GENERALIZED THETA FUNCTIONS, STRANGE DUALITY, AND ODD ORTHOGONAL BUNDLES ON CURVES

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ABSTRACT. This paper studies spaces of generalized theta functions for odd orthogonal bundles with nontrivial Stiefel-Whitney class and the associated space of twisted spin bundles. In particular, we prove a Verlinde type formula and a dimension equality that was conjectured by Oxbury-Wilson. Modifying Hitchin's argument, we also show that the bundle of generalized theta functions for twisted spin bundles over the moduli space of curves admits a flat projective connection. We furthermore address the issue of strange duality for odd orthogonal bundles, and we demonstrate that the naive conjecture fails in general. A consequence of this is the reducibility of the projective representations of spin mapping class groups arising from the Hitchin connection for these moduli spaces. Finally, we answer a question of Nakanishi-Tsuchiya about rank-level duality for conformal blocks on the pointed projective line with spin weights.

1. INTRODUCTION

Let C be a smooth projective curve of genus $g \geq 2$, and choose integers $n \geq 2$, $\ell \geq 1$. Let $M_{\mathsf{SL}(n)}$ denote the coarse moduli space of semistable vector bundles of rank n and trivial determinant on C, and let $\mathcal{L}$ be the ample generator of the Picard group $\mathrm{Pic}(M_{\mathsf{SL}(n)}) \simeq \mathbb{Z}$. Similarly, let $M_{\mathsf{GL}(\ell)}$ denote the moduli space of semistable vector bundles of rank ℓ and degree $\ell(g-1)$, and consider the locus $\Theta_\ell \subset M_{\mathsf{GL}(\ell)}$ of points $[\mathcal{E}] \in M_{\mathsf{GL}(\ell)}$ such that $H^0(C, \mathcal{E}) \neq 0$. It turns out that Θ_ℓ is a Cartier divisor in $M_{\mathsf{GL}(\ell)}$, and we use the same notation for the associated line bundle. Tensor product defines a map:

$$s : M_{\mathsf{SL}(n)} \times M_{\mathsf{GL}(\ell)} \longrightarrow M_{\mathsf{GL}(n\ell)} \ ,$$

and by the "see-saw" principle it is easy to see that $s^*\Theta_{n\ell} \simeq \mathcal{L}^{\otimes \ell} \boxtimes \Theta_\ell^{\otimes n}$. The pull-back of the defining section of $\Theta_{n\ell}$ gives a map, well-defined up to a multiplicative constant,

$$s_{n\ell} : H^0(M_{\mathsf{SL}(n)}, \mathcal{L}^{\otimes \ell})^* \longrightarrow H^0(M_{\mathsf{GL}(\ell)}, \Theta_\ell^{\otimes n}) \ ,$$

known as the *strange duality map*. It was conjectured to be an isomorphism (cf. Donagi-Tu [22] and Beauville [9]), and this conjecture was confirmed independently by Belkale [15] and by Marian-Oprea [47] (cf. Beauville-Narasimhan-Ramanan [14] for $\ell = 1$). The analogous

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strange duality for symplectic bundles was conjectured by Beauville [11] and proven by Abe [1] (see also result of Belkale [17]). Strange duality for maximal subgroups of E_8 has been considered independently by Boysal-Pauly [19] and by the first author [19, 51]. However, a conjectural description of strange duality for other dual pairs, e.g. orthogonal bundles, has as yet not been formulated in the literature.

An approach to strange duality questions, and in fact the original motivation, comes from the study of the space $\mathcal{V}^*_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$ of *conformal blocks* (cf. Tsuchiya-Ueno-Yamada [65] and Definition 2.2 below). These are dual spaces to quotients of tensor products of level ℓ integrable highest weight modules of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$, and with weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ attached to the curve $\mathfrak{X} = (C, p_1, \dots, p_n)$ with marked points p_i. Isomorphisms between spaces of conformal blocks can sometimes arise from *conformal embeddings* of affine Lie algebras (cf. Kac–Wakimoto [38] and Definition 2.1 below), and this phenomenon is known in the conformal field theory literature as *rank-level duality* (cf. Naculich–Schnitzer [53] and Nakanishi–Tsuchiya [54]). By a factorization or sewing procedure (see Sections 2.3 and 9.7), one can often reduce strange duality questions for curves of positive genus to rank-level duality on $\mathbb{P}^1$ with marked points. Indeed, all known strange dualities can be proved using this approach. In [50], the first author proved a rank-level duality for $\mathfrak{g} = \mathfrak{so}(2r+1)$ conformal blocks on $\mathbb{P}^1$ with marked points and weights associated to representations of the group $\mathsf{SO}(2r+1)$. One would naturally like to investigate whether the result can be generalized to curves of positive genus to give a strange duality for orthogonal bundles. This question forms the starting point of the present work.

As we shall see below, any generalization of rank-level or strange dualities for orthogonal groups is complicated by the existence of spin representations (in the former case) and the fundamental group (in the latter). Spin weights cause difficulty in the branching rules for highest weight representations under embeddings. This issue was already raised in the discussion in [54], and for this reason only vector representations were considered in [50]. On the geometric side, since $\mathsf{SO}(m)$ is not simply connected the moduli spaces for orthogonal groups will be disconnected, and any reasonable approach to strange duality must take into account all components. It was this observation that led to the conjectural Verlinde type formula of Oxbury-Wilson [56], which is proved below.

In this paper, we discuss these issues for the conformal embeddings of the odd orthogonal algebras $\mathfrak{so}(2r+1)$. The next subsections summarize the results we have obtained.

1.1. **Twisted moduli spaces and uniformization.** For a complex reductive group G, let $\mathcal{M}_G$ denote the moduli stack of principal G-bundles on C. Consider the natural map $\mathsf{Spin}(m) \times \mathsf{Spin}(n) \to \mathsf{Spin}(mn)$ induced by tensor product of vector spaces of dimensions m and n, each endowed with a symmetric nondegenerate bilinear form. This map induces one between the corresponding moduli stacks $\mathcal{M}_{\mathsf{Spin}(m)} \times \mathcal{M}_{\mathsf{Spin}(n)} \to \mathcal{M}_{\mathsf{Spin}(mn)}$. If we pull back any section of $H^0(\mathcal{M}_{\mathsf{Spin}(mn)}, \mathcal{P})$, we get a map

$$H^0(\mathcal{M}_{\mathsf{Spin}(m)}, \mathcal{P}_1^{\otimes n})^* \longrightarrow H^0(\mathcal{M}_{\mathsf{Spin}(n)}, \mathcal{P}_2^{\otimes m}) \ ,$$

where $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are the ample generators of the respective Picard groups of the moduli stacks, which are given by Pfaffian line bundles. By the Verlinde formula (cf. [10, Cor. 9.8]), it is easy to find m and n for which

$$\dim_{\mathbb{C}} H^0(\mathcal{M}_{\mathsf{Spin}(m)}, \mathcal{P}_1^{\otimes n}) \neq \dim_{\mathbb{C}} H^0(\mathcal{M}_{\mathsf{Spin}(n)}, \mathcal{P}_2^{\otimes m}) \ , \tag{1.1}$$

and hence there can be no obvious strange duality for spin bundles. Nevertheless, following suggestions of Oxbury-Wilson [56], we can attempt to rectify this situation by considering orthogonal bundles that do not lift to spin.

Fix $p \in C$, and let $\mathcal{M}^-_{\mathsf{Spin}(m)}$, $m \geq 5$, denote the moduli stack of special Clifford bundles whose spinor norm is $\mathcal{O}_C(p)$ (cf. Section 3 and the discussion around eq. (3.2)). We refer to these objects as *twisted spin bundles*: their associated orthogonal bundles have nontrivial Stiefel-Whitney class. A uniformization theorem for these moduli stacks was proved in Beauville-Laszlo-Sorger [13], and there is again a Pfaffian line bundle $\mathcal{P} \to \mathcal{M}^-_{\mathsf{Spin}(m)}$ which generates the Picard group. Now if G is simply connected and $\mathcal{L} \to \mathcal{M}_G$ is the ample generator of $\mathrm{Pic}(\mathcal{M}_G)$, then $H^0(\mathcal{M}_G, \mathcal{L}^{\otimes \ell})$ is canonically identified with the space of conformal blocks $\mathcal{V}^*_{\omega_0}(\mathfrak{X}, \mathfrak{g}, \ell)$. We prove the analog of this result in the twisted case.

Theorem 1.1. *The space $H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell})$ is naturally isomorphic to the space of conformal blocks $\mathcal{V}^*_{\ell\omega_1}(\mathfrak{X}, \mathfrak{so}(m), \ell)$.*

In particular, from the Verlinde formula and results in [50], we obtain an expression for the dimension of $H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell})$ that was first conjectured to hold in [56] (see Theorem 4.9 below).

Next, we observe the following. Let

$$\mathcal{M}_{2r+1} = \mathcal{M}_{\mathsf{Spin}(2r+1)} \sqcup \mathcal{M}^-_{\mathsf{Spin}(2r+1)} \ , \tag{1.2}$$

and denote also by $\mathcal{P}$ the bundle which restricts to the Pfaffian on each component. Then we prove the following equality.

Corollary 1.2. $\dim_{\mathbb{C}} H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)}) = \dim_{\mathbb{C}} H^0(\mathcal{M}_{2s+1}, \mathcal{P}^{\otimes(2r+1)})$.

1.2. **Hecke transformations.** Let $\mathcal{M}^{par}_{\mathsf{Spin}(m)}$ be the moduli stack of pairs (S, P), where $S \to C$ is a $\mathsf{Spin}(m)$ bundle and P is a maximal parabolic subgroup of the fiber S_p preserving an isotropic line in the associated orthogonal bundle. A theorem of Laszlo-Sorger [45] states that $H^0(\mathcal{M}^{par}_{\mathsf{Spin}(m)}, \mathcal{P}(\ell))$ is naturally isomorphic to $\mathcal{V}^*_{\ell\omega_1}(\mathfrak{X}, \mathfrak{so}(m), \ell)$, for a suitable choice of line bundle $\mathcal{P}(\ell) \to \mathcal{M}^{par}_{\mathsf{Spin}(m)}$. Theorem 1.1 raises the question of whether $\mathcal{M}_{\mathsf{Spin}(m)}$ and $\mathcal{M}^-_{\mathsf{Spin}(m)}$ are related by a Hecke type elementary transformation.

Recall that an *oriented orthogonal bundle* on C is a pair (E, q), where $E \to C$ is a vector bundle with trivial determinant and a nondegenerate quadratic form $q : E \otimes E \to \mathcal{O}_C$ with obvious compatibility of trivialization with $\det E$ and $\det q$. In [2], T. Abe defines a transformation yielding a new orthogonal bundle E^ι from an orthogonal bundle E equipped with an isotropic line in the fiber E_p. Below we observe that the bundles E^ι and E have opposite Stiefel-Whitney classes, meaning that the ι-transform switches components of $\mathcal{M}_{\mathsf{SO}(m)}$. We then extend the ι-transform to a Hecke type elementary transformation on

Clifford bundles (see (5.6)). This enables us to give an alternative proof of Theorem 1.1. The advantage of this identification will be seen in Theorem 1.6 below. The details of this construction are contained in Section 5.1.

1.3. **Hitchin connection.** The locally free sheaf of conformal blocks associated to a family of smooth projective curves $\pi : \mathcal{C} \to B$ and any simple Lie algebra $\mathfrak{g}$ carries a flat projective connection known as the TUY connection (or the KZ connection in genus zero). The identification in Theorem 1.1 motivates a geometric description of this connection for twisted spin bundles. Indeed, Hitchin [34] introduced a flat projective connection on spaces of generalized theta functions as the underlying curve C varies over the Teichmüller space of Riemann surfaces (see also [7, 66, 3]). In [44], Laszlo showed that with this identification, and over the pointed Teichmüller space $\mathcal{T}_{g,1}$, the Hitchin connection coincides with the TUY connection on the space of conformal blocks. This statement also generalizes to the case of twisted spin bundles. More precisely, we prove the following.

Theorem 1.3. *As the pointed curve (C, p) varies in $\mathcal{T}_{g,1}$, the vector bundle with fiber $H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell})$ is endowed with a flat projective connection which we also call the Hitchin connection. Under the identification of $H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell})$ with $\mathcal{V}^*_{\ell\omega_1}(\mathfrak{X}, \mathfrak{g}, \ell)$, the Hitchin connection coincides with the TUY connection.*

Let $M^{-,reg}_{\mathsf{Spin}(m)}$ denote the moduli space of regularly stable twisted spin bundles (see Section 3.1). Then the Pfaffian line bundle descends to $M^{-,reg}_{\mathsf{Spin}(m)}$, and

$$H^0(M^{-,reg}_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell}) \simeq H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell}) \ . \tag{1.3}$$

We refer the reader to Proposition 4.4 for more details. Now the essential strategy in the proof of Theorem 1.3 is the same as in [34], but there are two key differences. These are as follows:

- The connectivity of the fibers of the Hitchin map from the moduli space M^θ_G of G-Higgs bundles to the Hitchin base is an essential ingredient in Hitchin's proof. In the untwisted case, the connectivity follows, for example, from a description of the fibers in terms of spectral data. It seems not to be known if the fiber of the Hitchin map for twisted Higgs bundles is connected in general. We circumvent this issue by reducing to the $\mathsf{SO}(m)$ moduli space, and then using results of Donagi-Pantev [21].
- The condition $H^1(M^{-,reg}_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell}) = \{0\}$, is sufficient to show that the symbol map of the projective heat operator is injective. In the untwisted case, one can again use Higgs bundles to establish this vanishing [34, 44]. For the same reason as above, this argument is unavailable in the twisted case. However, Kumar-Narasimhan proved such vanishing results directly without using Higgs bundles. In the present paper, we generalize the proof in [42] to the twisted setting.

The proof of the second statement in Theorem 1.3 is analogous to that in [44]. We refer the reader to Section 6 for more details.

1.4. **Level one sections.** Since the strange duality map arises by pulling back a level one section, we study these sections in detail. Let

$$\mathrm{Th}(C) := \{\kappa \in \mathrm{Pic}_{g-1}(C) \mid \kappa^{\otimes 2} = \omega_C\} \tag{1.4}$$

denote the set of *theta characteristics* of C. Furthermore, denote by $\mathrm{Th}^+(C) \subset \mathrm{Th}(C)$ (resp. $\mathrm{Th}^-(C) \subset \mathrm{Th}(C)$) the set of *even* (resp. *odd*) theta characteristics, i.e. those for which $h^0(C,\kappa)$ is even (resp. odd). We shall prove the following analog of a theorem of Belkale [17] and Pauly-Ramanan [57].

Theorem 1.4. *Let s_κ denote the canonical (up to scale) section of the Pfaffian line bundle for a theta characteristic κ (see Definition 3.6). Then the collection $\{s_\kappa \mid \kappa \in \mathrm{Th}^-(C)\}$ forms a basis of the space of the level one generalized theta functions $H^0(\mathcal{M}^-_{\mathsf{Spin}(2r+1)}, \mathcal{P})$.*

Remark 1.5. In [5], using TQFT methods, Andersen-Masbaum give a "brick decomposition" of the $\mathsf{SL}(m)$-conformal block bundles under the action of the Heisenberg group. The invariant Pfaffian sections and the decomposition of $H^0(\mathcal{M}^-_{\mathsf{Spin}(2r+1)}, \mathcal{P})$ (as well as $H^0(\mathcal{M}_{\mathsf{Spin}(2r+1)}, \mathcal{P})$) into Pfaffian sections should be considered as an analog of brick decompositions for these spaces.

By passing to a local étale cover, we can assume the torsor of theta characteristics is trivialized on $\mathcal{C} \to B$. We show the following.

Theorem 1.6. *For each $\kappa \in \mathrm{Th}^-(C)$, the Pfaffian section $s_\kappa \in H^0(\mathcal{M}^-_{\mathsf{Spin}(2r+1)})$ is projectively flat with respect to the Hitchin/TUY connection of Theorem 1.3.*

In the untwisted case this result appears in [17]. The proof of Theorem 1.6 uses the fact that the projective heat operator is invariant under the action of the group of two torsion points of the the Jacobian. Once the existence of the Hitchin connection is established, the rest of the proof is same as that in [17].

1.5. **Rank-level duality for genus zero.** For $r, s \geq 2$, let $d = 2rs + r + s$ (this notation will be used throughout the paper). The embedding

$$\mathfrak{so}(2r+1) \oplus \mathfrak{so}(2s+1) \longrightarrow \mathfrak{so}(2d+1) \tag{1.5}$$

extends to an embedding of affine Lie algebras. For integrable weights $\vec{\lambda}$, $\vec{\mu}$, and $\vec{\Lambda}$ of $\widehat{\mathfrak{so}}(2r+1)$ at level $2s+1$, $\widehat{\mathfrak{so}}(2s+1)$ at level $2r+1$, and $\widehat{\mathfrak{so}}(2d+1)$ at level 1, respectively, suppose that the pair $(\vec{\lambda}, \vec{\mu})$ appears in the affine branching of $\vec{\Lambda}$. This in turn gives rise to maps on dual conformal blocks

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1)) \to \mathcal{V}^*_{\vec{\mu}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \otimes \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1) \ .$$

We note that in case $\vec{\Lambda} = (\omega_{\varepsilon_1}, \ldots, \omega_{\varepsilon_{n-2}}, \omega_d, \omega_d)$, with $\varepsilon_i \in \{0,1\}$, then

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1) = 1 \ ,$$

and we have a rank-level duality map,

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \longrightarrow \mathcal{V}^*_{\vec{\mu}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \ , \tag{1.6}$$

which is well-defined up to a nonzero multiplicative constant. Recall that $\widehat{\mathfrak{so}}(2r+1)$ has a *diagram automorphism* σ which interchanges the nodes of the extended Dynkin diagram associated to the weights ω_0 and ω_1 (cf. (8.1)). In Section 9.1, we prove the following.

Theorem 1.7. *Let $C = \mathbb{P}^1$. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n)$, where λ_i is the highest weight of a representation of the group $\mathsf{SO}(2r+1)$ for $i \leq n-3$, λ_{n-1}, λ_n are spin representations that are not fixed by the diagram automorphism σ, and $\vec{\mu}, \vec{\Lambda}$ are as above. Then the rank-level duality map defined in (1.6) is injective.*

This answers a question of Nakanishi and Tsuchiya (cf. [54, Sec. 6]). It is important to note that the dimensions of the left and right hand sides of (1.6) are not equal in general: some explicit examples are given in Section 9.2 below. This fact is in stark contrast with the case of $\mathfrak{sl}(m)$ conformal blocks and demonstrates the subtlety of rank-level duality.

Remark 1.8. If $\lambda \in P_{2s+1}(\mathfrak{so}(2r+1))$, $\mu \in P_{2r+1}(\mathfrak{so}(2s+1))$, are such that $\sigma(\lambda) \neq \lambda$ and (λ, μ) appears in the branching of ω_d, then $\sigma(\mu) = \mu$.

Let $X_n = \{(z_1, \dots, z_n) \mid z_i \in \mathbb{P}^1 \ , \ z_i \neq z_j\}$ denote the configuration space of points on $\mathbb{P}^1$, and let $P_n = \pi_1(X_n)$. The conformal blocks form a vector bundle over X_n with a flat connection ∇_{KZ}, and one can define the rank-level duality map as a map of vector bundles over X_n. Moreover, the rank-level duality map commutes with ∇_{KZ}.

As a corollary of Theorem 1.7, by factorizing two Spin weights at a time, we also obtain a result asserted in [54].

Corollary 1.9. *Let $C = \mathbb{P}^1$ and m be a positive integer. The representations of the pure braid group P_n associated to the conformal block bundles $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1)$ with spin weights are reducible in general. More precisely, this occurs if $\vec{\lambda}$ is of the form $(\lambda_1, \dots, \lambda_{n-2m}, \mu_1, \dots, \mu_{2m})$, where $\lambda_1, \dots, \lambda_{n-2}$ are SO-weights and $\mu_1, \dots, \mu_{2m}$ are weights of $\mathrm{Spin}(2r+1)$ that are fixed by the Dynkin automorphism σ.*

1.6. **Strange duality maps in higher genus.** Let $\mathcal{M}_{2r+1}$ be as in (1.2). The equality of dimensions in Corollary 1.2 suggests the possibility of a strange duality isomorphism. To make this precise, note that we have the following map:

$$SD : H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)})^* \longrightarrow H^0(\mathcal{M}_{2s+1}, \mathcal{P}^{\otimes(2r+1)}) \otimes H^0(\mathcal{M}_{2d+1}, \mathcal{P})^* \ . \tag{1.7}$$

Since $\dim_{\mathbb{C}} H^0(\mathcal{M}_{2r+1}, \mathcal{P}) = 2^{2g}$, and we know that the Pfaffian sections $\{s_\kappa \mid \kappa \in \mathrm{Th}(C)\}$ form a basis (Theorem 1.4, [17], [57]), it is natural to consider $s_\Delta = \sum_\kappa s_\kappa$, and investigate whether the induced *strange duality map* is an isomorphism. Denote this map by

$$s_\Delta^* : H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)})^* \longrightarrow H^0(\mathcal{M}_{2s+1}, \mathcal{P}^{\otimes(2r+1)}) \ . \tag{1.8}$$

It is easy to arrange that the map (1.7) be equivariant with respect to the action of $J_2(C)$ permuting the theta characteristics. By taking invariants, for every $\kappa \in \mathrm{Th}(C)$ we get a map induced by the Pfaffian section s_κ:

$$s_\kappa^* : H^0(\mathcal{M}_{\mathsf{SO}(2r+1)}, \mathcal{P}_\kappa^{\otimes(2s+1)})^* \longrightarrow H^0(\mathcal{M}_{\mathsf{SO}(2s+1)}, \mathcal{P}_\kappa^{\otimes(2r+1)}) \ . \tag{1.9}$$

A simple argument shows that s_κ^* is an isomorphism for every κ if and only if the map s_Δ^* is an isomorphism. We refer the reader to Section 10.2 for more details. However,

the fact, mentioned above, that the rank-level duality map for spin weights fails to be an isomorphism may be taken as an indication that the strange duality map (1.8) might not be an isomorphism either. We shall prove that this is indeed the case.

Theorem 1.10. *The strange duality map* (1.8) (*resp.* (1.9)) *is not an isomorphism* (*resp. is not an isomorphism for every* κ).

The analysis passes through the sewing construction and detailed calculations involving the rank-level maps discussed above. Since the Pfaffian sections are projectively flat, there is a consequence for the holonomy representations of spin mapping class groups.

Corollary 1.11. *For some theta characteristic* κ *and any* $r, s \geq 2$*, the Hitchin connection in Theorem 1.3 has reducible holonomy representation.*

Remark 1.12. The holonomy representations of the Hitchin connection for $\mathcal{M}_{\mathsf{Spin}(2r+1)}$ and $\mathcal{M}^{-}_{\mathsf{Spin}(2r+1)}$ are easily seen to be reducible by noting the difference of dimensions of the Verlinde spaces for $\mathsf{Spin}(2r+1)$ and $\mathsf{Spin}(2s+1)$ (cf. (1.1)). However, for the SO moduli spaces and powers of the Pfaffian line bundle there is no known Verlinde type formula. Hence, simple arguments based on dimension do not work. Questions about irreducibility of mapping class group representations for $\mathsf{SL}(n)$ have been considered in [4].

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2. Conformal blocks and basic properties

Here we recall some definitions from [65]. Let $\mathfrak{g}$ be a simple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let $\Delta = \Delta_+ \sqcup \Delta_-$ be a positive/negative decomposition of the set of roots, and $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$, the decomposition into root spaces $\mathfrak{g}_\alpha$. Let $(\ ,\ )$ denote the Cartan-Killing form on $\mathfrak{g}$, normalized so that $(\theta,\theta) = 2$ for a longest root θ. We often identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using $(\ ,\ )$.

2.1. **Affine Lie algebras.** The *affine Lie algebra* $\widehat{\mathfrak{g}}$ is defined as a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((\xi))$. As a vector space $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c$, where c is central, and the Lie bracket is determined by

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X,Y] \otimes f(\xi)g(\xi) + (X,Y)\operatorname{Res}_{\xi=0}(gdf) \cdot c\ ,$$

where $X, Y \in \mathfrak{g}$ and $f(\xi), g(\xi) \in \mathbb{C}((\xi))$. Set $X(n) = X \otimes \xi^n$ and $X = X(0)$ for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

The theory of highest weight integrable irreducible modules for $\widehat{\mathfrak{g}}$ runs parallel to that of finite dimensional irreducible modules for $\mathfrak{g}$. Let us briefly recall the details for completeness. The finite dimensional irreducible $\mathfrak{g}$-modules are parametrized by the set of dominant integral weights $P_+(\mathfrak{g}) \subset \mathfrak{h}^*$. For each $\lambda \in P_+(\mathfrak{g})$, let V_λ denote the irreducible $\mathfrak{g}$-module

with highest weight λ. Fix a positive integer ℓ, called the *level*. The set of *dominant integral weights of level* ℓ is defined by:

$$P_\ell(\mathfrak{g}) := \{\lambda \in P_+(\mathfrak{g}) \mid (\lambda, \theta) \leq \ell\} \ .$$

For each $\lambda \in P_\ell(\mathfrak{g})$, there is a unique irreducible *integrable highest weight* $\widehat{\mathfrak{g}}$*-module* $\mathcal{H}_\lambda(\mathfrak{g}, \ell)$ which satisfies the following properties:

(1) $\mathcal{H}_\lambda(\mathfrak{g}, \ell)$ is generated by V_λ over $\widehat{\mathfrak{g}}$ (cf. [38]);
(2) $\mathcal{H}_\lambda(\mathfrak{g}, \ell)$ are infinite dimensional;
(3) $V_\lambda \subset \mathcal{H}_\lambda(\mathfrak{g}, \ell)$;
(4) The central element c of $\widehat{\mathfrak{g}}$ acts by the scalar ℓ.

When there are implicitly understood, we sometimes omit the notation $\mathfrak{g}$ or ℓ from $\mathcal{H}_\lambda(\mathfrak{g}, \ell)$.

We will also need the following quantity. For any $\lambda \in P_\ell(\mathfrak{g})$, define the *trace anomaly*

$$\Delta_\lambda(\mathfrak{g}, \ell) := \frac{(\lambda, \lambda + 2\rho)}{2(g^\vee + \ell)} \ , \tag{2.1}$$

where ρ is the half sum of positive roots, and $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Let $\widehat{\mathcal{L}}(\mathfrak{g})$ denote the untwisted affine Kac-Moody Lie algebra of $\mathfrak{g}$ (see [36, Sec. 7.2]). Explicitly, it can be defined as $\widehat{\mathcal{L}}\mathfrak{g} = \widehat{\mathfrak{g}} \oplus \mathbb{C}d$, where d is a derivation that commutes with c and acts on $\mathfrak{g} \otimes \mathbb{C}((\xi))$ by the formula $d = \xi \dfrac{d}{d\xi}$. Clearly, $\widehat{\mathfrak{g}}$ is a Lie subalgebra of $\widehat{\mathcal{L}}\mathfrak{g}$. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ (resp. $\omega_1, \ldots, \omega_m$) denote the affine fundamental weights of $\widehat{\mathcal{L}}\mathfrak{g}$ (resp. $\mathfrak{g}$), where $m = \operatorname{rank} \mathfrak{g}$. We observe the following:

- $\Lambda_i = \omega_i + a_i^\vee \Lambda_0$ for $1 \leq i \leq m$, where $a_i^\vee$ are the dual Coxeter labels ([36]).
- Any $\lambda \in P_\ell(\mathfrak{g})$ corresponds to the weight $\lambda + \ell\Lambda_0$ of $\widehat{\mathcal{L}}\mathfrak{g}$.
- Any highest weight integrable irreducible representation of $\widehat{\mathcal{L}}\mathfrak{g}$ is also irreducible as $\widehat{\mathfrak{g}}$-module.
- Λ_0 restricted to the Cartan subalgebra of $\widehat{\mathfrak{g}}$ is zero.

Since we are working with $\widehat{\mathfrak{g}}$ in this paper, we will denote by ω_i the fundamental weight for both $\mathfrak{g}$ and $\widehat{\mathcal{L}}\mathfrak{g}$. For uniformity of notation, we will denote by ω_0-the zero-th fundamental weight Λ_0 of $\widehat{\mathcal{L}}\mathfrak{g}$.

2.2. **Conformal embeddings.** Let $\phi : \mathfrak{s} \to \mathfrak{g}$ an embedding of simple Lie algebras, and let $(\ ,\ )_\mathfrak{s}$ and $(\ ,\ )_\mathfrak{g}$ be the Cartan-Killing forms, normalized as above. Then the *Dynkin index* of ϕ is the unique integer d_ϕ satisfying $(\phi(x), \phi(y))_\mathfrak{g} = d_\phi \cdot (x, y)_\mathfrak{s}$, for all $x, y \in \mathfrak{s}$. More generally, when $\mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_i$ simple, we define the *Dynkin multi-index of* $\phi = \phi_1 \oplus \phi_2 : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ to be $d_\phi = (d_{\phi_1}, d_{\phi_2})$.

Definition 2.1. Let $\phi = (\phi_1, \phi_2) : \mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be an embedding of Lie algebras with Dynkin multi-index $d_\phi = (d_{\phi_1}, d_{\phi_2})$. Then ϕ is said to be a *conformal embedding* if

$$\frac{d_{\phi_1} \dim \mathfrak{g}_1}{g_1^\vee + d_{\phi_1}} + \frac{d_{\phi_2} \dim \mathfrak{g}_2}{g_2^\vee + d_{\phi_2}} = \frac{\dim \mathfrak{g}}{g^\vee + 1} \ ,$$

where $g_1^\vee$, $g_2^\vee$, and $g^\vee$ are the dual Coxeter numbers of $\mathfrak{g}_1$, $\mathfrak{g}_2$, and $\mathfrak{g}$, respectively.

Many familiar and important embeddings are conformal: (1.5) is one family of such examples. For a complete list, see [8]. For the purposes of this paper, the key property of conformal embeddings that we need is the following: an embedding $\phi : \mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ is conformal if and only if any irreducible $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\Lambda(\mathfrak{g}, 1)$, $\Lambda \in P_1(\mathfrak{g})$, decomposes into a finite sum of irreducible $\widehat{\mathfrak{s}}$-modules of the form $\mathcal{H}_{\lambda_1}(\mathfrak{g}_1, \ell_1) \otimes \mathcal{H}_{\lambda_2}(\mathfrak{g}_2, \ell_2)$, where $\lambda_i \in P_{\ell_i}(\mathfrak{g}_i)$, $i = 1, 2$, and $(\ell_1, \ell_2) = d_\phi$, the Dynkin multi-index. See [38].

2.3. **Conformal blocks.** Let C be a smooth projective curve with marked points $\vec{p} = (p_1, \ldots, p_n)$ such that $(C, \vec{p})$ satisfies the Deligne-Mumford stability conditions. We furthermore assume a choice coordinates and formal neighborhoods around the p_i, which give isomorphisms $\widehat{\mathcal{O}}_{C,P_i} \xrightarrow{\sim} \mathbb{C}[\![\xi_i]\!]$. We will use the notation $\mathfrak{X} = (C; \vec{p})$ to denote this data. The *current algebra* is defined to be $\mathfrak{g}(\mathfrak{X}) := \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(*(p_1, \ldots, p_n)))$. By local expansion of functions using the chosen coordinates ξ_i, we get an embedding:

$$\mathfrak{g}(\mathfrak{X}) \hookrightarrow \widehat{\mathfrak{g}}_n := \bigoplus_{i=1}^{n} \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}(\!(\xi_i)\!) \oplus \mathbb{C} \cdot c \ .$$

Consider an n-tuple of weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in P_\ell^n(\mathfrak{g})$, and set

$$\mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell) = \mathcal{H}_{\lambda_1}(\mathfrak{g}, \ell) \otimes \cdots \otimes \mathcal{H}_{\lambda_n}(\mathfrak{g}, \ell) \ .$$

The algebra $\widehat{\mathfrak{g}}_n$ (and hence also the current algebra $\mathfrak{g}(\mathfrak{X})$) acts on $\mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ componentwise using the embedding above.

Definition 2.2. The space of *conformal blocks* is

$$\mathcal{V}^*_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell)/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell), \mathbb{C}) \ .$$

The space of *dual conformal blocks* is $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) = \mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell)/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell)$.

Conformal blocks are finite dimensional vector spaces, and their dimensions are given by the *Verlinde formula* [26, 63, 65]. We now discuss some important properties of the spaces of conformal blocks.

- (Flat projective connection) Consider a family

  $$\mathcal{F} = (\pi : \mathcal{C} \to B; \sigma_1, \ldots, \sigma_n; \xi_1, \ldots, \xi_n)$$

  of nodal curves on a base B with sections σ_i and formal coordinates ξ_i. In [65], a locally free sheaf $\mathcal{V}^*_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g}, \ell)$ known as the sheaf of conformal blocks is constructed over the base B. Moreover, if $\mathcal{F}$ is a family of smooth projective curves, then the sheaf $\mathcal{V}^*_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g}, \ell)$ carries a flat projective connection known as the *TUY connection*. We refer the reader to [65] for more details. In genus zero, the TUY connection is a flat connection and is also known as *KZ connection*.
- (Propagation of vacua) Let C be any curve with n-marked points satisfying the Deligne-Mumford stability conditions and C be the same curve with $n + 1$ marked points. Assume that the weights attached to the n marked points are

$\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ and we associate the vacuum representation $(\mathcal{H}_{\omega_0})$ at the $(n+1)$-st point. Then there is a canonical isomorphism $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) \simeq \mathcal{V}_{\vec{\lambda}\cup\omega_0}(\mathfrak{X}', \mathfrak{g}, \ell)$, where $\mathfrak{X}$ (resp. $\mathfrak{X}'$) denote the data associated to the n (resp. $n+1$) pointed curve C.

- (Gauge symmetry) Let $f \in H^0(C, \mathcal{O}_C(*(p_1, \dots, p_n)))$ and $\langle \Psi \mid \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$, then $\langle \Psi \mid (X \otimes f) = 0$. More precisely, for any $\mid \phi_1 \otimes \cdots \otimes \phi_n \rangle \in \mathcal{H}_{\vec{\lambda}}(\mathfrak{g}, \ell)$,

$$\sum_{i=1}^{n} \langle \Psi \mid \phi_1 \otimes \cdots \otimes (X \otimes f(\xi_i))\phi_i \otimes \cdots \otimes \phi_n \rangle = 0 \ .$$

Let $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[\![t]\!]$ be a family of curves of genus g with n marked points with chosen coordinates such that the special fiber $\mathcal{X}_0$ is a curve X_0 over $\mathbb{C}$ with exactly one node, and the generic fiber $\mathcal{X}_t$ is a smooth curve. Let $\widetilde{X}_0$ be the normalization of X_0. For $\lambda \in P_\ell(\mathfrak{g})$, the following isomorphism is constructed in [65]:

$$\oplus \iota_\lambda : \bigoplus_{\lambda \in P_\ell(\mathfrak{g})} \mathcal{V}^*_{\vec{\lambda},\lambda,\lambda^\dagger}(\widetilde{\mathfrak{X}}_0, \mathfrak{g}, \ell) \to \mathcal{V}^*_{\vec{\lambda}}(\mathcal{X}_0, \mathfrak{g}, \ell) \ ,$$

where $\widetilde{\mathfrak{X}}_0$ is the data associated to the $(n+2)$ points of the smooth pointed curve $\widetilde{X}_0$ with chosen coordinates and $\lambda^\dagger$ is the highest weight of the contragredient representation of V_λ. This is commonly referred to as *factorization of conformal blocks*.

In the same paper [65], a sheaf theoretic version of the above isomorphism was also constructed which is commonly referred to as the *sewing construction*. This provides for each $\lambda \in P_\ell(\mathfrak{g})$, a map of $\mathbb{C}[\![t]\!]$-modules: $s_\lambda(t) : \mathcal{V}^*_{\vec{\lambda},\lambda,\lambda^\dagger}(\widetilde{\mathfrak{X}}_0, \mathfrak{g}, \ell) \otimes \mathbb{C}[\![t]\!] \to \mathcal{V}^*_{\vec{\lambda}}(\mathcal{X}, \mathfrak{g}, \ell)$. Then $s_\lambda(t)$ extends the map ι_λ in families such that $\oplus_{\lambda \in P_\ell(\mathfrak{g})} s_\lambda(t)$, is an isomorphism of locally free sheaves over $\operatorname{Spec} \mathbb{C}[[t]]$. We refer the reader to [50, 65] for exact details.

3. Twisted moduli stacks

3.1. **Uniformization.** In this section we recall the construction of the twisted moduli stacks for spin bundles as in [13] (see also [55, 56]). First, let us fix some notation.

Definition 3.1. Let G be a connected complex reductive Lie group. Then

(1) $\mathcal{M}_G :=$ the moduli stack of G-bundles on C;
(2) $M_G :=$ the Ramanathan coarse moduli space of S-equivalence classes of semistable G-bundles on C;
(3) a G-bundle is *regularly stable* if it is stable and its automorphism group is equal to the center $Z(G)$. We denote by $M_G^{reg} \subset M_G$ the moduli space of regularly stable bundles.

Recall the exact sequence $1 \to \mathbb{Z}/2 \to \mathsf{Spin}(m) \to \mathsf{SO}(m) \to 1$. Identify $\mathbb{Z}/2$ with the subgroup $\{\pm 1\} \subset \mathbb{C}^\times$, and define the *special Clifford group*

$$\mathsf{SC}(m) := \mathsf{Spin}(m) \times_{\mathbb{Z}/2} \mathbb{C}^\times \ . \tag{3.1}$$

The *spinor norm* is the group homomorphism $\mathrm{Nm} : \mathsf{SC}(m) \longrightarrow \mathbb{C}^\times$, which induces a morphism of stacks $\mathcal{M}_{\mathsf{SC}(m)} \to \mathcal{M}_{\mathbb{C}^\times}$. We will denote this stack morphism also by Nm.

Let p be a fixed point of the curve C. Throughout the paper we will denote the punctured curve by $C^* = C - \{p\}$. Consider bundles $\mathcal{O}_C(dp)$, where $d \in \mathbb{Z}$. Then the preimage by Nm of the class of $[\mathcal{O}_C(dp)] \in \mathcal{M}_{\mathbb{C}^\times}$ depends only on the parity of d (cf. [55, Prop. 3.4]). We will denote by $\mathcal{M}^{\pm}_{\mathsf{SC}(m)}$ the inverse images of the Jacobian $J(X)$ and $\mathrm{Pic}_1(C)$, respectively. Let $\mathcal{M}^{\pm}_{\mathsf{Spin}(m)}$ be the inverse images of the points $\mathcal{O}_C(dp)$, for $d = 0, 1$, respectively. Therefore, while by definition $\mathcal{M}^{+}_{\mathsf{Spin}(m)} = \mathcal{M}_{\mathsf{Spin}(m)}$, the space $\mathcal{M}^{-}_{\mathsf{Spin}(m)}$ is a "twisted" component that does not correspond to a stack of G-bundles for any complex reductive G.

The components $\mathcal{M}^{\pm}_{\mathsf{SO}(m)}$ of $\mathcal{M}_{\mathsf{SO}(m)}$ are labeled by $\delta \in \pi_1(\mathsf{SO}(m)) \simeq \mathbb{Z}/2$ (cf. [13, Prop. 1.3]). The map $\mathsf{SC}(m) \to \mathsf{SO}(m)$, coming from the projection of (3.1) on the first factor, induces a morphism of stacks

$$p : \mathcal{M}^{\pm}_{\mathsf{Spin}(m)} \longrightarrow \mathcal{M}^{\pm}_{\mathsf{SO}(m)} \ . \tag{3.2}$$

Definition 3.2. For G as in Definition 3.1, let

(1) $LG := G(\!(\xi)\!)$ be the algebraic loop group of G;
(2) $L^+G := G[\![\xi]\!]$ be the group of positive loops;
(3) $\mathcal{Q}_G := LG/L^+G$ be the *affine Grassmannian*;
(4) $L_CG := G(\mathcal{O}_{C^*}) \hookrightarrow LG$.

The following result, proved in [13], gives a uniformization for the twisted moduli stacks and determines their Picard groups. We only state it in the case $G = \mathsf{Spin}(m)$.

Proposition 3.3. *Let $\delta \in \{\pm 1\} = \pi_1(\mathsf{SO}(m))$ and $\zeta \in (L\mathsf{SO}(m))^\delta(\mathbb{C})$. Then*

$$\mathcal{M}^{\delta}_{\mathsf{Spin}(m)} = (\zeta^{-1} \cdot L_C(\mathsf{Spin}(m)) \cdot \zeta) \backslash \mathcal{Q}_{\mathsf{Spin}(m)} \ ,$$

where $\mathcal{Q}_{\mathsf{Spin}(m)}$ is the affine Grassmannian of $\mathsf{Spin}(m)$. The torsion subgroup of $\mathrm{Pic}(\mathcal{M}^{\pm}_{\mathsf{Spin}(m)})$ is trivial, and in fact, $\mathrm{Pic}(\mathcal{M}^{\pm}_{\mathsf{Spin}(m)}) \simeq \mathbb{Z}$.

As we have done with stacks, we may also define the coarse moduli spaces $M^{-}_{\mathsf{SC}(m)}$ and $M^{-}_{\mathsf{Spin}(m)}$ of semistable twisted bundles on C.

3.2. **Pfaffian divisors.** The set $\mathrm{Th}(C)$ of theta characteristics forms a torsor for the 2-torsion points $J_2(C)$ of the Jacobian of C. Note the cardinalities [52, Sec. 4]: $|J_2(C)| = |\mathrm{Th}(C)| = 2^{2g}$, $|\mathrm{Th}^{\pm}(C)| = 2^{g-1}(2^g \pm 1)$. Recall from the introduction that by an oriented orthogonal bundle on C we mean a pair (E, q) consisting of a bundle $E \to C$ with trivial determinant, and a nondegenerate quadratic form $q : E \otimes E \to \mathcal{O}_C$. Then q induces a nondegenerate quadratic form $\hat{q} : (E \otimes \kappa) \otimes (E \otimes \kappa) \to \omega_C$. We recall the following from [45].

Proposition 3.4. *Let B be a locally noetherian scheme, $\pi : C \times B \to B$, $\mathrm{pr} : C \times B \to C$, the projections, and $(\mathcal{E}, \hat{q}) \to C \times B$ a vector bundle equipped with an ω_C-valued nondegenerate quadratic form $\hat{q}$. Then the choice of a theta characteristic $\kappa \to C$ gives a canonical square root $\mathcal{P}_{\mathcal{E},\hat{q},\kappa}$ of the determinant of cohomology $\mathcal{D}_{\mathcal{E}} = [\mathrm{Det}R\pi_*(\mathcal{E} \otimes \mathrm{pr}^*\kappa)]^*$. Moreover, if $f : B' \to B$ is a morphism of locally noetherian schemes, then the Pfaffian functor commutes with base change, i.e. $f^*\mathcal{P}_{\mathcal{E},\hat{q}} = \mathcal{P}_{f^*\mathcal{E},f^*\hat{q}}$.*

Next, we recall the definition of the *Pfaffian divisor*, following [13, 45]. Let $(\mathcal{E}, q) \to C \times \mathcal{M}_{\mathsf{SO}(m)}$ be the universal quadratic bundle. For $\kappa \in \mathrm{Th}(C)$, consider the substack defined by: $\Theta_\kappa := \mathrm{div}(R\pi_*(\mathcal{E} \otimes pr^*\kappa))$. It is shown in [45, eq. (7.10)] that Θ_κ is a divisor on $\mathcal{M}^+_{\mathsf{SO}(m)}$ if and only if either m or κ is even. We postpone the proof of the following proposition to Section 5.3.

Proposition 3.5. *The substack Θ_κ is a divisor on $\mathcal{M}^-_{\mathsf{SO}(m)}$ if and only if both m and κ are odd.*

Definition 3.6. It follows from the above that there is a nonzero section s_κ (canonical up to scale) of $\mathcal{P}_\kappa \to \mathcal{M}_{\mathsf{SO}(2r+1)}$, supported on $\mathcal{M}^+_{\mathsf{SO}(2r+1)}$ (resp. $\mathcal{M}^-_{\mathsf{SO}(2r+1)}$) if κ is even (resp. odd). We call s_κ the *Pfaffian section.*

Recall the projection (3.2). For $\kappa, \kappa' \in \mathrm{Th}(C)$, the line bundles $p^*\mathcal{P}_\kappa$, $p^*\mathcal{P}_{\kappa'}$ are isomorphic. We therefore set $\mathcal{P} = p^*\mathcal{P}_\kappa$, which is well-defined up to this isomorphism. On each component $\mathcal{M}^\pm_{\mathsf{Spin}(m)}$, $\mathcal{P}$ is the ample generator of $\mathrm{Pic}(\mathcal{M}^\pm_{\mathsf{Spin}(m)})$ [13].

Let A be the group of principal $\mathbb{Z}/2$-bundles on C, where $\mathbb{Z}/2$ is identified with the kernel of the map $\mathsf{Spin}(m) \to \mathsf{SO}(m)$. Then $A \simeq J_2(X)$. Let $\widehat{A}$ denote the set of characters of A. Let $Y = M^{-,reg}_{\mathsf{SO}(m)}$ (the notion of regularly stable extends directly to the twisted setting), and $\widetilde{Y} = p^{-1}(Y)$. Here $p : M^-_{\mathsf{Spin}(m)} \to M^-_{\mathsf{SO}(m)}$ is the projection map. By [13, Prop. 13.5], the Galois covering p is étale over $M^{-,reg}_{\mathsf{SO}(m)}$. Since $M^-_{\mathsf{SO}(m)} - Y$ has codimension ≥ 2 (see [25, Thm. II.6], [44, Appendix] and recall $m \geq 5$), and p is finite and dominant, we conclude that $M^-_{\mathsf{Spin}(m)} - \widetilde{Y}$ has codimension ≥ 2 as well. The moduli spaces $M^-_{\mathsf{SO}(m)}$ and $M^-_{\mathsf{Spin}(m)}$ can be constructed as GIT quotients of a smooth scheme by a reductive group ([13, Lemma 7.3]), hence they are normal. Therefore, by normality of the moduli spaces $M^-_{\mathsf{SO}(m)}$ and $M^-_{\mathsf{Spin}(m)}$, we get $H^0(Y, \mathcal{O}_Y) = H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) = \mathbb{C}$. There is a decomposition of sheaves $p_*\mathcal{O}_{\widetilde{Y}} = \oplus_{\chi \in \widehat{A}} L_\chi$, where as a presheaf $L_\chi(U) = \{s \in \mathcal{O}_{\widetilde{Y}}(p^{-1}(U)) \mid gs = \chi(g)s, \forall g \in A\}$.

Proposition 3.7. *We have the following properties:*

(1) $H^0(Y, L_\chi) = \begin{cases} \mathbb{C} & \chi = 1 \\ 0 & \chi \neq 1 \end{cases}$;
(2) *for any χ, $p^*L_\chi = \mathcal{O}_{\widetilde{Y}}$;*
(3) $L_\chi \otimes L_{\chi'} = L_{\chi\chi'}$;
(4) $L_\chi \simeq L_{\chi'} \iff \chi = \chi'$.

It is well-known [13] that Y is smooth, and since the map $p : \widetilde{Y} \to Y$ is Galois and étale, this implies that $\widetilde{Y}$ is also smooth and is contained in $M^{-,reg}_{\mathsf{Spin}(m)}$. We will also need the following fact.

Lemma 3.8. $\pi_1(\widetilde{Y}) = \{1\}$.

Proof. The proof is essentially the same as in Atiyah-Bott [6, Thm. 9.12]. Let $K \subset \mathsf{SC}(m)$ be a maximal compact subgroup. Fix a topologically nontrivial smooth principal $\mathsf{SC}(m)$-bundle $P \to C$, and let P_K be a reduction to K. Let $\mathcal{A}(P_K)$ be the space of connections

P_K. Then $\mathcal{A}(P_K)$ can be identified with the space of holomorphic structures on P, i.e. holomorphic principal $\mathsf{SC}(m)$-bundles. Let $\mathcal{G}(P)$ denote the group of $\mathsf{SC}(m)$ gauge transformations, and $\overline{\mathcal{G}}(P)$ the quotient of $\mathcal{G}(P)$ by the constant central gauge transformations (recall that $Z(\mathsf{SC}(m)) = \mathbb{C}^\times$ for m odd, and $Z(\mathsf{SC}(m)) = \mathbb{C}^\times \times \mathbb{Z}/2$ for m even). By a standard argument, $\pi_0(\mathcal{G}(P)) \simeq H^1(C, \pi_1(\mathsf{SC}(m)))$. Since $\pi_1(\mathsf{SC}(m)) = \mathbb{Z}$, we conclude that $\pi_0(\mathcal{G}(P)) \simeq H^1(C, \mathbb{Z})$. From the fibration $Z(\mathsf{SC}(m)) \to \mathcal{G}(P) \to \overline{\mathcal{G}}(P)$, we find $\pi_0(\overline{\mathcal{G}}(P)) \simeq H^1(C, \mathbb{Z})$, as well. From [6, Sec. 10], the complement of the stable points $\mathcal{A}^s(P_K) \subset \mathcal{A}(P_K)$ has complex codimension at least 2. As noted by Faltings (see the comment in the proof of part (ii) of Theorem II.6 in [25]), the same proof applies to show that the set of regularly stable points $\mathcal{A}^{reg}(P_K) \subset \mathcal{A}(P_K)$ has complex codimension at least 2. Since $\mathcal{A}(P_K)$ is smooth and contractible, this implies in particular that $\mathcal{A}^{reg}(P_K)$ is simply connected. Note also that $\overline{\mathcal{G}}(P)$ acts freely on $\mathcal{A}^{reg}(P_K)$ with quotient $M^{-,reg}_{\mathsf{SC}(m)}$. It follows that

$$\pi_1\big(M^{-,reg}_{\mathsf{SC}(m)}\big) = \pi_1(\mathcal{A}^{reg}(P_K)/\overline{\mathcal{G}}(P)) \simeq \pi_0(\overline{\mathcal{G}}(P)) \simeq H^1(C, \mathbb{Z}) \ . \tag{3.3}$$

Now consider the fibration:

$$\begin{array}{ccc} M^{-,reg}_{\mathsf{Spin}(m)} & \longrightarrow & M^{-,reg}_{\mathsf{SC}(m)} \\ & & \big\downarrow \mathrm{Nm} \\ & & \mathrm{Pic}_1(C) \end{array} \tag{3.4}$$

By the associated exact sequence of fundamental groups,

$$1 \longrightarrow \pi_1\big(M^{-,reg}_{\mathsf{Spin}(m)}\big) \longrightarrow \pi_1\big(M^{-,reg}_{\mathsf{SC}(m)}\big) \longrightarrow \pi_1(\mathrm{Pic}_1(C)) \longrightarrow 1 \ ,$$

and (3.3), we see immediately that $\pi_1\big(M^{-,reg}_{\mathsf{Spin}(m)}\big) = \{1\}$. Now both $\widetilde{Y}$ and $M^{-,reg}_{\mathsf{Spin}(m)}$ are smooth with complement of codimension ≥ 2. Therefore, $\pi_1(\widetilde{Y}) \simeq \pi_1\big(M^{-,reg}_{\mathsf{Spin}(m)}\big) = \{1\}$. □

Proposition 3.9. *Given $\kappa \in \mathrm{Th}(C)$ and $\alpha \in J_2(C)$, then $\mathcal{P}_{\kappa\otimes\alpha} \otimes \mathcal{P}_\kappa^{\otimes(-1)}$ is isomorphic to a unique L_χ, where $\chi \in \widehat{A}$.*

Proof. By the proof of [13, Prop. 5.2], there is an injective homomorphism $\lambda : \widehat{A} \to \mathrm{Pic}(\mathcal{M}^\delta_{\mathsf{SO}(m)})$, and $\mathcal{P}_{\kappa\otimes\alpha} \otimes \mathcal{P}_\kappa^{\otimes(-1)}$ equals $\lambda(W(\alpha))$, where W is the Weil pairing on $J_2(C) \otimes J_2(C) \to \mu_2 = \{1, -1\}$. Now if $\alpha \neq \alpha'$, we get $\lambda(W(\alpha)) \neq \lambda(W(\alpha'))$. This proves the uniqueness. By Lemma 3.8 we get that $\pi_1(Y)$ is isomorphic to J_2 and all torsion line bundles on Y are of the form L_χ for some $\chi \in \widehat{A}$. We know that $\mathcal{P}_{\kappa\otimes\alpha} \otimes \mathcal{P}_\kappa^{\otimes(-1)}$ is torsion and hence $\mathcal{P}_{\kappa\otimes\alpha} \otimes \mathcal{P}_\kappa^{\otimes(-1)}$ is isomorphic to some L_χ.

□

Using the above, we have the following decomposition of A-modules:

$$H^0(\widetilde{Y}, \mathcal{P}) = \bigoplus_{\chi\in\widehat{A}} H^0(Y, \mathcal{P}_\kappa \otimes L_\chi) \ . \tag{3.5}$$

In the next section we will prove the following.

Proposition 3.10. *Suppose m is odd. Then,*

(1) $\dim_{\mathbb{C}} H^0(\widetilde{Y}, \mathcal{P}) = 2^{g-1}(2^g - 1)$;
(2) *each $H^0(Y, \mathcal{P}_\kappa)$, κ odd, is 1-dimensional and is spanned by the Pfaffian section s_κ;*
(3) *the set $\{s_\kappa \mid \kappa \in \mathrm{Th}^-(C)\}$, is a basis for $H^0(\widetilde{Y}, \mathcal{P})$.*

This result should be compared with [17, Props. 2.3 and 2.4] in the even case.

4. Uniformization

4.1. **Conformal blocks via uniformization.** The main result in this section is the identification of generalized theta functions on $\mathcal{M}_{\mathsf{Spin}(m)}$ at any level with the space of conformal blocks. Let V be a countable dimensional complex vector space. We define the tautological line bundle L_V on $\mathbb{P}(V)$ as the Zariski closed subset: $L_V := \{(x, v) \in \mathbb{P}(V) \times V \mid v \in x\}$. The space $\mathbb{P}(V) \times V$ has a natural ind-scheme structure. We induce a ind-scheme structure on L_V such that $L_V \to \mathbb{P}(V) \times V$ is a closed immersion. Hence by projecting onto the first factor gives an algebraic line bundle L_V on $\mathbb{P}(V)$. We denote its dual by L_V^*.

For a simply connected complex Lie group G, we denote by $\mathcal{Q}_G$ the affine Grassmannian associated to G with Lie algebra $\mathfrak{g}$. The affine Grassmannian is a ind-projective scheme [45, 40, 49]. Let ω_0 be the affine fundamental weight of affine Kac-Moody algebra associated to G and for any positive integer ℓ consider $\mathcal{H}_{\omega_0}(\mathfrak{g}, \ell)$ be the corresponding irreducible, integrable representation with highest weight ω_0 at level ℓ. Let $v_0(\ell)$ be a highest weight vector of $\mathcal{H}_{\omega_0}(\mathfrak{g}, \ell)$. The natural map $i : \mathcal{Q}_G \to \mathbb{P}(\mathcal{H}_{\omega_0}(\mathfrak{g}, \ell))$; $g[L^+G] \to gv_0(\ell)$ is an embedding of ind-schemes (see Appendix C in [41] or [40]). For any $\ell > 0$, we define $\mathcal{L}(\ell\chi)$ to be the pull back $L^*_{\mathcal{H}_{\omega_0}(\mathfrak{g},\ell)}$ under the map i. We can extend the definition to non-negative integers by taking the dual bundles. By [61, Sec. 2.7], it is easy to see that $\mathcal{L}_{\ell\chi} = \mathcal{L}_\chi^{\otimes \ell}$. The line bundle $\mathcal{L}_\chi$ can also be realized in a *line bundle associated to a character* as follows (see [45, Lemma 4.1]): Consider the representation $\mathcal{H}_{\omega_0}(\mathfrak{g}, 1)$ of the affine Lie algebra $\widehat{\mathfrak{g}}$. This representation can be ([26]) integrated to a projective representation of the loop group $\mathrm{L}\,G$. We pullback the exact sequence to $\mathrm{L}\,G$

$$0 \longrightarrow \mathbb{G}_m \longrightarrow GL(\mathcal{H}_{\omega_0}(\mathfrak{g}, 1)) \longrightarrow PGL(\mathcal{H}_{\omega_0}(\mathfrak{g}, 1)) \longrightarrow 1,$$

defines a central extension which splits ([45, Lemma 4.9]) over $\mathrm{L}^+ G$:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{\mathrm{L}\,G} \xrightarrow{p} \mathrm{L}\,G \longrightarrow 1.$$

Let $\widehat{\mathrm{L}^+ G}$ be the inverse image of $\mathrm{L}^+ G$ via the map p, then $\widehat{\mathrm{L}^+ G} \simeq \mathrm{L}\,G \times \mathbb{G}_m$. The projection onto the second factor defines a character χ_0 of $\widehat{\mathrm{L}^+ G}$. Then $\mathcal{L}_\chi$ can be described ([45, Lemma 4.11]) as the line bundle on the $\mathcal{Q}_G = \widehat{\mathrm{L}\,G}/\widehat{\mathrm{L}^+ G}$ associated to the character χ_0^{-1}. We will henceforth refer to $\mathcal{L}_\chi$ as the line bundle associate to the character χ of the affine fundamental weight ω_0.

We have the following proposition.

Proposition 4.1. *Let $\pi : \mathcal{Q}_{\mathsf{Spin}(m)} \to \mathcal{M}^-_{\mathsf{Spin}(m)}$ be the projection from Proposition 3.3, and χ be the character corresponding to the affine fundamental weight ω_0. Then we have: $\pi^*\mathcal{P} = \mathcal{L}_\chi$.*

Proof. Consider the map $\mathsf{Spin}(m) \to \mathsf{SL}(m)$ that comes from the standard embedding. This induces a map between the affine Grassmannians $\mathcal{Q}_{\mathsf{Spin}(m)} \to \mathcal{Q}_{\mathsf{SL}(m)}$. Let $\mathcal{L}_\chi^0$ denote the pull-back of the determinant of cohomology line bundle on $\mathcal{M}_{\mathsf{SL}(m)}$ to the affine Grassmannian $\mathcal{Q}_{\mathsf{SL}(m)}$. By a result in [43], we know that the pull-back of $\mathcal{L}_\chi^0$ is $\mathcal{L}_{2\chi}$, where χ is the character and 2 is the Dynkin index of the embedding $\mathfrak{so}(m) \to \mathfrak{sl}(m)$. Now the pull-back of the determinant of cohomology to $\mathcal{Q}_{\mathsf{SL}(m)}$ is $\mathcal{L}_\chi^0$. Since the Picard group of $\mathcal{M}^-_{\mathsf{Spin}(m)}$ is torsion-free, we see that $\mathcal{P}$ pulls back to $\mathcal{L}_\chi$ on $\mathcal{Q}_{\mathsf{Spin}(m)}$. □

Let V be a vector space of dimension $2m$ (resp. $2m+1$) endowed with a symmetric nondegenerate bilinear form $(\,,)$. Let $e_1, \ldots, e_{2m}$ (resp. e_{2m+1}) be a basis of V such that $(e_i, e_{2m+1-j}) = \delta_{ij}$ (resp. $(e_i, e_{2m+2-j}) = \delta_{ij}$. The elements $H_i = E_{i,i} - E_{2m-i,2m-i}$ (resp. $H_i = E_{i,i} - E_{2m+1-i,2m+1-i}$) span a basis of the Cartan subalgebra of $\mathfrak{so}(2m)$ (resp. $\mathfrak{so}(2m+1)$. The normalized Cartan-Killing form is given by $(A, B) = \frac{1}{2}\operatorname{Tr}(AB)$. Let L_i be the dual of H_i where $L_i(H_j) = \delta_{ij}$ and $\omega_i = \sum_{a=1}^i L_a$ for $1 \le i \le m-1$ be the first $m-1$ fundamental weights of both $\mathfrak{so}(2m)$ and $\mathfrak{so}(2m+1)$.

For $\zeta \in L\mathsf{SO}(m)$, following [26], we define an automorphism $\mathrm{Ad}(\zeta)$ of $\widehat{\mathfrak{so}}(m)$ by the following formula. Let $A(z)$ be an element of $\widehat{\mathfrak{so}}(m)$.

$$\mathrm{Ad}(\zeta)(A(z), s) := (\mathrm{Ad}(\zeta)A(z), s + \mathrm{Res}_{z=0}\, \tfrac{1}{2}\operatorname{Tr}(\zeta^{-1}(d\zeta/dt)A(z))\ . \tag{4.1}$$

Let

$$\zeta = \begin{pmatrix} z & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & z^{-1} \end{pmatrix}, \tag{4.2}$$

regarded as an element of $L\mathsf{SO}(m)$.

Lemma 4.2. *Let $\pi : \widehat{\mathfrak{so}}(m) \to \mathrm{End}(\mathcal{H}_{\omega_0}(\mathfrak{so}(m), \ell))$ be an integrable representation of $\widehat{\mathfrak{so}}(m)$ and $\mathrm{Ad}(\zeta) : \widehat{\mathfrak{so}}(m) \to \widehat{\mathfrak{so}}(m)$ is the automorphism defined by formula 4.1, for ζ as above. Then the representation $\tilde{\pi} : \widehat{\mathfrak{so}}(m) \to \mathrm{End}(\mathcal{H}_{\omega_0}(\mathfrak{so}(m), \ell))$ defined by $\pi \circ \mathrm{Ad}(\zeta)$ is isomorphic to $\mathcal{H}_{\ell\omega_1}(\mathfrak{so}(m), \ell)$.*

Proof. Since $\mathcal{H}_{\omega_0}(\mathfrak{so}(m), \ell)$ is irreducible under the representation π, this implies that the representation $\tilde{\pi}$ is also irreducible. Let $A(z) = \sum A_i \otimes z^i$, then by a direct computation we can check that

$$\mathrm{Res}_{z=0}\, \tfrac{1}{2}\operatorname{Tr}(\zeta^{-1}(d\zeta/dt)A(z)) = \omega_1(H_0)\ , \tag{4.3}$$

where H_0 is the diagonal part of A_0. From a direct calculation, we can check that if X_α is a generator of the root space of α, then $\mathrm{Ad}(\zeta)X_\alpha(n) = X_\alpha(n + \omega_1(H_\alpha))$, where H_α is the coroot of α. In particular, this shows that positive nilpotent part $\widehat{\mathfrak{n}}_+$ of $\widehat{\mathfrak{so}}(m)$ is preserved under the automorphism $\mathrm{Ad}(\zeta)$. This implies that if $v_0 \in \mathcal{H}_{\omega_0}(\mathfrak{so}(m), \ell)$ is the highest weight vector for the representation π, then v_0 is also the highest weight vector for the representation $\tilde{\pi}$. Thus, it remains to determine the weight of the vector v_0

under the representation $\tilde{\pi}$. By (4.3), we get for H in the Cartan subalgebra of $\mathfrak{so}(m)$, $\tilde{\pi}(H,s)v_0 = \ell(s+\omega_1(H))v_0$. This completes the proof. □

Theorem 4.3. *There is a canonical isomorphism:* $H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes\ell}) \simeq \mathcal{V}^*_{\ell\omega_1}(\mathfrak{X}, \mathfrak{so}(m), \ell)$.

Proof. The essential idea of the proof is the same as in [12, Thm. 9.1]. Let δ be the generator of $\pi_1(\mathsf{SO}(m))$. It is easy to check that the element ζ defined in (4.2) lies in the component $L\mathsf{SO}(m)^\delta$. By Proposition 3.3, we get

$$\mathcal{M}^-_{\mathsf{Spin}(m)} = (\zeta^{-1}L_C(\mathsf{Spin}(m))\zeta)\backslash\Omega_{\mathsf{Spin}(m)} \ .$$

By Proposition 4.1, the line bundle $\mathcal{L}_\chi$ has a $\zeta^{-1}L_C(\mathsf{Spin}(m))\zeta$ linearization. In particular, the map $A(\alpha) \to \zeta^{-1}A(\alpha)\zeta$ extends to the Kac-Moody group $\widehat{L\mathsf{Spin}}(m)$. Now it is easy to check that this map is, on the level of the Lie algebra, given by the following: $\mathrm{Ad}(\zeta^{-1})\circ\iota$, where ι is the canonical embedding of $\mathfrak{so}(m)$ into $\widehat{\mathfrak{so}}(m)$. By [12, Prop. 7.4] and [43], the space of global sections $H^0(\mathcal{M}^-_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes\ell})$ is canonically isomorphic to the space of linear forms on $\mathcal{H}_{\omega_0}(\mathfrak{so}(m),\ell)$ that vanish on the image $\zeta^{-1}L_C(\mathsf{Spin}(m))\zeta$. By Lemma 4.2, this is same as the $L_C(\mathsf{Spin}(m))$-invariant sections $\mathcal{H}_{\ell\omega_1}(\mathfrak{so}(m),\ell)$. This, by definition, is the space of conformal blocks $\mathcal{V}^*_{\ell\omega_1}(\mathfrak{X},\mathfrak{so}(m),\ell)$. □

The following proposition compares global section of the line bundles on the moduli stack to the stack of the corresponding regularly stable moduli space.

Proposition 4.4. *Let $g \geq 2$, then there is a canonical isomorphism between*

$$H^0(M^{-,reg}_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell}) \simeq \mathcal{V}^*_{\ell\omega_1}(\mathfrak{X},\mathfrak{so}(m),\ell)$$

Proof. Applying Theorem 4.3, we are reduced to show that there are canonical identifications $H^0(M^{-,reg}_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell}) \simeq H^0(\mathcal{M}^-_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell})$. The codimension of the complement of the stack $\mathcal{M}^{-,reg}_{\mathrm{Spin}(m)}$ in $\mathcal{M}^-_{\mathrm{Spin}(m)}$ is at least 2 ([25, Theorem II.6]). Hence by Hartog's theorem $H^0(\mathcal{M}^-_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell}) \simeq H^0(\mathcal{M}^{-,reg}_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell})$. Now we observe that $\mathcal{M}^{-,reg}_{\mathrm{Spin}(m)}$ (respectively $M^{-,reg}_{\mathrm{Spin}(m)}$) has a presentation ([13]) of the form as a quotient stack $[R/\Gamma]$(respectively GIT quotient R/Γ), where R is a smooth scheme and Γ is a reductive group. Applying Proposition 4.1 in [43], we get that $H^0(\mathcal{M}^{-,reg}_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell})$ can be identified with Γ-variant section of $\mathcal{P}^{\otimes\ell}$ on the scheme R. The later is canonically identified with $H^0(\mathcal{M}^{-,reg}_{\mathrm{Spin}(m)}, \mathcal{P}^{\otimes\ell})$. This completes the proof. □

For a genus g curve with marked points $\mathfrak{X}$, let us denote: $N_g(\mathfrak{g},\vec{\lambda},\ell) := \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g},\ell)$. We sometimes omit the notation of the Lie algebra when it is clear. In the following, $\mathfrak{g} = \mathfrak{so}(2r+1)$, and we want to compute $N_g(\omega_1,1)$ and $N_g(\vec{\omega}_r,1)$, where $\vec{\omega}_r$ is an n-tuple of ω_r's for n even. Let σ denote the diagram automorphism $\omega_0 \leftrightarrow \omega_1$.

Lemma 4.5. *Let σ denote the Dynkin diagram automorphism that switches the 0-th node with the first node of the affine Dynkin diagram. Then $N_g(\vec{\sigma}\vec{\lambda},\ell) = N_g(\vec{\lambda},\ell)$, where $\vec{\sigma}\vec{\lambda} = (\sigma_1\lambda_1,\ldots,\sigma_n\lambda_n)$, $\sigma_i =$ either σ or 1, and $\sigma_1\cdots\sigma_n = 1$.*

Proof. The proof of the above follows from factorization (cf. Section 9.7), the genus 0 [28] result, and the fact that σ induces a permutation of $P_\ell(\mathfrak{so}(2r+1))$. □

Proposition 4.6. *For $n > 0$, let $\vec{\omega}_r^{(n)}$ denote a $2n$-tuple of ω_r's. Then $N_g(\vec{\omega}_r^{(n)}, 1) = 2^{2g+n-1}$.*

Proof. If $g = 0$, then the above is a result of N. Fakhruddin [24]. We will prove this using factorization (cf. Section 9.7) and induction on g. Therefore, suppose that the result holds for genus $g-1$ and all n. Then since the level one weights are precisely, ω_0, ω_1, and ω_r, and using Lemma 4.5, factorization and induction,

$$\begin{aligned} N_g(\vec{\omega}_r^{(n)}, 1) &= N_{g-1}(\omega_0, \omega_0; \vec{\omega}_r^{(n)}, 1) + N_{g-1}(\omega_1, \omega_1; \vec{\omega}_r^{(n)}, 1) + N_{g-1}(\omega_r, \omega_r; \vec{\omega}_r^{(n)}, 1) \\ &= 2N_{g-1}(\vec{\omega}_r^{(n)}, 1) + N_{g-1}(\vec{\omega}_r^{(n+1)}, 1) \quad \text{(By Lemma 4.5)} \\ &= 2 \cdot 2^{2(g-1)+n-1} + 2^{2(g-1)+n} = 2^{2g+n-1} \ . \end{aligned}$$

□

Now by factorization and Lemma 4.5 we get, $N_g(\omega_1, 1) = 2N_{g-1}(\omega_1, 1) + N_{g-1}(\omega_r, \omega_r, 1)$. By induction on g, the expression for $N_{g-1}(\omega_1, 1)$, and the above calculation it follows:

Corollary 4.7. $N_g(\omega_1, 1) = 2^{g-1}(2^g - 1)$.

4.1.1. *Proof of Proposition 3.10.* Combining Theorem 4.3, Proposition 4.4 and Proposition 4.6, along with the decomposition (3.5), we obtain Proposition 3.10. Reformulated in terms of the stack, we have the following.

Theorem 4.8. *For any $r \geq 1$, $\dim_{\mathbb{C}} H^0(\mathcal{M}^-_{\mathsf{Spin}(2r+1)}, \mathcal{P}) = 2^{g-1}(2^g - 1) = |\mathrm{Th}^-(C)|$. Moreover, the Pfaffian sections $\{s_\kappa \mid \kappa \in \mathrm{Th}^-(C)\}$ give a basis.*

4.2. **Oxbury-Wilson conjecture.** Let $\mathcal{P}$ be the line bundle which restricts on each component of $\mathcal{M}_{2r+1}$ to the ample generator of the Picard group (cf. (1.2)). We now prove a Verlinde formula for powers of $\mathcal{P}$.

Theorem 4.9. *Let*

$$N_g^0(\mathfrak{so}(2r+1), \ell) := (4(\ell + 2r - 1)^r)^{g-1} \sum_{\mu \in P_\ell(\mathsf{SO}(2r+1))} \prod_{\alpha > 0} \left(2 \sin \pi \frac{(\mu + \rho, \alpha)}{\ell + 2r - 1} \right)^{2-2g} .$$

where $P_\ell(\mathsf{SO}(2r+1))$ denotes the set of level ℓ weights of $\mathfrak{so}(2r+1)$ that exponentiate to a representation of the group $\mathsf{SO}(2r+1)$. Then

$$\dim_{\mathbb{C}} H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes \ell}) = 2N_g^0(\mathfrak{so}(2r+1), \ell) \ . \tag{4.4}$$

Proof. By Theorem 4.3, $H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes \ell}) \simeq \mathcal{V}^*_{\omega_0}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell) \oplus \mathcal{V}^*_{\ell\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$. Now the Verlinde formula tells us the following:

$$\begin{aligned} &\dim_{\mathbb{C}}(\mathcal{V}^*_{\omega_0}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)) = \\ &(4(\ell + 2r - 1)^r)^{g-1} \Big\{ \sum_{\mu \in P_\ell(\mathsf{SO}(2r+1))} \prod_{\alpha > 0} \big(2 \sin \pi \frac{(\mu + \rho, \alpha)}{\ell + 2r - 1} \big)^{2-2g} \end{aligned}$$

$$+\sum_{\mu\in P_\ell(\mathsf{SO}(2r+1))^c}\prod_{\alpha>0}\Big(2\sin\pi\frac{(\mu+\rho,\alpha)}{\ell+2r-1}\Big)^{2-2g}\Big\}\ ,$$

where $P_\ell(\mathsf{SO}(2r+1))^c := P_\ell(\mathfrak{so}(2r+1))\setminus P_\ell(\mathsf{SO}(2r+1))$ is the set of level ℓ weights that do not exponentiate to representations of $\mathsf{SO}(2r+1)$. Similarly

$$\begin{aligned}\dim_{\mathbb{C}}(\mathcal{V}^*_{\ell\omega_1}(\mathfrak{X},\mathfrak{so}(2r+1),\ell)) &= (4(\ell+2r-1)^r)^{g-1}\times\Big\{\\ &\sum_{\mu\in P_\ell(\mathsf{SO}(2r+1))}\mathrm{Tr}_{V_{\ell\omega_1}}(\exp 2\pi i\frac{\mu+\rho}{\ell+2r-1})\prod_{\alpha>0}\Big(2\sin\pi\frac{(\mu+\rho,\alpha)}{\ell+2r-1}\Big)^{2-2g}\\ &+\sum_{\mu\in P_\ell(\mathsf{SO}(2r+1))^c}\mathrm{Tr}_{V_{\ell\omega_1}}(\exp 2\pi i\frac{\mu+\rho}{\ell+2r-1})\prod_{\alpha>0}\Big(2\sin\pi\frac{(\mu+\rho,\alpha)}{\ell+2r-1}\Big)^{2-2g}\Big\}\ .\end{aligned}$$

It follows from [50, Lemmas 10.6 and 10.7] that

$$\mathrm{Tr}_{V_{\ell\omega_1}}\left(\exp\left(2\pi\sqrt{-1}\frac{\mu+\rho}{\ell+2r-1}\right)\right)=\begin{cases}1 & \mu\in P_\ell(\mathsf{SO}_{2r+1})\\ -1 & \text{otherwise.}\end{cases}$$

Using this, the proof follows by taking the sum of the expressions above. □

Remark 4.10. The formula (4.4) was conjectured in Oxbury-Wilson [56, Conj. 5.2]. Theorem 4.9 resolves this conjecture.

For any $r,s\geq 2$, the following result is proved in [56].

Lemma 4.11. $N^0_g(\mathfrak{so}(2r+1),2s+1)=N^0_g(\mathfrak{so}(2s+1),2r+1)$.

Proof of Corollary 1.2. Combine Lemma 4.11 and Theorem 4.9. □

Remark 4.12. The equality of dimensions in Corollary 1.2 also holds if either $r,s=1$. In this case, $\mathsf{SC}(3)=\mathsf{GL}(2)$, and so the moduli stack $\mathcal{M}_3$ is the disjoint union of the moduli stacks of rank 2 vector bundles with fixed trivial determinant and determinant $=\mathcal{O}_C(p)$. The Verlinde formula for these spaces is due to Thaddeus [64]. Also in this case, the equality of dimensions in Lemma 4.11 is mentioned in [56, Prop. 4.16].

5. Hecke transformations for orthogonal bundles

5.1. **The ι-transform on orthogonal bundles.** In this section we review a Hecke type elementary transformation called the ι-transform introduced by T. Abe [2]. This exchanges one orthogonal bundle with a choice of isotropic line at a point for another. As we shall see, this operation flips the Stiefel-Whitney class.

Let B be a scheme, $\mathcal{X}:=C\times B$, and $\pi:\mathcal{X}\to B$ the projection. Let $\sigma:B\to\mathcal{X}$ be a constant section of π. A *parabolic structure* on an orthogonal bundle $(\mathcal{E},q)\to\mathfrak{X}$ at σ is a choice of isotropic line subbundle of $\sigma^*\mathcal{E}$. If we let $\mathsf{OG}(\sigma^*\mathcal{E})\to B$ denote the bundle of Grassmannians of isotropic lines of $\sigma^*\mathcal{E}$, and $\tau\to\mathsf{OG}(\sigma^*\mathcal{E})$ the tautological line bundle,

then the data of an orthogonal bundle with parabolic structure on $\mathcal{X}$ may be summarized in the following diagram:

$$\begin{array}{ccc} (\mathcal{E},q) & \longrightarrow & \mathcal{X} \\ & & \pi \downarrow \uparrow \sigma \\ \sigma^*\mathcal{E} & \longrightarrow & B \end{array} \qquad\qquad \begin{array}{ccc} \tau & \longrightarrow & \mathsf{OG}(\sigma^*\mathcal{E}) \\ & & \downarrow \quad \circlearrowright s \\ s^*\tau \subset \sigma^*\mathcal{E} & \longrightarrow & B \end{array}$$

Let $\tau^\perp \to \mathsf{OG}(\sigma^*\mathcal{E})$ be the bundle orthogonal to τ in the quadratic form q, and let $\tau_1 = \sigma^*\mathcal{E}/s^*\tau^\perp$ be the quotient line bundle on B. Then we may define the locally free sheaf $\mathcal{E}^\flat$ by the *elementary transformation* (cf. [48]),

$$0 \longrightarrow \mathcal{E}^\flat \to \mathcal{E} \to \sigma_*(\tau_1) \longrightarrow 0\ . \tag{5.1}$$

Next, let $\mathcal{E}^\sharp = (\mathcal{E}^\flat)^*$. Since the normal bundle to $\sigma(B)$ is trivial and the orthogonal structure gives an isomorphism $\mathcal{E}^* \simeq \mathcal{E}$, dualizing (5.1) gives

$$0 \longrightarrow \mathcal{E} \to \mathcal{E}^\sharp \to \sigma_*(\tau_1^*) \longrightarrow 0\ . \tag{5.2}$$

Now q induces maps

$$q : \mathcal{E}^\sharp \otimes \mathcal{E}^\sharp \longrightarrow \mathcal{O}_\mathcal{X}(\sigma(B))\ ,\ q : \mathcal{E}^\flat \otimes \mathcal{E}^\sharp \longrightarrow \mathcal{O}_\mathcal{X}\ . \tag{5.3}$$

Consider the subsheaf $\mathcal{E}^\flat \hookrightarrow \mathcal{E}^\sharp$ coming from (5.2). Then $\mathcal{E}^\sharp/\mathcal{E}^\flat$ is a torsion sheaf supported on $\sigma(B)$, and along $\sigma(B)$ it is locally free of rank 2 with trivial determinant and an orthogonal structure. Since $\mathcal{E}/\mathcal{E}^\flat$ is isotropic, $\mathcal{E}^\sharp/\mathcal{E}^\flat \simeq \mathcal{E}/\mathcal{E}^\flat \oplus \sigma_*(\tau_1^*)$. Finally, we define $\mathcal{E}^\iota \subset \mathcal{E}^\sharp$ to be the kernel of the map $\mathcal{E}^\sharp \to \mathcal{E}/\mathcal{E}^\flat$. Equivalently, there is an exact sequence

$$0 \longrightarrow \mathcal{E}^\flat \to \mathcal{E}^\iota \to \sigma_*(\tau_1^*) \longrightarrow 0\ . \tag{5.4}$$

Then $\mathcal{E}^\iota$ inherits an orthogonal structure q^ι from (5.3). Moreover, the exact sequence (5.4) determines an isotropic line $s^\iota \subset \sigma^*(\mathcal{E}^\iota)$. Finally, from (5.1) and (5.4), the trivialization of $\det \mathcal{E}$ induces one for $\det \mathcal{E}^\iota$.

Definition 5.1. The ι-transform is the map: $(\mathcal{E},q,s) \mapsto (\mathcal{E}^\iota, q^\iota, s^\iota)$.

Remark 5.2. It is clear that the ι-transform is functorial with respect to base change. Moreover T. Abe [2] has shown that the ι-transform is an involution.

5.2. **The ι-transform on Clifford bundles.** We now show that the ι-transform sends a bundle in one component of $\mathcal{M}_{\mathsf{SO}(m)}$ to the other one. Fix a point $p \in C$, and recall that $C^* = C - \{p\}$. Let $\mathcal{M}^{par}_{\mathsf{Spin}(m)}$ denote the moduli stack of pairs $(\mathcal{S},\mathsf{P})$, where $\mathcal{S}$ is a $\mathsf{Spin}(m)$-bundle on C and P is a maximal parabolic subgroup of the fiber $\sigma^*\mathcal{S}$ preserving an isotropic line s in the fiber of the associated orthogonal bundle at p. Similarly, let $\mathcal{M}^{par}_{\mathsf{SO}(m)}$ be the moduli stack of tuples $(\mathcal{E},q,s)$, where $(\mathcal{E},q)$ is a rank m orthogonal bundle, and s is an isotropic line in the fiber $\mathcal{E}_p$. We then have a map $\mathcal{M}^{par}_{\mathsf{Spin}(m)} \to \mathcal{M}^{par}_{\mathsf{SO}(m)}$, $(\mathcal{S},\mathsf{P}) \mapsto (\mathcal{E},q,s)$. Forgetting the parabolic structure gives a morphism $\mathcal{M}^{par}_{\mathsf{Spin}(m)} \to \mathcal{M}_{\mathsf{Spin}(m)} \to \mathcal{M}^{+}_{\mathsf{SO}(m)}$.

We wish to define a morphism $\mathcal{M}^{par}_{\mathsf{Spin}(m)} \to \mathcal{M}^{-}_{\mathsf{Spin}(m)} \to \mathcal{M}^{-}_{\mathsf{SO}(m)}$. Associated to $(\mathcal{E},q,s)$ we obtain a new orthogonal bundle with isotropic line $(\mathcal{E}^\iota, q^\iota, s^\iota)$ defined in the previous

section. By Remark 5.2, this gives an involution of stacks: $\iota : \mathcal{M}^{par}_{\mathsf{SO}(m)} \to \mathcal{M}^{par}_{\mathsf{SO}(m)}$. This can be described explicitly in terms of transition functions as follows. First, since the result we wish to prove is topological it suffices to work locally in the analytic topology, and in fact at a closed point of B. We therefore let $\mathcal{S}$ be a spin bundle and $(\mathcal{E}, q)$ the associated orthogonal bundle; $\mathcal{S} = \mathsf{Spin}(\mathcal{E}, q)$. Let $\Delta \subset C$ be a disk centered at p, and $\sigma : \Delta \to \mathcal{S}$ a section. This gives a trivialization of $\mathcal{S}$ and a local frame $e_1, \dots, e_m$ for $\mathcal{E}$ on Δ with respect to which the quadratic structure is, say, of the form $q_{ij} = \delta_{i+j-1,m}$. Similarly, we may choose a section of $\mathcal{S}|_{C^*}$. Set $\Delta^* = C^* \cap \Delta$. Let $\hat{\varphi} : \Delta^* \to \mathsf{Spin}(\mathcal{E}, q)$ denote the transition function gluing the bundles $\mathcal{S}|_{\Delta}$ and $\mathcal{S}|_{C^*}$, and let $\varphi : \Delta^* \to \mathsf{SO}((\mathcal{E}, q)|_{\Delta^*})$ be the quotient transition function for $(\mathcal{E}, q)$. The transformed bundle $\mathcal{E}^\iota$ (cf. Section 5.1) is defined by modifying φ via $\zeta : \Delta^* \to \mathsf{SO}((\mathcal{E}, q)|_{\Delta^*})$, where ζ is as in (4.2). Write $z = \exp(2\pi i \xi)$. Then there is a well-defined lift $\widehat{\zeta} : \Delta^* \to \mathsf{SC}((\mathcal{E}, q)|_{\Delta^*})$, given by

$$\widehat{\zeta}(z) = \exp(\pi i \xi) \exp\left((\pi i \xi/2)(e_1 e_m - e_m e_1)\right) \ . \tag{5.5}$$

One checks that $\widehat{\zeta}$ is well-defined under $\xi \mapsto \xi + 1$, and the projection of $\widehat{\zeta}$ under the map $\mathsf{SC}(\mathcal{E}, q) \to \mathsf{SO}(\mathcal{E}, q)$ recovers ζ. Gluing the trivial SC-bundles over Δ and C^* via $\hat{\varphi}(z)\widehat{\zeta}(z)$, we define a new Clifford bundle $\mathcal{S}^\iota$. The associated orthogonal bundle (with transition function $\varphi(z)\zeta(z)$) coincides with $\mathcal{E}^\iota$. With this understood, the main observation is the following.

Proposition 5.3. *The ι-transform switches the Stiefel-Whitney class, i.e. if $\mathcal{E} \in \mathcal{M}^{+}_{\mathsf{SO}(m)}$ then $\mathcal{E}^\iota \in \mathcal{M}^{-}_{\mathsf{SO}(m)}$, and vice-versa.*

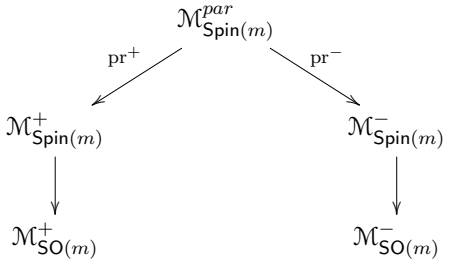
Proof. It suffices to check the spinor norm of $\mathcal{S}^\iota$. But from (5.5), $\mathrm{Nm}(\mathcal{S}^\iota)$ is a line bundle with transition function on Δ^* given by:

$$\mathrm{Nm}(\hat{\varphi}\widehat{\zeta}) = \exp(2\pi i \xi)\, \mathrm{Nm}(\hat{\varphi}(z))\, \mathrm{Nm}\left(\exp\left((\pi i \xi/2)(e_1 e_m - e_m e_1)\right)\right) = z \ ,$$

since $\hat{\varphi}(z)$ and $\exp\left((\pi i \xi/2)(e_1 e_m - e_m e_1)\right) \in \mathsf{Spin}(\mathcal{E}, q)$. Therefore, $\mathrm{Nm}(\mathcal{S}^\iota) \simeq \mathcal{O}_C(p)$. □

It will be useful to keep in mind the following diagram:

(5.6)



Here, pr^+ is the forgetful map that discards the parabolic structure, and pr^- is the ι-transform described above.

Remark 5.4. As in the case of $\mathsf{SO}(m)$ bundles, the ι-transform on $\mathsf{SC}(m)$ bundles is reversible. In particular, the fiber of pr^- is a copy of OG, and so it is connected and projective.

5.3. **The ι-transform and the Pfaffian bundle.** We first use the ι-transform to prove the following.

Proof of Proposition 3.5. For an orthogonal bundle, we have (cf. [55, Prop. 4.6]) $w_2(\mathcal{E}) \equiv h^0(C, \mathcal{E}\otimes\kappa) + mh^0(C,\kappa) \bmod 2$. Hence, for $\mathcal{E}\in\mathcal{M}^-_{\mathsf{SO}(m)}$, if either m or κ are even, then $h^0(C,\mathcal{E}\otimes\kappa)$ is odd. If both m and κ are odd, then by [55, Prop. 4.6], $h^0(C,\mathcal{E}\otimes\kappa)=0$ for generic $\mathcal{E}$. On the other hand, choose any theta characteristic κ_0 with $h^0(C,\kappa_0)\neq 0$. Write $m=2r+1$, and let: $\mathcal{E}_0 = (\kappa_0\otimes\kappa^{-1})^{\oplus r}\oplus\mathcal{O}_C\oplus(\kappa\otimes\kappa_0^{-1})^{\oplus r}$, with the obvious orthogonal structure. Then

$$h^0(C,\mathcal{E}_0\otimes\kappa) = (m-1)h^0(C,\kappa_0)+h^0(C,\kappa)\geq m-1\geq 2\ . \tag{5.7}$$

Pick an isotropic line of $\mathcal{E}_0$ at a point, and perform the elementary transformation in (5.1). Then by (5.7), $h^0(C,\mathcal{E}^\flat\otimes\kappa)\neq 0$, which by (5.4) implies that the ι-transform $\mathcal{E}=\mathcal{E}_0^\iota\in\mathcal{M}^-_{\mathsf{SO}(m)}$ has $h^0(C,\mathcal{E}\otimes\kappa)\neq 0$. This completes the proof. □

Recall the notation from Section 5.1. Choose $\kappa\in\mathrm{Th}(C)$ and denote the pull-back to $\mathcal{X}$ by $\mathrm{pr}^*\kappa$. Then we have the next result.

Proposition 5.5. *For a family of orthogonal bundles $(\mathcal{E},q)\to\mathcal{X}$, and $\mathcal{E}^\iota$ the ι-transform,*

$$\mathrm{Det}\, R\pi_*(\mathcal{E}^\iota\otimes\mathrm{pr}^*\kappa)\simeq\mathrm{Det}\, R\pi_*(\mathcal{E}\otimes\mathrm{pr}_1^*\kappa)\otimes(s^*\tau)^{\otimes 2}\ .$$

Proof. First, notice that the quadratic form gives an isomorphism $\tau_1\simeq s^*(\tau^*)$. Let $\kappa_\sigma = \sigma^*\mathrm{pr}_1^*\kappa$. Then using (5.1) and (5.4),

$$\mathrm{Det}\, R\pi_*(\mathcal{E}\otimes\mathrm{pr}_1^*\kappa)\simeq\mathrm{Det}\, R\pi_*(\mathcal{E}^\flat\otimes\mathrm{pr}_1^*\kappa)\otimes s^*(\tau^*)\otimes\kappa_\sigma\ ,$$

$$\mathrm{Det}\, R\pi_*(\mathcal{E}^\iota\otimes\mathrm{pr}_1^*\kappa)\simeq\mathrm{Det}\, R\pi_*(\mathcal{E}^\flat\otimes\mathrm{pr}_1^*\kappa)\otimes s^*\tau\otimes\kappa_\sigma\ .$$

The result follows. □

Corollary 5.6. *When pulled back to $\mathcal{M}^{par}_{\mathsf{Spin}(m)}$, the Pfaffian bundles on $\mathcal{M}^{\pm}_{\mathsf{Spin}(m)}$ for any theta characteristic κ are related by $(\mathrm{pr}^-)^*\mathcal{P}_\kappa\simeq(\mathrm{pr}^+)^*\mathcal{P}_\kappa\otimes s^*(\tau^*)$.*

Proof. From Propositions 3.4 and 5.5, we see that $[(\mathrm{pr}^-)^*\mathcal{P}_\kappa]^{\otimes 2}\simeq[(\mathrm{pr}^+)^*\mathcal{P}_\kappa\otimes s^*(\tau^*)]^{\otimes 2}$. The result follows from the fact that $\mathrm{Pic}(\mathcal{M}^{par}_{\mathsf{Spin}(m)})$ is torsion-free (cf. [45, Thm. 1.1]). □

5.4. **Geometric version of Theorem 4.3.** By Corollary 5.6 and Remark 5.4, we have

$$H^0(\mathcal{M}^-_{\mathsf{Spin}(m)},\mathcal{P}^{\otimes\ell}) = H^0(\mathcal{M}^{par}_{\mathsf{Spin}(m)},(\mathrm{pr}^-)^*\mathcal{P}^{\otimes\ell}) = H^0(\mathcal{M}^{par}_{\mathsf{Spin}(m)},(\mathrm{pr}^+)^*\mathcal{P}^{\otimes\ell}\otimes s^*(\tau^*)^{\otimes\ell})\ .$$

By the Borel-Weil theorem, the highest weight representation $V_{\ell\omega_1}$ of $\mathsf{Spin}(m)$ is given by the global sections of $(\tau^*)^{\otimes\ell}\to\mathsf{OG}$. It then follows as in [45, Thm. 1.2] or [58, Props. 6.5 and 6.6] that the space of sections of $\mathcal{P}^{\otimes\ell}\to\mathcal{M}^-_{\mathsf{Spin}(m)}$ is isomorphic to the space of conformal blocks. This gives an alternative proof of Theorem 4.3.

6. Hitchin connection for twisted spin bundles

6.1. **Higgs bundles.** Let $M^{\theta}_{\mathsf{SO}(m)}$ denote the coarse quasi-projective moduli space of semistable $\mathsf{SO}(m)$-*Higgs bundles* on C, and let $M^{\theta,reg}_{\mathsf{SO}(m)} \subset M^{\theta}_{\mathsf{SO}(m)}$ denote the regularly stable locus, i.e. pairs $(\mathcal{E},\theta)$, $\theta \in H^0(C, \mathrm{Ad}(\mathcal{E}) \otimes \omega_C)$, such that $\mathrm{Aut}(E,\theta) = Z(\mathsf{SO}(m))$, where $Z(\mathsf{SO}(m))$ denotes the center of $\mathsf{SO}(m)$. We assume in this section that C has genus $g \geq 2$. Then $M^{\theta,reg}_{\mathsf{SO}(m)}$ is smooth ([20, Proposition 2.14]), with complement of codimension ≥ 2 ([25, Theorem II.6]). Let

$$B(m) = \begin{cases} \bigoplus_{i=1}^{r} H^0(C, \omega_C^{\otimes 2r}) & m = 2r+1 \\ \bigoplus_{i=1}^{r-1} H^0(C, \omega_C^{\otimes 2r}) \oplus H^0(C, \omega_C^{r}) & m = 2r\ . \end{cases} \tag{6.1}$$

The Hitchin map $H : M^{\theta}_{\mathsf{SO}(m)} \to B(m)$ is a dominant, proper morphism. Away from the discriminant locus $\Delta \subset B_{\mathsf{SO}(m)}$, H is a smooth fibration by abelian varieties, and $M^{\theta,ns}_{\mathsf{SO}(m)} := M^{\theta}_{\mathsf{SO}(m)}\big|_{B(m)-\Delta} \subset M^{\theta,reg}_{\mathsf{SO}(m)}$ (see [33, 35], and also [21, Lemma 4.2]). There is an action of $\mathbb{C}^\times$ on $M^{\theta}_{\mathsf{SO}(m)}$, and H is equivariant with respect to multiplication on $H^0(C, \omega_C^{\otimes k})$ with weight k. We will need the following.

Proposition 6.1. *Let $V \to M^{\theta,reg}_{\mathsf{SO}(m)}$ be a flat bundle. If V restricted to a general fiber of h is trivial, then V is trivial.*

The proposition is an immediate consequence of the following.

Lemma 6.2. *If A is a general fiber of h, then the inclusion $A \hookrightarrow M^{\theta,reg}_{\mathsf{SO}(m)}$ induces a surjection $\pi_1(A) \to \pi_1(M^{\theta,reg}_{\mathsf{SO}(m)})$.*

Proof. Since Δ is codimension 1, inclusion induces a surjection $\imath_* : \pi_1(M^{\theta,ns}_{\mathsf{SO}(m)}) \to \pi_1(M^{\theta,reg}_{\mathsf{SO}(m)})$. Hence, we have the diagram:

$$\begin{array}{ccccccc} \pi_1(A) & \xrightarrow{a_*} & \pi_1(M^{\theta,ns}_{\mathsf{SO}(m)}) & \xrightarrow{h_*} & \pi_1(B(m)-\Delta) & \longrightarrow & \{1\} \\ & \searrow & \downarrow{\scriptstyle \imath_*} & & & & \\ & & \pi_1(M^{\theta,reg}_{\mathsf{SO}(m)}) & & & & \\ & & \downarrow & & & & \\ & & \{1\} & & & & \end{array}$$

Notice that every element of $\pi_1(B(m)-\Delta)$ is represented by the boundary of a transverse disk. More precisely, for $\gamma \in \pi_1(B(m)-\Delta)$ and $D \subset \mathbb{C}$ the unit disk, there is an embedding $D \hookrightarrow B(m)$, $D \cap \Delta = \{0\}$, and such that ∂D represents γ.

Next, consider the fiber in $M^{\theta,reg}_{\mathsf{SO}(m)}$ of h over $\{0\}$. Since the image in $B(m)$ of fibers contained in the critical locus of h lies in a set of codimension 2, we may assume without loss of generality that there is a regular point $x \in h^{-1}(0)$. There are therefore local smooth

coordinates about x with respect to which h is given by projection. By shrinking the disk if necessary, it follows that there is a local section $\sigma : D \to M^{\theta,reg}_{\mathsf{SO}(m)}$ of h. The image $\sigma(\partial D)$ is therefore a loop in $M^{\theta,ns}_{\mathsf{SO}(m)}$, contractible in $M^{\theta,reg}_{\mathsf{SO}(m)}$, that projects to ∂D. We conclude that for any $\gamma \in \pi_1(B(m) - \Delta)$ there is $\beta \in \pi_1(M^{\theta,ns}_{\mathsf{SO}(m)})$ such that $h_*(\beta) = \gamma$ and $\imath_*(\beta) = 1$.

The lemma now follows easily. For if $\alpha \in \pi_1(M^{\theta,reg}_{\mathsf{SO}(m)})$, then by surjectivity of $\imath_*$ there is $\tilde{\alpha} \in \pi_1(M^{\theta,ns}_{\mathsf{SO}(m)})$ such that $\imath_*(\tilde{\alpha}) = \alpha$. By the discussion in the previous paragraph, we can find $\beta \in \pi_1(M^{\theta,ns}_{\mathsf{SO}(m)})$ such that $h_*(\beta) = h_*(\tilde{\alpha})$ and $\imath_*(\beta) = 1$. But then $\beta^{-1}\tilde{\alpha}$ is in the kernel of h_*, and so is in the image of a_*, while at the same time it projects by $\imath_*$ to α. Therefore $\imath_* \circ a_*$ is surjective. □

The application of the previous result that we need is the following

Corollary 6.3. *Let $\chi : \pi_1(M^{reg}_{\mathsf{SO}(m)}) \to \{\pm 1\}$ be a nontrivial character with associated line bundle L_χ. Then the pullback of L_χ to $M^{\theta,reg}_{\mathsf{SO}(m)}$ is nontrivial on generic fibers of the Hitchin map.*

Proof. Let $M^{\theta,s}_{\mathsf{SO}(m)}$ be the locus of stable $\mathsf{SO}(m)$-Higgs bundles. By [25, Thm. II.6], the set $M^{\theta,hs}_{\mathsf{SO}(m)}$ consisting of stable Higgs bundles (E, θ) where E is not stable as a principal $\mathsf{SO}(m)$ bundle has codimension at least 2 in $M^{\theta,s}_{\mathsf{SO}(m)}$. Clearly, $M^{\theta,reg}_{\mathsf{SO}(m)} \backslash T^*M^{reg}_{\mathsf{SO}(m)}$ is contained in $M^{\theta,hs}_{\mathsf{SO}(m)}$. Moreover by [25, Theorem II.6], the complement of $M^{\theta,reg}_{\mathsf{SO}(m)}$ in $M^{\theta,s}_{\mathsf{SO}(m)}$ has codimension at least 4. This implies that the complement of $T^*M^{reg}_{\mathsf{SO}(m)} \subset M^{\theta,reg}_{\mathsf{SO}(m)}$ has positive codimension. Hence, there is a surjection $\pi_1(M^{\theta,reg}_{\mathsf{SO}(m)}) \to \pi_1(M^{reg}_{\mathsf{SO}(m)})$. The result then follows from Proposition 6.1. □

6.2. **A vanishing theorem.** We use the notation $Y = M^{-,reg}_{\mathsf{SO}(m)}$ and $\widetilde{Y} = p^{-1}(Y) \subset M^{-,reg}_{\mathsf{Spin}(m)}$ from Section 3.2. The following is a key assumption in the construction of the Hitchin connection.

Proposition 6.4. *With the notation above,*

(1) $H^0(\widetilde{Y}, T\widetilde{Y}) = \{0\}$;
(2) $H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) = \{0\}$.

Proof. (1) Since $\widetilde{Y} \to Y$ is étale it suffices to prove the vanishing of

$$H^0(\widetilde{Y}, T\widetilde{Y}) = \bigoplus_{\chi \in J_2(C)} H^0(Y, TY \otimes L_\chi) \ .$$

Fix a character χ, and suppose $\sigma \in H^0(Y, TY \otimes L_\chi)$ is nonzero. Then pulling back and contracting with the fibers gives a section f of the flat line bundle (also denoted L_χ) corresponding to χ on T^*Y. By Corollary 6.3, it immediately follows that σ vanishes if $\chi \neq 1$, for since L_χ is torsion it could have no sections on the generic fiber unless it is trivial there. For $\chi = 1$, the argument is exactly as in [34, p. 373]. Namely, by Hartog's theorem

f extends to a function on $M^{\theta}_{\mathsf{SO}(m)}$. Since the Hitchin map $H : M^{\theta}_{\mathsf{SO}(m)} \to B(m)$ is proper with connected fibers, it follows that f is the pull-back of a function g on $B(m)$. On the other hand, f is homogeneous of degree 1, whereas the minimal degree of homogeneity for the action on $B(m)$ is 2 (see the discussion following (6.1)). Since H is equivariant we conclude that f, and hence also σ, must vanish.

For (2), we first note that provided $g \geq 2$, the complement of $\widetilde{Y} \subset M^{-}_{\mathsf{Spin}(m)}$ is of codimension at least 3. This follows from the fact that the complement of Y in $M^{-}_{\mathsf{SO}(m)}$ of codimension at least 3 if $g \geq 2$ and the map $p : M^{-}_{\mathsf{Spin}(m)} \to M^{-}_{\mathsf{SO}(m)}$ is finite and dominant (see [25, Thm. II.6], [44, Appendix] and recall $m \geq 5$). The proof of part (2) now follows from Scheja's theorem [60]. To prove the result, it then suffices by Scheja's theorem to prove that $H^1(M^{-}_{\mathsf{Spin}(m)}, \mathcal{O}) = \{0\}$. For this we closely follow the proof in [42, Thm. 2.8]. First, observe that by [13, Lemma 7.3] (see also [59]) the moduli space $M^{-}_{\mathsf{Spin}(m)}$ is a good quotient of a projective scheme R by a reductive group Γ. From [18] it then follows that $M^{-}_{\mathsf{Spin}(m)}$ is Cohen-Macaulay, normal and has rational singularities. Let ω_M denote the dualizing sheaf.

Let $\widehat{\mathcal{E}} \to C \times R$ be the universal bundle. By construction of the GIT quotient it follows that the adjoint vector bundle $\mathrm{Ad}(\widehat{\mathcal{E}})$ descends to a vector bundle on $C \times \widetilde{Y}$, which we also denote by $\mathrm{Ad}(\widehat{\mathcal{E}})$. Since $\widetilde{Y}$ is an étale cover of $M^{-,reg}_{\mathsf{SO}(m)}$, deformation theory tells us that $T_{[\mathcal{E}]}\widetilde{Y}$ can be identified with $H^1(C, \mathrm{Ad}(\mathcal{E}))$. Moreover, since $\mathcal{E}$ is regularly stable, $H^0(C, \mathrm{Ad}(\mathcal{E})) = \{0\}$. In particular, it follows from the definition of determinant of cohomology that $\mathrm{Det}(R\pi_* \operatorname{Ad} \widehat{\mathcal{E}})^*|_{\widetilde{Y}} = \omega_M|_{\widetilde{Y}}$. But $\mathrm{Det}(R\pi_* \operatorname{Ad} \widehat{\mathcal{E}})$ extends to an invertible sheaf on the entire moduli space $M^{-}_{\mathsf{Spin}(m)}$, and hence in particular it is reflexive. Furthermore, dualizing sheaves on Cohen-Macaulay normal varieties are also reflexive. Since the complement of $\widetilde{Y}$ is codimension ≥ 2, then by [42, Lemma 2.7] we get that ω_M is locally free, and hence by definition $M^{-}_{\mathsf{Spin}(m)}$ is Gorenstein. Now $\mathrm{Det}(R\pi_* \operatorname{Ad} \widehat{\mathcal{E}})$ is ample and the Picard group of $M^{-}_{\mathsf{Spin}(m)}$ is isomorphic to $\mathbb{Z}$ ([13]). It follows from Serre duality [31, Cor. III.7.7] and the Grauert-Riemenschneider vanishing theorem [30] that $H^i(M^{-}_{\mathsf{Spin}(m)}, \mathcal{O}) = \{0\}$ for $i > 0$. □

6.3. **The Hitchin connection.** Let $\mathcal{C} \to B$ be a smooth family of genus g curves, B smooth. Let $\pi : \underline{M}^{-,reg}_{\mathsf{Spin}(m)} \to B$ denote the universal moduli space of regularly stable twisted $\mathsf{Spin}(m)$ bundles on the fibers of $\mathcal{C}$, with universal Pfaffian bundle $\underline{\mathcal{P}}$. Then the direct image sheaf $\pi_*\underline{\mathcal{P}}^{\ell}$ is a holomorphic vector bundle over B with fiber $H^0(M^{-,reg}_{\mathsf{Spin}(m)}, \mathcal{P}^{\otimes \ell})$. We wish to find a connection on the projective bundle $\mathbb{P}(\pi_*\underline{\mathcal{P}}^{\otimes \ell}) \to B$. Following the method of Hitchin [34], the connection is constructed as a "heat operator" on the smooth sections of $\underline{\mathcal{P}}^{\otimes \ell} \to \underline{M}^{-,reg}_{\mathsf{Spin}(m)}$. As noted in that reference (see also [3]), the procedure can be applied to the open moduli space of (regularly) stable points.

Given $[\mathcal{E}] \in \widetilde{Y} \subset M^{-,reg}_{\mathsf{Spin}(m)}$, recall that $T\widetilde{Y}|_{[\mathcal{E}]} \simeq H^1(C, \operatorname{Ad}\mathcal{E})$. By Serre duality,

$$H^0(C, \operatorname{Ad}\mathcal{E} \otimes \omega_C) \simeq H^1(C, \operatorname{Ad}\mathcal{E})^* \ .$$

Combine this with the cup product:

$$H^1(C, TC) \otimes H^0(C, \operatorname{Ad}\mathcal{E} \otimes \omega_C) \to H^1(C, \operatorname{Ad}\mathcal{E})$$

and the identification above to obtain a map: $\tau : H^1(C, TC) \to H^0(\widetilde{Y}, S^2T\widetilde{Y})$. Let $\mathcal{D}^i(\mathcal{P}^{\otimes\ell})$ denote the sheaf of differential operators of order i on $\widetilde{Y}$. Given a nonzero section $s \in H^0(\widetilde{Y}, \mathcal{P}^{\otimes\ell})$, evaluation on s gives a length 2 complex: $\mathcal{D}^i(\mathcal{P}^{\otimes\ell}) \to \mathcal{P}^{\otimes\ell}$, from which we obtain a hypercohomology group $\mathbb{H}^1_s(\widetilde{Y}, \mathcal{D}^1(\mathcal{P}^{\otimes\ell}))$. Let $\delta : H^0(\widetilde{Y}, S^2T\widetilde{Y}) \to \mathbb{H}^1_s(\widetilde{Y}, \mathcal{D}^1(\mathcal{P}^{\otimes\ell}))$ be given by the coboundary associated to

$$0 \longrightarrow \mathcal{D}^1(\mathcal{P}^{\otimes\ell}) \longrightarrow \mathcal{D}^2(\mathcal{P}^{\otimes\ell}) \longrightarrow S^2T\widetilde{Y} \longrightarrow 0 \ . \tag{6.2}$$

The main result is the following.

Theorem 6.5. *With the above notation*

(1) *Given a deformation $[\mu] \in H^1(C, TC)$, let $\dot{I}$ denote the variation of the almost complex structure on $\widetilde{Y}$. The association of the class $A(\dot{I}, s) := \dfrac{\delta\tau[\mu]}{2i(\ell + g^\vee)} \in \mathbb{H}^1_s(\widetilde{Y}, \mathcal{D}^1(\mathcal{P}^{\otimes\ell}))$ to each section $s \in H^0(\widetilde{Y}, \mathcal{P}^{\otimes\ell})$ defines a connection on $\mathbb{P}(\pi_*\underline{\mathcal{P}}^{\otimes\ell}) \to B$.*
(2) *The connection commutes with the action of $J_2(C)$.*
(3) *The connection is projectively flat.*
(4) *Under the identification (1.3), the connection agrees with the TUY connection.*

Proof. The existence (1) of the connection follows if we show that the hypotheses (i) and (ii) of [34, Thm. 1.20] are satisfied. Condition (i) follows trivially from the vanishing result Proposition 6.4 above. For (ii), we must show that $-iA(\dot{I}, s)$ projects to the Kodaira-Spencer class under the map $\sigma : \mathbb{H}^1_s(\widetilde{Y}, \mathcal{D}^1(\mathcal{P}^{\otimes\ell})) \to H^1(\widetilde{Y}, T\widetilde{Y})$. This follows exactly as in [34, p. 365], where in [34, Lemma 2.13] one replaces $\operatorname{End}_0 E$ with $\operatorname{Ad} E$, and in [34, Prop. 2.16] one uses the cup product defined above.

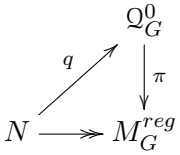
Item (2) follows by the same argument as in [16, Cor. 4.2].

The flatness (3) proved in [34, Thm. 4.9] is a consequence of two other results of that paper: the fact that certain trace functions Poisson commute ([34, Prop. 4.2]) follows in our case from the integrable system structure for orthogonal bundles (cf. [33]); and by (2) above the version of [34, Prop. 4.4] that we need states that the map

$$f : H^1(C, TC) \longrightarrow H^0(Y, S^2TY) \quad , \quad f([\mu])(\alpha, \alpha) = \int_C (\alpha, \alpha)\, \mu$$

is an isomorphism. This also follows as in [34, bottom of p. 373]. Indeed, for $(G^{ij}) \in H^0(Y, S^2T\widetilde{Y})$, then by the same argument as in the proof of Proposition 6.4 (1), the pull-back of (G^{ij}) via the Hitchin map extends as a function on $M^\theta_{\mathsf{SO}(m)}$, and thus descends to a function on $B(m)$, homogeneous of degree 2. The homogeneous degree 2 functions on $B(m)$ are identified in (6.1) with $H^0(C, \omega_C)^* \simeq H^1(C, TC)$.

Finally, (4) follows from the proof in [44] where the only properties of the Hitchin connection used are the ones stated above and the uniformization theorem. For the case of (simply connected) G-bundles, Laszlo first considers the uniformization $\pi : \mathcal{Q}_G = LG/L^+G \to \mathcal{M}_G$ of the moduli stack of G-bundles over a curve C and restricts to the regularly stable locus $\mathcal{Q}^0_G$ of $\mathcal{Q}_G$. The ample generator $\mathcal{P}$ of the Picard group of $\mathcal{M}_G$ pulls back to give a line bundle $\mathcal{L}_\chi$, where χ is the character of the affine fundamental weight ω_0. The Sugawara construction then provides a second order differential operator T (see [44, eq. (8.8)]) on $\mathcal{L}_\chi^{\otimes \ell}$ (see Section 4.1 for notation). Following [23], Laszlo takes an étale quasi-section $r : N \twoheadrightarrow M_G^{reg}$ and $q : N \to \mathcal{Q}^0_G$ of the map $\pi : \mathcal{Q}^0_G \to M_G^{reg}$



and defines an action of the differential operator T on the line bundle $r^*\mathcal{P}^{\otimes \ell}$(also isomorphic to $q^*\mathcal{L}_\chi^{\otimes \ell}$). Once this is done, then the symbol of the "heat operator" in the Hitchin connection is shown to be equal to the operator T coming from the Sugawara construction (see [44, Sec. 8.14]).

In the present situation for $\mathcal{M}^-_{\mathrm{Spin}(m)}$, uniformization (Proposition 3.3) and the existence of étale quasi-section follows from [13, 23]. Further the pull back of the Pfaffian line bundle $\mathcal{P}$ on $\mathcal{Q}_{\mathrm{Spin}(m)}$ is also given by the same character χ of the affine fundamental weight ω_0 (Section 4.1). Thus the set-up is exactly as in [44].

□

We will call the projective connection constructed in Theorem 6.5 the *Hitchin connection*. The main consequence of the existence of a Hitchin connection is the following.

Proposition 6.6. *The Pfaffian sections s_κ are projectively flat with respect to the Hitchin connection.*

Proof. The action of $J_2(C)$ commutes with the projective heat operator. Hence, the Hitchin connection preserves the spaces $H^0(M^{reg}_{\mathsf{SO}(m)}, \mathcal{P}_\kappa)$. Now these one dimensional spaces are spanned by Pfaffian sections, making them projectively flat. □

Remark 6.7. If we consider the isomorphism of $\mathrm{SC}(3) \simeq \mathsf{GL}(3)$, then $\mathcal{M}^-_{\mathrm{Spin}(3)}$ is just the stack of rank two bundles on a curve with determinant $\mathcal{O}_C(p)$. The corresponding moduli space in this case is smooth (GIT stability and GIT semistability coincide) and the Hitchin connection is constructed in [34]. The case $\mathcal{M}^-_{\mathsf{Spin}(4)}$ can be defined and handled similarly using the isomorphism of $\mathsf{Spin}(4)$ with $\mathsf{SL}(2) \times \mathsf{SL}(2)$.

7. Fock space realization of level one modules

In this section, following work of I. Frenkel [27] and Kac-Petersen [37], we first recall the explicit construction of level one modules of $\widehat{\mathfrak{so}}(2d+1)$ using infinite dimensional Clifford algebras. We also give explicit expressions for the space of invariants.

7.1. **Clifford algebra.** Let W be a vector space (not necessarily finite dimensional) with a nondegenerate bilinear form $\{\ ,\ \}$. Let $T(W)$ denote the tensor algebra of W, and define the *Clifford algebra* $Cl(W)$ to be the quotient of $T(W)$ by the ideal generated by elements of the form $v\otimes w + w\otimes v - \{v,w\}$. Let $W = W^+ \oplus W^- \oplus \mathbb{C}\cdot e^0$ be a quasi-isotropic decomposition of W such that e^0 is either orthonormal with respect to $W^\pm$, or zero. Then the Clifford algebra $Cl(W)$ acts on $\bigwedge W^-$ by setting $w^+\cdot 1 = 0$ for all $w^+\in W^+$ and letting W^- act by wedge product on the left. If $e^0\neq 0$ and $v\in \bigwedge^p W^-$, then we set $\sqrt{2}e^0\cdot v = (-1)^p v$.

7.2. **Level one modules.** Now suppose $W = W_d$ is $(2d+1)$-dimensional. We choose an ordered basis $\phi^1,\dots\phi^r,\phi_0=\phi^0,\phi^{-r},\dots,\phi^{-1}$ of W_d such that $\{\phi^a,\phi^b\} = \delta_{a+b,0}$. Define operators $E^i_j(\phi^k) := \delta_{j,k}\phi^i$, and set $B^i_j := E^i_j - E^{-j}_{-i}$. It follows that elements of $\mathfrak{so}(2d+1)$ are of the form B^i_j (cf. [32, 29]).

For $h\in\{0,\frac{1}{2}\}$, let $W_d^{\mathbb{Z}+h} := W_d\otimes t^h\mathbb{C}[t,t^{-1}]$. We extend the bilinear form on W_d to $W_d^{\mathbb{Z}+h}$ by setting $\{w_1(a_1),w_2(a_2)\} := \{w_1,w_2\}\delta_{a_1+a_2,0}$, where $w(a) = w\otimes t^a$. As above, choose a quasi-isotropic decomposition:

$$W_d^{\mathbb{Z}+h} = \begin{cases} W_d^{\mathbb{Z}+h,+}\oplus W_d^{\mathbb{Z}+h,-}\oplus\mathbb{C}\cdot e^0 & ,\ \text{if } h=1/2\ ,\\ W_d^{\mathbb{Z}+h,+}\oplus W_d^{\mathbb{Z}+h,-} & ,\ \text{if } h=0\ , \end{cases}$$

where $e^0=\phi_0(0)$. Similarly, $W_d^{\mathbb{Z}+h,\pm}$ is given by the following:

- If $h=0$, then $W_d^{\mathbb{Z}+h,\pm} := W_d\otimes t^{\pm1}\mathbb{C}[t^{\pm1}]\oplus W_d^\pm\otimes t^0$;
- If $h=\frac{1}{2}$, then $W_d^{\mathbb{Z}+h,\pm} := W_d\otimes t^{\pm\frac{1}{2}}\mathbb{C}[t^{\pm1}]$.

We define the *normal ordering* $\mathbf{:}\ \mathbf{:}$ for products in $Cl(W_d^{\mathbb{Z}+h})$ as follows: For any element $w_1(a_1)$ and $w_2(a_2)$ in $W_d^{\mathbb{Z}+h}$, we define

$$\mathbf{:}w_1(a_1)w_2(a_2)\mathbf{:} = \begin{cases} -w_2(a_2)w_1(a_1) & \text{if } a_1>0>a_2\\ \frac{1}{2}(w_1(a_1)w_2(a_2)-w_2(a_2)w_1(a_1)) & \text{if } a_1=a_2=0\\ w_1(a_1)w_2(a_2) & \text{otherwise.}\end{cases} \tag{7.1}$$

For $X\in\mathfrak{so}(2d+1)$, we denote $X(m) := X\otimes t^m$. Now for any i and j, we can define an action of $B^i_j(m)$ on $\bigwedge W_d^{\mathbb{Z}+h,-}$ by the following formula:$B^i_j(m)w := \sum_{a+b=m}\mathbf{:}\phi^i(a)\phi_j(b)\mathbf{:}\,w$, where the action on w is given by Clifford multiplication. Then we have the following important result.

Proposition 7.1 (Frenkel [27], Kac-Petersen [37])**.** *The above action $B^i_j(m)$ gives an isomorphism of the following $\widehat{\mathfrak{so}}(2d+1)$-modules at level one.*

- $\mathcal{H}_{\omega_0}(\mathfrak{so}(2d+1),1)\oplus\mathcal{H}_{\omega_1}(\mathfrak{so}(2d+1),1)\simeq\bigwedge W_d^{\mathbb{Z}+\frac{1}{2},-}$;
- $\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1),1)\simeq\bigwedge W_d^{\mathbb{Z},-}$.

7.3. **Clifford multiplication and the invariant form.** In this section, we give explicit expressions for conformal blocks in $\mathcal{V}^*_{\omega_0,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ and $\mathcal{V}^*_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ in terms of Clifford multiplication. In the following, let W_d be as in Section 7.2.

7.3.1. *The case* $\vec{\Lambda} = (\omega_0, \omega_d, \omega_d)$. From representation theory it follows that the conformal block $\mathcal{V}^*_{\omega_0,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ is a subspace of $\mathrm{Hom}_{\mathfrak{so}(2d+1)}(V_{\omega_d} \otimes V_{\omega_d}, \mathbb{C})$. But since both spaces are 1-dimensional they are isomorphic. Since we know that V_{ω_d} is isomorphic as an $\mathfrak{so}(2d+1)$-module to $\bigwedge W_d^-$, then taking the opposite Borel we can express it as $\bigwedge W_d^+$. Hence, the invariant bilinear form (unique up to constants) is given by

$$B(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}, \phi^{j_1} \wedge \cdots \wedge \phi^{j_q}) := \begin{cases} \prod_{a=1}^{p} \{\phi_{i_a}, \phi^{j_a}\} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

7.3.2. *The case* $\vec{\Lambda} = (\omega_1, \omega_d, \omega_d)$. As in the previous case, we know that the conformal block $\mathcal{V}^*_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ is a subspace of $\mathrm{Hom}_{\mathfrak{so}(2d+1)}(V_{\omega_1} \otimes V_{\omega_d} \otimes V_{\omega_d}, \mathbb{C})$. The $\mathfrak{so}(2d+1)$-module V_{ω_1} appears with multiplicity one (see Kempf-Ness [39]) in the tensor product of module $V_{\omega_d} \otimes V_{\omega_d}$, thus we have

$$\mathrm{Hom}_{\mathfrak{so}(2d+1)}(W_d \otimes \bigwedge W_d^- \otimes \bigwedge W_d^+, \mathbb{C}) \simeq \mathbb{C} \ .$$

We will denote a nonzero element of the left hand side above as $\langle \widetilde{\Psi} \,|$. Consider the Clifford multiplication map from $m : W_d \otimes \wedge W_d^- \to \wedge W_d^-$. Then we define

$$\langle \widetilde{\Psi} \mid a \otimes v \otimes w^* \rangle := B(m(a \otimes v), w^*) \ .$$

We will show that $\langle \widetilde{\Psi} \,|$ is a nonzero element of $\mathcal{V}_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ and hence it is unique up to constants. First we prove that $\langle \widetilde{\Psi} \,|$ is $\mathfrak{so}(2d+1)$-invariant.

There is an isomorphism $\wedge^2 W_d \simeq \mathfrak{so}(2d+1)$. Any $X \in \mathfrak{so}(2d+1)$ may be regarded as an element of the Clifford algebra as follows: for $a, b \in W_d$, we get an element of the Clifford algebra $a \cdot b - \frac{1}{2}\{a, b\}$. First we show that the Clifford multiplication map m defined above is $\mathfrak{so}(2d+1)$-invariant; i.e $X \cdot m(a, w) = m(X \cdot a, w) + m(a, X \cdot w)$, where $a \in W_d$ and $w \in \bigwedge W_d^-$. Without loss of generality assume that X is of the form $a \cdot b - \frac{1}{2}\{a, b\}$. By a direct calculation we see that

$$\begin{aligned} m((X \cdot a) \otimes w) + m(a \otimes (X \cdot w)) &= (((ab - \tfrac{1}{2}\{a,b\})v) \cdot w + v \cdot \big((ab - \tfrac{1}{2}\{a,b\}) \cdot w\big) \\ &= (a \cdot b - \tfrac{1}{2})v \cdot w = X \cdot (v \cdot w) \ . \end{aligned}$$

Thus the Clifford multiplication map m is $\mathfrak{so}(2d+1)$-invariant. By a direct calculation, we get the following:

$$\begin{aligned} \langle \widetilde{\Psi} \cdot X \mid a \otimes v \otimes w^* \rangle &:= \langle \widetilde{\Psi} \mid Xa \otimes v \otimes w^* \rangle + \langle \widetilde{\Psi} \mid a \otimes Xv \otimes w^* \rangle + \langle \widetilde{\Psi} \mid a \otimes v \otimes Xw^* \rangle \\ &= B(Xm(a \otimes v) \otimes w^*) + B(m(a \otimes v) \otimes Xw^*) \\ &= B \cdot X(m(a \otimes v) \otimes w^*) \\ &= 0 \qquad \text{(since } B \text{ is } \mathfrak{so}(2d+1) \text{ invariant).} \end{aligned}$$

This shows that $\langle\widetilde{\Psi}\mid$ is $\mathfrak{so}(2d+1)$-invariant.

It will actually be more convenient to express $\langle\widetilde{\Psi}\mid$ in terms of an $\mathfrak{so}(2d+1)$-equivariant map $f : \bigwedge W_d^- \otimes \bigwedge W_d^+ \to W_d^*$, that will be unique up to constants; the relationship is $\langle\widetilde{\Psi}\mid a\otimes v\otimes w^*\rangle = f(v\otimes w^*)(a)$. We want to write f explicitly with respect to the given choice of basis. Let $\mathcal{I}_p = (1\le i_1<\cdots<i_p\le d)$ be a set of p tuples of distinct ordered integers from the set $\{1,\dots,d\}$ and similarly let $\mathcal{J}_q = ((1\le j_1<\cdots<j_q\le d)$ be a set of q tuples of distinct ordered integers. We are now ready to define the function f. This will be defined in several steps. First of all $f(v,w)=0$ if $v\in\wedge^p W_d^-$ and $w\in\wedge^q W_d^+$ and $|p-q|>1$.

7.3.3. *Case I, p=q.* This is divided into the following subcases. If $\mathcal{I}_p\neq\mathcal{J}_p$, then we declare $f(\phi_{i_1}\wedge\cdots\wedge\phi_{i_p},\phi^{j_1}\wedge\cdots\wedge\phi^{j_p}) := 0$. If $\mathcal{I}_p=\mathcal{J}_p$, then we define $f(\phi_{i_1}\wedge\cdots\wedge\phi_{i_p},\phi^{i_1}\wedge\cdots\wedge\phi^{i_p})$ is up to a constant equal to $\{\phi_0,-\}$.

7.3.4. *Case II, q=p-1.* Then $f(\phi_{i_1}\wedge\cdots\wedge\phi_{i_p},\phi^{j_1}\wedge\cdots\wedge\phi^{j_{p-1}})$ is up to a constant equal to

$$\begin{cases}\{\phi_{i_k},-\} & \text{if } \mathcal{J}_{p-1}\cup\{i_k\}=\mathcal{I}_p\\ 0, & \text{otherwise.}\end{cases}$$

7.3.5. *Case III, q=p+1.* Then $f(\phi_{i_1}\wedge\cdots\wedge\phi_{i_p},\phi^{j_1}\wedge\cdots\wedge\phi^{j_{p+1}})$ is up to a constant equal to

$$\begin{cases}\{\phi^{j_k},-\} & \text{if } \mathcal{I}_p\cup\{j_k\}=\mathcal{J}_{p+1}\\ 0, & \text{otherwise.}\end{cases}$$

This shows that f, and hence also $\langle\widetilde{\Psi}\mid$, is nonzero and $\mathfrak{so}(2d+1)$-invariant.

8. Highest weight vectors for branching of basic modules

We give an explicit description of highest weight vectors in the branching rule in "Kac-Moody" form i.e. as product of operators from the affine Lie algebra acting on the level one representations. Our guideline is the paper of Hasegawa [32].

8.1. **Tensor products.** Let W_s be a $(2s+1)$-dimensional $\mathbb{C}$-vector space with a nondegenerate bilinear form $\{,\}$, and let $\{e_p\}_{p=-s}^{s}$ be an orthonormal basis of W_s. Let $\phi^1,\dots,\phi^s,\phi^0,\phi^{-s},\dots,\phi^{-1}$ be an ordered quasi-isotropic basis of W_s. The tensor product of $W_d = W_r\otimes W_s$ carries a nondegenerate symmetric bilinear form $\{,\}$ given by the product of the forms on W_r and W_s. Clearly the elements $\{e_{j,p} := e_j\otimes e_p \mid -r\le j\le r \text{ and } -s\le p\le s\}$ form an orthonormal basis of W_d. By $(j,p)>0$, we mean $j>0$ or $j=0, p>0$. Set

$$\phi^{j,p} = \frac{1}{\sqrt{2}}(e_{j,p}-\sqrt{-1}e_{-j,-p})\ , \qquad \phi^{-j,-p} = \frac{1}{\sqrt{2}}(e_{j,p}+\sqrt{-1}e_{-j,-p})\ ,$$

for $(j,p)>0$. The form $\{,\}$ on W_d is given by the formula $\{\phi^{j,p},\phi^{-k,-q}\} = \delta_{j,k}\delta_{p,q}$, for $-r\le j,k\le r$, $-s\le p,q\le s$. Let as before $W_d^{\pm} = \bigoplus_{(j,p)>0}\mathbb{C}\cdot\phi^{\pm j,\pm p}$ and $\phi^{0,0}=e_{0,0}$. The quasi-isotropic decomposition of W_d is given by $W_d = W_d^+\oplus W_d^-\oplus\mathbb{C}\cdot\phi^{0,0}$.

Define the operators $E^{j,p}_{k,q}$ by the formula $E^{j,p}_{k,q}(\phi^{i,l}) = \delta_{i,k}\delta_{l,q}\phi^{j,p}$. We get a matrix in $\mathfrak{so}(2d+1)$ by the formula $B^{j,p}_{k,q} = E^{j,p}_{k,q} - E^{-k,-q}_{-j,-p}$. Clearly the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2d+1)$ is generated by the diagonal matrices $B^{j,p}_{j,p}$ for $(j,p) > 0$. The dual basis is denoted by $L_{j,p}$. Hence $\mathfrak{h}^* = \oplus_{(j,p)>0}\mathbb{C}\cdot L_{j,p}$.

The tensor product $W_d = W_r \otimes W_s$ gives the embedding (1.5). If B^i_j be an element of $\mathfrak{so}(2r+1)$, then the action of $B^i_j(m)$ on $W_d^{\mathbb{Z}+h}$ is given by

$$L(B^i_j(m)) = \sum_{q=-s}^{s} \sum_{a+b=m} {:}\phi^{i,q}(a)\phi_{j,q}(b){:} \, .$$

Similarly for B^i_j is an element of $\mathfrak{so}(2s+1)$, then the action of $B^i_j(m)$ is given by

$$R(B^i_j(m)) = \sum_{p=-r}^{r} \sum_{a+b=m} {:}\phi^{p,i}(a)\phi_{p,j}(b){:} \, .$$

8.2. **Notation for weights.** The Cartan algebra $\mathfrak{h}$ of $\mathfrak{so}(2r+1)$ is generated by elements of the form B^i_i for $1 \le i \le r$. Let L_i denote the dual of B^i_i. The fundamental weights of $\mathfrak{so}(2r+1)$ are given by $\omega_i = \sum_{k=1}^{i} L_i$ for $1 \le i \le r-1$ and $\omega_r = \frac{1}{2}(L_1 + \cdots + L_r)$. Let us denote by $\mathcal{Y}_r$ the set of Young diagrams with at most r rows. Any integral dominant weight λ of $\mathfrak{so}(2r+1)$ is of the form $\lambda = \sum_{i=1} a_i\omega_i$, $a_i \ge 0$ for all i.

(1) If a_r is even, then the representation λ induces a representation of the group $\mathsf{SO}(2r+1)$. By using the expression of ω_i in terms of L_i's, we get $\lambda = \sum_i b_i L_i$, and $b_1 \ge \cdots \ge b_r$, give a Young diagram in $\mathcal{Y}_r$.
(2) If a_r is odd, then we can rewrite $\lambda = \lambda' + \omega_r$. Then the coefficient of ω_r in λ' is even and by repeating the same process for λ', we can write $\lambda = Y + \omega_r$, where Y is an element of $\mathcal{Y}_r$.

The group of affine Dynkin-diagram automorphisms $\mathbb{Z}/2$ acts on the set of level $2s+1$ weights $P_{2s+1}(\mathfrak{so}(2r+1))$ by interchanging the affine fundamental weight ω_0 with ω_1. Let $\lambda = \sum_{i=1}^{r} a_i\omega_i$ and σ be the generator of $\mathbb{Z}/2$, then

$$\sigma(\lambda) := (2s+1-(a_1+2(a_2+\cdots+a_{r-1})+a_r))\omega_1 + a_2\omega_2 + \cdots + a_r\omega_r \, . \tag{8.1}$$

Let $\mathcal{Y}_{r,s}$ denote the set of Young diagrams with at most r rows and s columns. Then the orbits of the group action with the cardinality are given below [56]:

- $Y \in \mathcal{Y}_{r,s}$ and the orbit length is 2.
- $Y + \omega_r$, where $Y \in \mathcal{Y}_{r,s-1}$ and the orbit length is 2.
- $Y + \omega_r$, where $Y \in \mathcal{Y}_{r,s} \backslash \mathcal{Y}_{r,s-1}$ and the orbit length is 1.

For a Young diagram Y, we denote by $Y^T \in \mathcal{Y}_{s,r}$, the diagram obtained by interchanging the rows and columns of Y, by $Y^c \in Y_{r,s}$ the complement of Y in a box of size $r \times s$, and by $Y^* \in \mathcal{Y}_{s,r}$ the Young diagram $(Y^T)^c$ obtained by first taking the transpose and then taking the complement in a box of size $(s \times r)$.

8.3. **Branching rules.** For reference, we state here some of the components that appear in the branching rule for the embedding (1.5) (recall Section 2.2). Let σ be as in the previous section. Let λ, μ, and Λ be integrable highest weights for $\mathfrak{so}(2r+1)$ at level $2s+1$, $\mathfrak{so}(2s+1)$ at level $2r+1$, and $\mathfrak{so}(2d+1)$ at level 1, respectively. We say that $(\lambda,\mu)\in B(\Lambda)$ if $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_\mu(\mathfrak{so}(2s+1))$ appears in the branching of $\mathcal{H}_\Lambda(\mathfrak{so}(2d+1))$. Note that here (and for the rest of the paper) unless specified otherwise the levels $2s+1, 2r+1$, and 1, have been (wiil be) suppressed from the notation of highest weight modules. Then the branching rules we need are the following:

- $(Y,Y^T)\in B(\omega_0)$ if $|Y|$ is even;
- $(\sigma(Y),Y^T)$ and $(Y,\sigma(Y^T))\in B(\omega_0)$ if $|Y|$ is odd;
- $(Y,Y^T)\in B(\omega_1)$ if $|Y|$ is odd;
- $(\sigma(Y),Y^T)$ and $(Y,\sigma(Y^T))\in B(\omega_1)$ if $|Y|$ is even;
- for $Y\in\mathcal{Y}_{r,s-1}$, $(Y+\omega_r,Y^*+\omega_s)$ and $(\sigma(Y+\omega_r),Y^*+\omega_s)$ are in $B(\omega_d)$;
- for $Y\in\mathcal{Y}_{r,s}\setminus\mathcal{Y}_{r,s-1}$, $(Y+\omega_r,Y^*+\omega_s)$ and $(Y+\omega_r,\sigma(Y^*+\omega_s))$ are in $B(\omega_d)$.

We refer the reader to [32] for a proof.

8.4. **Highest weight vectors of branching.** An explicit description of the highest weight vectors for the components of the branching can be found in [32]. In this section, in those cases that will be convenient for our applications, we express them as products of operators in $\widehat{\mathfrak{so}}(2d+1)$ acting on the level one representations of $\widehat{\mathfrak{so}}(2d+1)$. Recall the following from [50].

Proposition 8.1. *Let $\lambda'\in\mathcal{Y}_{r,s}$ be obtained from λ by removing two boxes with coordinates (a,b) and (c,d) and $\epsilon\in\{0,1\}$. Assume that $(a,b)<(c,d)$ under the lexicographic ordering. If $v_{\lambda'}\in\mathrm{End}(\mathcal{H}_{\omega_\epsilon}(\mathfrak{so}(2d+1)))$ is the highest weight vector of the component $\mathcal{H}_{\lambda'}\otimes\mathcal{H}_{\lambda'^T}$, then the highest weight vector v_λ of the component $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$ is given by $v_\lambda=B^{a,b}_{-c,-d}(-1)v_{\lambda'}$.*

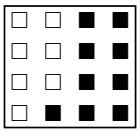
From [32] we have:

Proposition 8.2. *The element $\bigwedge_{j=-r}^{r}\phi^{j,1}(-\frac{1}{2})\in\bigwedge W_d^{\mathbb{Z}+\frac{1}{2},-}$ gives the highest weight vector for the component $\mathcal{H}_{\omega_0}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{(2r+1)\omega_1}(\mathfrak{so}(2s+1))$.*

Next, we describe the highest weight vectors for the branching of the Spin module at level one. First, we need some notation. Given $Y\in\mathcal{Y}_{r,s}$, we view it pictorially as an $r\times s$ box with the white boxes carving out the Young diagram (see below). We associate to Y another diagram as follows:

$$\widetilde{Y}_{j,p}=\begin{cases}\blacksquare & \text{if } Y \text{ has an empty box in the } (j,p+s+1)\text{-th position,}\\ \square & \text{otherwise.}\end{cases}$$

Here in the matrix $\widetilde{Y}$, $j=0,1,\dots,r$, $p=1,\dots,s,-s,\dots,-1$. This is illustrated as:

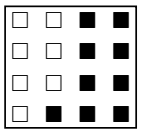
$Y =$  $\longleftrightarrow$ $\widetilde{Y}_{j,p} =$

$j\backslash p$	1	..	..	s	$-s$	..	..	-1
-1	□	..	..	□	□	..	..	□
0	□	..	..	□	□	..	..	□
1	□	..	..	□	□	□	■	■
:	:			:	□	□	■	■
:	:			:	□	□	■	■
r	□	..	..	□	□	■	■	■

With this notation, we can state the branching rules. If $Y \in \mathcal{Y}_{r,s}$, then $\mathcal{H}_{Y+\omega_r}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{Y^*+\omega_s}(\mathfrak{so}(2s+1))$ appears in the decomposition of $\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1))$. For a proof of the following, we refer the reader to [32].

Proposition 8.3. *The highest weight vector v_Y of the component $\mathcal{H}_{Y+\omega_r}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{Y^*+\omega_s}(\mathfrak{so}(2s+1))$ is given by $\bigwedge_{\widetilde{Y}_{j,p}=\blacksquare} \phi_{j,p}$.*

From Section 8.3, if $Y \in \mathcal{Y}_{r,s-1}$, the component $\mathcal{H}_{\sigma(Y+\omega_r)}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{Y^*+\omega_s}(\mathfrak{so}(2s+1))$ appears in the decomposition of $\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1))$. We describe the highest weight vectors. We define a new diagram $\sigma(\widetilde{Y})$, obtained by first considering $\widetilde{Y}$ and then interchanging the black boxes in the 1-st row by the corresponding white boxes in the (-1)-st row, and keeping the columns invariant.

$Y =$  $\longleftrightarrow$ $\sigma(\widetilde{Y})_{j,p} =$

$j\backslash p$	1	..	..	s	$-s$	..	..	-1
-1	□	..	..	□	□	..	■	■
0	□	..	..	□	□	..	..	□
1	□	..	..	□	□	□	□	□
:	:			:	□	□	■	■
:	:			:	□	□	■	■
r	□	..	..	□	□	■	■	■

With the above notation, we have the following from [32].

Proposition 8.4. *The highest weight vector of the component $\mathcal{H}_{\sigma(Y+\omega_r)}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{Y^*+\omega_s}(\mathfrak{so}(2s+1))$ is given by $\bigwedge_{\sigma(\widetilde{Y})_{j,p}=\blacksquare} \phi_{j,p}(\epsilon)$, where $\epsilon = -1$ if $j = -1$, and $\epsilon = 0$ otherwise.*

We can rewrite the result above in the "Kac-Moody" form.

Corollary 8.5. *Let Y' be the Young diagram obtained from Y by changing the black boxes in the first row to white. Let $v_{Y'}$ be the highest weight vector of the component $\mathcal{H}_{Y'+\omega_r}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{(Y')^*+\omega_s}(\mathfrak{so}(2s+1))$. Then a highest weight vector of the component $\mathcal{H}_{\sigma(Y+\omega_r)}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{Y^*+\omega_s}(\mathfrak{so}(2s+1))$ is given by:* $\prod_{\sigma(\widetilde{Y})_{-1,p}=\blacksquare} B_{0,0}^{1,-p}(-1) v_{Y'}$.

Proof. Use Proposition 8.4 and the definition of the action of $B_{0,0}^{1,-p}(-1)$. □

9. Rank-level duality in genus zero

9.1. **General context of rank-level duality.** Let $\phi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be a conformal embedding with Dynkin multi-index $d_\phi = (\ell_1, \ell_2)$ (see Section 2.2). Then ϕ extends to a homomorphism of affine Lie algebras $\widehat{\phi} : \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \to \widehat{\mathfrak{g}}$. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ (resp. $\vec{\mu} = (\mu_1, \ldots, \mu_n)$) and $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be n-tuples of level ℓ_1 (resp. ℓ_2) and level one integrable highest weights such that for each $1 \leq i \leq n$, $(\lambda_i, \mu_i) \in B(\Lambda_i)$. Taking the n-fold tensor product, we get a map: $\bigotimes_{i=1}^n \mathcal{H}_{\lambda_i}(\mathfrak{g}_1) \otimes \mathcal{H}_{\mu_i}(\mathfrak{g}_2) \to \bigotimes_{i=1}^n \mathcal{H}_{\Lambda_i}(\mathfrak{g})$. Let $\mathfrak{X}$ be the data associated to curve C with n marked points and chosen coordinates. Taking coinvariants, we get the following map of dual conformal blocks:$\alpha : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$. We call a triple $(\vec{\lambda}, \vec{\mu}, \vec{\Lambda}) \in P^n_{\ell_1}(\mathfrak{g}_1) \times P^n_{\ell_2}(\mathfrak{g}_2) \times P^n_1(\mathfrak{g})$ *admissible*, if they are connected by a map as above by the branching of level one modules. If $\mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$ is one dimensional we get a map: $\alpha^* : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \to \mathcal{V}^*_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2)$, which is determined up to a nonzero multiplicative constant. Then α^* is known as the *rank-level duality map*. We say that *rank-level duality holds* if α^* is an isomorphism.

Let $\mathcal{F} = (\pi : \mathcal{C} \to B; \sigma_1, \ldots, \sigma_n; \xi_1, \ldots, \xi_n)$ be a family of nodal curves on a base B with sections σ_i and local coordinates ξ_i. The map α^* can be extended to a map of sheaves $\alpha(\mathcal{F}) : \mathcal{V}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\mu}}(\mathcal{F}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}_{\vec{\Lambda}}(\mathcal{F}, \mathfrak{g}, 1)$. Furthermore, if the embedding is $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ is conformal [38], as in the case of the odd orthogonal groups considered here, it follows that the rank-level duality map is flat with respect to the TUY connection. We refer the reader to [16] for a proof.

9.2. **Failure of rank-level duality over $\mathbb{P}^1$ with spin weights.** Rank-level duality isomorphisms for odd orthogonal groups on $\mathbb{P}^1$ with SO-weights was proved in [50]. In this section, we give explicit examples where the rank-level duality map over $\mathbb{P}^1$ with four marked points is well-defined, but fails to be an isomorphism.

9.2.1. *Example 1.* Consider the embedding $\mathfrak{so}(5) \oplus \mathfrak{so}(7) \to \mathfrak{so}(35)$. From the branching rules in Section 8.3, we know that $\mathcal{H}_{2\omega_1+\omega_2}(\mathfrak{so}(5)) \otimes \mathcal{H}_{\omega_1+3\omega_3}(\mathfrak{so}(7))$ appears in the branching of the spin module $\mathcal{H}_{\omega_{17}}(\mathfrak{so}(35))$. By functoriality we get the following map of conformal blocks: $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(5), 7) \otimes \mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(7), 5) \to \mathcal{V}_{\omega_{17},\omega_{17},\omega_1,\omega_1}(\mathbb{P}^1, \mathfrak{so}(35), 1)$, where $\vec{\lambda} = (2\omega_1 + \omega_2, 2\omega_1 + \omega_2, \omega_1, \omega_1)$ and $\vec{\mu} = (\omega_1 + 3\omega_3, \omega_1 + 3\omega_3, \omega_1, \omega_1)$. One checks (e.g. by [62]) that $\dim_{\mathbb{C}} \mathcal{V}_{\omega_{17},\omega_{17},\omega_1,\omega_1}(\mathbb{P}^1, \mathfrak{so}(35), 1) = 1$. Hence, we get a rank-level duality map between $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(5), 7)^*$ and $\mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(7), 5)$. But this map cannot be an isomorphism since $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(7), 5) = 5$, whereas $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(5), 7) = 4$.

9.2.2. *Example 2.* Consider the embedding $\mathfrak{so}(7) \oplus \mathfrak{so}(9) \to \mathfrak{so}(63)$. Then

$$\dim_{\mathbb{C}} \mathcal{V}_{\omega_{31},\omega_{31},\omega_1}(\mathbb{P}^1, \mathfrak{so}(63), 1) = 1 \ ,$$

and following the branching rules in Section 8.3, there is a well-defined rank-level duality map $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(7), 9) \otimes \mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(9), 7) \to \mathcal{V}_{\omega_{31},\omega_{31},\omega_1}(\mathbb{P}^1, \mathfrak{so}(63), 1)$, where $\vec{\lambda} = (\omega_2 + 3\omega_3, \omega_2 + 3\omega_3, \omega_1 + \omega_2)$, $\vec{\mu} = (\omega_1 + 2\omega_3 + \omega_4, \omega_1 + 2\omega_3 + \omega_4, \omega_1 + \omega_2)$. But this map is not an isomorphism, since the dimensions of $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(7), 9)$ and $\mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(9), 7)$ are 3 and 4, respectively.

9.2.3. *Example 3.* Consider the embedding $\mathfrak{so}(9) \oplus \mathfrak{so}(7) \to \mathfrak{so}(63)$. Then

$$\dim_{\mathbb{C}} \mathcal{V}_{\omega_{31},\omega_{31},\omega_1}(\mathbb{P}^1, \mathfrak{so}(63), 1) = 1 \ ,$$

and following the branching rules in Section 8.3, there is a well-defined rank-level duality map $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(9), 7) \otimes \mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(7), 9) \to \mathcal{V}_{\omega_{31},\omega_{31},\omega_1}(\mathbb{P}^1, \mathfrak{so}(63), 1)$, where $\vec{\lambda} = (\omega_2 + 3\omega_4, \omega_2 + 3\omega_4, 2\omega_1 + 2\omega_4)$, $\vec{\mu} = (2\omega_1 + 2\omega_2 + \omega_3, 2\omega_1 + 2\omega_2 + \omega_3, 3\omega_1 + 2\omega_3)$. But this map is not an isomorphism, since the dimensions of $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \mathfrak{so}(9), 7)$ and $\mathcal{V}_{\vec{\mu}}(\mathbb{P}^1, \mathfrak{so}(7), 9)$ are 8 and 14, respectively.

Remark 9.1. The above examples show that even if the rank-level duality map is well-defined it may not be an isomorphism due to the inequality of dimensions of the source and the target spaces. However, we can still ask if α^* is injective? The next section gives a positive answer to that question. This is distinct from the issue in higher genus, where we will see that there is an equality of dimensions but still the rank-level duality map is not an isomorphism (see Section 10).

9.3. **Rank-level duality for 3-pointed $\mathbb{P}^1$ with spin weights.** Consider the embedding (1.5). The only interesting cases for 3-points with spin weights are the tuples $\vec{\Lambda} = (\omega_0, \omega_d, \omega_d)$ and $\vec{\Lambda} = (\omega_1, \omega_d, \omega_d)$. We also observe that the action of the automorphism of the affine Dynkin diagram fixes ω_d and interchanges ω_0 with ω_1. We first fix some notation. Let $Y_1 \in \mathcal{Y}_{r,s}$ and $Y_2, Y_3 \in \mathcal{Y}_{r,s-1}$ and we consider $\vec{\lambda} = (Y_1, Y_2 + \omega_r, Y_3 + \omega_r)$. Let $\vec{\mu} = (Y_1^T, Y_2^* + \omega_s, Y_3^* + \omega_s)$ and $\vec{\Lambda} = (\omega_\epsilon, \omega_d, \omega_d)$, where ϵ is zero or one depending on the even or odd parity of the number of boxes of the Young diagram of Y_1. From the branching rules of Section 8.3, we get the following map of conformal blocks

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1)) \to \mathcal{V}^*_{\vec{\mu}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \otimes \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1) \ , \tag{9.1}$$

Here, $\mathfrak{X}$ denotes the data associated to $\mathbb{P}^1$ with three marked points and chosen coordinates. The following is the main statement of this section.

Theorem 9.2. *The rank-level duality map defined in* (9.1) *is injective.*

The proof of this theorem is broken up into several steps and can be reduced to the case when both Y_2 and Y_3 are empty and Y_1 is just a Young diagram with one column, in which case the corresponding conformal blocks are one dimensional for $\mathfrak{so}(2r+1)$. We now describe the steps in the reduction.

9.4. **Reduction to the one dimensional case.** The main tools used here are factorization/sewing of conformal blocks (cf. Sections 2.3 and 9.7), and the fact that certain Littlewood Richardson coefficients are one.

9.4.1. *Step I.* Clearly, we may assume that the rank of the conformal block in the source of (9.1) is nonzero. Now consider a new tuple $\vec{\lambda}' = (\vec{\nu}_2, \omega_r, Y_1, \vec{\nu}_3, \omega_r)$, where $\vec{\nu}_2$ (resp. $\vec{\nu}_3$) is a tuple of ω_1 of cardinality $|Y_2|$ (resp. $|Y_3|$). Similarly let $\vec{\mu}' = (\vec{\nu}_2^T, (2r+1)\omega_s, Y_1^T, \vec{\nu}_3^T, (2r+1)\omega_s)$ and $\vec{\Lambda}' = (\vec{\omega}_1, \omega_d, \omega_\epsilon, \vec{\omega}_1, \omega_d)$, where ω_ϵ is ω_1 or ω_0's, depending on the number of boxes of Y_1. It is easy to see that the triple $(\vec{\lambda}', \vec{\mu}', \vec{\Lambda}')$ is admissible.

9.4.2. *Step II.* Let $\mathfrak{X}$ denote the data associated to $\mathbb{P}^1$ with $|Y_2|+|Y_3|+3$ marked points with chosen coordinates. The rank of the conformal block $\mathcal{V}_{\vec{\Lambda}'}(\mathfrak{X},\mathfrak{so}(2d+1),1)$ is one and the rank of the conformal block $\mathcal{V}_{\vec{\lambda}'}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)$ is nonzero. The first assertion can be easily checked via factorization (cf. Section 9.7) since the only nontrivial three point cases with spin weights up to permutation are $(\omega_0,\omega_d,\omega_d)$ and $(\omega_1,\omega_d,\omega_d)$ both of which are rank one. For the second assertion, we get by the factorization theorem that the dimension of the conformal block $\mathcal{V}_{\vec{\lambda}'}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)$ is greater than equal to the dimension of the following product of conformal blocks:

$$\mathcal{V}_{\vec{\nu}_2,\omega_r,Y_2+\omega_r}(\mathfrak{X}_1,\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{Y_1,Y_2+\omega_r,Y_3+\omega_r}(\mathfrak{X}_2,\mathfrak{so}(2r+1),2s+1)$$
$$\otimes\mathcal{V}_{\vec{\nu}_3,\omega_r,Y_3+\omega_r}(\mathfrak{X}_3,\mathfrak{so}(2r+1),2s+1)\ .$$

Here, $\mathfrak{X}_1$ (resp. $\mathfrak{X}_3$) denote the data associated to a $\mathbb{P}^1$ with $|Y_2|+2$ (resp. $|Y_3|+2$) marked points and $\mathfrak{X}_2$ denote the data associated to a $\mathbb{P}^1$ with three marked points and chosen coordinates. The nonvanishing of the dimensions on the first and third factors in the above expression follows from Proposition 9.4.

9.4.3. *Step III.* Assume that the injectivity of the rank-level duality map for the admissible pairs $(\vec{\lambda}',\vec{\mu}',\vec{\Lambda}')$ holds, then Theorem 9.2 holds, where $\vec{\lambda}'$, $\vec{\mu}'$ and $\vec{\Lambda}'$ be as in Step II. The basic idea is that we split up the rank-level duality map into a direct sum of several rank-level duality maps. Now the injectivity of the rank-level duality map for the bigger space implies the injectivity of the rank-level duality map for the components, and vice-versa. The key geometric input is Lemma 9.7 in Section 9.7. The conditions in Lemma 9.7 are guaranteed by the fact that the dimensions of the two conformal blocks on $\mathbb{P}^1$ with weights $\vec{\nu}_2,\omega_r,\sigma(Y_2+\omega_r)$ and $\vec{\nu}_3,\omega_r,\sigma(Y_3+\omega_r)$ are zero (cf. Proposition 9.4). This is where we use that $Y_2,Y_3\in\mathcal{Y}_{r,s-1}$.

9.4.4. *Step IV.* By the previous discussion, it enough to prove the injectivity of the following rank-level duality map:

$$\mathcal{V}_{\vec{\lambda}'}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\longrightarrow\mathcal{V}^*_{\vec{\mu}'}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\vec{\Lambda}'}(\mathfrak{X},\mathfrak{so}(2d+1),1)\ .$$

We now consider a degeneration of $\mathbb{P}^1$ into nodal curve $C=C_1\cup C_2$, where C_1 is a copy of $\mathbb{P}^1$ with two smooth marked points and the weights ω_r, ω_r are the weights attached to the markings. The other component C_2 is $\mathbb{P}^1$ with rest of the marked points. The components C_1 and C_2 meet at a point p. The normalization $\widetilde{C}$ of C is a disjoint union of C_1 and C_2 with one extra marked point on each component. Since the two marked points of C_1 have spin weights, it follows that the weight associated to the new marked point on C_1, considered as a component of the normalization of $\widetilde{C}$, is marked by an $\mathsf{SO}(2r+1)$ weight. Hence, by repeating the process discussed in Section 9.7, we are reduced to the case where $\vec{\lambda}=(\omega_r,\omega_r,Y)$, $Y\in P_{2s+1}(\mathsf{SO}(2r+1))$, and the case where $\vec{\lambda}=(Y,Y_1,\vec{\omega}_1)$. The rank-level duality in the latter case is a Theorem in [50]. Hence, we are only left with the admissible triples of the form $(\vec{\lambda},\vec{\mu},\vec{\Lambda})$, when $\vec{\lambda}=(Y,\omega_r,\omega_r)$, $\vec{\mu}=(Y^T,(2r+1)\omega_s,(2r+1)\omega_s)$ and $\vec{\Lambda}=(\omega_\epsilon,\omega_d,\omega_d)$. We now determine which Y are possible.

9.4.5. *Step V.* As in Section 2.1, let V_λ denote the finite dimensional irreducible representation of $\mathsf{Spin}(2r+1)$ with highest weight λ. By a theorem of Kempf-Ness [39], we know that V_λ appears in the tensor product decomposition of $V_{\omega_r} \otimes V_{\omega_r}$ if and only if $\lambda \in \{\omega_0, \omega_1, \dots, \omega_{r-1}, 2\omega_r\}$. It follows from [10, Prop. 4.3] that the conformal blocks $\mathcal{V}_{Y,\omega_r,\omega_r}(\mathbb{P}^1, \mathfrak{so}(2r+1), 2s+1)$ are one dimensional, where $Y \in \{\omega_0, \dots \omega_{r-1}, 2\omega_r\}$, and trivial otherwise. We refer to these as the *minimal cases*.

9.5. **Rank-level duality for the minimal cases.** The minimal case can be further subdivided into the case when $|Y|$ is odd or even. After dividing into these cases, we will approach the minimal cases by induction. The basic strategy is similar to the strategy of the minimal cases in [50]. We will refer to the Appendix for some of the details of the formulas.

To show that the rank-level duality map is injective, it is enough to find vectors

$$v_1 \otimes v_2 \otimes v_3 \in \mathcal{H}_Y \otimes \mathcal{H}_{Y^T} \otimes \mathcal{H}_{\omega_r} \otimes \mathcal{H}_{(2r+1)\omega_s} \otimes \mathcal{H}_{\omega_r} \otimes \mathcal{H}_{(2r+1)\omega_s}$$

such that $\langle \Psi \mid v_1 \otimes v_2 \otimes v_3 \rangle \neq 0$, where $\langle \Psi \mid$ is the (up to scalars) unique nonzero element of $\mathcal{V}^*_{\omega_\epsilon,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$, and ϵ is either 0 or 1, depending on the parity of $|Y|$.

9.5.1. *The case* $|Y| = 0$. In this case we choose $v_1 = 1$, $v_2 = \bigwedge_{1\le i\le r, -1\le j\le -s} \phi_{i,j}$, and $v_3 = \bigwedge_{1\le i\le r, -1\le j\le -s} \phi^{i,j}$. It is then clear (cf. Section 7.3.1) that $\langle \Psi \mid v_1 \otimes v_2 \otimes v_2 \rangle \neq 0$.

9.5.2. *The case* $Y = \omega_1$. Choose $v_1 = \phi^{1,0}(-\frac{1}{2}) = R(B_1^0)\phi^{1,1}(-\frac{1}{2})$ (cf. Lemma A.2 of the Appendix), $v_2 = L(B_1^0) \bigwedge_{1\le i\le r, -1\le j\le -s} \phi_{i,j}$, and $v_3 = \bigwedge_{1\le i\le r, -1\le j\le -s} \phi^{i,j}$. Now by a direct computation (cf. Proposition A.1), we get $v_2 = \phi_{1,0} \wedge \bigwedge_{1\le i\le r, -1\le j\le -s} \phi_{i,j}$. We are now left to evaluate $\langle \Psi \mid v_1 \otimes v_2 \otimes v_3 \rangle$, and from the discussion in Section 7.3.2, it is nonzero.

9.5.3. *The case* $Y = \omega_2$. We need to choose v_1, v_2 and v_3 as before. In this case, take $v_1 = R^2(B_1^0)\phi^{1,1}(-\frac{1}{2})\phi^{2,1}(-\frac{1}{2})$ and $v_2 = L(B_2^{-1})v$, where as before $v = \bigwedge_{1\le i\le r, -1\le j\le -s} \phi_{i,j}$. By Proposition A.1, we get $v_2 = \phi_{1,0} \wedge \phi_{2,0} \wedge v$ and $v_3 = v^{opp} = \bigwedge_{1\le i\le r, -1\le j\le -s} \phi^{i,j}$. Now by Proposition A.4, we get

$$R^2(B_1^0)\phi^{1,1}(-\tfrac{1}{2})\phi^{2,2}(-\tfrac{1}{2}) \cdot 1 = \left(2B^{1,0}_{-2,0}(-1) + B^{2,1}_{-1,1}(-1) + B^{2,-1}_{-1,-1}(-1) \right) \cdot 1 \ .$$

Let the three points be $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty$ and let z be the local coordinate at the point 0. Consider f defined by the equation $1/z$. Around P_1, the functions f has a pole of order one and hence a zero of order one around p_3. Let $\xi = z - 1$ be a coordinate at the point $p_2 = 1$ and around p_2, the function f has the following form $f_1(\xi) = 1 - \xi + \xi^2 - \xi^3 + \cdots$. This follows by formally expanding

$$f(z) = \frac{1}{1 + (z-1)} = 1 - (z-1) + (z-2)^2 - (z-3)^3 + \cdots$$

We now use gauge symmetry (cf. Section 2.3) to finish the argument

$$\begin{aligned} &\langle \Psi \mid R^2(B_1^0)\phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) \otimes L(B_2^{-1})v \otimes v^{opp} \rangle \\ &\qquad = \langle \Psi \mid (2B^{1,0}_{-2,0}(-1) + B^{2,1}_{-1,1}(-1) + B^{2,-1}_{-1,-1}(-1)).1 \otimes \phi_{1,0} \wedge \phi_{2,0} \wedge v \otimes v^{opp} \rangle \end{aligned}$$

$$
\begin{aligned}
&= 2\langle\Psi \mid 1\otimes(-B^{1,0}_{-2,0}+B^{1,0}_{-2,0}(1)-\cdots)(\phi_{1,0}\wedge\phi_{2,0}\wedge v)\otimes v^{opp}\rangle\\
&\quad+\langle\Psi \mid 1\otimes(-B^{2,1}_{-1,1}+B^{2,1}_{-1,1}(1)-\cdots)(\phi_{1,0}\wedge\phi_{2,0}\wedge v)\otimes v^{opp}\rangle\\
&\quad+\langle\Psi \mid 1\otimes(-B^{2,-1}_{-1,-1}+B^{2,-1}_{-1,-1}(1)-\cdots)(\phi_{1,0}\wedge\phi_{2,0}\wedge v)\otimes v^{opp}\rangle\\
&= -2\langle\Psi \mid 1\otimes B^{1,0}_{-2,0}\phi_{1,0}\wedge\phi_{2,0}\wedge v\otimes v^{opp}\rangle\ .
\end{aligned}
$$

In the above calculation, we use the fact $B^{2,-1}_{-1,-1}(\phi_{1,0}\wedge\phi_{2,0}\wedge v)=B^{2,1}_{-1,1}(\phi_{1,0}\wedge\phi_{2,0}\wedge v)=0$. This is justified by Lemma A.5. But now $B^{1,0}_{-2,0}(\phi_{1,0}\wedge\phi_{2,0}\wedge v)=-v$. Hence $\langle\Psi \mid 1\otimes B^{1,0}_{-2,0}(\phi_{1,0}\wedge\phi_{2,0}\wedge v\otimes v^{opp})\rangle\neq 0$ (cf. Section 7.3.1). Thus we are done in this case.

9.5.4. *The case $Y=\omega_3$.* The highest weight vector for the component $\mathcal{H}_{\omega_3}\otimes\mathcal{H}_{3\omega_1}$ is given by $\phi^{1,1}(-\frac{1}{2})\phi^{2,1}(-\frac{1}{2})\phi^{3,1}(-\frac{1}{2})$. We choose $v_1=R^3(B^0_1)\phi^{1,1}(-\frac{1}{2})\phi^{2,1}(-\frac{1}{2})\phi^{3,1}(-\frac{1}{2})$, $v_2=L(B^0_1)L(B^{-2}_3)v$, and $v_3=v^{opp}$. Now by Proposition A.1, we get $L(B^0_1)L(B^{-2}_3)v=\phi_{1,0}\wedge\phi_{2,0}\wedge\phi_{3,0}\wedge v$, and by Proposition A.6 we get

$$
\begin{aligned}
&R^3(B^0_1)\phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2})=3!\big[\phi^{1,0}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2})\big]\\
&-3\big[\phi^{1,-1}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2})+\phi^{1,0}(-\tfrac{1}{2})\phi^{2,-1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2})\\
&+\phi^{1,-1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2})+\phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,-1}(-\tfrac{1}{2})\\
&+\phi^{1,1}(-\tfrac{1}{2})\phi^{2,-1}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2})+\phi^{1,1}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,-1}(-\tfrac{1}{2})\big]\ .
\end{aligned}
$$

We can rewrite the above expression in the following "Kac-Moody" form.

$$
\begin{aligned}
&R^3(B^0_1)\phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2})=3!B^{2,0}_{-3,0}(-1)\phi^{1,0}(-\tfrac{1}{2})\\
&-3\big[-B^{1,-1}_{-3,-1}(-1)\phi^{2,0}(-\tfrac{1}{2})+B^{2,-1}_{-3,-1}(-1)\phi^{1,0}(-\tfrac{1}{2})+B^{1,-1}_{-2,-1}(-1)\phi^{3,0}(-\tfrac{1}{2})\\
&-B^{3,-1}_{-2,-1}(-1)\phi^{1,0}(-\tfrac{1}{2})-B^{2,-1}_{-1,-1}(-1)\phi^{3,0}(-\tfrac{1}{2})+B^{3,-1}_{-1,-1}(-1)\phi^{2,0}(-\tfrac{1}{2})\big]\ .
\end{aligned}
$$

We now evaluate $\langle\Psi \mid v_1\otimes v_2\otimes v_3\rangle$ using gauge symmetry (cf. Section 2.3) as before. Choose P_1, P_2 and P_3 to be $(0,1,\infty)$ with the obvious coordinates. Expanding $f(z)=1/z$ around 1 and ∞ and applying gauge symmetry, we get that $\langle\Psi \mid v_1\otimes v_2\otimes v_3\rangle$ is (up to a sign) equal to $3!\langle\Psi \mid \phi^{1,0}(-\frac{1}{2})\otimes B^{2,0}_{-3,0}(\phi_{1,0}\wedge\phi_{2,0}\wedge\phi_{3,0}\wedge v)\otimes v^{opp}\rangle$, which is nonzero by the discussion in Section 7.3.2. Explicitly,

Lemma 9.3. *Let a and b be both nonzero integers and $i\neq j$ are both positive integers, then $B^{2,0}_{-3,0}(\phi_{1,0}\wedge\phi_{2,0}\wedge\phi_{3,0}\wedge v)=-v$, and $B^{i,a}_{-j,b}(\phi_{1,0}\wedge\phi_{2,0}\wedge\phi_{3,0}\wedge v)=0$.*

9.5.5. *The general case: $Y=\omega_k$ or $2\omega_r$.* The strategy for the general case is the same as for the previous special case. We choose the points $(p_1,p_2,p_3)=(0,1,\infty)$. We choose $v_1=R^k(B^0_1)\phi^{1,1}(-\frac{1}{2})\wedge\phi^{2,1}(-\frac{1}{2})\wedge\cdots\wedge\phi^{k,1}(-\frac{1}{2})$, $v_2=\phi_{1,0}\wedge\cdots\wedge\phi_{k,0}\wedge v$ and $v_3=v^{opp}$. Using gauge symmetry (cf. Section 2.3), the expression $\langle\Psi \mid v_1\otimes v_2\otimes v_3\rangle$ is equal (up to a sign) to,

$$
k!\langle\Psi \mid \phi^{1,0}(-\tfrac{1}{2})\wedge\cdots\wedge\phi^{k,0}(-\tfrac{1}{2})\otimes v_2\otimes v_3\rangle\ . \tag{9.2}
$$

The above step uses Proposition A.7 and a calculation similar to Lemma 9.3. We can rewrite the right hand side of (9.2) as follows:

- If k is odd,

$$\langle \Psi \mid \phi^{1,0}(-\tfrac{1}{2}) \wedge \cdots \wedge \phi^{k,0}(-\tfrac{1}{2}) \otimes v_2 \otimes v_3 \rangle = \langle B^{2,0}_{-3,0}(-1) \cdots B^{k-1,0}_{-k,0}(-1) \phi^{1,0}(-\tfrac{1}{2}) \otimes v_2 \otimes v_3 \rangle \ ;$$

- If k is even,

$$\langle \Psi \mid \phi^{1,0}(-\tfrac{1}{2}) \wedge \cdots \wedge \phi^{k,0}(-\tfrac{1}{2}) \otimes v_2 \otimes v_3 \rangle = \langle B^{1,0}_{-2,0}(-1) \cdots B^{k-1,0}_{-k,0}(-1) \otimes v_2 \otimes v_3 \rangle \ .$$

By using gauge symmetry (cf. Section 2.3) we get that up to a sign $\langle \Psi \mid v_1 \otimes v_2 \otimes v_3 \rangle$ is the following:

- If k is odd,

$$k! \langle \Psi \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes B^{2,0}_{-3,0} \cdots B^{k-1,0}_{-k,0} v_2 \otimes v_3 = k! \langle \Psi \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes \phi_{1,0} \wedge v \otimes v^{opp} \rangle = k! \ ;$$

- If k is even,

$$k! \langle \Psi \mid 1 \otimes \otimes B^{1,0}_{-2,0} \cdots B^{k-1,0}_{-k,0} v_2 \otimes v_3 = k! \langle \Psi \mid 1 \otimes v \otimes v^{opp} \rangle = k!.$$

This completes the proof in the general case.

9.6. **Key Littlewood-Richardson coefficients.** In this section, we prove some basic facts on dimensions of conformal blocks and apply this to reduce the general case of rank-level duality to the minimal cases.

Proposition 9.4. *Let $\lambda \in P_\ell(\mathfrak{so}(2r+1))$ and let $\lambda = Y + \omega_r$ and $Y \in \mathcal{Y}_{r,s}$, then $\dim_{\mathbb{C}} \mathcal{V}^*_{\lambda,\vec{\omega}_1,\omega_r}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell) \neq 0$, where $\vec{\omega}_1$ is a $|Y|$-tuple of ω_1's at level ℓ.*

Proof. The proof is by induction on $|Y|$. If Y is zero or one, then it is easy to see that $\mathcal{V}_{\omega_r,\omega_r,\omega_0}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$ and $\mathcal{V}_{\omega_r,\omega_r,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$ are both one dimensional. Now the inductive step follows by factorization (cf. Section 9.7). By factorization and Lemma 9.6, we know that the dimension of $\mathcal{V}_{\lambda,\omega_r,\vec{\omega}_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$ is greater than equal to the dimensions $\mathcal{V}_{\lambda,\omega_1,\lambda'}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell) \otimes \mathcal{V}_{\lambda',\omega_r,\vec{\omega}'_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$. Here $\lambda' = Y' + \omega_r$ and $|Y'| = |Y| - 1$ and $\vec{\omega}'_1$ is an $|Y|$-tuple of ω_1. □

We now determine which three point $\mathfrak{so}(2r+1)$, level $2s+1$, conformal blocks with weights $\vec{\lambda}$ are nonzero. First, we compute the Littlewood-Richardson numbers following Littlemann [46].

Lemma 9.5. *Let $\lambda \in P_+(\mathfrak{so}(2r+1))$ and assume that λ is of the form $Y + \omega_r$, where $Y \in \mathcal{Y}_{r,s}$. Then the dimension of the space $\mathrm{Hom}_{\mathfrak{so}(2r+1)}(V_{\omega_1} \otimes V_\lambda \otimes V_\mu, \mathbb{C})$ is nonzero if μ is either λ, or is of the form $Y' + \omega_r$, where Y' is obtained by adding or deleting a box of Y.*

We use the above proposition to calculate the following dimensions.

Lemma 9.6. *Let $\lambda = \sum_{i=1}^r a_i \omega_i + \omega_r \in P_\ell(\mathfrak{so}(2r+1))$. Then,*

$$\dim_{\mathbb{C}} \mathcal{V}^*_{\lambda,\mu,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell) = \begin{cases} 1 & \text{if } \mu \in P_\ell(\mathfrak{so}(2r+1)) \ , \ \mu \text{ as in Lemma 9.5;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof follows directly from the explicit description of three pointed description of conformal blocks on $\mathbb{P}^1$ as the space of invariants (cf. [10, Prop. 4.3]). □

9.7. **Sewing and injectivity.** In this section, we discuss the key induction steps in the proof of rank-level duality maps. This strategy has already been used in [19, 51]. We recall the details here for completeness. We begin with an important lemma.

Let $B = \operatorname{Spec} \mathbb{C}[\![t]\!]$. Suppose $\mathcal{V}$ and $\mathcal{W}$ are two coherent sheaves such that $\operatorname{rank} \mathcal{V} \leq \operatorname{rank} \mathcal{W}$ and $\mathcal{L}$ be a line bundle on B. Suppose $f : \mathcal{V} \to \mathcal{W} \otimes \mathcal{L}$ be a morphism of vector bundles over B. Assume that over B there are isomorphisms: $\oplus s_i : \mathcal{V} \xrightarrow{\sim} \oplus_{i \in I} \mathcal{V}_i$, and $\oplus t_j : \oplus_{j \in I} \mathcal{W}_j \xrightarrow{\sim} \mathcal{W}$, so that if $f_{i,j} : \mathcal{V}_i \to \mathcal{W}_j \otimes \mathcal{L}$, then

- For each $i \in I$, $f_{i,j} = 0$ unless $i = j$.
- The map $f = \sum_i t^{m_i} t_i \circ f_{i,i} \circ s_i$, where m_i are nonnegative integers.

With the above notation and hypotheses, we have the following easy lemma.

Lemma 9.7. *The map f is injective on $B^* = B \backslash \{t = 0\}$ if and only if the maps $f_{i,i}$'s are injective for all $i \in I$.*

Remark 9.8. We will sometimes need to use a slightly generalized version of Lemma 9.7. Suppose in the above situation there is an isomorphism $\oplus t_j : \oplus_{j \in J} \mathcal{W}_j \xrightarrow{\sim} \mathcal{W}$, and an injective map $\delta : I \to J$ such that $f_{i,j} = 0$ unless $j = \delta(i)$. Then f is injective if and only if for each $i \in I$, the maps $f_{i,\delta(i)}$'s are injective. For our applications, the role of I will often be played by the set $\mathcal{Y}_{r,s}$ or $\mathcal{Y}_{r,s-1}$ and the role of J by $\mathcal{Y}_{s,r}$.

Consider a conformal embedding $\mathfrak{s} \to \mathfrak{g}$. Assume that all level one highest weight integrable modules of $\widehat{\mathfrak{g}}$ decompose with multiplicity one as $\widehat{\mathfrak{s}}$-modules. Let $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be an n-tuple of level one highest weights of $\mathfrak{g}$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple of level ℓ weights that appear in the branching of $\vec{\Lambda}$. By functoriality of the embedding of $\mathfrak{s} \to \mathfrak{g}$, we get a $\mathbb{C}[\![t]\!]$-linear map $\alpha(t) : \mathcal{V}^*_{\vec{\Lambda}}(\mathcal{X}, \mathfrak{g}, 1) \to \mathcal{V}^*_{\vec{\lambda}}(\mathcal{X}, \mathfrak{s}, \ell)$. For λ appearing the branching of Λ, we denote by $\alpha_{\Lambda,\lambda}(t)$ the rank-level duality map for the smooth curve X_0. as follows:

$$\alpha_{\Lambda,\lambda}(t) : \mathcal{V}^*_{\vec{\Lambda},\Lambda,\Lambda^\dagger}(\widetilde{\mathfrak{X}}_0, \mathfrak{g}, 1) \otimes \mathbb{C}[\![t]\!] \to \mathcal{V}^*_{\vec{\lambda},\lambda,\lambda^\dagger}(\widetilde{\mathfrak{X}}_0, \mathfrak{s}, \ell) \otimes \mathbb{C}[\![t]\!] \ .$$

We recall the following proposition from [19].

Proposition 9.9. *On B, the map $\alpha(t)$ decomposes under factorization/sewing as follows*

$$\alpha(t) \circ s_\Lambda(t) = \sum_{\lambda \in B(\Lambda)} t^{m_\lambda} \cdot s_\lambda(t) \circ \alpha_{\Lambda,\lambda}(t) \ ,$$

where m_λ are positive integers given by the formula: $m_\lambda = \Delta_\lambda(\mathfrak{s}, \ell) - \Delta_\Lambda(\mathfrak{g}, 1)$ (see (2.1)).

Remark 9.10. For the Lie algebra $\mathfrak{so}(2r+1)$, it is easy to see that V_λ is isomorphic to its dual as an $\mathfrak{so}(2r+1)$-module. Hence $\lambda^\dagger = \lambda$.

9.8. **Proof of Theorem 1.7.** The proof now follows from factorization as in the previous section, Lemma 9.7, the rank-level duality for SO-weights in [50], and Theorem 9.2.

10. Strange duality maps in higher genus

10.1. **Formulation of the problem.** As mentioned in the introduction, the natural map between the special Clifford groups obtained by the tensor product of vector spaces induces

one between the moduli stacks $p : \mathcal{M}_{2r+1} \times \mathcal{M}_{2s+1} \to \mathcal{M}_{2d+1}$. By a direct calculation we can check that $p^*(\mathcal{P}) \simeq \mathcal{P}^{\otimes 2s+1} \boxtimes \mathcal{P}^{\otimes 2r+1}$. Hence, we obtain the map SD defined in (1.7). Recall that $\dim_{\mathbb{C}} H^0(\mathcal{M}_{2d+1}, \mathcal{P}) = 2^{2g}$, so by Corollary 1.2 the map SD cannot be an isomorphism. However, it is natural to ask the following:

Question 10.1. *For $r, s \geq 1$, is the map SD injective?*

We shall show that the answer to this question is actually negative for all r, s, and in all genus.

10.2. **Action of $J_2(C)$ and the strange duality map.** If W_r and W_s are vector spaces each with a nondegenerate symmetric bilinear form, then the tensor product $W_d = W_r \otimes W_s$ inherits one as well. This gives an embedding $\mathsf{SO}(W_r) \times \mathsf{SO}(W_s) \longrightarrow \mathsf{SO}(W_d)$. If $\dim_{\mathbb{C}} W_r = 2r+1$, $\dim_{\mathbb{C}} W_s = 2s+1$, the map above in turn induces one between the corresponding moduli stacks,

$$m : \mathcal{M}_{\mathsf{SO}(2r+1)} \times \mathcal{M}_{\mathsf{SO}(2s+1)} \longrightarrow \mathcal{M}_{\mathsf{SO}(2d+1)} \ . \tag{10.1}$$

The embedding of orthogonal groups lifts to one on spin groups. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}/2 \times \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ \mathsf{Spin}(2r+1) \times \mathsf{Spin}(2s+1) & \longrightarrow & \mathsf{Spin}(2d+1) \\ \downarrow & & \downarrow \\ \mathsf{SO}(2r+1) \times \mathsf{SO}(2s+1) & \longrightarrow & \mathsf{SO}(2d+1) \end{array} \tag{10.2}$$

where the map $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ is multiplication. By results of [13], we know that $\mathcal{M}_{2r+1}$ forms a $J_2(C)$ torsor over $\mathcal{M}_{\mathsf{SO}(2r+1)}$. Hence, from (10.2) we get the following commutative diagram of moduli stacks:

$$\begin{array}{ccc} \mathcal{M}_{2r+1} \times \mathcal{M}_{2s+1} & \longrightarrow & \mathcal{M}_{2d+1} \\ \downarrow{\scriptstyle J_2(C)\times J_2(C)} & & \downarrow{\scriptstyle J_2(C)} \\ \mathcal{M}_{\mathsf{SO}(2r+1)} \times \mathcal{M}_{\mathsf{SO}(2s+1)} & \xrightarrow{m} & \mathcal{M}_{\mathsf{SO}(2d+1)} \end{array} \tag{10.3}$$

which is equivariant with respect to the action of $J_2(C) \times J_2(C)$ under the multiplication map $J_2(C) \times J_2(C) \to J_2(C)$.

The natural inclusion of $\mathsf{SO}(2r+1) \subset \mathsf{SL}(2r+1)$ gives the following commutative diagram of moduli stacks:

$$\begin{array}{ccc} \mathcal{M}_{\mathsf{SO}(2r+1)} \times \mathcal{M}_{\mathsf{SO}(2s+1)} & \xrightarrow{m} & \mathcal{M}_{\mathsf{SO}(2d+1)} \\ \downarrow{\scriptstyle f_1 \otimes f_2} & & \downarrow{\scriptstyle f} \\ \mathcal{M}_{\mathsf{SL}(2r+1)} \times \mathcal{M}_{\mathsf{SL}(2s+1)} & \xrightarrow{p} & \mathcal{M}_{\mathsf{SL}(2d+1)} \end{array}$$

Let $\mathcal{D}$ be the determinant of cohomology on $\mathcal{M}_{\mathsf{SL}(2d+1)}$ (cf. Proposition 3.4). Also, denote by $\widetilde{\mathcal{D}}$ the pull-back of $\mathcal{D}$ under f. Since we know that $p^*\mathcal{D} = \mathcal{D}^{\otimes(2s+1)} \boxtimes \mathcal{D}^{\otimes(2r+1)}$, it

follows that $m^*\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_1^{\otimes(2s+1)} \boxtimes \widetilde{\mathcal{D}}_2^{\otimes(2r+1)}$, where $\mathcal{D}_i$ and (resp. $\widetilde{\mathcal{D}}_i$) denote the determinants of cohomology and their respective pull-backs. From Proposition 3.4, it follows that if we fix a theta characteristic κ, the pull-back of $\mathcal{P}_\kappa$ under the map m in (10.1) is $\mathcal{P}_\kappa^{\otimes(2s+1)} \boxtimes \mathcal{P}_\kappa^{\otimes(2r+1)}$.

As mentioned before, given $\kappa \in \mathrm{Th}(C)$ we get an action of $J_2(C)$ on the space of global sections $H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)})$, and the above diagram of moduli stacks commutes and is equivariant with respect to $J_2(C) \times J_2(C)$. We have the following.

Lemma 10.2. *Let X and Y be two spaces with an action of a group G actions and $f : X \to Y$ be a G-equivariant map. Suppose $\mathcal{L}$ is a line bundle on Y and suppose both $\mathcal{L}$ and $f^*\mathcal{L}$ are G-linearized. Then the map of global sections is G-equivariant*

$$f^* : H^0(Y, \mathcal{L}) \longrightarrow H^0(X, f^*(\mathcal{L})) \ .$$

Proof. Let s be a global section of X and we consider $gf^*(s)$. Let x be any element of X. Then we get $gf^*(s)(x) = f^*(s)(g^{-1}x) = s(f(g^{-1}x) = s(g^{-1}(f(x))$. On the other hand $f^*(g(s))(x) = (g(s))(f(x) = s(g^{-1}f(x))$. Thus we have the equality $gf^*(s) = f^*(gs)$. □

The commutativity of the diagram 10.3 and Lemma 10.2 implies that the following map of global sections is $J_2(C) \times J_2(C)$ equivariant.

$$H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)})^* \otimes H^0(\mathcal{M}_{2s+1}, \mathcal{P}^{\otimes(2r+1)})^* \longrightarrow H^0(\mathcal{M}_{2d+1}, \mathcal{P})^* \ . \tag{10.4}$$

Lemma 10.3. *Let V_1, V_2 and W be three vector spaces endowed with an action of a finite abelian group A. Let $f : V_1 \otimes V_2 \to W$ be a $A \times A$ equivariant map, where the action of $A \times A$ on W is via multiplication map $A \times A \to A$. Then $f : V_1^{\chi_1} \otimes V_2^{\chi_2} \to W^{\chi_3}$ is zero unless $\chi_1 = \chi_2 = \chi_3$, where V^χ denotes the χ-character spaces of a vector space V with respect to A and $\chi \in \widehat{A}$.*

Proof. The proof follows directly by taking character subspaces of f with respect to $A \times A$. This implies that $\chi_1(a_1)\chi_2(a_2) = \chi_3(a_1.a_2)$ for any a_1 and a_2 in A. This is only possible when $\chi_1 = \chi_2 = \chi_3$. □

Let V be a finite dimensional vector space and A a finite abelian group acting on V. The invariant subspace V^A is a A-submodule of V, and since A is finite, V admits an orthogonal splitting of the form $V = \oplus_{\chi \in \widehat{A}} V^\chi$, i.e. there is a symmetric nondegenerate bilinear A-invariant bilinear form $\{,\}$ such that $\{V^{\chi_1}, V^{\chi_2}\} = 0$ unless $\chi_1 = \chi_2$. Hence $s : (V^\chi)^* \simeq (V^*)^\chi$ canonically. The map is given as if $f \in (V^\chi)^*$, then extend f to $\tilde{f}$ by the obvious rule $\tilde{f}(v) = 0$ unless $v \in V^\chi$. Clearly $\tilde{f} \in (V^*)^\chi$.

Lemma 10.4. *Let V_1, V_2 and W be as in Lemma 10.3 and further assume that $f^* : V_1 \to V_2^* \otimes W$ is injective. Then the induced map between the χ-character spaces $g : V_1^\chi \to (V_2^\chi)^* \otimes W^\chi$ is also injective.*

Proof. Since the map f is $A \times A$ equivariant under the multiplication map, then it is also equivariant with respect to the subgroup $B = A \times \mathrm{id}$. Taking invariants with respect to B implies that the map $f_\chi^* : V_1^\chi \to V_2^* \otimes W^\chi$ is also injective. Now recall that there is a canonical isomorphism $s : (V^\chi)^* \xrightarrow{\simeq} (V^*)^\chi$. Consider the map $V_1^\chi \to V_2^* \otimes W^\chi$ given

by the following: $V_1^\chi \xrightarrow{g} (V_2^\chi)^* \otimes W^\chi \xrightarrow{s} (V_2^*)^\chi \otimes W^\chi \hookrightarrow V_2^* \otimes W^\chi$. It follows from the definition that the composition of the above map coincides with f_χ^*, and since the latter is injective so must be g. □

By taking invariants in (10.4) with respect to the $J_2(C) \times J_2(C)$ action, Lemma 10.4 and "yes" to Question 10.1 would imply that the following map is injective:

$$\left[H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)})^*\right]^{J_2(C)} \longrightarrow \left[H^0(\mathcal{M}_{2s+1}, \mathcal{P}^{\otimes(2r+1)})\right]^{J_2(C)} \otimes \mathbb{C} \cdot s_\kappa^* \ .$$

Now since r and s are arbitrary, and $\left[H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2s+1)})\right]^{J_2(C)} = H^0(\mathcal{M}_{\mathsf{SO}(2r+1)}, \mathcal{P}_\kappa^{2s+1})$, we get the following :

Question 10.5. *Let $\kappa \in \mathrm{Th}(C)$ and consider the Pfaffian section s_κ in $H^0(\mathcal{M}_{\mathsf{SO}(2d+1)}, \mathcal{P}_\kappa)$. The pull-back of the Pfaffian divisor induces a strange duality map:*

$$s_\kappa^* : H^0(\mathcal{M}_{\mathsf{SO}(2r+1)}, \mathcal{P}_\kappa^{\otimes(2s+1)})^* \longrightarrow H^0(\mathcal{M}_{\mathsf{SO}(2s+1)}, \mathcal{P}_\kappa^{\otimes(2r+1)}) \ .$$

Is s_κ^* an isomorphism for all κ?

The discussion above tells us that an affirmative answer to Question 10.1 implies an affirmative answer to Question 10.5. We now show the converse.

Proposition 10.6. *Questions 10.1 and 10.5 are equivalent.*

Proof. Since the $H^0(\mathcal{M}_{2d+1}, \mathcal{P})^{J_2(C)}$ is one dimensional and generated by s_κ, it follows from Lemma 10.3 that the map:$s_\kappa^* : H^0(\mathcal{M}_{\mathsf{SO}(2r+1)}, \mathcal{P}_{\kappa_1}^{\otimes(2s+1)})^* \longrightarrow H^0(\mathcal{M}_{\mathsf{SO}(2s+1)}, \mathcal{P}_{\kappa_2}^{\otimes(2r+1)})$ is zero unless $\kappa = \kappa_1 = \kappa_2$. Now if the answer to Question 10.5 is yes, this implies that for $\kappa = \kappa_1 = \kappa_2$, then s_κ^* is injective. Since the level is odd, we have $L_\chi^{2r+1} = L_\chi$, where L_χ is the two torsion line bundle corresponding to the character χ, and so

$$H^0(\mathcal{M}_{2r+1}, \mathcal{P}^{\otimes(2r+1)}) \simeq \bigoplus_{\kappa \in \mathrm{Th}(C)} H^0(\mathcal{M}_{\mathsf{SO}(2r+1)}, \mathcal{P}_\kappa^{\otimes(2r+1)}) \ .$$

It follows that the map s_Δ^* from (1.8) is also injective. This implies an affirmative answer to Question 10.1. In the above decomposition, it is crucial that we are working with odd levels. □

10.3. **Comparison of dualities and reduction to genus one.** In this section, we reformulate Question 10.1 in terms of conformal blocks. We use the factorization/sewing theorem of conformal blocks to reduce Question 10.1 to a rank-level duality of conformal blocks on elliptic curves with one marked point (cf. Sections 2.3 and 9.7).

Let C be a stable curve of genus g with one marked point, and let $\mathfrak{X}$ be the data associated to the additional choice of a formal neighborhood around the point. Introduce:

$$\widetilde{\mathcal{V}}_{\omega_0}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) := \mathcal{V}_{\omega_0}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \oplus \mathcal{V}_{(2s+1)\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \ .$$

We have the following diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{V}}_{\omega_0}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\otimes\widetilde{\mathcal{V}}_{\omega_0}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1) & \longrightarrow & \widetilde{\mathcal{V}}_{\omega_0}(\mathfrak{X},\mathfrak{so}(2d+1),1) \\ \downarrow & & \downarrow \\ H^0(\mathcal{M}_{2r+1},\mathcal{P}^{\otimes(2s+1)})^*\otimes H^0(\mathcal{M}_{2s+1},\mathcal{P}^{\otimes(2r+1)})^* & \longrightarrow & H^0(\mathcal{M}_{2d+1},\mathcal{P})^* \end{array} \tag{10.5}$$

Here, the vertical arrows are given by Theorem 4.3, and the horizontal arrow on the top is given by the rank-level duality map induced by the branching rule in Section 8.3. The other horizontal arrow is the strange duality map. With the above notation, we have the following.

Proposition 10.7. *The rank-level duality and strange duality maps are the same under uniformization, i.e. the diagram* (10.5) *commutes.*

Proof. The proof follows from the uniformization theorem of the moduli stacks and is similar to the proof of [16, Prop. 5.2]. We omit the details. □

Recall the notation $B(\Lambda)$ from Section 8.3.

Question 10.8. *Let E be any elliptic curve and $\mathfrak{X}$ associated to E with a formal neighborhood at one marked point. Let $\lambda \in P_{2s+1}(\mathsf{SO}(2r+1))$ (resp. $\mu \in P_{2r+1}(\mathsf{SO}(2s+1))$ and $\Lambda \in P_{2d+1}(\mathsf{SO}(2r+1))$ such that Λ is either ω_0 or ω_1, and $(\lambda,\mu)\in B(\Lambda)$. Is the following map of conformal blocks injective:*

$$\begin{aligned}\mathcal{V}_\lambda(\mathfrak{X},\mathfrak{so}(2r+1),2s+1) \to \mathcal{V}^*_\mu(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_\Lambda(\mathfrak{X},\mathfrak{so}(2d+1),1)\\ \oplus\mathcal{V}^*_{\sigma(\mu)}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\sigma(\Lambda)}(\mathfrak{X},\mathfrak{so}(2d+1),1)\ ?\end{aligned}$$

Proposition 10.9. *An affirmative answer to Question 10.8 implies one for Question 10.1.*

Proof. By Proposition 10.7, it is enough to prove it for conformal blocks. Let C_0 be a nodal curve with g elliptic tails attached to a $\mathbb{P}^1$. Consider a one parameter family of $\mathcal{C}\to \mathrm{Spec}(\mathbb{C}[\![t]\!])$ such that the generic fiber is smooth and the special fiber is C_0. The normalization of C_0 is a $\mathbb{P}^1$ with g marked points, and g-elliptic curves each with one marked point. By factorization (cf. Section 9.7), it follows that $\mathcal{V}_{\omega_0}(C_0,\mathfrak{so}(2r+1),2s+1)$ splits up as a direct sum where each component looks like

$$\Big(\bigotimes_{i=1}^{g}\mathcal{V}_{\lambda_i}(E,\mathfrak{so}(2r+1),2s+1)\Big)\otimes\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)\ ,$$

and the direct sum is indexed by g-tuples $\vec{\lambda}=(\lambda_1,\dots,\lambda_g)$, $\lambda_i\in P_{2s+1}(\mathsf{SO}(2r+1)$. These weights have the special property that given μ and Λ, there exists at most one λ such that $(\lambda,\mu)\in B(\Lambda)$ (cf. Section 8.3). This guarantees that the conditions in Lemma 9.7 are satisfied, and the map in Question 10.1 splits as a direct sum (up to a nonnegative power of the parameter t) indexed by the set of g-tuple of points in $P_{2s+1}(\mathsf{SO}(2r+1))$.

Now [50, Prop. 9.11] and the compatibility of factorization/sewing with rank-level duality (cf. Sections 2.3 and 9.7, and also [65]), imply that Question 10.1 holds in the affirmative if this is true for Question 10.8 and if there is a rank-level duality isomorphism on $\mathbb{P}^1$

with g marked points and weights coming from $P_{2s+1}(\mathsf{SO}(2r+1))$. The rank-level duality isomorphism on $\mathbb{P}^1$ with n-marked points and weights coming from $P_{2s+1}(\mathsf{SO}(2r+1))$ has been proved in [50]. Hence, "yes" in Question 10.8 implies "yes" in Question 10.1. □

Remark 10.10. By the same strategy, it is natural to continue by degenerating the elliptic curves and applying factorization to reduce to the case of $\mathbb{P}^1$. However, serious issues occur due to the appearance of spin weights, which were avoided before. The problems are twofold. They are:

(1) The property that given Λ and μ, there exists an unique weights λ such that $(\lambda,\mu)\in B(\Lambda)$ fails. The failure of this property means the factorization of rank-level duality maps falls into nondiagonal blocks, so we can not do induction.
(2) Rank-level duality isomorphisms on $\mathbb{P}^1$ with spin weights fail to hold. This was explained in Section 9.2.

11. The case of elliptic curves

In this section, we study the following maps and investigate whether they are injective. An affirmative answer to both would give an affirmative answer to the strange duality question for elliptic curves. However, we shall see that this is in fact not the case (Proposition 11.2). Let E be an elliptic curve with one marked point and a choice of formal coordinate. The maps are:

$$\begin{aligned}\mathcal{V}_{\omega_0}(E,\mathfrak{so}(2r+1),2s+1)\longrightarrow\big(&\mathcal{V}^*_{\omega_0}(E,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_0}(E,\mathfrak{so}(2d+1),1)\\&\oplus\mathcal{V}^*_{(2r+1)\omega_1}(E,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_1}(E,\mathfrak{so}(2d+1),1)\big)\ ;\end{aligned}\tag{11.1}$$

$$\begin{aligned}\mathcal{V}_{(2s+1)\omega_1}(E,\mathfrak{so}(2r+1),2s+1)\longrightarrow\big(&\mathcal{V}^*_{\omega_0}(E,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_1}(E,\mathfrak{so}(2d+1),1)\\&\oplus\mathcal{V}^*_{(2r+1)\omega_1}(E,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_0}(E,\mathfrak{so}(2d+1),1)\big)\ .\end{aligned}\tag{11.2}$$

11.1. **Factorization for elliptic curves.** We will use factorization to further reduce to the case of $\mathbb{P}^1$ with three marked points. Let us first focus on (11.2). By definition of the diagram automorphism σ, we know that $(2r+1)\omega_1=\sigma(\omega_0)$. Hence, by factorization (cf. Section 9.7):

$$\begin{aligned}&\dim_{\mathbb{C}}\mathcal{V}_{(2r+1)\omega_1}(E,\mathfrak{so}(2s+1),2r+1)\\&\qquad=\sum_{\lambda\in P_{2s+1}(\mathfrak{so}(2r+1))}\dim_{\mathbb{C}}\mathcal{V}_{(2r+1)\omega_1,\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)\\&\qquad=\sum_{\lambda\in P_{2s+1}(\mathfrak{so}(2r+1))}\dim_{\mathbb{C}}\mathcal{V}_{\omega_0,\sigma(\lambda),\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)\qquad\text{(cf. [28])}\\&\qquad=|\{\lambda\in P_{2s+1}(\mathfrak{so}(2r+1))\mid\sigma(\lambda)=\lambda\}|\\&\qquad=|\mathcal{Y}_{r,s}\backslash\mathcal{Y}_{r,s-1}|\ .\end{aligned}$$

By the above calculation, the next result proves injectivity of (11.2):

Proposition 11.1. *The rank-level duality map between the following one dimensional conformal blocks is an isomorphism.*

$$\mathcal{V}_{(2s+1)\omega_1,\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)\longrightarrow \\ \mathcal{V}^*_{\omega_0,\lambda^*,\lambda^*}(\mathbb{P}^1,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)\ ,$$

where $\lambda = Y+\omega_r$ *and* $Y\in \mathcal{Y}_{r,s}\backslash\mathcal{Y}_{r,s-1}$ *and* $\lambda^* = Y^*+\omega_s$ *and* $Y^*\in\mathcal{Y}_{s,r}$ *obtained by taking the transpose of* Y *and taking the complement in an* $r\times s$ *box.*

Proof. This is a consequence of Theorem 9.2. □

It remains to investigate the injectivity of (11.1). The following appear in the factorizations of the conformal block $\mathcal{V}_{\omega_0}(E,\mathfrak{so}(2r+1),2s+1)$:

(1) $\mathcal{V}_{\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)$ for $\lambda\in\mathcal{Y}_{r,s}$.
(2) $\mathcal{V}_{\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)$ and $\mathcal{V}_{\sigma(\lambda),\sigma(\lambda)}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)$ for $\lambda\in\mathcal{Y}_{r,s-1}+\omega_r$.
(3) $\mathcal{V}_{\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)$ for $\lambda\in\mathcal{Y}_{r,s}\backslash\mathcal{Y}_{r,s-1}$.

Thus we need for check injectivity for each of the factors. For the factors of the form in (1), λ is a weight of $\mathsf{SO}(2r+1)$, and we only need that the rank-level duality map is an isomorphism for $\lambda\in\mathcal{Y}_{r,s}$:

$$\mathcal{V}_{\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)\to\mathcal{V}_{\lambda^T,\lambda^T}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)^*\otimes\mathcal{V}_{\omega_\epsilon,\omega_\epsilon}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)\ ,$$

where ϵ is zero or one depending on the parity of $|\lambda|$. This is done in [50]. The argument for $\lambda\in\mathcal{Y}_{r,s}\backslash\mathcal{Y}_{r,s-1}$ follows from Theorem 1.7. Thus, we are only left with the case when $\lambda\in\mathcal{Y}_{r,s-1}+\omega_r$. For every $\lambda = Y+\omega_r$, $Y\in\mathcal{Y}_{r,s-1}$, consider the map:

$$\begin{aligned}&\mathcal{V}_{\omega_0,\lambda,\lambda}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)\oplus\mathcal{V}_{\omega_0,\sigma(\lambda),\sigma(\lambda)}(\mathbb{P}^1,\mathfrak{so}(2r+1),2s+1)\\ &\longrightarrow \mathcal{V}^*_{\omega_0,\lambda^*,\lambda^*}(\mathbb{P}^1,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_0,\omega_d,\omega_d}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)\\ &\quad\oplus\ \mathcal{V}^*_{(2r+1)\omega_1,\lambda^*,\lambda^*}(\mathbb{P}^1,\mathfrak{so}(2s+1),2r+1)\otimes\mathcal{V}_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)\ .\end{aligned}\tag{11.3}$$

The following is the main result of this section.

Proposition 11.2. *The rank-level duality map in* (11.3) *is not injective.*

Proof of Theorem 1.10. By Propositions 10.9 and 11.2, it follows that the answer to Question 10.1 is negative. Then Proposition 10.6 completes the proof. □

Remark 11.3. It is easy to see that the dimensions of both the source and target of (11.3) is two. Since there are maps between all components that appear in map Question 10.8, it follows from Theorem 9.2 that all entries of the (2×2)-matrix are nonzero. The proof of Proposition 11.2 is broken up to into several steps. We will fix an explicit basis to compute the matrix of the map (11.3) and show that the determinant vanishes.

11.2. **Tensor decompositions.** We digress to give an explicit expression for highest weight vectors that is compatible with branching and tensor products associated to the marked points. As above, let $\mathfrak{X}$ refer to the data of a curve C with marked points and a choice of local coordinates. To make the notation more transparent, let us denote the inclusion of an abstract highest weight $\widehat{\mathfrak{so}}(2r+1)\oplus\widehat{\mathfrak{so}}(2s+1)$ module appearing in the branching of a highest weight $\widehat{\mathfrak{so}}(2d+1)$ module by

$$\beta_\Lambda^{\lambda\mu} : \mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_\mu(\mathfrak{so}(2s+1))\hookrightarrow\mathcal{H}_\Lambda(\mathfrak{so}(2d+1)) \ .$$

For an element $v_1\otimes\cdots\otimes v_n\in\mathcal{H}_{\lambda_1}(\mathfrak{g})\otimes\cdots\otimes\mathcal{H}_{\lambda_n}(\mathfrak{g})$, let $[v_1\otimes\cdots\otimes v_n]$ denote the vector in the quotient space of dual conformal blocks, e.g.

$$[v_1\otimes\cdots\otimes v_n]\in\mathcal{V}_{\vec\lambda}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)=\mathcal{H}_{\vec\lambda}(\mathfrak{so}(2s+1))/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec\lambda}(\mathfrak{so}(2s+1)) \ ,$$

where $\vec\lambda=(\lambda_1,\dots,\lambda_n)$. Since $\mathfrak{X}$ is fixed, we will drop the notation for the curve $\mathfrak{X}$ from the notation of conformal blocks. It is easy to check (and we have already used!) the fact that the map

$$\mathcal{V}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{\mu_1,\mu_2,\mu_3}(\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\Lambda_1,\Lambda_2,\Lambda_3}(\mathfrak{so}(2d+1),1) \ ,$$
$$[v_1\otimes v_2\otimes v_3]\otimes[w_1\otimes w_2\otimes w_3]\mapsto\left[\beta_{\Lambda_1}^{\lambda_1\mu_1}(v_1\otimes w_1)\otimes\beta_{\Lambda_2}^{\lambda_2\mu_2}(v_2\otimes w_2)\otimes\beta_{\Lambda_3}^{\lambda_3\mu_3}(v_3\otimes w_3)\right] \ ,$$

is well-defined.

Let $v_\lambda\in\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1))$ be the highest weight vector of the component $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda^*}(\mathfrak{so}(2s+1))$, and $\bar v_\lambda\in\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1))$ the highest weight vector of the component $\mathcal{H}_{\sigma(\lambda)}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda^*}(\mathfrak{so}(2s+1))$ as expressed explicitly as an element of $\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1))$ as in Section 8.4. We denote v^λ and $\bar v^\lambda$ to be the corresponding opposite highest weights again expressed explicitly.

Choose highest weight vectors v_1 and v_2 of $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))$ and $\mathcal{H}_{\lambda^*}(\mathfrak{so}(2s+1))$ such that $\beta_{\omega_d}^{\lambda\lambda^*}(v_1\otimes v_2)=v_\lambda$. Similarly choose v^1 and v^2 for the opposite highest weight such that $\beta_{\omega_d}^{\lambda\lambda^*}(v^1\otimes v^2)=v^\lambda$. Let $\bar v_1$ be such that $\beta_{\omega_d}^{\sigma(\lambda)\lambda^*}(\bar v_1\otimes v_2)=\bar v_\lambda$ and similarly choose $\bar v^1$ for the corresponding $\bar v^\lambda$.

Let $\tilde v$ be a vector in $\mathcal{H}_{\omega_1}(\mathfrak{so}(2d+1))$ which is equal (up to a scalar) to $R^{2r+1}(B_1^0)$ acting on the highest weight vector of the component $\mathcal{H}_{\omega_0}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{(2r+1)\omega_1}(\mathfrak{so}(2s+1))$ expressed explicitly in Clifford algebra terms. Since we get the vector $\tilde v$ by acting only on the right component in the tensor decomposition, it follows that $\tilde v$ is a pure tensor in $\mathcal{H}_{\omega_0}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{(2r+1)\omega_1}(\mathfrak{so}(2s+1))$. Hence, we can choose $x\in\mathcal{H}_{(2r+1)\omega_1}(\mathfrak{so}(2s+1))$ such that $\beta_{\omega_1}^{0,(2r+1)\omega_1}(1\otimes x)=\tilde v$.

We are interested in the following classes.

- $\mathcal{V}_{\omega_0,\lambda,\lambda}(\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{\omega_0,\lambda^*,\lambda^*}(\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\omega_0,\omega_d,\omega_d}(\mathfrak{so}(2d+1),1)$

$$[1\otimes v_1\otimes v^1]\otimes[1\otimes v_2\otimes v^2]\mapsto[\beta_0^{0,0}(1\otimes1)\otimes\beta_{\omega_d}^{\lambda\lambda^*}(v_1\otimes v_2)\otimes\beta_{\omega_d}^{\lambda\lambda^*}(v^1\otimes v^2)]$$
$$=[1\otimes v_\lambda\otimes v^\lambda] \ .$$

- $\mathcal{V}_{\omega_0,\sigma(\lambda),\sigma(\lambda)}(\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{\omega_0,\lambda^*,\lambda^*}(\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\omega_0,\omega_d,\omega_d}(\mathfrak{so}(2d+1),1)$

$$[1\otimes\bar v_1\otimes\bar v^1]\otimes[1\otimes v_2\otimes v^2]\mapsto[\beta_0^{0,0}(1\otimes 1)\otimes\beta_{\omega_d}^{\sigma(\lambda)\lambda^*}(\bar v_1\otimes v_2)\otimes\beta_{\omega_d}^{\sigma(\lambda)\lambda^*}(\bar v^1\otimes v^2)]$$
$$=[1\otimes\bar v_\lambda\otimes\bar v^\lambda]\ .$$

- $\mathcal{V}_{\omega_0,\lambda,\lambda}(\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{(2r+1)\omega_1,\lambda^*,\lambda^*}(\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\omega_1,\omega_d,\omega_d}(\mathfrak{so}(2d+1),1)$

$$[1\otimes v_1\otimes v^1]\otimes[x\otimes v_2\otimes v^2]\mapsto[\beta_{\omega_1}^{0,(2r+1)\omega_1}(1\otimes x)\otimes\beta_{\omega_d}^{\lambda\lambda^*}(v_1\otimes v_2)\otimes\beta_{\omega_d}^{\lambda\lambda^*}(v^1\otimes v^2)]$$
$$=[\tilde v\otimes v_\lambda\otimes v^\lambda]\ .$$

- $\mathcal{V}_{\omega_0,\sigma(\lambda),\sigma(\lambda)}(\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{(2r+1)\omega_1,\lambda^*,\lambda^*}(\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\omega_1,\omega_d,\omega_d}(\mathfrak{so}(2d+1),1)$

$$[1\otimes\bar v_1\otimes\bar v^1]\otimes[x\otimes v_2\otimes v^2]\mapsto[\beta_{\omega_1}^{0,(2r+1)\omega_1}(1\otimes x)\otimes\beta_{\omega_d}^{\sigma(\lambda)\lambda^*}(\bar v_1\otimes v_2)\otimes\beta_{\omega_d}^{\sigma(\lambda)\lambda^*}(\bar v^1\otimes v^2)]$$
$$=[\tilde v\otimes\bar v_\lambda\otimes\bar v^\lambda]\ .$$

11.3. **Case by case analysis.**

11.3.1. *The case* $(\omega_0,\lambda,\lambda)\times(\omega_0,\lambda^*,\lambda^*)\to(\omega_0,\omega_d,\omega_d)$. Let $\lambda=Y+\omega_r$, $Y\in\mathcal{Y}_{r,s}$. Let $v_\lambda=\bigwedge_{\widetilde{Y}_{i,j}=\blacksquare}\phi_{i,j}$ as in Proposition 8.3. Similarly let $v^\lambda=\bigwedge_{\widetilde{Y}_{i,j}=\blacksquare}\phi^{i,j}$. Let $\langle\Psi|$ denote the unique up to constants nonzero element of $\mathcal{V}^*_{\omega_0,\omega_d,\omega_d}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)$. This was discussed in Section 7.3.1. Let $B(\ ,\ )$ be the nondegenerate bilinear form on W_d. The choice of v_λ (resp. v^λ) implies that $\langle\Psi|1\otimes v_\lambda\otimes v^\lambda\rangle$ is up to a sign equal to $\prod_{\widetilde{Y}_{i,j}=\blacksquare}B(\phi_{i,j},\phi^{i,j})$ which is nonzero.

11.3.2. *The case* $(\omega_0,\lambda,\lambda)\times((2r+1)\omega_1,\lambda^*,\lambda^*)\to(\omega_1,\omega_d,\omega_d)$. Let $\lambda=Y+\omega_r$ and further assume that $\lambda\in\mathcal{Y}_{r,s-1}$. We choose v_λ and v^λ as above in the previous case. We need to choose a vector in $\mathcal{H}_{\omega_0}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{(2r+1)\omega_1}(\mathfrak{so}(2s+1))$ as an explicit element in $\mathcal{H}_{\omega_1}(\mathfrak{so}(2d+1))$. We choose the vector: $\tilde v:=B^{2,0}_{2,0}(-1)\cdots B^{r,0}_{r,0}(-1)B^{0,0}_{1,0}(-1)\phi^{1,0}(-\frac12)$. Let $\langle\widetilde{\Psi}\mid$ be the unique nonzero vector of $\mathcal{V}^*_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1,\mathfrak{so}(2d+1),1)$ normalized such that it is equal to the one induced from Clifford multiplication (cf. Section 7.3.2). We now evaluate the following using gauge symmetry (cf. Section 2.3) and choosing the points to be $(1,0,\infty)$ with the obvious local coordinates.

$$\begin{aligned}
\langle\widetilde{\Psi}\mid\tilde v\otimes v_\lambda\otimes v^\lambda\rangle &= \langle\widetilde{\Psi}\mid B^{2,0}_{2,0}(-1)\cdots B^{r,0}_{r,0}(-1)B^{0,0}_{1,0}(-1)\phi^{1,0}(-\tfrac12)\otimes v_\lambda\otimes v^\lambda\rangle\\
&= (-1)\langle\widetilde{\Psi}\mid B^{r,0}_{r,0}(-1)B^{0,0}_{1,0}(-1)\phi^{1,0}(-\tfrac12)\otimes B^{2,0}_{2,0}v_\lambda\otimes v^\lambda\rangle\\
&= \frac12\langle\widetilde{\Psi}\mid B^{r,0}_{r,0}(-1)B^{0,0}_{1,0}(-1)\phi^{1,0}(-\tfrac12)\otimes v_\lambda\otimes v^\lambda\rangle\\
&= \frac{1}{2^{r-1}}\langle\widetilde{\Psi}\mid B^{0,0}_{1,0}(-1)\phi^{1,0}(-\tfrac12)\otimes v_\lambda\otimes v^\lambda\rangle\\
&= \frac{1}{2^{r-1}}\langle\widetilde{\Psi}\mid \phi^{1,0}(-\tfrac12)\otimes B^{0,0}_{1,0}v_\lambda\otimes v^\lambda\rangle\\
&= \frac{1}{2^{r-1}}\frac{(-1)^{rs-|\lambda|+1}}{\sqrt{2}}\langle\widetilde{\Psi}\mid \phi^{1,0}(-\tfrac12)\otimes\phi_{1,0}\wedge v_\lambda\otimes v^\lambda\rangle\ .
\end{aligned}$$

We used the following in the calculation above.

Lemma 11.4. *With the same notation,*

(1) *for $i > 0$, we get $B^{i,0}_{i,0} v_\lambda = \frac{1}{2} v_\lambda$;*
(2) $B^{0,0}_{1,0} v_\lambda = \frac{(-1)^{rs-|\lambda|+1}}{\sqrt{2}} \phi_{1,0} \wedge v_\lambda$.

Proof. The proof is as usual by a direct calculation. The most important observation is that $\sqrt{2}\phi^{0,0} v = (-1)^p v$, where p is the degree of v in $\bigwedge W_d^-$. □

11.3.3. *The case* $(\omega_0, \sigma(\lambda), \sigma(\lambda)) \times (\omega_0, \lambda^*, \lambda^*) \to (\omega_0, \omega_d, \omega_d)$. Let $\lambda = Y + \omega_r$ and further assume that the number of boxes in the first row of Y is $s-1$. Assume that $\overline{Y} = Y + y_{1,-1}$, so that the number of boxes of $\overline{Y}$ is s. Let $\bar\lambda = \overline{Y} + \omega_r$ and $v_{\bar\lambda} = \bigwedge_{\widetilde{\overline{Y}}_{i,j} = \blacksquare} \phi_{i,j}$ (for notation, see Section 8.4).

We need to choose a highest weight vector of $\mathcal{H}_{\sigma(\lambda)}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^*}(\mathfrak{so}(2s+1))$ as an explicit element in $\mathcal{H}_{\omega_d}(\mathfrak{so}(2d+1))$. Applying Corollary 8.5, we choose $\bar v_\lambda := B^{1,1}_{0,0}(-1) v_{\bar\lambda}$. Let $\langle \Psi \mid$ be the unique element of the $\mathcal{V}^*_{\omega_0,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ We want to evaluate the following:

$$
\begin{aligned}
\langle \Psi \mid 1 \otimes \bar v_\lambda \otimes \bar v^\lambda \rangle &= \langle \Psi \mid 1 \otimes B^{1,1}_{0,0}(-1) v_{\bar\lambda} \otimes B^{0,0}_{1,1}(-1) v^{\bar\lambda} \rangle \\
&= -\langle \Psi \mid 1 \otimes B^{0,0}_{1,1}(1) B^{1,1}_{0,0}(-1) v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= -\langle \Psi \mid 1 \otimes \big([B^{0,0}_{1,1}, B^{1,1}_{0,0}] + (B^{0,0}_{1,1}, B^{1,1}_{0,0}) c \big) v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= -\langle \Psi \mid 1 \otimes [B^{0,0}_{1,1}, B^{1,1}_{0,0}] v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle - \langle \Psi \mid 1 \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= \langle \Psi \mid 1 \otimes B^{1,1}_{1,1} v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle - \langle \Psi \mid 1 \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= \frac{1}{2} \langle \Psi \mid 1 \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle - \langle \Psi \mid 1 \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= -\frac{1}{2} \langle \Psi \mid 1 \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \ .
\end{aligned}
$$

11.3.4. *The case* $(\omega_0, \sigma(\lambda), \sigma(\lambda)) \times ((2r+1)\omega_1, \lambda^*, \lambda^*) \to (\omega_1, \omega_d, \omega_d)$. Let λ be such that the Young diagram associated to the weight λ has exactly $s-1$ boxes in the first row. Let $\langle \widetilde{\Psi} \mid$ be the unique nonzero element of $\mathcal{V}^*_{\omega_1,\omega_d,\omega_d}(\mathbb{P}^1, \mathfrak{so}(2d+1), 1)$ normalized such that it is equal to the Clifford multiplication. We want to evaluate

$$
\begin{aligned}
\langle \widetilde{\Psi} \mid \tilde v \otimes \bar v_\lambda \otimes \bar v^\lambda \rangle &= \frac{1}{2} \langle \Psi \mid \tilde v \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle = \frac{1}{2^r} \langle \widetilde{\Psi} \mid B^{0,0}_{1,0}(-1) \phi^{1,0}(-\tfrac{1}{2}) \otimes v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= \frac{1}{2^r} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes B^{0,0}_{1,0} v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= \frac{1}{2^r} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes {:}\, \phi^{0,0} \phi_{1,0} \,{:} \, v_{\bar\lambda} \otimes v^{\bar\lambda} \rangle \\
&= \frac{1}{2^r} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes \phi^{0,0}(\phi_{1,0} \wedge v_{\bar\lambda}) \otimes v^{\bar\lambda} \rangle
\end{aligned}
$$

$$= \frac{(-1)^{(rs-|\bar{\lambda}|+1)}}{(\sqrt{2})^{2r+1}} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes \phi_{1,0} \wedge v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle$$
$$= \frac{(-1)^{(rs-|\lambda|)}}{(\sqrt{2})^{2r+1}} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes \phi_{1,0} \wedge v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle \ .$$

11.4. **Proof of Proposition 11.2.** Let $\lambda = Y + \omega_r$, $Y \in \mathcal{Y}_{r,s-1}$, and we further assume that the number of boxes in the first row of Y is exactly $s-1$. The previous calculations tell us that the matrix of the map (11.3) is the following: $A_\lambda :=$

$$\begin{bmatrix} \langle \Psi \mid 1 \otimes v_\lambda \otimes v^\lambda \rangle & -\frac{1}{a^2} \langle \Psi \mid 1 \otimes v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle \\ \frac{(-1)^{rs+1-|\lambda|}}{a^{2r-1}} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\frac{1}{2}) \otimes \phi_{1,0} \wedge v_\lambda \otimes v^\lambda \rangle & \frac{(-1)^{(rs-|\lambda|)}}{a^{2r+1}} \langle \widetilde{\Psi} \mid \phi^{1,0}(-\frac{1}{2}) \otimes \phi_{1,0} \wedge v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle \end{bmatrix}.$$

Then the determinant is (up to a constant): $\det A_\lambda \simeq$

$$\langle \Psi \mid 1 \otimes v_\lambda \otimes v^\lambda \rangle \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes \phi_{1,0} \wedge v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle - \langle \Psi \mid 1 \otimes v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle \langle \widetilde{\Psi} \mid \phi^{1,0}(-\tfrac{1}{2}) \otimes \phi_{1,0} \wedge v_\lambda \otimes v^\lambda \rangle \ .$$

But now by the construction of $\langle \widetilde{\Psi} \mid$ and $\langle \Psi \mid$ in Sections 7.3.2 and 7.3.1, we get

- $\langle \widetilde{\Psi} \mid \phi^{1,0}(-\frac{1}{2}) \otimes \phi_{1,0} \wedge v_\lambda \otimes v^\lambda \rangle = \langle \Psi \mid 1 \otimes v_\lambda \otimes v^\lambda \rangle$;
- $\langle \widetilde{\Psi} \mid \phi^{1,0}(-\frac{1}{2}) \otimes \phi_{1,0} \wedge v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle = \langle \Psi \mid 1 \otimes v_{\bar{\lambda}} \otimes v^{\bar{\lambda}} \rangle$.

It follows that $\det A_\lambda = 0$. This completes the proof.

Appendix A. Computations in the Clifford algebra

In this section, we compute some vectors in the highest weight modules as explicit elements in the infinite dimensional Clifford algebra.

A.1. **Action of $L(B^i_j)$.** Consider the rectangle $r \times s$ as a Young diagram Y where the rows are indexed by integer in $\{1, \dots, r\}$ and the columns by $\{-s, \dots, -1\}$. Let (i, j) be the coordinates of Y and let $v := \bigwedge_{\tilde{Y}_{i,j} = \blacksquare} \phi_{i,j}$ (cf. Section 8.4).

Proposition A.1. *Let v as before be the highest weight vector of the component with highest weight $(\omega_r, (2r+1)\omega_s)$. Let $0 \le k \le m \le r$, then*

$$L(B^{-k}_{(k+1)}) L(B^{-(k+2)}_{k+3}) \cdots L(B^{-(m-3)}_{m-2}) L(B^{-(m-1)}_m) \cdot v = \phi_{k,0} \wedge \phi_{k+1,0} \wedge \cdots \wedge \phi_{m,0} \wedge v \ .$$

A.2. **Action of $R^k(B^0_1)$.** Let $v_k = \phi^{1,1}(-\frac{1}{2}) \phi^{2,1}(-\frac{1}{2}) \cdots \phi^{k,1}(-\frac{1}{2}) \cdot 1$. In this section, we want to give explicit expressions for $R^k(B^i_j) v_k$. First, consider the case when $k = 1$.

Lemma A.2. *Consider the highest weight vector $\phi^{1,1}(-\frac{1}{2}) \cdot 1$ of the component with highest weight (ω_1, ω_1). Then $R(B^0_1) \phi^{1,1}(-\frac{1}{2}) \cdot 1 = \phi^{1,0}(-\frac{1}{2})$.*

Now we want to compute $R^2(B^0_1) v_2$. We first have the following lemma

Lemma A.3. *For the component* $(\omega_2, 2\omega_1)$, $\phi^{1,1}(-\frac{1}{2})\phi^{2,2}(-\frac{1}{2})\cdot 1$, *is a highest weight vector. Moreover,*

$$R(B_1^0)v_2 = \phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) + \phi^{1,1}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2}) \ . \tag{A.1}$$

Proof. The proof is by a direct computation. As before, we know that $R(B_1^0)$ acts as $\sum_{-r\le q\le r} :\phi^{q,0}\phi_{q,1}:$ on the infinite dimensional Clifford algebra for $\widehat{\mathfrak{so}}(2d+1)$. Hence,

$$\begin{aligned} R(B_1^0)\phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) &= \sum_{-r\le q\le r} \big[\phi^{q,0}(-\tfrac{1}{2})\phi_{q,1}(\tfrac{1}{2}) - \phi_{q,1}(-\tfrac{1}{2})\phi^{q,0}(\tfrac{1}{2})\big]\phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) \\ &= \sum_{-r\le q\le r} \big[\phi^{q,0}(-\tfrac{1}{2})\phi_{q,1}(\tfrac{1}{2})\big]\phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) \\ &= \phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) - \phi^{2,0}(-\tfrac{1}{2})\phi^{1,1}(-\tfrac{1}{2}) \ . \end{aligned}$$

□

Proposition A.4. *We have:*

$$R^2(B_1^0)v_2 = 2\left[\phi^{1,0}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\right) - \left(\phi^{1,-1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) + \phi^{1,1}(-\tfrac{1}{2})\phi^{2,-1}(-\tfrac{1}{2})\right] \ .$$

Proof. Compute using (A.1),

$$\begin{aligned} R(B_1^0)\phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) &= \sum_{-r\le q\le r} \big[\phi^{q,0}(-\tfrac{1}{2})\phi_{q,1}(\tfrac{1}{2}) - \phi_{q,1}(-\tfrac{1}{2})\phi^{q,0}(\tfrac{1}{2})\big]\phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) \\ &= -\phi^{2,0}(-\tfrac{1}{2})\phi^{1,0}(-\tfrac{1}{2}) - \phi_{-1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2}) \\ &= \phi^{1,0}(-\tfrac{1}{2})\phi_{-2,0}(-\tfrac{1}{2}) + \phi^{2,1}(-\tfrac{1}{2})\phi_{-1,1}(-\tfrac{1}{2}) \\ R(B_1^0)\phi^{2,0}(-\tfrac{1}{2})\phi^{1,1}(-\tfrac{1}{2}) &= \sum_{-r\le q\le r} \big[\phi^{q,0}(-\tfrac{1}{2})\phi_{q,1}(\tfrac{1}{2}) - \phi_{q,1}(-\tfrac{1}{2})\phi^{q,0}(\tfrac{1}{2})\big]\phi^{2,0}(-\tfrac{1}{2})\phi^{1,1}(-\tfrac{1}{2}) \\ &= -\phi^{1,0}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2}) - \phi_{-2,1}(-\tfrac{1}{2})\phi^{1,1}(-\tfrac{1}{2}) \\ &= -\phi^{1,0}(-\tfrac{1}{2})\phi_{-2,0}(-\tfrac{1}{2}) - \phi^{2,-1}(-\tfrac{1}{2})\phi_{-1,-1}(-\tfrac{1}{2}) \ . \end{aligned}$$

□

We use the following calculation in the proof of strange duality for the pair $(\omega_2, \omega_r, \omega_r)$ and $(2\omega_1, (2r+1)\omega_s, (2r+1)\omega_s)$.

Lemma A.5. *Let* $w = \phi_{1,0}\wedge\phi_{2,0}\wedge\bigwedge_{1\le i\le r,-s\le j\le -1}\phi_{i,j}$. *Then the following hold in* $\mathcal{H}_{\omega_r}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{(2s+1)\omega_s}(\mathfrak{so}(2s+1))$*:*

$$B^{2,1}_{-1,1}w = B^{2,-1}_{-1,-1}w = 0 \ ; \ B^{1,0}_{-2,0}w = \bigwedge_{1\le i\le r,-s\le j\le -1}\phi_{i,j} \ .$$

Next we compute $R^3(B_1^0)v_3$. Our strategy is same as the previous steps.

Proposition A.6. *We have:*

$$R^3(B_1^0)v_3 = 6\left[\phi^{1,0}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2})\right] - 3\big[\phi^{1,-1}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2})$$

$$
\begin{aligned}
&+ \phi^{1,0}(-\tfrac{1}{2})\phi^{2,-1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2}) + \phi^{1,-1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2}) \\
&+ \phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,-1}(-\tfrac{1}{2}) + \phi^{1,1}(-\tfrac{1}{2})\phi^{2,-1}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2}) \\
&+ \phi^{1,1}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,-1}(-\tfrac{1}{2})\big] \ .
\end{aligned}
$$

Proof. The proof follows by applying the expression for $R(B_1^0)$ successively:

$$
\begin{aligned}
R(B_1^0)v_3 &= \phi^{1,0}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2}) + \phi^{1,1}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2}) \\
&\quad + \phi^{1,1}(-\tfrac{1}{2})\phi^{2,1}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2}) \ . \\
R^2(B_1^0)v_3 &= 2\big(\phi^{1,0}(-\tfrac{1}{2})\phi^{2,0}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2}) + \phi^{1,0}(-\tfrac{1}{2}\phi^{2,1}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2}) \\
&\quad + \phi^{1,1}(-\tfrac{1}{2}\phi^{2,0}(-\tfrac{1}{2})\phi^{3,0}(-\tfrac{1}{2})\big) \\
&\quad - \big(\phi^{1,1}(-\tfrac{1}{2}\phi^{2,1}(-\tfrac{1}{2})\phi^{3,-1}(-\tfrac{1}{2}) + \phi^{1,1}(-\tfrac{1}{2})\phi^{2,-1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2}) \\
&\quad + \phi^{1,-1}(-\tfrac{1}{2}\phi^{2,1}(-\tfrac{1}{2})\phi^{3,1}(-\tfrac{1}{2})\big).
\end{aligned}
$$

and acting once more by $R(B_1^0)$. □

We now gather these calculations into the following algorithm:

- If $v_k = \phi^{1,1}(-\frac{1}{2})\cdots\phi^{k,1}(-\frac{1}{2})$, then the $\mathfrak{h}_2$-weight of $R^k(B_1^0)v_k$ is zero, where $\mathfrak{h}_2$ is the Cartan subalgebra of $\mathfrak{so}(2s+1)$.
- The expression for $R(B_1^0)$, viewed as an operator on the Clifford module for $\widehat{\mathfrak{so}}(2d+1)$, implies that
  - if $v = \phi^{1,a_1}(-\frac{1}{2})\cdots\phi^{k,a_k}(-\frac{1}{2})$, where $0 \le a_1 + \cdots + a_k \le k$, and each $a_i \in \{-1, 0, 1\}$, then the action of $R(B_1^0)$ on v is a sum of expressions of the form $\phi^{1,b_1}(-\frac{1}{2})\cdots\phi^{k,b_k}(-\frac{1}{2})$, where exactly one of the b_i's is different from a_i;
  - the operator $R(B_1^0)$ can change an $a_i = 1$ to $b_i = 0$, or $a_i = 0$ to $b_i = -1$. In the latter case, this introduces a minus sign in front of the new expression. In particular for each expression $\phi^{1,b_1}(-\frac{1}{2})\cdots\phi^{k,b_k}(-\frac{1}{2})$ appearing in $R(B_1^0)v$, we get $b_1+\cdots+b_k+1 = a_1+\cdots+a_k$. For examples, see the previous lemmas.
- Thus, applying the operator $R(B_1^0)$ to v_k, k-times, we get an expression which is a sum of terms of the form $(-1)^m\phi^{1,c_1}(-\frac{1}{2})\cdots\phi^{k,c_k}(-\frac{1}{2})$, with multiplicities, where $c_1 + \cdots + c_k = 0$, and each $-1 \le c_i \le 1$, and m is the number of (-1)'s appearing among the c_i's.
- The multiplicity of the expression $\phi^{1,0}(-\frac{1}{2})\cdots\phi^{k,0}(-\frac{1}{2})$ is $k!$.

To summarize, we have the following.

Proposition A.7. *As an element of $\mathcal{H}_{\omega_k}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{k\omega_1}(\mathfrak{so}(2s+1))$, the vector $R^k(B_1^0)v_k$ is of the form $k!\phi^{1,0}(-\frac{1}{2})\cdots\phi^{k,0}(-\frac{1}{2})$, plus a sum of terms of the form $B^{i,a}_{-j,b}(-1)w$, where $i \neq j$ are positive integers and a, b are nonzero.*

References

1. Takeshi Abe, *Strange duality for parabolic symplectic bundles on a pointed projective line*, Int. Math. Res. Not. IMRN (2008), Art. ID rnn121, 47.

2. ———, *Moduli of oriented orthogonal sheaves on a nodal curve*, Kyoto J. Math. **53** (2013), no. 1, 55–90.
3. Jørgen Ellegaard Andersen, *Hitchin's connection, Toeplitz operators, and symmetry invariant deformation quantization*, Quantum Topol. **3** (2012), no. 3-4, 293–325.
4. Jørgen Ellegaard Andersen and Jens Fjelstad, *Reducibility of quantum representations of mapping class groups*, Lett. Math. Phys. **91** (2010), no. 3, 215–239.
5. Jørgen Ellegaard Andersen and Gregor Masbaum, *Involutions on moduli spaces and refinements of the Verlinde formula*, Math. Ann. **314** (1999), no. 2, 291–326.
6. Michael Atiyah and Raoul Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A **308** (1982), 523–615.
7. Scott Axelrod, Steve Della Pietra, and Edward Witten, *Geometric quantization of Chern-Simons gauge theory*, J. Differential Geom. **33** (1991), no. 3, 787–902.
8. F. Alexander Bais and Peter G. Bouwknegt, *A classification of subgroup truncations of the bosonic string*, Nuclear Phys. B **279** (1987), no. 3-4, 561–570.
9. Arnaud Beauville, *Vector bundles on curves and generalized theta functions: recent results and open problems*, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), Math. Sci. Res. Inst. Publ., vol. 28, Cambridge Univ. Press, Cambridge, 1995, pp. 17–33.
10. ———, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 75–96.
11. ———, *Orthogonal bundles on curves and theta functions*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 5, 1405–1418.
12. Arnaud Beauville and Yves Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), no. 2, 385–419.
13. Arnaud Beauville, Yves Laszlo, and Christoph Sorger, *The Picard group of the moduli of G-bundles on a curve*, Compositio Math. **112** (1998), no. 2, 183–216.
14. Arnaud Beauville, M. S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor*, J. Reine Angew. Math. **398** (1989), 169–179.
15. Prakash Belkale, *The strange duality conjecture for generic curves*, J. Amer. Math. Soc. **21** (2008), no. 1, 235–258 (electronic).
16. ———, *Strange duality and the Hitchin/WZW connection*, J. Differential Geom. **82** (2009), no. 2, 445–465.
17. ———, *Orthogonal bundles, theta characteristics and symplectic strange duality*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 185–193.
18. Jean-François Boutot, *Singularités rationnelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987), no. 1, 65–68.
19. Arzu Boysal and Christian Pauly, *Strange duality for Verlinde spaces of exceptional groups at level one*, Int. Math. Res. Not. IMRN (2010), no. 4, 595–618.
20. Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen, *Higgs bundles for the non-compact dual of the special orthogonal group*, Geom. Dedicata **175** (2015), 1–48.
21. R. Donagi and T. Pantev, *Langlands duality for Hitchin systems*, Invent. Math. **189** (2012), no. 3, 653–735.
22. Ron Donagi and Loring W. Tu, *Theta functions for* $\mathrm{SL}(n)$ *versus* $\mathrm{GL}(n)$, Math. Res. Lett. **1** (1994), no. 3, 345–357.
23. V. G. Drinfeld and Carlos Simpson, *B-structures on G-bundles and local triviality*, Math. Res. Lett. **2** (1995), no. 6, 823–829.
24. Najmuddin Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 145–176.

25. Gerd Faltings, *Stable G-bundles and projective connections*, J. Algebraic Geom. **2** (1993), no. 3, 507–568.
26. ———, *A proof for the Verlinde formula*, J. Algebraic Geom. **3** (1994), no. 2, 347–374.
27. I. B. Frenkel, *Spinor representations of affine Lie algebras*, Proc. Nat. Acad. Sci. U.S.A. **77** (1980), no. 11, part 1, 6303–6306.
28. Jürgen Fuchs and Christoph Schweigert, *The action of outer automorphisms on bundles of chiral blocks*, Comm. Math. Phys. **206** (1999), no. 3, 691–736.
29. William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.
30. Hans Grauert and Oswald Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.
31. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.
32. Koji Hasegawa, *Spin module versions of Weyl's reciprocity theorem for classical Kac-Moody Lie algebras—an application to branching rule duality*, Publ. Res. Inst. Math. Sci. **25** (1989), no. 5, 741–828.
33. Nigel Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.
34. ———, *Flat connections and geometric quantization*, Comm. Math. Phys. **131** (1990), no. 2, 347–380.
35. ———, *Langlands duality and G_2 spectral curves*, Q. J. Math. **58** (2007), no. 3, 319–344.
36. Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.
37. Victor G. Kac and Dale H. Peterson, *Spin and wedge representations of infinite-dimensional Lie algebras and groups*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), no. 6, part 1, 3308–3312.
38. Victor G. Kac and Minoru Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Adv. in Math. **70** (1988), no. 2, 156–236.
39. George Kempf and Linda Ness, *Tensor products of fundamental representations*, Canad. J. Math. **40** (1988), no. 3, 633–648.
40. Shrawan Kumar, *Demazure character formula in arbitrary Kac-Moody setting*, Invent. Math. **89** (1987), no. 2, 395–423.
41. ———, *Infinite Grassmannians and moduli spaces of G-bundles*, Vector bundles on curves—new directions (Cetraro, 1995), Lecture Notes in Math., vol. 1649, Springer, Berlin, 1997, pp. 1–49.
42. Shrawan Kumar and M. S. Narasimhan, *Picard group of the moduli spaces of G-bundles*, Math. Ann. **308** (1997), no. 1, 155–173.
43. Shrawan Kumar, M. S. Narasimhan, and A. Ramanathan, *Infinite Grassmannians and moduli spaces of G-bundles*, Math. Ann. **300** (1994), no. 1, 41–75.
44. Yves Laszlo, *Hitchin's and WZW connections are the same*, J. Differential Geom. **49** (1998), no. 3, 547–576.
45. Yves Laszlo and Christoph Sorger, *The line bundles on the moduli of parabolic G-bundles over curves and their sections*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 4, 499–525.
46. Peter Littelmann, *A generalization of the Littlewood-Richardson rule*, J. Algebra **130** (1990), no. 2, 328–368.
47. Alina Marian and Dragos Oprea, *The level-rank duality for non-abelian theta functions*, Invent. Math. **168** (2007), no. 2, 225–247.
48. Masaki Maruyama, *Elementary transformations in the theory of algebraic vector bundles*, Algebraic geometry (La Rábida, 1981), Lecture Notes in Math., vol. 961, Springer, Berlin, 1982, pp. 241–266.
49. Olivier Mathieu, *Formules de caractères pour les algèbres de Kac-Moody générales*, Astérisque (1988), no. 159-160, 267.
50. Swarnava Mukhopadhyay, *Rank-level duality of conformal blocks for odd orthogonal Lie algebras in genus 0*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6741–6778.
51. ———, *Strange duality of Verlinde spaces for G_2 and F_4*, Math. Z. **283** (2016), no. 1-2, 387–399.

52. David Mumford, *Theta characteristics of an algebraic curve*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 181–192.
53. Stephen G. Naculich and Howard J. Schnitzer, *Duality relations between* $\mathrm{SU}(N)_k$ *and* $\mathrm{SU}(k)_N$ *WZW models and their braid matrices*, Phys. Lett. B **244** (1990), no. 2, 235–240.
54. Tomoki Nakanishi and Akihiro Tsuchiya, *Level-rank duality of WZW models in conformal field theory*, Comm. Math. Phys. **144** (1992), no. 2, 351–372.
55. W. M. Oxbury, *Spin Verlinde spaces and Prym theta functions*, Proc. London Math. Soc. (3) **78** (1999), no. 1, 52–76.
56. W. M. Oxbury and S. M. J. Wilson, *Reciprocity laws in the Verlinde formulae for the classical groups*, Trans. Amer. Math. Soc. **348** (1996), no. 7, 2689–2710.
57. C. Pauly and S. Ramanan, *A duality for spin Verlinde spaces and Prym theta functions*, J. London Math. Soc. (2) **63** (2001), no. 3, 513–532.
58. Christian Pauly, *Espaces de modules de fibrés paraboliques et blocs conformes*, Duke Math. J. **84** (1996), no. 1, 217–235.
59. A. Ramanathan, *Moduli for principal bundles over algebraic curves. II*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), no. 4, 421–449.
60. Günter Scheja, *Riemannsche Hebbarkeitssätze für Cohomologieklassen*, Math. Ann. **144** (1961), 345–360.
61. Peter Slodowy, *On the geometry of Schubert varieties attached to Kac-Moody Lie algebras*, Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, pp. 405–442.
62. Dave Swinarski, `conformalblocks:` *a macaulay2 package for computing conformal block divisors*, http://www.math.uiuc.edu/Macaulay2/ (2010).
63. Constantin Teleman, *Lie algebra cohomology and the fusion rules*, Comm. Math. Phys. **173** (1995), no. 2, 265–311.
64. Michael Thaddeus, *Conformal field theory and the cohomology of the moduli space of stable bundles*, J. Differential Geom. **35** (1992), no. 1, 131–149.
65. Akihiro Tsuchiya, Kenji Ueno, and Yasuhiko Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Integrable systems in quantum field theory and statistical mechanics, Adv. Stud. Pure Math., vol. 19, Academic Press, Boston, MA, 1989, pp. 459–566.
66. Bert van Geemen and Aise Johan de Jong, *On Hitchin's connection*, J. Amer. Math. Soc. **11** (1998), no. 1, 189–228.

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