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On the blow-up set of the Yang–Mills flow on Kähler surfaces

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Abstract The Yang–Mills flow on a Kähler surface with holomorphic initial data converges smoothly away from a singular set determined by the Harder–Narasimhan–Seshadri filtration of the initial holomorphic bundle.

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1 Introduction

Let (X, ω) be a compact Kähler surface. Given a holomorphic bundle (E, D_0'') on X, let $\{E_{ij}\}$ denote the Harder–Narasimhan–Seshadri (double) filtration of E, and let $\mathrm{Gr}(E) = \oplus_{ij} Q_{ij}$, where $Q_{ij} = E_{ij}/E_{i,j-1}$ are the stable successive quotients of the filtration (see [6] for more details). Then $\mathrm{Gr}(E)$ is a torsion-free sheaf on X whose double dual $\mathrm{Gr}(E)^{**}$ is locally free. Denote by $\mathcal{Z}^{\mathrm{alg.}}$ the singularity set of $\mathrm{Gr}(E)$, i.e. the support of the torsion sheaf $\mathrm{Gr}(E)^{**}/\,\mathrm{Gr}(E)$. Associated to each point $p \in \mathcal{Z}^{\mathrm{alg.}}$ is an integral multiplicity $\mu_p^{\mathrm{alg.}}$ (see (2)).

Given a hermitian metric on E, let

$$\frac{\partial D}{\partial t} = -D^* F_D\ , \quad D(0) = D_0\ , \tag{1}$$

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denote the Yang–Mills flow with initial condition given by the hermitian connection associated to D_0'' and H. Let D_∞ denote the Yang–Mills connection associated to the holomorphic vector bundle $\mathrm{Gr}(E)^{**}$, i.e. the hermitian connection for the direct sum of Hermitian–Einstein metrics on the stable factors of $\mathrm{Gr}(E)^{**}$.

For a sequence $t_j \to \infty$, it follows from the work of Uhlenbeck and Sedlacek (see [10,16,18]) that there exists a subsequence t_{j_k}, and a finite set of points $\mathcal{Z}^{\text{an.}}$, depending a priori on the subsequence, such that $D_{t_{j_k}}$ converges (modulo real gauge transformations) weakly in $L^p_{1,\text{loc.}}(X - \mathcal{Z}^{\text{an.}})$ to a limiting Yang–Mills connection. Furthermore, by [19] this connection extends to a Yang–Mills connection on a bundle E_∞ on X. Call this connection an *Uhlenbeck limit*. The weak $L^p_{1,\text{loc.}}$ convergence can be improved to strong $C^\infty_{\text{loc.}}$ convergence using parabolic estimates (cf. [4,11,17]), and by the main theorem of [6] any Uhlenbeck limit can be identified with the Yang–Mills connection D_∞ on $\mathrm{Gr}(E)^{**}$.

Associated to each point p in the blow-up set $\mathcal{Z}^{\text{an.}}$ is an integral multiplicity $\mu_p^{\text{an.}}$ measuring the concentration of the curvature of the sequence of connections $D_{t_{j_k}}$ (see Lemma 5). The first goal of this note is to show that the analytic and algebraic singular sets coincide, along with their multiplicities.

Theorem 1 *Let $(E, D_0'') \to X$ be a holomorphic hermitian vector bundle over a compact Kähler surface, and let $\mathcal{Z}^{\text{alg.}}$ denote the algebraic singular set as above. Modulo real gauge transformations, the Yang–Mills flow D_t with initial condition D_0 converges smoothly as $t \to \infty$ to the Yang–Mills connection D_∞ on the vector bundle $\mathrm{Gr}(E, D_0'')^{**}$ away from $\mathcal{Z}^{\text{alg.}}$. Moreover, for any sequence $t_j \to \infty$ defining an Uhlenbeck limit with blow-up set $\mathcal{Z}^{\text{an.}}$, $\mathcal{Z}^{\text{an.}} = \mathcal{Z}^{\text{alg.}}$ as sets with $\mu_p^{\text{an.}} = \mu_p^{\text{alg.}}$ for all p.*

To state the second result of this paper, consider $E \to X$ as a fixed smooth hermitian complex vector bundle, and let $\mathcal{A}^s$ (resp. $\mathcal{A}^{ss}$) denote the space of stable (resp. semistable) holomorphic structures on E with the smooth topology. Here, stability is in the sense of Mumford–Takemoto with respect to the Kähler class ω. Let $\overline{\mathcal{M}}$ denote the Uhlenbeck compactification of the space of equivalence classes of ω anti-self-dual connections on E. We have

Theorem 2 *The Yang–Mills flow defines a continuous map from the closure of $\mathcal{A}^s$ in $\mathcal{A}^{ss}$ to $\overline{\mathcal{M}}$.*

In the case of Riemann surfaces it was conjectured in [1], and proved in [5,15], that the map from $\mathcal{A}^{ss}$ to the minimum of the Yang–Mills functional is a deformation retract. It follows that the equivariant cohomology of $\mathcal{A}^{ss}$ (by the action of the gauge group) is equal to the equivariant cohomology of the moduli space of projectively flat connections. In the case of a Kähler surface, the corresponding map fails to be a retraction due to the bubbling described above. Theorem 2 serves as a substitute. It would be interesting to use this result to obtain more information about the topology of $\overline{\mathcal{M}}$. This space, for arbitrary rank, has recently been extensively studied in [13].

2 Preliminaries

Let (X, ω) be a Kähler surface, normalized so that $\mathrm{vol}(X) = 2\pi$. The ω-slope $\mu(E)$ of a torsion-free sheaf $E \to X$ is defined by

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{1}{\mathrm{rk}(E)} \int_X c_1(E) \wedge \omega \,.$$

The sheaf is called *stable* (resp. *semistable*) if $\mu(F) < \mu(E)$ (resp. $\leq$) for all subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$. Following [6], if $E \to X$ is a holomorphic vector bundle there is an associated double filtration $\{E_{ij}\}$ of holomorphic vector bundles E_{ij}, called the Harder–Narasimhan–Seshadri filtration, which is defined as follows. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

be the Harder–Narasimhan filtration of E, where $Q_i = E_i/E_{i-1}$ is torsion-free, semistable, and $\mu(Q_i) > \mu(Q_{i+1})$ for $i = 1, \ldots, \ell - 1$. Then for each i, let

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,\ell_i} = E_i$$

be the Jordan–Hölder filtration, where the successive quotients $Q_{ij} = E_{ij}/E_{i,j-1}$ are torsion-free, stable, and $\mu(Q_{ij}) = \mu(Q_i)$, $j = 1, \ldots, \ell_i$. Furthermore, the associated graded object

$$\mathrm{Gr}(E) = \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{\ell_i} Q_{ij} \,,$$

is uniquely determined by the isomorphism class of E (see [6, Sect. 2.1] for details).

Let $\xi \to X$ be a torsion-free sheaf. Since X is a surface ξ^{**} is locally free. The *singular set* $\mathfrak{Z}^{\mathrm{alg.}}(\xi)$, defined to be the support of the torsion sheaf ξ^{**}/ξ, is a finite collection of points (cf. [12, Corollaries V.5.15 and V.5.20]). For $p \in X$, define the *algebraic multiplicity* $\mu_p^{\mathrm{alg.}}(\xi)$ of ξ at p to be the dimension of $(\xi^{**}/\xi)_p$ as a $\mathbb{C}$-vector space. Since the support is at isolated points, this may be reformulated as

$$\mu_p^{\mathrm{alg.}}(\xi) = h^0(U, \xi^{**}/\xi) \,, \tag{2}$$

where U is an open set chosen so that $U \cap \mathfrak{Z}^{\mathrm{alg.}} = \{p\}$ if $p \in \mathfrak{Z}^{\mathrm{alg.}}$ and empty otherwise. It follows that $\mu_p^{\mathrm{alg.}}(\xi)$ is a non-negative integer that is strictly positive if and only if $p \in \mathfrak{Z}^{\mathrm{alg.}}$.

Definition 3 *Let (E, D'') be a holomorphic vector bundle. Define the* algebraic singular set $\mathfrak{Z}^{\mathrm{alg.}}$*, to be the singular set of* $\mathrm{Gr}(E, D'')$ *with multiplicities as in* (2).

Next, we briefly review the Yang–Mills flow and the main result of [6]. Let $E \to X$ be a smooth complex vector bundle of rank r with Chern classes $c_1(E)$ and $c_2(E)$. For later reference, recall that:

$$-\,\mathrm{ch}_2(E)[X] = \left\{ c_2(E) - \frac{1}{2} c_1^2(E) \right\} [X] = \frac{1}{8\pi^2} \int_X \mathrm{Tr}\, F_D \wedge F_D \,, \tag{3}$$

where D is any connection on E with curvature F_D.

Fix a hermitian metric H on E and denote by $\mathcal{A}^{1,1}$ the space of integrable hermitian connections on E, i.e. those for which the curvature is of type $(1,1)$. Let $\mathfrak{G}$ (resp. $\mathfrak{G}^{\mathbb{C}}$) denote the space of real (resp. complex) gauge transformations. The Yang–Mills functional is given by

$$\mathrm{YM}(D) = \int_X |F_D|^2 \mathrm{d}\,\mathrm{vol}_\omega\,,$$

and is invariant under the action of $\mathfrak{G}$. Its L^2-gradient flow on $\mathcal{A}^{1,1}/\mathfrak{G}$ with initial condition D_0 is given by (1). By the results of [8], solutions of (1) for $D_0 \in \mathcal{A}^{1,1}$ exist for all $t > 0$. Furthermore, using the main theorem of [6] we have

Theorem 4 *Given $D_0 \in \mathcal{A}^{1,1}$ and any sequence $t_j \to \infty$, let $D_j = D_{t_j}$. Then there is*

1. *a subsequence (also denoted j);*
2. *a finite subset $\mathcal{Z}^{\mathrm{an.}} \subset X$ and a nonnegative integer valued multiplicity $\mu_p^{\mathrm{an.}}$ which is positive if and only if $p \in \mathcal{Z}^{\mathrm{an.}}$;*
3. *a smooth hermitian vector bundle (E_∞, H_∞) on X, smoothly isomorphic to* $\mathrm{Gr}(E, D_0'')^{**}$*;*
4. *a smooth Yang–Mills connection D_∞ on (E_∞, H_∞) such that D_∞'' induces a holomorphic structure isomorphic to* $\mathrm{Gr}(E, D_0'')^{**}$*;*
5. *a smooth isometry*

$$\tau : (E_\infty, H_\infty)\big|_{X-\mathcal{Z}^{\mathrm{an.}}} \longrightarrow (E,H)\big|_{X-\mathcal{Z}^{\mathrm{an.}}},$$

such that $\tau(D_j) \to D_\infty$ smoothly on compact subsets of $X - \mathcal{Z}^{\mathrm{an.}}$;

6. *the Yang–Mills energy densities (with respect to H_∞):*

$$|F_{\tau(D_j)}|^2\,\mathrm{d}\,\mathrm{vol}_\omega \longrightarrow |F_{D_\infty}|^2\,\mathrm{d}\,\mathrm{vol}_\omega + \sum_{p\in\mathcal{Z}^{\mathrm{an.}}} \mu_p^{\mathrm{an.}}\delta_p,$$

in the weak-$*$-topology, where δ_p is the Dirac measure at p.

Note that by [9, Proposition 6.2.14], $\|D_t^* F_{D_t}\|_{L^2} \to 0$ as $t \to \infty$; hence, any Uhlenbeck limit is Yang–Mills. The contribution of [6] was to identify the limiting connections with the Yang–Mills connection on $\mathrm{Gr}(E, D_0'')^{**}$. The convergence stated in [6] was weakly in $L^p_{1,\mathrm{loc.}}(X - \mathcal{Z}^{\mathrm{an.}})$, however this can be improved to local C^∞ convergence using parabolic estimates analogous to the ones developed by Struwe [17] for the harmonic map flow (cf. [4, Sects. 5 and 6] and also [11]).

Lemma 5 *The* analytic multiplicity *in Theorem 4 is given by*

$$\mu_p^{\mathrm{an.}} = \lim_{j\to\infty} \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F_{D_j} \wedge F_{D_j} - \mathrm{Tr}\, F_{D_\infty} \wedge F_{D_\infty} \right\}. \tag{4}$$

where $B_\varepsilon(p)$ is a ball about p chosen so that $B_\varepsilon(p) \cap \mathcal{Z}^{\mathrm{an.}} = \{p\}$ if $p \in \mathcal{Z}^{\mathrm{an.}}$ and empty otherwise.

Proof By [12, IV.4.5],

$$|F_{D_j}|^2\,\mathrm{d}\,\mathrm{vol}_\omega = |\Lambda F_{D_j}|^2\,\mathrm{d}\,\mathrm{vol}_\omega + \mathrm{Tr}\, F_{D_j} \wedge F_{D_j}\,.$$

By [6, Lemma 2.17], ΛF_{D_j} converges in any L^p. Hence, the result follows. As in [9, Sect. 4.4], $\mu_p^{\mathrm{an.}}$ is a non-negative integer that is strictly positive if and only if $p \in \mathcal{Z}^{\mathrm{an.}}$ (see also the discussion in the next section). □

3 Proof of Theorem 1

In order to prove Theorem 1 we have to show that for any Uhlenbeck limit $\mu_p^{\text{an.}} = \mu_p^{\text{alg.}}$ for all $p \in X$. The key result is the following

Lemma 6 *Let $E \to X$ be a complex vector bundle with connection D, and let $\xi \to X$ be a torsion-free sheaf. Fix $p \in X$. We assume there is a modification $\pi : \widehat{X} \to X$, supported away from p, and a sheaf $\hat{\xi} \to \widehat{X}$ such that*

1. *$\hat{\xi} \simeq \xi$ in a neighborhood of p;*
2. *$\hat{\xi}$ restricted to $\widehat{X} - \{p\}$ is the sheaf of holomorphic sections of a vector bundle that is smoothly isomorphic to $\pi^* E$;*
3. $\mathrm{ch}(\pi^* E) = \mathrm{ch}(\hat{\xi})$.

For any connection D_∞ on ξ^{**}, and for $B_{2\varepsilon}(p)$ disjoint from the center of π, it follows from (1) *and* (2) *that we may write $D = D_\infty + a$ on $B_\varepsilon(p) - \{p\}$. Then we have*

$$\mu_p^{\text{alg.}}(\xi) = \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \{\mathrm{Tr}\, F_D \wedge F_D - \mathrm{Tr}\, F_{D_\infty} \wedge F_{D_\infty}\}$$
$$+ \frac{1}{8\pi^2} \int\limits_{\partial B_\varepsilon(p)} \mathrm{Tr} \left\{ a \wedge D_\infty(a) + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_{D_\infty} \right\} .$$

Proof By (1) and (2), ξ is locally free on $B_\varepsilon(p) - \{p\}$, and $\hat{\xi}$ is locally free on $\widehat{X} - \{p\}$. Hence, if χ denotes the holomorphic Euler characteristic, it follows from the Hirzebruch–Riemann–Roch Theorem [14] that

$$\begin{aligned} \mu_p^{\text{alg.}}(\xi) &= h^0(B_\varepsilon(p), \xi^{**}/\xi) = h^0(B_\varepsilon(p), \hat{\xi}^{**}/\hat{\xi}) \\ &= h^0(\widehat{X}, \hat{\xi}^{**}/\hat{\xi}) = \chi(\hat{\xi}^{**}/\hat{\xi}) = \chi(\hat{\xi}^{**}) - \chi(\hat{\xi}) \\ &= \int\limits_{\widehat{X}} \{ \mathrm{ch}_2(\hat{\xi}^{**}) - \mathrm{ch}_2(\hat{\xi}) \} = \int\limits_{\widehat{X}} \{ \mathrm{ch}_2(\hat{\xi}^{**}) - \mathrm{ch}_2(\pi^* E) \} \quad \text{by (3)} . \end{aligned} \tag{5}$$

Now D induces a connection $\widehat{D}$ on $\pi^* E$. Extend the connection D_∞ over $B_\varepsilon(p)$ smoothly to a compatible connection $\widehat{D}_0$ on $\hat{\xi}^{**}$. Then from (3) and (5),

$$\begin{aligned} \mu_p^{\text{alg.}}(\xi) &= \frac{1}{8\pi^2} \int\limits_{\widehat{X}} \left\{ \mathrm{Tr}\, F_{\widehat{D}} \wedge F_{\widehat{D}} - \mathrm{Tr}\, F_{\widehat{D}_\infty} \wedge F_{\widehat{D}_\infty} \right\} \\ &= \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \{\mathrm{Tr}\, F_D \wedge F_D - \mathrm{Tr}\, F_{D_\infty} \wedge F_{D_\infty}\} \\ &\quad + \frac{1}{8\pi^2} \int\limits_{\widehat{X} - B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F_{\widehat{D}} \wedge F_{\widehat{D}} - \mathrm{Tr}\, F_{\widehat{D}_\infty} \wedge F_{\widehat{D}_\infty} \right\} . \end{aligned}$$

Now by the assumption that $\hat{\xi}$ and $\pi^* E$ are smoothly isomorphic on $\widehat{X} - B_\varepsilon(p)$, the second term on the right hand side can be transferred to the boundary via the secondary characteristic class. □

Proposition 7 *Let D_j be a sequence along the Yang–Mills flow for an initial integrable connection D_0 on E, and let $\mathcal{Z}^{\text{alg.}}$ denote the algebraic singular set of the associated*

Harder–Narasimhan–Seshadri filtration of (E, D_0''). *Suppose there is an Uhlenbeck limit* $D_j \to D_\infty$ *with bubbling set* $\mathcal{Z}^{\text{an.}}$. *Then* $\mathcal{Z}^{\text{an.}} = \mathcal{Z}^{\text{alg.}}$, *with multiplicities.*

Proof Write $a_j = D_j - D_\infty$ on $X - \mathcal{Z}^{\text{an.}}$. For any $p \in X$, choose ε sufficiently small such that $B_\varepsilon(p)$ intersects $\mathcal{Z}^{\text{an.}}$ and $\mathcal{Z}^{\text{alg.}}$ in at most $\{p\}$. Given (E, D_0'') with Harder–Narasimhan–Seshadri filtration $\{E_{ij}\}$ we can perform a sequence of monoidal transformations $\pi : \widehat{X} \to X$, supported away from p and with exceptional divisor $\mathbf{e}$, such that $\pi^* E$ has a double filtration $\{\widehat{E}_{ij}\}$, where $\widehat{E}_{ij}$ are *subbundles* except possibly at p. We can also arrange that $\pi(\mathbf{e}) = \mathcal{Z}^{\text{alg.}} - \{p\}$. The proof of this statement follows from a slight modification of [6, Proposition 3.7] (see also [2]). Note that $\{\widehat{E}_{ij}\}$ is not necessarily the Harder–Narasimhan–Seshadri filtration of $\pi^* E$ for any Kähler metric on $\widehat{X}$. With this understood, set $\xi = \mathrm{Gr}(E, D_0'')$, and let $\hat{\xi}$ be the associated graded object for the filtration $\{\widehat{E}_{ij}\}$ of $\pi^*(E)$. Then $\hat{\xi}$ is smoothly isomorphic to $\pi^*(E)$ away from p. Applying Lemma 6 to E with connections D_j and ξ with connection D_∞ on ξ^{**}, we have

$$\mu_p^{\text{alg.}} = \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F_{D_j} \wedge F_{D_j} - \mathrm{Tr}\, F_{D_\infty} \wedge F_{D_\infty} \right\}$$
$$+ \frac{1}{8\pi^2} \int_{\partial B_\varepsilon(p)} \mathrm{Tr} \left\{ a_j \wedge D_\infty(a_j) + \tfrac{2}{3} a_j \wedge a_j \wedge a_j + 2 a_j \wedge F_{D_\infty} \right\}.$$

By (4), the limit of the first term on the right hand side as $j \to \infty$ is $\mu_p^{\text{an.}}$, and by Theorem 4(5) the limit of the second term vanishes. Hence, $\mu_p^{\text{alg.}} = \mu_p^{\text{an.}}$ for all $p \in X$. This completes the proof. □

Proof of Theorem 1 By Proposition 7, the bubbling set for any Uhlenbeck limit $D_{t_j} \to D_\infty$ is fixed. In particular, $\|F_{D_t}\|_{L^2(B_r(x))}$ is uniformly small on small balls $B_r(x)$ in the complement of $\mathcal{Z}^{\text{alg.}}$. Indeed, if this were not the case then for a given $\varepsilon > 0$ we could find a sequence of radii $r_j \to 0$ and times $t_j \to \infty$ such that

$$\int_{B_{r_j}(x)} |F_{D_{t_j}}|^2 \, \mathrm{d}\,\mathrm{vol} \geq \varepsilon\,.$$

According to Theorem 4(6), this would force x to be in the singular set of any subsequential Uhlenbeck limit of D_{t_j}, contradicting the assumption that $x \notin \mathcal{Z}^{\text{alg.}}$. By [8, Proposition 16] the Hermitian–Einstein tensors of D_t are uniformly bounded. On the other hand, the ε-regularity result in [21, Theorem 5.1] and [20, Theorems 2.2 and 3.5], which is a generalization of the result in [19], states that there are positive constants c_1, c_2 and ε_0 such for any connection D satisfying $\mathcal{E}_D(x,r) < \varepsilon_0$ for r sufficiently small, where

$$\mathcal{E}_D(x,r) = c_1 r^4 \|\Lambda F_D\|_{L^\infty}^2 + \int_{B_r(x)} |F_D|^2 \, \mathrm{d}\,\mathrm{vol}\,,$$

then we have

$$\sup_{B_{r/4}(x)} |F_D|^2 \leq \frac{c_2}{r^4} \mathcal{E}_D(x,r)\,. \tag{6}$$

Applying this to the flow, it follows that F_{D_t} is locally bounded away from $\mathcal{Z}^{\mathrm{alg.}}$ uniformly as $t \to \infty$. Then the local smooth convergence follows as discussed above (cf. [4]). □

4 Proof of Theorem 2

Let D_i be a sequence of integrable unitary connections on $E \to X$ inducing stable holomorphic structures, and assume that $D_i \to D$ smoothly, where (E, D'') is semi-stable. Associated to (E, D'') is a Seshadri filtration by subsheaves and an algebraic singular set $\mathcal{Z}^{\mathrm{alg.}}$. Since D_i are stable, by Donaldson's theorem [8] the Yang–Mills flow with initial conditions D_i converges to an irreducible Hermitian–Yang–Mills connection $\widetilde{D}_i$. By Uhlenbeck compactness, after passing to a subsequence, there is an Uhlenbeck limit $(E, \widetilde{D}_i) \to (\widetilde{E}_\infty, \widetilde{D}_\infty)$ with a bubbling set $\widetilde{\mathcal{Z}}^{\mathrm{an.}}$. On the other hand, by Theorem 1, the Yang–Mills flow with initial condition D converges smoothly away from $\mathcal{Z}^{\mathrm{alg.}}$ to the Hermitian–Yang–Mills connection D_∞ on $E_\infty = \mathrm{Gr}(E, D'')^{**}$. The proof of Theorem 2 is complete if we show that $\widetilde{D}_\infty = D_\infty$ and $\widetilde{\mathcal{Z}}^{\mathrm{an.}} = \mathcal{Z}^{\mathrm{alg.}}$ (with multiplicity). The first statement follows from a modification of the argument in [6] and the second by an argument similar to the one used above.

Let $D_{i,t}$ and D_t denote the time t solutions to the YM flow with initial conditions D_i and D, respectively. Fix $t_j \to \infty$ with $\|D_{t_j}^* F_{D_{t_j}}\|_{L^2} \to 0$. By Theorem 1 it follows that any subsequential Uhlenbeck limit of D_{t_j} converges to D_∞. For each t_j we may choose i_j sufficiently large so that $\|D_{i_j,t_j} - D_{t_j}\| \to 0$ in any norm. In particular, any subsequential Uhlenbeck limit of D_{i_j,t_j} also converges to D_∞.

With this understood, relabel the sequences $D_j = D_{i_j,t_j}$, $\widetilde{D}_j = \widetilde{D}_{i_j}$. Then the first claim is a consequence of the following general uniqueness result.

Lemma 8 *Let $D_j \to D_\infty$, $\widetilde{D}_j = g_j(D_j) \to \widetilde{D}_\infty$ converge weakly in $L^p_{1,\mathrm{loc.}}$ away from a finite set of points $Z \subset X$. Assume a uniform bound on $\|\Lambda F_{\widetilde{D}_j}\|_{L^\infty}$, and suppose that (E_∞, D_∞) and $(\widetilde{E}_\infty, \widetilde{D}_\infty)$ are Hermitian–Einstein connections. Then $(E_\infty, D_\infty) \simeq (\widetilde{E}_\infty, \widetilde{D}_\infty)$.*

Proof Since Hermitian–Einstein metrics are unique, it suffices to prove that (E_∞, D''_∞) and $(\widetilde{E}_\infty, \widetilde{D}''_\infty)$ are holomorphically isomorphic. Let S be an irreducible factor of (E_∞, D_∞) and π_S the projection to S. Let U be a finite union of disjoint balls containing Z, and set $X^* = X - \overline{U}$. Let $\tau_j = \mathrm{Tr}\,(g_j^* g_j \pi_S)$, and suppose g_j has been rescaled so that

$$\int_{X^*} \tau_j^2 \, \mathrm{d}\,\mathrm{vol}_X = 1\,. \tag{7}$$

Fix a ball $B_R \subset X^*$. By the main result of [7], we may find local holomorphic frames for $(E, D''_j)\big|_{B_R}$ converging in C^1 to a holomorphic frame for $(E, D''_\infty)\big|_{B_R}$. In particular, there exist complex gauge transformations φ_j on B_R such that

(i) $\varphi_j \to \mathbf{I}$ uniformly;
(ii) $g_j \varphi_j \pi_S : S\big|_{B_R} \to (E, \widetilde{D}''_j)\big|_{B_R}$ is holomorphic.

Setting $\hat{\tau}_j = \mathrm{Tr}\,((g_j\varphi_j)^* g_j \varphi_j \pi_S)$, it follows that (cf. [8, p. 23])

$$\left(\sup_{B_r} \hat{\tau}_j\right)^2 \le C_r \int_{B_R} \hat{\tau}_j^2 \, \mathrm{d}\,\mathrm{vol}_X\,,$$

for any $r < R$. By (7) and (i), the right hand side above is uniformly bounded. Hence, again referring to the argument in [8], we can extract a subseqence of $g_j\varphi_j\pi_S$ converging to a holomorphic map on B_R. By (i), it follows that $g_j\pi_S$ itself converges to this limit. By a diagonalization argument we conclude there is a subsequence, also denoted by $g_j\pi_S$, converging uniformly on compact subsets of X^* to a holomorphic map $g_\infty : S\big|_{X^*} \to (\widetilde{E}_\infty, \widetilde{D}''_\infty)\big|_{X^*}$, which then extends to X by Hartogs' Theorem. By the argument in [3, Lemma 2.2], g_∞ is nonzero. Now since S is stable and $(\widetilde{E}_\infty, \widetilde{D}''_\infty)$ is polystable with the same slope as S, it follows that g_∞ is an isomorphism onto one of the irreducible factors of $(\widetilde{E}_\infty, D''_\infty)$.

Assume that we have found in this way an isomorphism of $S_{(k)} = S_1 \oplus \cdots \oplus S_k \subset (E_\infty, D''_\infty)$ with $\widetilde{S}_{(k)} = \widetilde{S}_1 \oplus \cdots \oplus \widetilde{S}_k \subset (\widetilde{E}_\infty, \widetilde{D}''_\infty)$. Let $\widetilde{Q}$ denote the orthogonal complement of $\widetilde{S}_{(k)}$, and suppose S_{k+1} is an irreducible factor of (E_∞, D''_∞) in the complement of $S_{(k)}$. By the previous construction, $(\pi_{\widetilde{Q}})g_j\pi_{S_{k+1}}$ is injective for j sufficiently large (again identifying sequences and subsequences). We now proceed as above. Set $\tau_j = \mathrm{Tr}\,(g_j^*(\pi_{\widetilde{Q}})g_j\pi_{S_{k+1}})$ with the same normalization (7). On B_R we find complex gauge transformations φ_j, ψ_j such that

(iii) $\varphi_j \to \mathbf{I}$ and $\psi_j \to \mathbf{I}$ uniformly;
(iv) $(\pi_{\widetilde{Q}})\psi_j g_j\varphi_j\pi_{S_{k+1}} : S_{k+1}\big|_{B_R} \to (\widetilde{Q}, \widetilde{D}''_\infty\big|_{\widetilde{Q}})\big|_{B_R}$ is holomorphic.

As above we conclude the existence of a subsequence of $(\pi_{\widetilde{Q}})g_j\pi_{S_{k+1}}$ converging uniformly on compact subsets of X^* to a nonzero holomorphic map $S_{k+1} \to (\widetilde{Q}, \widetilde{D}''_\infty\big|_{\widetilde{Q}})$. Since S_{k+1} is stable and $(\widetilde{Q}, \widetilde{D}''_\infty\big|_{\widetilde{Q}})$ is polystable with the same slope, S_{k+1} is isomorphic to an irreducible factor $\widetilde{S}_{k+1} \subset \widetilde{Q}$. Hence, we obtain an isomorphism $S_{(k+1)} \simeq \widetilde{S}_{(k+1)}$, completing the inductive step and the proof of the lemma. □

We will also require the following

Lemma 9 *Let $D_j \to D_\infty$ be an Uhlenbeck limit for a sequence of integrable holomorphic structures on E with bubbling set $\mathcal{Z}^{\mathrm{an.}}$, and write $a_j = D_j - D_\infty$ on $X - \mathcal{Z}^{\mathrm{an.}}$. Furthermore, we assume that the Hermitian–Einstein tensors ΛF_{D_j} are uniformly bounded in j. Then away from $\mathcal{Z}^{\mathrm{an.}}$,*

1. *$a_j \to 0$ in $C^\alpha_{\mathrm{loc.}}$ for any $0 < \alpha < 1$;*
2. *$D_\infty a_j$ is locally bounded, uniformly in j.*

Proof Since $a_j \to 0$ weakly in $L^p_{1,\mathrm{loc.}}$ for any p, the first statement follows from Sobolev embedding. For the second statement, note that

$$D_\infty a_j = F_{D_j} - F_{D_\infty} - a_j \wedge a_j\,.$$

By the assumption on Hermitian–Einstein tensors and the ε-regularity theorem [20], F_{D_j} is locally bounded away from $\mathcal{Z}^{\mathrm{an.}}$ [cf. (6)]. The result follows. □

Proof of Theorem 2 Choose $p \in X$ and ε so small that $B_{2\varepsilon}(p)$ intersects $\widetilde{\mathcal{Z}}^{\mathrm{an.}} \cup \mathcal{Z}^{\mathrm{alg.}}$ in at most p. On $B_\varepsilon(p) - \{p\}$, we may define: $D_\infty = D_T + a_T$ and $D_j^\infty = D_T + a_{j,T}$. We assume T has been chosen sufficiently large so that the following two assumptions hold. First,

$$\left|\mu_p^{\mathrm{alg.}} - \frac{1}{8\pi^2}\int\limits_{B_\varepsilon(p)} \left\{\mathrm{Tr}\, F_{D_T}^2 - \mathrm{Tr}\, F_{D_\infty}^2\right\}\right| < \frac{1}{4}\,. \tag{8}$$

This is possible by Proposition 7. Second, by Theorem 1, we may assume for large T that

$$\left| \frac{1}{8\pi^2} \int\limits_{\partial B_\varepsilon(p)} \mathrm{Tr}\left\{ a_T \wedge D_T(a_T) + \tfrac{2}{3} a_T \wedge a_T \wedge a_T + 2a_T \wedge F_{D_T} \right\} \right| < \frac{1}{4} \tag{9}$$

With this understood, we now show $\tilde{\mu}_p^{\text{an.}} = \mu_p^{\text{alg.}}$. The connections D_j^∞ are Hermitian–Einstein; hence, it follows as in the proof of Lemma 5 that

$$\begin{aligned}
\tilde{\mu}_p^{\text{an.}} &= \lim_{j\to\infty} \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F^2_{D_j^\infty} - \mathrm{Tr}\, F^2_{\tilde{D}_\infty} \right\} \\
&= \lim_{j\to\infty} \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F^2_{D_j^\infty} - \mathrm{Tr}\, F^2_{D_\infty} \right\} \quad \text{by Lemma 8}, \\
&= \lim_{j\to\infty} \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F^2_{D_j^\infty} - \mathrm{Tr}\, F^2_{D_T} \right\} + \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F^2_{D_T} - \mathrm{Tr}\, F^2_{D_\infty} \right\}.
\end{aligned}$$

Now D_j^∞ and D_T are connections on the same bundle. Hence,

$$\begin{aligned}
&\frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F^2_{D_j^\infty} - \mathrm{Tr}\, F^2_{D_T} \right\} \\
&\quad = \frac{1}{8\pi^2} \int\limits_{\partial B_\varepsilon(p)} \mathrm{Tr}\left\{ a_{j,T} \wedge D_T(a_{j,T}) + \tfrac{2}{3} a_{j,T} \wedge a_{j,T} \wedge a_{j,T} + 2a_{j,T} \wedge F_{D_T} \right\}.
\end{aligned}$$

By Lemma 9, we may assume that $a_{j,T} \to a_T$ in $C^\alpha_{\text{loc.}}$ and that $D_T(a_{j,T})$ is locally bounded, uniformly in j, in $B_{2\varepsilon}(p) - \{p\}$. Letting $j \to \infty$, we deduce that

$$\begin{aligned}
\tilde{\mu}_p^{\text{an.}} &= \frac{1}{8\pi^2} \int\limits_{B_\varepsilon(p)} \left\{ \mathrm{Tr}\, F^2_{D_T} - \mathrm{Tr}\, F^2_{D_\infty} \right\} \\
&\quad + \frac{1}{8\pi^2} \int\limits_{\partial B_\varepsilon(p)} \mathrm{Tr}\left\{ a_T \wedge D_T(a_T) + \tfrac{2}{3} a_T \wedge a_T \wedge a_T + 2a_T \wedge F_{D_T} \right\}.
\end{aligned}$$

By (8) and (9), $|\tilde{\mu}_p^{\text{an.}} - \mu_p^{\text{alg.}}| < 1/2$. Since the multiplicities are integers, they must coincide. □

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