline{\mathcal{H}}_{\mathrm{Dol}} : \overline{\mathcal{M}}_{\mathrm{Dol}} \to \overline{\mathcal{B}}$ and $\overline{\mathcal{H}}_{\mathrm{Hit}} : \overline{\mathcal{M}}_{\mathrm{Hit}} \to \overline{\mathcal{B}}$ such that $\overline{\mathcal{H}}_{\mathrm{Hit}} \circ \overline{\Xi} = \overline{\mathcal{H}}_{\mathrm{Dol}}$.

In summary, there are commutative diagrams

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Dol}} \arrow[r, "\Xi"] \arrow[d, hook] & \mathcal{M}_{\mathrm{Hit}} \arrow[d, hook] \\ \overline{\mathcal{M}}_{\mathrm{Dol}} \arrow[r, "\overline{\Xi}"] & \overline{\mathcal{M}}_{\mathrm{Hit}} \end{tikzcd} \ , \qquad \begin{tikzcd} \overline{\mathcal{M}}_{\mathrm{Dol}} \arrow[r, "\overline{\Xi}"] \arrow[dr, "\overline{\mathcal{H}}_{\mathrm{Dol}}"'] & \overline{\mathcal{M}}_{\mathrm{Hit}} \arrow[d, "\overline{\mathcal{H}}_{\mathrm{Hit}}"] \\ & \overline{\mathcal{B}} \end{tikzcd} \ .$$

We now turn to the analysis of some properties of the compactified Kobayashi-Hitchin map. Define

$$\overline{\mathcal{B}}^{\mathrm{reg}} = \{[(q,w)] \in \overline{\mathcal{B}} \mid q \neq 0 \text{ has simple zeros}\} \ .$$

This is the compactified space of quadratic differentials with simple zeros. Let $\overline{\mathcal{B}}^{\mathrm{sing}} = \overline{\mathcal{B}} \setminus \overline{\mathcal{B}}^{\mathrm{reg}}$ be its complement. Additionally, define the open sets $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} = \overline{\mathcal{H}}_{\mathrm{Dol}}^{-1}(\overline{\mathcal{B}}^{\mathrm{reg}})$ and $\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}} = \overline{\mathcal{H}}_{\mathrm{Hit}}^{-1}(\overline{\mathcal{B}}^{\mathrm{reg}})$ as the collections of elements with regular spectral curves. Set $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} = \overline{\mathcal{H}}_{\mathrm{Dol}}^{-1}(\overline{\mathcal{B}}^{\mathrm{sing}})$ and $\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}} = \overline{\mathcal{H}}_{\mathrm{Hit}}^{-1}(\overline{\mathcal{B}}^{\mathrm{sing}})$ to be the sets of singular fibers. We can write $\overline{\Xi} = \overline{\Xi}^{\mathrm{reg}} \cup \overline{\Xi}^{\mathrm{sing}}$, where

$$\overline{\Xi}^{\mathrm{reg}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} \longrightarrow \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}, \quad \overline{\Xi}^{\mathrm{sing}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} \longrightarrow \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}} \ .$$

Proposition 8.1. *The map $\overline{\Xi}^{\mathrm{reg}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$ is bijective, whereas $\overline{\Xi}^{\mathrm{sing}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}}$ is neither surjective nor injective.*

Proof. The bijectivity of $\overline{\Xi}^{\mathrm{reg}}$ is established by Theorem 4.9. The non-surjectivity and non-injectivity of $\overline{\Xi}^{\mathrm{sing}}$ follow from Theorems 6.17 and 7.11. □

8.2 Continuity properties of the compactified Kobayashi-Hitchin map

In this subsection, we prove that the continuity of the compactified Kobayashi-Hitchin map (30) is fully determined by the continuity of the analytic Mochizuki map.

Let $(\mathcal{E}_i, t_i\varphi_i)$ be a sequence of Higgs bundles with real numbers $t_i \to +\infty$, $\det(\varphi_i) = q_i$, $Z_i = q_i^{-1}(0)$, and $\|q_i\|_{L^2} = 1$. We denote $\xi_i = [(\mathcal{E}_i, t_i\varphi_i)] \in \mathcal{M}_{\mathrm{Dol}}$. By the compactness of $\overline{\mathcal{M}}_{\mathrm{Dol}}$, after passing to a subsequence, we may assume there is $\xi_\infty \in \partial\overline{\mathcal{M}}_{\mathrm{Dol}}$ such that $\lim_{i\to\infty} \xi_i = \xi_\infty$. Since $\partial\overline{\mathcal{M}}_{\mathrm{Dol}} \cong (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*$, we can select a representative $(\mathcal{E}_\infty, \varphi_\infty)$ of ξ_∞. By Lemma 4.4, we have that $(\mathcal{E}_i, \varphi_i)$ converges to $(\mathcal{E}_\infty, \varphi_\infty)$ in $\mathcal{M}_{\mathrm{Dol}}$, and q_i converges to q_∞. We write $Z_\infty = q_\infty^{-1}(0)$. For each q_i, recall that we have the decomposition $\mathcal{M}_{q_i} = \bigcup_{D_i} \mathcal{M}_{q_i, D_i}$.

By Proposition 4.6, $\lim_{i\to\infty} \overline{\Xi}(\mathcal{E}_i, t_i\varphi_i)$ exists. The following result establishes the continuity of this map with respect to the analytic Mochizuki map Υ^{Moc}.

Proposition 8.2. *Under the previous conventions, and suppose q_i, q_∞ are irreducible. Consider $(\mathcal{E}_i, \varphi_i) \in \mathcal{M}_{q_i, D_i = 0}$ positioned in the top stratum. Under these conditions, $\lim_{i\to\infty} \overline{\Xi}(\xi_i) = \overline{\Xi}(\xi_\infty)$ if and only if*

$$\lim_{i\to\infty} \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i) = \Upsilon^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty).$$

In other words, $\overline{\Xi}$ is continuous at ξ_∞ if and only if Υ^{Moc} is continuous at ξ_∞.

Proof. Set

$$\overline{\Xi}(\mathcal{E}_i, t_i\varphi_i) = \Xi(\mathcal{E}_i, t_i\varphi_i) = A_i + t_i\phi_i \ ,$$
$$\Upsilon^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i) = (A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}}) \ ,$$
$$\Upsilon^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty) = (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}}) \ .$$

By Proposition 4.6, there exists a limiting configuration $(A_\infty, \phi_\infty) := \lim_{i\to\infty}(A_i, \phi_i)$ over $\Sigma\setminus Z_\infty$. Let K be any compact set in $\Sigma \setminus Z_\infty$. Note that by the convergence assumption, there exists i_0 such that $Z_i \cap K = \emptyset$ for all $i \geqslant i_0$. Moreover, from Theorems 6.16 and 7.10, there exist t-independent constants $C, C' > 0$ such that (up to gauge transformations)

$$|(A_i, \phi_i) - (A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}})|_{\mathcal{C}^k(K)} \leqslant Ce^{-C' t_i}.$$

The convergence is uniform and exponential for fixed K. Therefore, over K, the size of $|(A_\infty, \phi_\infty) - (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}})|_{\mathcal{C}^k(K)}$ is the same as the size of $|(A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}}) - (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}})|_{\mathcal{C}^k(K)}$. This proves the Proposition. □

8.2.1 *Continuity along rays.* We now investigate the behavior of the compactified Kobayashi-Hitchin map restricted to a singular fiber. Specifically, fix $0 \neq q \in H^0(K^2)$, and denote by $[q]$ the $\mathbb{C}^*$-orbit of $q \times 1$ in the compactified Hitchin base $\overline{\mathcal{B}}$. Define $\overline{\mathcal{M}}_{\mathrm{Dol},[q]} := \overline{\mathcal{H}}_{\mathrm{Dol}}^{-1}([q])$, $\overline{\mathcal{M}}_{\mathrm{Hit},[q]} := \overline{\mathcal{H}}_{\mathrm{Hit}}^{-1}([q])$. Then the restriction of $\overline{\Xi}$ on $\overline{\mathcal{M}}_{\mathrm{Dol},[q]}$ defines a map $\overline{\Xi}_{[q]} : \overline{\mathcal{M}}_{\mathrm{Dol},[q]} \to \overline{\mathcal{M}}_{\mathrm{Hit},[q]}$.

Theorem 8.3. *Let q be an irreducible quadratic differential.*

(i) If q contains only zeroes of odd order, then $\overline{\Xi}_{[q]}$ is continuous.

(ii) If q contains a zero of even order, let $\mathcal{M}_q = \cup_D \mathcal{M}_{q,D}$ be the stratification defined earlier. Then for each $D \neq 0$, there exists an integer $n_D > 0$ such that for any Higgs bundle $(\mathcal{F}, \psi) \in \mathcal{M}_{q,D}$, there exist $2n_D$ sequences of Higgs bundles $(\mathcal{E}_i^k, \varphi_i^k)$ with $k = 1, \ldots, 2n_D$ such that

(a) $\lim_{i\to\infty}(\mathcal{E}_i^k, \varphi_i^k) = (\mathcal{F}, \psi)$ for $k = 1, \ldots, 2n_D$,

(b) if $\lim_{i\to\infty} t_i = \infty$, and we write

$$\eta^k := \lim_{i\to\infty} \overline{\Xi}_{[q]}(\mathcal{E}_i^k, t_i\varphi_i^k),\ \xi := \lim_{i\to\infty} \overline{\Xi}_{[q]}(\mathcal{F}, t_i\psi).$$

Suppose $(\mathcal{F}, \psi)$ doesn't lie in the real locus, then $\xi, \eta^1, \cdots, \eta^{2n_D}$ are $2n_D + 1$ different limiting configurations in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. Suppose $(\mathcal{F}, \psi)$ lies in the real locus, then we obtain $n_D + 1$ different limiting configurations in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$.

Proof. This follows from Theorem 6.17, Proposition 8.2 and Proposition 6.15. □

8.2.2 *Varying fiber.* With the conventions above, suppose $(\mathcal{E}_i, \varphi_i)$ converges to $(\mathcal{E}_\infty, \varphi_\infty)$ with q_∞ having only simple zeros, and $\xi_i = (\mathcal{E}_i, t_i\varphi_i)$ converges to ξ_∞ on $\overline{\mathcal{M}}_{\mathrm{Dol}}$. Since the condition of having only simple zeros is open, the q_i also have simple zeros for i sufficiently large.

Proposition 8.4 (cf. [OSWW, Thm. 2.12]). *Suppose q_∞ has only simple zeros. Then, $\lim_{i\to\infty} \overline{\Xi}(\xi_i) = \overline{\Xi}(\xi_\infty)$. In particular, the map $\overline{\Xi}^{\mathrm{reg}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$ is continuous.*

Proof. Let S_i denote the spectral curve of $(\mathcal{E}_i, \varphi_i)$ with branching locus Z_i. Also, let $L_i := \chi_{BNR}^{-1}(\mathcal{E}_i, \varphi_i)$ be the eigenline bundles. By the construction in Section 6, we have $\Upsilon^{\mathrm{Moc}}(\xi_i) = \mathcal{F}_*(L_i, \chi_i)$, where $\chi_i = -\frac{1}{2}\chi_{Z_i}$. Our assumption implies that $\mathcal{F}_*(L_i, \chi_i)$ converges to $\mathcal{F}_*(L_\infty, \chi_\infty)$

in the sense of Definition 3.2. Thus, by Theorem 3.3, we obtain the convergence of the limiting configurations: $\lim_{i\to\infty}\Upsilon^{\mathrm{Moc}}(\xi_i)=\Upsilon^{\mathrm{Moc}}(\xi_\infty)$. The claim follows from Proposition 8.2. □

Theorem 8.5. *The map $\Xi^{\mathrm{reg}}:\overline{\mathcal{M}}^{\mathrm{reg}}_{\mathrm{Dol}}\to\overline{\mathcal{M}}^{\mathrm{reg}}_{\mathrm{Hit}}$ is a homeomorphism.*

Proof. By Theorem 4.9, Ξ^{reg} is a bijection. Moreover, by Proposition 8.4, Ξ^{reg} is continuous. Finally, that $(\Xi^{\mathrm{reg}})^{-1}$ is continuous follows directly from the construction in [MSWW19]. □

Appendix A. Classification of rank 1 torsion modules for A_n singularities

In this appendix, we review the classification result for rank 1 torsion free modules at A_n singularities, as given in [GK85]. We compute the integer invariants defined in Subsection 5.3.

Let S be the spectral curve of an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle, and x a singular point with local defining equation given by $r^2-s^{n+1}=0$; this is an A_n singularity. Let $p:\widetilde{S}\to S$ be the normalization, where $p^{-1}(x)=\{\tilde{x}_+,\tilde{x}_-\}$ if n is odd and $p^{-1}(x)=\tilde{x}$ if n is even. We use R to denote the completion of the local ring $\mathcal{O}_x$, K its field of fractions, and $\widetilde{R}$ its normalization.

A.1 A_{2n} singularity

The local equation is $r^2-s^{2n+1}=0$. The normalization induces a map between coordinate rings, and we can write

$$\psi:\mathbb{C}[r,s]/(r^2-s^{2n+1})\longrightarrow\mathbb{C}[t],\quad \psi(f(r,s))=f(t^{2n},t^2),$$

where $\widetilde{R}=\mathbb{C}[[t]]$ and $R=\mathbb{C}[[t^2,t^{2n+1}]]\subset\widetilde{R}$. According to [GK85, Anh. (1.1)], any rank 1 torsion free R-module can be written as

$$M_k=R+R\cdot t^k\subset\widetilde{R},\quad k=1,3,\dots,2n+1.$$

Here, M_k is a fractional ideal that satisfies $R\subset M_k\subset\widetilde{R}$, with $M_1=\widetilde{R}$ and $M_{2n+1}=R$. We may express any $f\in M_k$ as $f=\sum_{i=0}^{\frac{k-1}{2}}f_{2i}t^{2i}+\sum_{i\geqslant k}f_it^i$, where $f_i\in\mathbb{C}$.

We are interested in the integers $\ell_x:=\dim_{\mathbb{C}}(M_k/R)$, $a_{\tilde{x}}:=\dim_{\mathbb{C}}(\widetilde{R}/C(M_k))$ and $b_x=\dim_{\mathbb{C}}(T(M_k\otimes_R\widetilde{R}))$ (where T denotes the torsion submodule). Thus, as a $\mathbb{C}$-vector space, M_k/R is generated by $t^k,t^{k+2},\dots,t^{2n-1}$, implying that $\ell_x=\frac{2n+1-k}{2}$.

The conductor of M_k is given by $C(M_k)=\{u\in K\mid u\cdot\widetilde{R}\subset M_k\}$. By the expression of M_k and a straightforward computation, we have $C(M_k)=(t^{k-1})$, where (t^{k-1}) is the ideal in $\widetilde{R}$ generated by t^{k-1}. Thus, $1,t,\dots,t^{k-2}$ will form a basis for $\widetilde{R}/C(M_k)$, and we have $a_{\tilde{x}}=k-1$. Therefore, we have $a_{\tilde{x}}=2n-2\ell_x$.

For $i=0,1,\dots,\frac{2n-1-k}{2}$, we define $s_i=t^{k+2i}\otimes_R1-1\otimes t^{k+2i}\in M_k\otimes_R\widetilde{R}$. As k is odd, $t^{2n+1-k-2i}\in R$ and $t^{2n+1-k-2i}s_i=t^{2n+1}\otimes_R1-1\otimes_Rt^{2n+1}=0$, where the last equality is becasue $t^{2n+1}\in R$. Moreover, $\{s_1,\dots,s_{\frac{2n-1-k}{2}}\}$ form a basis of $T(M_k\otimes_R\widetilde{R})$, thus $b_x=\frac{2n+1-k}{2}=\ell_x$.

A.2 A_{2n-1} singularity

The local equation is $r^2-s^{2n}=0$. The normalization induces a map between the coordinate rings:

$$\psi:\mathbb{C}[r,s]/(r^2-s^{2n})\longrightarrow\mathbb{C}[t]\oplus\mathbb{C}[t],\quad \psi(f(r,s))=(f(t^n,t),f(-t^n,t))\ ,$$

where $\widetilde{R}=\mathbb{C}[[t]]\oplus\mathbb{C}[[t]]$ and $R=\mathbb{C}[[(t,t),(t^n,-t^n)]]\cong\mathbb{C}[[(t,t),(t^n,0)]]$. By [GK85, Anh. (2.1)], any rank 1 torsion free R-module can be written as:

$$M_k=R+R\cdot(t^k,0)\subset\widetilde{R},\quad k=0,1,\dots,n.$$

Then, M_k is also a fractional ideal with $R \subset M_k \subset \widetilde{R}$. Moreover, $M_n = R$, and $M_0 = \widetilde{R}$.

As $p^{-1}(x) = \{\tilde{x}_+, \tilde{x}_-\}$, $\widetilde{R}$ contains two maximal ideals, $\mathfrak{m}_+ = ((t,1))$, $\mathfrak{m}_- = ((1,t))$. For $f \in M_k$, we can express f as:

$$f = \sum_{i=0}^{k-1} f_{ii}(t^i, t^i) + \sum_{l \geqslant 0} f_{l0}(t^{k+l}, 0) + f_{0l}(0, t^{k+l}),$$

where $f_{ij} \in \mathbb{C}$. Therefore, $\ell_x = \dim_{\mathbb{C}}(M_k/R) = n - k$. Moreover, using the expression, we can compute the conductor $C(M_k) = ((t^k, 1)) \cdot ((1, t^k))$, which implies $a_{\tilde{x}_\pm} = k$. Similarly, for $i = k, \dots, n-1$, we define $s_i = (t^i, 0) \otimes_R (1,1) - (1,1) \otimes_R (t^i, 0)$, then $(t,t)^{n-i} \cdot s_i = 0$ and $\{s_k, \dots, s_{n-1}\}$ will be a basis for $T(M_k \otimes_R \widetilde{R})$ and $b_x = \ell_x$.

In summary, we have the following:

Proposition A.1. *For the integers defined above, we have:*

(i) for the A_{2n} singularity, we have $a_{\tilde{x}} = 2n - 2\ell_x$ and $b_x = \ell_x$,

(ii) for the A_{2n-1} singularity, we have $a_{\tilde{x}_\pm} = n - \ell_x$ and $b_x = \ell_x$.

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