THE ACTION OF THE MAPPING CLASS GROUP ON REPRESENTATION VARIETIES

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1. Introduction

These notes are based on lectures given at Zhejiang University, July 14–20, 2008. My goal was to present some analytic techniques that can be used to study the action of the mapping class group on the representations varieties of surface groups. The subject has been quite active of late. For more information – and more complete coverage of recent work – I refer the interested reader to the introductory article to the 2007 AIM conference [AIM] and to the survey article [G06]. I am very grateful to the organizers of the conference for their extraordinary hospitality, and especially to Lizhen Ji for his work in preparing this volume. I would also like to thank Bill Goldman for comments.

Let S be a closed oriented surface of genus $g \geq 2$ with fundamental group $\pi = \pi_1(S, p)$. If G is a connected Lie group, we will be interested in the space

$$\mathfrak{R}(\pi, G) = \mathrm{Hom}(\pi, G) /\!\!/ G$$

where G acts on the space of representations by conjugation:

$$g \cdot \rho = g\rho g^{-1}$$

The double slash essentially means the quotient by this action. If G is noncompact, some orbits may not be closed, and so to get a Hausdorff moduli space we must identify a given orbit with those orbits in its closure. This will be explained with a particular example in Section 4. Note that $\mathrm{Aut}(\pi)$ acts on $\mathrm{Hom}(\pi, G)$ by precomposition. Since inner automorphisms act by overall conjugation, this gives an action of the outer automorphism group $\mathrm{Out}(\pi)$ on $\mathfrak{R}(\pi, G)$.

The spaces $\mathfrak{R}(\pi, G)$ arise in a variety of contexts, from geometric structures to applications in low dimensional topology. Here are some motivating questions about $\mathfrak{R}(\pi, G)$:

- How many components does it have, and can they be characterized in terms of properties of the representations (e.g. discreteness)?
- What are the dimensions, topology of the components?
- Is the space compact, or does it admit a natural compactification (natural in the sense that the action of $\mathrm{Out}(\pi)$ extends to the compactification)?

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- Do the representations correspond to geometric structures?
- Describe the action of $\mathrm{Out}(\pi)$ (e.g. does it act properly discontinuously? ergodically with respect to natural invariant measures?)

When G is compact, there are answers to essentially all of these questions.

- Connected components in 1-1 correspondence with $\pi_1(G)$ (here, G is semisimple; see J. Li [Li]).
- Topology (e.g. Poincaré polynomial computed by Harder-Narasimhan, Atiyah-Bott; [HN, AB]).
- Components are projective varieties (in particular compact; [GIT, NS]).
- The Narasimhan-Seshadri-Ramanathan theorem establishes a 1-1 correspondence between representations and semistable $G^{\mathbb{C}}$ bundles [NS, Ram].
- $\mathrm{Out}(\pi)$ acts ergodically on $\mathfrak{R}(\pi, G)$ (Goldman [G97], Pickrell-Xia [PX]).

When $G = \mathsf{PSL}(2, \mathbb{R})$:

- Discrete embeddings $\rho : \pi \to \mathsf{PSL}(2, \mathbb{R})$ are called *Fuchsian.* These form entire contractible components of the representation variety (Teichmüller space).
- Compactified by certain actions on $\mathbb{R}$-trees.
- The Euler class $\tau(\rho)$ of the associated $\mathbb{R}P^1$ bundle satisfies the inequality
$$|\tau(\rho)| \leq |\chi(S)| = 2g - 2$$
(Milnor-Wood [Mil, Wo]; Kneser [Kn]).
- Components of $\mathrm{Hom}(\pi, \mathsf{PSL}(2, \mathbb{R}))$ are in 1-1 correspondence with the possible value of $\tau(\rho)$ above (Goldman [G87] and Hitchin [Hi87]) and they retract onto symmetric products of S.
- The *maximal* representations, i.e. $|\tau(\rho)| = 2g - 2$, are precisely the Fuchsian ones (Goldman [G87]).
- $\mathrm{Out}(\pi)$ acts properly on the maximal component (Fricke).

Many of the results above have been generalized to (1) representations into isometry groups of hermitian symmetric spaces; and (2) the *Hitchin-Teichmüller component* of representations to split real forms [Hi92].

In these notes we will focus on the action of $\mathrm{Out}(\pi)$ in two directions. First, we consider basic properties of the action itself, and in the second part we study the "linearization" of the action on cohomology. More precisely,

- In Section 2, we will see how some fundamental facts about the action of the mapping class group on Teichmüller space, specifically properness and Mumford compactness, generalize to a particular class of representations to arbitrary isometry groups.
- In Section 3, we give results on the action of the mapping class group on the cohomology of representation varieties of unitary groups.

- In Section 4, we describe how to extend this result to the cohomology of the $\mathsf{SL}(2,\mathbb{C})$ representation variety.

2. Action of $\mathrm{Out}(\pi)$ on representation varieties

2.1. **Teichmüller space.** We first recall some basic definitions.

Definition 1. *The Teichmüller space $T(S)$ is the space of equivalence classes of complex structures on S. The equivalence is given by biholomorphisms connected to the identity.*

There are several equivalent models for $T(S)$. As we have described it

$$T(S) = \mathfrak{J}(S)/\mathsf{Diff}_0(S)$$

where $\mathfrak{J}(S)$ is the space of smooth complex structures on S (inducing the given orientation) and $\mathsf{Diff}_0(S)$ denotes diffeomorphisms that are connected to the identity. It is a result of Earle and Eells [EE] that $\mathsf{Diff}_0(S)$ acts properly on $\mathfrak{J}(S)$, and by the contractibility of $T(S)$ this realizes $\mathfrak{J}(S)$ as a trivial $\mathsf{Diff}_0(S)$ principal bundle.

A second description is given by the Fricke space $\mathfrak{F}(S)$ which is a connected component of discrete embeddings $\pi \to \mathsf{PSL}(2,\mathbb{R})$; i.e.

$$\mathfrak{F}(S) \subset \mathrm{Hom}(\pi, \mathsf{PSL}(2,\mathbb{R}))/\mathsf{PSL}(2,\mathbb{R})$$

The uniformization theorem states that every complex structure admits a compatible hyperbolic metric, and it follows essentially from this fact that $\mathfrak{F}(S) \simeq T(S)$.

Definition 2. *The mapping class group of S is $\mathrm{Mod}(S) = \mathsf{Diff}^+(S)/\mathsf{Diff}_0(S)$.*

Here the superscript $+$ means orientation preserving. Notice that $\mathrm{Mod}(S)$ acts on $T(S)$, and the quotient $\mathfrak{M}(S) = T(S)/\mathrm{Mod}(S)$, is the Riemann moduli space.

We have the following key classical results:

- (Dehn-Nielsen-Baer) $\mathrm{Mod}(S) = \mathrm{Out}(\pi)$.
- (Fricke) $\mathrm{Mod}(S)$ acts properly discontinuously on $T(S)$.
- (Mumford-Mahler Compactness [M]) If $\mathfrak{M}_\varepsilon(S)$ denotes the subspace of $\mathfrak{M}(S)$ where the lengths, with respect to the conformal hyperbolic metric, of all closed geodesics are at least $\varepsilon > 0$, then the set $\mathfrak{M}_\varepsilon(S)$ is compact.

Here are some properties of individual mapping classes:

- (Thurston) Mapping classes are either finite order, reducible, or pseudo-Anosov.
- (Yamada [Y], Daskalopoulos-Wentworth [DW], Wolpert [Wol]) From the point of view of Weil-Peterson geometry, the Thurston classification above is precisely the classification in terms of elliptic, parabolic, loxodromic. In particular, infinite irreducible mapping classes have unique Weil-Petersson axes.
- (Thurston) A pseudo-Anosov has a fixed point in the $\mathsf{SL}(2,\mathbb{C})$ representation variety which is a discrete and faithful representation.

2.2. **Generalizing to representation varieties.** We will see how the classical theorems of Fricke and Mumford-Mahler can be bootstrapped to obtain statements for more general embeddings. Let X be a complete, simply connected length space with nonpositive curvature (an *NPC space*; see [BH]). Recall that nonpositive curvature is defined in terms of triangle comparisons. Examples include simply connected Riemannian manifolds of nonpositive sectional curvature (i.e. *Cartan-Hadamard* manifolds). Let $G = \mathrm{Iso}(X)$.

Definition 3. *A subgroup $\Gamma \subset G$ is called discrete if for each metric ball B in X,*

$$\#\{\gamma \in \Gamma : \gamma B \cap B \neq \emptyset\} < \infty$$

Γ is called convex cocompact if there is a Γ-invariant closed convex subset $N \subset X$ such that N/Γ is compact. A homomorphism $\rho : \pi \to G$ is called if $\rho(\Gamma)$ is discrete (resp. convex cocompact). The homomorphism is faithful if it is injective.

Examples of discrete, faithful, convex cocompact representations of surface groups:

- When $G = \mathsf{PSL}(2, \mathbb{R})$ this is just Teichmüller space;
- When $G = \mathsf{PSL}(2, \mathbb{C})$ these are the *quasi-Fuchsian* representations.

We note the following

Theorem 4. $\mathrm{Mod}(S)$ *acts properly on the space of quasi-Fuchsian representations.*

Proof. By Bers' simultaneous uniformization theorem, $\mathfrak{QF}(S) \simeq T(S) \times \overline{T}(S)$ in a $\mathrm{Mod}(S)$-equivariant way. Since $\mathrm{Mod}(S)$ acts properly on $T(S)$, the result follows. □

This result does not extend to all discrete faithful representations (cf. [SS]). Let $\mathcal{C}(\pi, G)$ denote the set of discrete, faithful, convex cocompact representations to G. This space is in some sense a generalization of Teichmüller space, and our aim is to see how the two properties of Teichmüller space stated above extend to this situation. The first is the following

Theorem 5. $\mathrm{Mod}(S)$ *acts properly on* $\mathcal{C}(\pi, G)$.

Using different techniques, Labourie has shown the following (see [La06, La07a, La08]).

Theorem 6. *The action of* $\mathrm{Mod}(S)$ *is proper on the Hitchin-Teichmüller component of* $\mathsf{SL}(n, \mathbb{R})$ *representations.*

More generally, $\mathrm{Mod}(S)$ acts properly on the set of representations satisfying the "well-displacing" property. This includes representations in the Hitchin component, and maximal symplectic representations (see [La08]). A particular case of the results in [DGLM] shows that for surface groups the well-displacing condition is equivalent to the orbit maps being quasi-isometric embeddings.

To describe the analog of Mumford compactness, let

$$|g| = \inf_{x \in X} d_X(x, gx)$$

denote the *translation length*. For example, if $\rho : \pi \to \mathsf{PSL}(2, \mathbb{R})$ is a Fuchsian representation, then the translation length of an element $\rho(\gamma)$ is essentially the hyperbolic length of the closed geodesic in the free homotopy class defined by γ. Let $\mathfrak{R}_\varepsilon(\pi, G)$ denote the subset of representations with all translation lengths (other than the identity) bounded from below by $\varepsilon > 0$. We then have the following

Theorem 7. *Let X be a simply connected complete Riemannian manifold with curvature bounded above by a negative constant. Suppose also that G acts cocompactly on X. Then*

$$\mathfrak{M}_\varepsilon(S, G) = \mathfrak{R}_\varepsilon(\pi, G)/\mathrm{Mod}(S)$$

is compact.

If G is a rank 1 Lie group of noncompact type and X the associated symmetric space, then this result is a special case of a theorem of Sela [Se]. Below we give a different proof. We note the following

Corollary 8. *A convex cocompact group discrete group in a Cartan-Hadamard manifold with strict negative curvature contains at most finitely many conjugacy classes of surface groups of genus g.*

These results are proven using a variant of the energy function for harmonic maps defined by Sacks-Uhlenbeck [SU] and Schoen-Yau [SY], which we describe next.

2.3. **Harmonic maps and the energy functional.** Let X be an NPC space with G as above. Given a representation $\rho : \pi \to G$, we define a functional

$$E_\rho : T(S) \longrightarrow \mathbb{R}$$

as follows. Let $\widetilde{S}$ be the universal cover of S. For a fixed complex structure σ on S, we define the energy of a ρ-equivariant map $u : \widetilde{S} \to X$ to be

$$E(u) = \int_S e(u)\, dvol = \int_S |du|^2\, dvol$$

When X is a Riemannian manifold, the energy density $e(u) = |du|^2$ may be defined using the Riemannian metric on X and any conformal metric on S (and therefore, $\widetilde{S}$). Notice, however, that the energy itself depends only on the complex structure of S, not the choice of conformal metric. When X is an arbitrary length space, the definition of the density and energy is far more subtle (see [GS, KS93, KS97, J] for details). It is still true however, that the energy depends only on the complex structure σ.

With this understood, we define

$$E_\rho(\sigma) = \inf\{E(u) : \text{where } u \text{ is } \rho\text{-equivariant}\}$$

This is well defined on Teichmüller space. A map u realizing the infimum is called a *harmonic map*. There are various criteria for the existence of harmonic maps. The details are not important, and we will always assume harmonic maps exist.

We will need two facts.

- The Hopf differential $\varphi = u^*(ds_X^2)^{(2,0)}$ of a harmonic map is a *holomorphic* quadratic differential. In a local conformal coordinate $z = x + iy$ (and when X is Riemannian)

$$\varphi = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle \tag{1}$$

- The derivative of E_ρ is given by

$$\left.\frac{d}{dt}\right|_{t=0} E_\rho(\sigma_t) = -4\,\mathfrak{Re}\left(\int \varphi\mu\right)$$

  Here, φ is the Hopf differential of the harmonic map at σ, and μ is the Beltrami differential for the family σ_t.

The pairing of quadratic differentials with harmonic Beltrami differentials is nondegenerate, so critical points of E_ρ correspond to *conformal* harmonic maps, i.e. those for which φ vanishes identically. A critical point for E_ρ is guaranteed if the function is proper. However, there are nontrivial examples where the minima are not unique. Here is one. Let $M \to S^1$ be a fibered 3-manifold with fiber S and pseudo-Anosov monodromy φ. By Thurston, M has a hyperbolic structure. Hence, by the presentation of $\pi_1(M)$ as a mapping cylinder, we get a representation $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{C})$ such that $\rho \circ \varphi_*$ is conjugate to ρ. It follows that E_ρ is not proper in this case (but it is proper on $T(S)/\langle\varphi\rangle$).

Problem. *For which ρ is E_ρ proper? For which ρ does E_ρ have a unique critical point?*

2.4. **Convex cocompact representations.** We now prove

Proposition 9 ([GW])**.** *If ρ is a discrete, faithful, convex cocompact representation, then E_ρ is proper.*

Remark. *For $G = \mathsf{SL}(2, \mathbb{R})$, properness was first shown by Schoen-Yau, and uniqueness of the critical point easily follows. Properness for the Hitchin-Teichmüller component of representations to $G = \mathsf{SL}(n, \mathbb{R})$ was shown by Labourie. It follows by work of Labourie* [La07b] *and Loftin* [Ln] *that uniqueness of the minimum of the energy functional holds for $n = 3$. It is not known whether critical points are unique for $n \geq 4$.*

The proof of Proposition 9 is essentially the same as that given by Schoen and Yau [SY]. We will give the details of the argument below.

Step 1: Bounded geometry. The first observation is the following general result. Suppose $\Gamma \subset G$ is discrete, convex cocompact, and torsion-free. Then there is some $\varepsilon_0 > 0$ such that $|\gamma| \geq \varepsilon_0$ for all $\gamma \in \Gamma$, $\gamma \neq 1$. Suppose not. Then there is a sequence $\{\gamma_i\} \in \Gamma$, $\gamma_i \neq 1$, with

$|\gamma_i| \to 0$. Let N be a closed convex Γ-invariant subset such that N/Γ is compact. Since N is convex, projection to N is distance nonincreasing. Since it is also Γ invariant, it follows that this projection is equivariant. Hence, we may assume there is some sequence $\{x_i\} \subset N$ such that $d(x_i, \gamma_i x_i) \to 0$. Since N/Γ is compact there are $\lambda_i \in \Gamma$ and $x \in N$ such that (after passing to a subsequence) $\lambda_i x_i \to z$. Then

$$d(\lambda_i x_i, (\lambda_i \gamma_i \lambda_i^{-1})\lambda_i x_i) \longrightarrow 0$$

Discreteness applied to balls centered at the point z implies that $\lambda_i \gamma_i \lambda_i^{-1}$ is some fixed element $\gamma_z \in \Gamma$ for all but finitely many i. Moreover, z is a fixed point of γ_z. Since Γ was assumed to be torsion free, discreteness again implies that $\gamma_z = 1$, and hence $\gamma_i = 1$ for all but finitely many i, which contradicts the assumptions.

Step 2. Use of Mumford compactness. By the collar lemma, if c is a short geodesic on (S, σ) of length ℓ, there is an embedded annulus

$$A_\ell = \left\{(r,\theta) : 1 \leq r \leq e^\ell \ , \ \theta_\ell \leq \theta \leq \pi - \theta_\ell\right\}$$

where $\theta_\ell \simeq \ell$. Now suppose $u : \widetilde{S} \to X$ is ρ-equivariant, and let $\gamma = \rho(c)$. By Step 1 we have

$$\varepsilon_0 \leq |\gamma| \leq \int_1^{e^\ell} \|u_*(\partial/\partial r)\| \frac{dr}{r\sin\theta} \leq \int_1^{e^\ell} e(u)^{1/2} \frac{dr}{r\sin\theta}$$

$$\varepsilon_0^2 \leq \int_1^{e^\ell} \frac{dr}{r} \int_1^{e^\ell} e(u) \frac{dr}{r\sin^2\theta}$$

$$\frac{\varepsilon_0^2}{\ell}(\pi - 2\theta_\ell) \leq \int_{A_\ell} e(u) dvol \leq E(u)$$

We conclude that $E_\rho(\sigma) \leq B$ implies a uniform lower bound on the length of geodesics for the hyperbolic metric defined by σ.

Step 3. Use of convex cocompactness. It follows from Step 2 and Mumford compactness that if $\sigma_i \in T(S)$ is an unbounded sequence with $E_\rho(\sigma_i) \leq B$, then there exist $\varphi_i \in \mathrm{Mod}(S)$ such that $\varphi_i^* \sigma_i \to \sigma_\infty$ (after passing to a subsequence). Let u_i be a ρ-equivariant harmonic map $(\widetilde{S}, \sigma_i) \to X$. Then $v_i = u_i \circ \varphi_i$ are harmonic and equivariant with respect to $\rho_i = \rho \circ (\varphi_i)_*$. Fix a point $x \in \widetilde{S}$. By convex cocompactness, there are $\lambda_i \in \rho(\pi)$ such that (up to passing to a subsequence) $\lambda_i v_i(x)$ converges. Now the v_i are uniformly Lipschitz, so by Ascoli's theorem we may arrange that $\tilde{v}_i = \lambda_i v_i$ converges uniformly on compact sets. Notice that $\tilde{v}_i(x)$ is equivariant with respect to $\tilde{\rho}_i = \lambda_i \rho_i \lambda_i^{-1}$.

Step 4. Completion of the argument. Fix $x \in \widetilde{S}$ and $c \in \pi$. Then for i, j sufficiently large we have

$$d(\rho_i(c) v_i(x), \rho_j(c) v_j(x)) \leq d(v_i(cx), v_j(cx)) = d(\tilde{v}_i(cx), \tilde{v}_j(cx)) < \varepsilon_0/2$$

and also

$$d(v_i(x), v_j(x)) = d(\tilde{v}_i(x), \tilde{v}_j(x)) < \varepsilon_0/2$$

Let $g = \rho_j(c)^{-1}\rho_i(c)$. Then

$$d(gv_i(x), v_i(x)) \leq d(gv_i(x), v_j(x)) + d(v_j(x), v_i(x)) < \varepsilon_0$$

It follows from Step 1 that $\rho_j(c) = \rho_i(c)$ for all i, j sufficiently large. Since ρ is injective, we conclude that $(\varphi_i)_*(c) = (\varphi_j)_*(c)$ for i, j sufficiently large. Doing this on a finite generating set of π, we conclude that $\varphi_i = \varphi_j$ for all i, j sufficiently. But this contradicts the divergence of $\sigma_i \in T(S)$.

Proof of Theorem 5. Let $\mathcal{K}(T(S))$ denote the set of compact subsets of $T(S)$, with the Hausdorff metric. Since the energy functional E_ρ is proper for $\rho \in \mathcal{C}(\pi, G)$, we have a map

$$\min : \mathcal{C}(\pi, G) \longrightarrow \mathcal{K}(T(S))$$

which associates to ρ its set of minima. This is clearly equivariant with respect to $\mathrm{Mod}(S)$. Now $\mathrm{Mod}(S)$ acts properly on $\mathcal{K}(T(S))$, since it acts properly on $T(S)$. Since min is equivariant, this implies that Mod acts properly on $\mathcal{C}(\pi, G)$ as well. □

Convex cocompactness is a rather restrictive condition. Kleiner and Leeb [KL] have shown that for symmetric spaces, essentially all convex cocompact representations come from rank 1 subspaces (see also Quint [Q]). Convex cocompactness is not the only condition one can impose. Labourie has shown, for example, that *Anosov representations* have the same property [La06, La08]. In particular, he gives a different proof of Theorem 5 and Proposition 9. He also shows that $\mathrm{Mod}(S)$ acts properly on the Hitchin-Teichmüller component of $\mathsf{SL}(n, \mathbb{R})$ representations. In another direction, Wienhard [Wien] has proven properness on the maximal components of representations into a large number of isometry groups of hermitian symmetric spaces.

2.5. **Compactness.** We now discuss the analog of Mumford compactness, Theorem 7. For simplicity, assume that the representations are convex cocompact, so that by Theorem 5 there exist minima for E_ρ, and hence conformal harmonic maps. The key point is to show that for any $\rho \in \mathfrak{R}_\varepsilon(\pi, G)$ the set of minima of E_ρ project into a compact subset of

$$\mathfrak{M}(S, G) = \mathfrak{R}(\pi, G)\big/\mathrm{Mod}(S)$$

independent of ρ. By Mumford-Mahler compactness, this is guaranteed if we can show that for $\sigma \in \min(E_\rho)$, the length of the shortest geodesic of (S, σ) is bounded from below.

This follows from the Bochner formula for harmonic maps of Eells-Sampson [ES]. Namely, for u harmonic we have

$$\tfrac{1}{2}\Delta|du|^2 = |\nabla du|^2 + \mathsf{Ric}_S(du, du) - \mathsf{Riem}_X(du, du, du, du)$$

Let us assume a hyperbolic metric on S, and assume that the sectional curvature of X is ≤ -1. Then for a *conformal* harmonic map

$$\Delta|du|^2 \geq |du|^2(|du|^2 - 2) \tag{2}$$

(see (1)). By the maximum principle $|du|^2 \leq 2$, pointwise. It follows that for any $c \in \pi$,

$$\varepsilon \leq |\rho(c)|_X \leq \ell_\sigma(c)$$

Hence, σ projects to $\mathfrak{M}_\varepsilon(S, G)$.

Proof of Theorem 7. Suppose we have a sequence $\{\rho_j\} \subset \mathfrak{R}_\varepsilon(\pi, G)$. Assume again that the ρ_j are convex cocompact. By the previous argument we can find σ_j and ρ_j-equivariant harmonic maps u_j such that (2) holds. After composing with elements of $\mathrm{Mod}(S)$, we may assume $\sigma_j \to \sigma$. Since the u_j are uniformly Lipschitz, there is a constant C such that

$$d(u_j(x), \rho_j(c)u_j(x)) \leq C\ell_\sigma(c) \tag{3}$$

for all j and all $c \in \pi$. Let $\Sigma \subset \pi$ be a generating set. Define the displacement function

$$D_\Sigma(\rho) = \inf_{x \in X} \max_{s \in S} d(x, \rho(s)x)$$

It follows from (3) that $D_\Sigma(\rho_j)$ is uniformly bounded for all j. Since G acts cocompactly on X, it follows easily that ρ_j converges up to conjugation. With a little more work – essentially using the assumption that G acts cocompactly – one may drop the assumption of convex cocompactness. □

The argument above uses strongly the assumption that π is a closed surface group, whereas Sela's result [Se] holds more generally. It would be interesting to use recent work on harmonic maps from 2-complexes to prove the result in this larger context. Also, the Bochner technique requires X to be a manifold.

Problem. *Extend the proof above to more general finitely presented groups π and target NPC spaces X.*

3. Action on the cohomology of the space of flat unitary connections

In the next two sections we consider the action of the mapping class group on the cohomology of representation varieties. First, since representation varieties $\mathfrak{R}(\pi, G)$ arise as quotients, they have essentially two different natural cohomologies associated to them; the ordinary singular cohomology, and the equivariant cohomology. Recall that if G acts on a space X, the equivariant cohomology is given by the Borel construction:

$$H_G^*(X) = H^*(X \times_G EG)$$

where EG is a contractible principal G bundle over the classifying space BG. As is often noted, the two extreme cases are when G acts freely (then $H_G^*(X) = H^*(X/G)$) and when G acts trivially (then $H_G^*(X) = H^*(X) \otimes H^*(BG)$). Applied to the representation varieties, in addition to the ordinary cohomology $H^*(\mathfrak{R}(\pi, G))$ we have the equivariant cohomology $H_G^*(\mathrm{Hom}(\pi, G))$.

We have the Birman exact sequence

$$1 \longrightarrow \pi_1(S,p) \longrightarrow \mathrm{Mod}(S,p) \longrightarrow \mathrm{Mod}(S) \longrightarrow 1$$

where $\mathrm{Mod}(S,p)$ is the mapping class group of isotopy classes of orientation preserving diffeomorphisms fixing a point p (the isotopy also fixes p). Choosing p as the basepoint for $\pi = \pi_1(S,p)$, we have an action of $\mathrm{Mod}(S,p)$ on $\mathrm{Hom}(\pi, G)$, and hence on the equivariant cohomology $H^*_G(\mathrm{Hom}(\pi,G))$. In fact, one can show that this descends to an action of $\mathrm{Mod}(S)$ (inner automorphisms act trivially on the cohomology).

For compact groups, the arguments of Atiyah-Bott [AB] can be used to show that the action of the mapping class group on equivariant cohomology factors through the action on the cohomology of S (see below, and also Frohman [F]). In particular, this gives a result on the action of the Torelli group, and we will describe this result in detail. For simplicity, we restrict the discussion below to unitary groups.

First, recall the

Definition 10. *The Torelli group $\mathcal{I}(S)$ is the normal subgroup of* $\mathrm{Mod}(S)$ *that acts trivially on the homology of S.*

Then we have the following

Theorem 11. *Let G be $\mathsf{U}(n)$ or $\mathsf{SU}(n)$. Then $\mathcal{I}(S)$ acts trivially on the G-equivariant cohomology of* $\mathrm{Hom}(\pi, G)$.

3.1. **Harder-Narasimhan stratification.** In order to prove Theorem 11, we first digress to describe Atiyah-Bott's formulation of the Harder-Narasimhan stratification of the space of holomorphic bundles. Let $\mathcal{A}$ be the space of unitary connections on a trivial rank n hermitian vector bundle E over S. To be explicit, given a base connnection D_0, we have

$$\mathcal{A} = \left\{ D_0 + a : a \in \Omega^1(\mathrm{ad}\, E) \right\} \tag{4}$$

where $\mathrm{ad}\, E$ denotes the bundle of skew-hermitian endomorphisms of E. Let $\mathcal{G}$ denote the group of unitary gauge transformations, and $\mathcal{G}^{\mathbb{C}}$ its complexification. We may think of $\mathcal{A}$ as the space of holomorphic structures on E by taking $(0,1)$ parts of the connection. That is to say, given $D \in \mathcal{A}$, a $\bar\partial$-operator on E is obtained by setting $\bar\partial_E = D^{(0,1)}$. Conversely, given a $\bar\partial$-operator there is a unique unitary *Chern connection* inducing the given $\bar\partial$-operator. When we want to emphasize the holomorphic structure, we write $(E, \bar\partial_E)$. In particular, $\mathcal{G}^{\mathbb{C}}$ has an action on $\mathcal{A}$. Notice that since $\mathcal{A}$ is contractible:

$$H^*_{\mathcal{G}}(\mathcal{A}) \simeq H^*(B\mathcal{G})$$

Recall that the degree $\deg(F)$ of a bundle F is the integral of its first Chern class over S. We define the normalized degree, or *slope*, by $\mu(F) = \deg(F)/\operatorname{rk}(F)$. Since E is trivial, it has slope zero. We call a holomorphic bundle $(E, \bar\partial_E)$ is called *semistable* if for every proper holomorphic subbundle $F \subset E$ we have $\mu(F) \le 0$. By the correspondence described above, we have a notion of a sermistable connection as well. The relationship with representations is given by the famous Narasimhan-Seshadri theorem.

Theorem 12. *A connection is semistable if and only if there is a flat connection in the closure of its $\mathcal{G}^{\mathbb{C}}$ orbit.*

This is a slightly different formulation of the result which frames it in analogy to the situation in finite dimensional geometric invariant theory. See [Do83, Do87b] for more details. The $\mathcal{G}^{\mathbb{C}}$ orbit may not itself contain a flat connection. For this, the extra condition of *polystability* is required. The result extends more generally to compact Kähler manifolds, where it is known as the Donaldson-Uhlenbeck-Yau Theorem [Do85, UY, Do87b].

Given a holomorphic bundle, there is a filtration by subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

called the *Harder-Narasimhan filtration of* E, such that the quotients $Q_i = E_i/E_{i-1}$ are semistable. Moreover, $\mu(Q_i) > \mu(Q_{i+1})$, and the associated graded object $\oplus_{i=1}^{\ell} Q_i$ is uniquely determined by the isomorphism class of E (cf. [Ko]). The collection of slopes $\mu(Q_i)$ is an important invariant of the isomorphism class of the bundle. We construct an n-tuple of numbers $\vec{\mu}(E) = (\mu_1, \ldots, \mu_n)$ from the Harder-Narasimhan filtration by repeating the μ_i's according to the ranks of the Q_i's. Note that since E is topologically trivial, $\sum_{i=1}^n \mu_i = 0$. We then get a vector $\vec{\mu}(E)$, called the Harder-Narasimhan type of E. There is a natural partial ordering on vectors of this type that is key to the stratification we desire. For a pair $\vec{\mu}$, $\vec{\lambda}$ of n-tuple's satisfying $\mu_1 \geq \cdots \geq \mu_n$, $\lambda_1 \geq \cdots \geq \lambda_n$, and $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i$, we define

$$\vec{\lambda} \leq \vec{\mu} \quad \Longleftrightarrow \quad \sum_{j \leq k} \lambda_j \leq \sum_{j \leq k} \mu_j \quad \text{for all } k = 1, \ldots, n \ .$$

The importance of this ordering is that it defines a stratificaton of the space of holomorphic structures on a given complex vector bundle over a Riemann surface which may be described as follows (see [AB, §7]). Let $Y_{\vec{0}} = \mathcal{A}^{ss} \subset \mathcal{A}$, denote the subset of semistable bundles. This is the open stratum. Let $Y_{\vec{\mu}}$ denote the set of bundles of Harder-Narasimhan type $\vec{\mu}$, and let

$$X_{\vec{\mu}} = \bigcup_{\vec{\lambda} \leq \vec{\mu}} Y_{\vec{\lambda}}$$

Then the collection $\{X_{\vec{\mu}}\}$ gives a smooth $\mathcal{G}$-invariant stratification of $\mathcal{A}$. The fundamental result of Atiyah-Bott states that the inclusions $X_{\vec{\lambda}} \hookrightarrow X_{\vec{\mu}}$ for $\vec{\lambda} \leq \vec{\mu}$ induce surjections on $\mathcal{G}$-equivariant cohomology. In particular, it follows that the inclusion $\mathcal{A}^{ss} \hookrightarrow \mathcal{A}$ induces a surjection in equivariant cohomology

$$H^*(B\mathcal{G}) \simeq H^*_{\mathcal{G}}(\mathcal{A}) \longrightarrow H^*_{\mathcal{G}}(\mathcal{A}^{ss}) \tag{5}$$

3.2. **The Yang-Mills flow.** Let YM denote the Yang-Mills functional:

$$\mathrm{YM}(D) = \int_S \|F_D\|^2 \, dvol$$

where F_D is the curvature of the connection D. Then YM is invariant under the action of $\mathcal{G}$. The minimum of YM consists of flat connections $\mathcal{A}^{flat}$. If $\mathcal{G}_p \subset \mathcal{G}$ is the subgroup of unitary gauge transformations that are the identity at p, then there homeomorphisms

$$\mathcal{A}^{flat}/\mathcal{G}_p \simeq \mathrm{Hom}(\pi, \mathsf{U}(n)) \qquad \mathcal{A}^{flat}/\mathcal{G} \simeq \mathfrak{R}(\pi, \mathsf{U}(n))$$

Explicitly, these are given by the holonomy representation based at the point p. Notice that by this identification:

$$H^*_{\mathcal{G}}(\mathcal{A}^{flat}) \simeq H^*_{\mathsf{U}(n)}(\mathcal{A}^{flat}/\mathcal{G}_p) \simeq H^*_{\mathsf{U}(n)}(\mathrm{Hom}(\pi, \mathsf{U}(n)))$$

We note here that if we want to consider representations into $\mathsf{SU}(n)$, then we restrict to connections which induce the trivial connection on the determinant line bundle $\det(E)$. This amounts to taking traceless skew-hermitian endormorphisms in (4).

The higher critical sets consist of reducible Yang-Mills connections, i.e. splittings of the bundle $E = Q_1 \oplus \cdots \oplus Q_\ell$, where some Q_i has nonzero slope. The Yang-Mills flow

$$\frac{\partial A}{\partial t} = -d_A^* F_A$$

gives a stratification of the space $\mathcal{A}$ in terms of the stable manifolds of these critical sets. By a theorem of Daskalopoulos and Rade, the flow converges and identifies this Morse stratification with the Harder-Narasimhan stratification described above. In particular, we have the following result of Daskalopoulos [D] (see also Råde [R]).

Theorem 13. *The inclusion $\mathcal{A}^{flat} \hookrightarrow \mathcal{A}^{ss}$ is a $\mathcal{G}$-equivariant deformation retract. As a consequence,*

$$H^*_{\mathcal{G}}(\mathcal{A}^{ss}) \simeq H^*_{\mathsf{U}(n)}(\mathrm{Hom}(\pi, \mathsf{U}(n)))$$

The result also holds for $\mathsf{SU}(n)$.

Combining this result with (5) we have the following important

Corollary 14. *Let G be $\mathsf{U}(n)$ or $\mathsf{SU}(n)$. Then the map*

$$k_{\mathcal{G}} : H^*(B\mathcal{G}) \simeq H^*_{\mathcal{G}}(\mathcal{A}) \longrightarrow H^*_G(\mathrm{Hom}(\pi, G))$$

is surjective.

The map $k_{\mathcal{G}}$ is called the *Kirwan map*, and the statement above is an example of *Kirwan surjectivity*. This is an infinite dimensional version of a general result in finite dimensions. Recall that if G is a compact connected Lie group acting on a compact symplectic manifold X with a moment map $\mu : X \to \mathfrak{g}^*$, then the symplectic reduction is by definition $\mu^{-1}(0)/G$. A fundamental theorem of Kirwan [Ki] is that the inclusion $\mu^{-1}(0) \hookrightarrow X$ induces a surjection on

G-equivariant cohomology. In the infinite dimensional setting above, the curvature F_D may be interpreted as the moment map for $\mathcal{G}$ acting on $\mathcal{A}$ with its natural symplectic structure.

We note that Atiyah-Bott also show that the relative equivariant cohomology groups for strata associated to successive types are computed in terms of split bundles, and this allows one to inductively compute the Betti numbers of $H^*_{\mathcal{G}}(\mathcal{A}^{ss})$, and hence, by the above, the equivariant cohomology of $\mathrm{Hom}(\pi, G)$.

3.3. **Topology of the gauge group.** Now let G be either $\mathsf{SU}(n)$ or $\mathsf{U}(n)$. The gauge group $\mathcal{G}$ can be identified with $\mathrm{Map}_0(S, G)$, where the subscript indicates that the induced G-bundle from the maps is trivial. Then $B\mathcal{G} = \mathrm{Map}_0(S, BG)$.

Proposition 15. *The action of* $\mathcal{I}(S)$ *on* $H^*(B\mathcal{G})$ *is trivial.*

Proof. Let $V \to BG$ denote the universal bundle. We have the evaluation map $\mathsf{ev} : S \times B\mathcal{G} \to BG$. Let c_j be the Chern classes of $\widehat{V} = \mathsf{ev}^*(V)$. Since ev is invariant under the action of ϕ, it follows that $\phi^*\widehat{V} = \widehat{V}$, so $\phi^* c_j = c_j$. By the Kunneth formula we may write

$$c_j = \alpha_j + \sum_{k=1}^{2g} \beta_j^k \otimes b_k + \gamma_j$$

where $\{b_k\}$ is a basis of $H^1(S)$. If ϕ acts trivially on the homology of S, then it preserves α_j, β_j^k, and γ_j. Finally, a key result of Atiyah-Bott is that these classes generate $H^*(B\mathcal{G})$. □

The main result, Theorem 11, follows from Proposion 15 and Corollary 14. One needs to verify that the Kirwan map is equivariant with respect to the action of the mapping class group (see [F]).

3.4. **Results of Cappell-Lee-Miller.** It turns out that the Torelli group does *not* act trivially on the *ordinary* cohomology of $\mathfrak{R}(\pi, G)$ [CLM]. We will explain the case of $\mathsf{SU}(2)$, so let's temporarily set $\mathfrak{R} = \mathfrak{R}(\pi, \mathsf{SU}(2))$. Denote the reducibles and irreducibles by $\mathfrak{R}^{red.}$ and $\mathfrak{R}^{irr.}$. Then the inclusion $\imath : \mathfrak{R}^{red.} \hookrightarrow \mathfrak{R}$ gives a long exact sequence in equivariant cohomology

$$\longrightarrow H^*_{eq.}(\mathfrak{R}, \mathfrak{R}^{red.}) \longrightarrow H^*_{eq.}(\mathfrak{R}) \xrightarrow{\imath^*} H^*_{eq.}(\mathfrak{R}^{red.}) \longrightarrow$$

Here, the subscript indicates that we take the equivariant cohomology, i.e.

$$H^*_{eq.}(\mathfrak{R}) := H^*_{\mathsf{SU}(2)}(\mathrm{Hom}(\pi, \mathsf{SU}(2))$$

and similarly for the others. The exact sequence above comes from the $\mathcal{G}$-equivariant inclusion of reducible flat connections inside all flat connections. We have a similar long exact sequence in ordinary cohomology

$$\longrightarrow H^*(\mathfrak{R}, \mathfrak{R}^{red.}) \longrightarrow H^*(\mathfrak{R}) \xrightarrow{\imath^*} H^*(\mathfrak{R}^{red.}) \longrightarrow$$

Now the point is that the gauge group acts freely on the irreducibles $\mathfrak{R}^{irr.} = \mathfrak{R} \setminus \mathfrak{R}^{red.}$, so equivariant cohomology is the same as ordinary cohomology. This allows us to use the first sequence to compute the relative group, and plug that into the second long exact sequence. Also, a simple argument shows that the action of Torelli is nontrival on $H^*(\mathfrak{R})$ if and only if it is nontrivial on $H^*(\mathfrak{R}, \mathfrak{R}^{red.})$.

Now for each q we have a short exact sequence

$$0 \longrightarrow \operatorname{coker} \imath^* \longrightarrow H^q(\mathfrak{R}, \mathfrak{R}^{red.}) \longrightarrow \ker \imath^* \longrightarrow 0$$

The Torelli group acts trivially on $\ker \imath^*$ and $\operatorname{coker} \imath^*$, but it may act nontrivially on the middle group. Indeed, given

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

with a linear map ϕ acting trivially on A and C, we define an obstruction $\Psi : C \to A$ as follows: given $c \in C$ choose $b \in B$ such that $g(b) = c$. Then $g(b - \phi(b)) = 0$, so there is a unique $a \in A$ with $f(a) = b - \phi(b)$. Set $a = \Psi(c)$, and check that this is independent of the choice of b.

Cappell-Lee-Miller go on to compute this obstruction for particular elements. They first lift the computation to the cohomology of the gauge groups and then use the Johnson isomorphism

$$H^1(B\mathfrak{I}(S,p)) \simeq \Lambda^3 H^1(S)$$

to show that the obstruction is nonvanishing.

4. Action on the cohomology of the $\mathsf{SL}(2,\mathbb{C})$ character variety

The case of representations into $\mathsf{SL}(2,\mathbb{C})$ is different. The space $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ has the structure of an affine algebraic variety. As previously, set

$$\mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C})) = \mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})) /\!\!/ \mathsf{SL}(2,\mathbb{C})$$

where the double slash indicates the invariant theoretic quotient by overall conjugation of $\mathsf{SL}(2,\mathbb{C})$. As mentioned in the introduction, this means that nonclosed orbits are identified with orbits in their closure. The space $\mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C}))$, which is typically called the $\mathsf{SL}(2,\mathbb{C})$ *character variety* of the surface S (cf. [CS, LM]), is an irreducible affine variety of complex dimension $6g-6$. There is a surjective algebraic quotient map $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})) \to \mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C}))$, and this is a geometric quotient on the open set of irreducible (or simple) representations. Points of $\mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C}))$ are in 1-1 correspondence with conjugacy classes of semisimple (or reductive) representations, and every $\mathsf{SL}(2,\mathbb{C})$ orbit in $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ contains a semisimple representation in its closure.

Since $\mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C}))$ arises as a quotient, we will be interested in the associated equivariant cohomology. We shall use recent results on the Morse theory of the space of Higgs bundles to prove the following

Theorem 16 (see [DW[2]]). *$\mathcal{I}(S)$ acts trivially on the equivariant cohomology of* $\mathrm{Hom}(\pi, \mathsf{GL}(2,\mathbb{C}))$ *and nontrivially on the equivariant cohomology of* $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$.

In fact, we will get an explicit description in terms of *Prym representations* of the Torelli group. As with unitary groups, the character varieties may be realized through a gauge theoretic description. In particular, there is a canonically defined map

$$H^*(B\mathcal{G}) \longrightarrow H^*_{\mathsf{SL}(2,\mathbb{C})}(\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})))$$

which, by the result above, fails to be surjective (cf. Corollary 14). Hitchin observed that the space $\mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C}))$ is a *hyperkähler quotient*. While Kirwan surjectivity holds for finite dimensional symplectic reductions, it is still unknown for hyperkähler reductions. The $\mathsf{SL}(2,\mathbb{C})$ character variety is an example of the failure for infinite dimensional quotients.

4.1. **Higgs bundles.** In this lecture we will state a result on the equivariant cohomology of the space of flat $\mathsf{SL}(2,\mathbb{C})$ connections. The gauge theoretic extension to complex groups is due to Hitchin [Hi87], and here we give a very brief summary for the $\mathsf{SL}(2,\mathbb{C})$ case.

By a (rank 2) *Higgs bundle* on a Riemann surface S we mean a pair (E, Φ), where $E \to S$ is a holomorphic rank 2 vector bundle with fixed trivial determinant, and Φ is a holomorphic section of the associated bundle $\mathrm{End}_0(E) \otimes K$ of traceless endomorphisms with values in the canonical bundle. A Higgs bundle (E, Φ) is called semistable if $\deg(L) \leq 0$ for all holomorphic line subbundles preserved by Φ. Notice that this allows potentially some unstable bundles to be semistable Higgs bundles, with the right choice of Φ.

The relationship with representations is given by the following. Fix a hermitian metric on E, and consider the following equations

$$F_A + [\Phi, \Phi^*] = 0 \tag{6}$$

These should be regarded as a generalization of the flatness equations.

Then we have the following analog of Theorem 12. This is due to Hitchin [Hi87] and Simpson [Si].

Theorem 17. *A Higgs bundle (A, Φ) is semistable if and only if there is a solution to* (6) *in the closure of its $\mathcal{G}^{\mathbb{C}}$ orbit.*

Let $\mathcal{B}^{ss}$ denote the space of semistable Higgs bundles of rank 2 with fixed trivial determinant, and $\mathfrak{M}^{Higgs} = \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ the moduli space of equivalence classes of semistable rank 2 Higgs bundles with trival determinant.

To see the connection with representations, notice that holomorphicity of Φ is equivalent to the two equations

$$d_A \Psi = 0 \tag{7}$$

$$d_A(*\Psi) = 0 \tag{8}$$

where $\Psi = \Phi + \Phi^*$. If we define $D = d_A + \Psi$, then the equations (6) and (7) imply that D is a flat $\mathsf{SL}(2,\mathbb{C})$ connection. The following is a result of Corlette [Cor] and Donaldson [Do87a].

Theorem 18. *Every flat* $\mathsf{SL}(2,\mathbb{C})$ *connection has a solution of* (8) *in the closure of its* $\mathcal{G}^{\mathbb{C}}$ *orbit.*

As a consequence, there is a homeomorphism

$$\mathfrak{M}^{Higgs} \simeq \mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C})) \tag{9}$$

We will elaborate on this result in the next section.

4.2. **Hyperkähler reduction.** We view the cotangent bundle as follows:

$$T^*\mathcal{A} = \left\{(A,\Psi) : A \in \mathcal{A}\ ,\ \Psi \in \Omega^1(M, \sqrt{-1}\,\mathrm{ad}_0(E))\right\}$$

where $\mathrm{ad}_0(E)$ denotes the bundle of traceless skew-hermitian endomorphisms of E. Then $T^*\mathcal{A}$ is a hyperkähler manifold, and the action of the gauge group $\mathcal{G}$ has associated moment maps

$$\mu_1(A,\Psi) = F_A + \tfrac{1}{2}[\Psi,\Psi]\ ,\ \mu_2(A,\Psi) = d_A\Psi\ ,\ \mu_3(A,\Psi) = d_A(*\Psi) \tag{10}$$

Let $\mathfrak{M}^{Higgs}$ be the moduli space of semistable rank 2 Higgs bundles with trival determinant, introduced in the previous section, and let $\mathbf{m} = (\mu_1, \mu_2, \mu_3)$. Then $\mathfrak{M}^{Higgs}$ is the hyperkähler quotient

$$\mathfrak{M}^{Higgs} = \mathbf{m}^{-1}(0)/\mathcal{G} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathcal{G}$$

This is typically regarded as a reduction in steps in two different ways. The first is the point of view of Hitchin and Simpson described in the last lecture. Let

$$\mathcal{B} = \mu_2^{-1}(0) \cap \mu_3^{-1}(0) \subset T^*\mathcal{A}$$

denote the space of Higgs bundles. Then

$$\mathfrak{M}^{Higgs} = \mu_1^{-1}(0) \cap \mathcal{B}/\mathcal{G} \tag{11}$$

The second point of view (e.g. Corlette and Donaldson) is as the quotient

$$\mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C})) = \mu_3^{-1}(0) \cap (T^*\mathcal{A})^{flat}/\mathcal{G} \tag{12}$$

where

$$(T^*\mathcal{A})^{flat} = \{(A,\Psi) \in T^*\mathcal{A} : D = d_A + \Psi \text{ is a flat } \mathsf{SL}(2,\mathbb{C}) \text{ connection}\}$$

By (9), these two points of view give rise to homeomorphic spaces, but their descriptions as quotients are different. Nevertheless, the following shows that (11) and (12) give rise to the same equivariant cohomology:

Theorem 19. $H^*_{\mathsf{SL}(2,\mathbb{C})}(\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})) \simeq H^*_{\mathcal{G}}(\mathcal{B}^{ss})$.

We now describe the proof of this result. First, by studying the flow associated to the *Yang-Mills-Higgs* functional

$$\mathrm{YMH}(D,\Phi)=\int_S \|F_D+[\Phi,\Phi^*]\|\,dvol$$

on $\mathcal{B}$, Wilkin has shown the following analog of Theorem 13.

Theorem 20 ([Wil])**.** *The inclusion* $\mathbf{m}^{-1}(0)\hookrightarrow\mathcal{B}^{ss}$ *is a* $\mathcal{G}$*-equivariant deformation retract. In particular, it induces an isomorphism*

$$H^*_{\mathcal{G}}(\mathcal{B}^{ss})\simeq H^*_{\mathsf{SU}(2)}(\mathbf{m}^{-1}(0)/\mathcal{G}_p)$$

On the other hand, the holonomy map gives a proper embedding

$$\mathrm{hol}_p:\mathbf{m}^{-1}(0)/\mathcal{G}_p\hookrightarrow\mathrm{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$$

which is $\mathsf{SU}(2)$-equivariant. The second ingredient in the proof is

Theorem 21. *The inclusion* $\mathrm{hol}_p\left(\mathbf{m}^{-1}(0)/\mathcal{G}_p\right)\hookrightarrow\mathrm{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ *is an* $\mathsf{SU}(2)$*-equivariant deformation retract. In particular,*

$$H^*_{\mathsf{SU}(2)}(\mathbf{m}^{-1}(0)/\mathcal{G}_p)\simeq H^*_{\mathsf{SU}(2)}(\mathrm{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$$

Hence, Theorem 19 follows Theorems 20 and 21 (note that since $\mathsf{SL}(2,\mathbb{C})/\mathsf{SU}(2)$ is contractible, the equivariant cohomologies are the same).

The idea of proof of Theorem 21 is to use the harmonic map flow to define a flow on the space of representations. Fix a lift $\tilde p\in\widetilde S$ of p, and a point $z\in\mathbb{H}^3$ so that $\mathsf{PU}(2)$ is identified with the stabilizer of z in the isometry group $\mathsf{PSL}(2,\mathbb{C})$ of $\mathbb{H}^3$. Given $\rho\in\mathrm{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$, choose $D\in(T^*\mathcal{A})^{flat}$ with $\mathrm{hol}_p(D)=\rho$. The hermitian metric gives a unique ρ-equivariant lift $f:\widetilde S\to\mathbb{H}^3$ with $f(\tilde p)=z$. Let f_t, $t\geq 0$, denote the harmonic map flow with initial condition f. There is a unique continuous family $h_t\in\mathsf{SL}(2,\mathbb{C})$, $h_t^*=h_t$, such that $h_0=I$, and $h_tf_t(\tilde p)=z$. Notice that a different choice of flat connection $\widetilde D$ with $\mathrm{hol}_p(\widetilde D)=\rho$ will be related to D by a based gauge transformation g. The flow corresponding to $\widetilde D$ is $\tilde f_t=g\cdot f_t$, and since $g(\tilde p)=I$, $\tilde h_t=h_t$. Hence, h_t is well-defined by ρ. The flow we define is $\rho_t=h_t\rho h_t^{-1}$. The result states that this flow defines a continuous retraction to $\mathrm{hol}_p\left(\mathbf{m}^{-1}(0)/\mathcal{G}_p\right)$. When ρ is not semisimple, the flow converges precisely to the semisimplification.

4.3. **Singular Morse theory.** We can now attempt to compute the equivariant cohomology of $\mathrm{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ using Morse theory in the spirit of Atiyah-Bott. As before, we have a decomposition in terms of maximal destabilizing line subbundles

$$X_d=\bigcup_{d'=0}^{d}Y_{d'}$$

The major difference is that the space $\mathcal{B}$ is singular, and this leads to jumping in the dimension of the normal directions to $Y_d \subset X_d$. This lack of a well-defined normal bundle prevents the Atiyah-Bott lemma from working properly, and the corresponding exact sequences

$$\cdots \longrightarrow H^*_{\mathcal{G}}(X_d, X_{d-1}) \longrightarrow H^*_{\mathcal{G}}(X_d) \longrightarrow H^*_{\mathcal{G}}(X_{d-1}) \longrightarrow \cdots$$

do not necessarily split, i.e. the inclusion $X_d \hookrightarrow X_{d+1}$, unlike the Atiyah-Bott case, does not necessarily induce a surjection on equivariant cohomology.

Nevertheless, one does have some control (at least for $\mathsf{SL}(2,\mathbb{C})$), and one can still compute Poincaré polynomials (see also [HT]). As a consequence, though, the map

$$H^*(B\mathcal{G}) \longrightarrow H^*_{\mathcal{G}}(\mathcal{B}^{ss}) \tag{13}$$

is no longer surjective. Interestingly, surjectivity holds for $\mathsf{GL}(2,\mathbb{C})$ representations.

4.4. **Statement of the result.** The results above may be used to compute the action of the Torelli group on the equivariant cohomology of the space of flat $\mathsf{SL}(2,\mathbb{C})$ connections. Since the map in (13) is no longer surjective, we can no longer conlcude that this action is trivial, and in fact we will see that it is not. To simplify notation, let us set $\mathfrak{X}(\pi) = \mathfrak{R}(\pi, \mathsf{SL}(2,\mathbb{C}))$, and

$$H^*_{eq.}(\mathfrak{X}(\pi)) := H^*_{\mathsf{SL}(2,\mathbb{C})}(\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$$

Define

$$\Gamma_2 = H^1(S, \mathbb{Z}/2) \simeq \mathrm{Hom}(\pi, \{\pm 1\}) \tag{14}$$

Then Γ_2 acts on $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ by $(\gamma\rho)(x) = \gamma(x)\rho(x)$. This action commutes with conjugation by $\mathsf{SL}(2,\mathbb{C})$, and hence it defines an action on $\mathfrak{X}(\pi)$ and on the ordinary and equivariant cohomologies. We denote the Γ_2 invariant parts of the cohomology by $H^*(\mathfrak{X}(\pi))^{\Gamma_2}$ and $H^*_{eq.}(\mathfrak{X}(\pi))^{\Gamma_2}$.

Recall that the Torelli group $\mathcal{I}(S)$ is the subgroup of $\mathrm{Mod}(S)$ that acts trivially on the homology of S. In particular, the action of Γ_2 commutes with the action of $\mathcal{I}(S)$. The kernel of $\gamma \in \Gamma_2 \simeq \mathrm{Hom}(\pi, \{\pm 1\})$, $\gamma \neq 1$, defines an unramified double cover $S_\gamma \to S$ with involution σ. Let W^+_γ (resp. W^-_γ) denote the $2g$ (resp. $2g-2$) dimensional $+1$ (resp. -1) eigensubspaces of $H^1(S_\gamma)$ for σ. A lift of a diffeomorphism of S representing an element of $\mathcal{I}(S)$ that commutes with σ may or may not be in the Torelli group of S_γ; although it acts trivially on W^+_γ it may act nontrivially on W^-_γ. Since the two lifts differ by σ, there is thus defined a representation

$$\Pi_\gamma : \mathcal{I}(S) \longrightarrow \mathsf{Sp}(W^-_\gamma, \mathbb{Z})/\{\pm I\} \tag{15}$$

which is called the (degree 2) *Prym representation* of $\mathcal{I}(S)$ associated to γ. An element in $\ker \Pi_\gamma$ has a lift which lies in $\mathcal{I}(S_\gamma)$. By a theorem of Looijenga [Lo], the image of Π_γ has finite index for $g > 2$. Note that the representations for various $\gamma \neq 1$ are isomorphic via outer automorphisms of $\mathcal{I}(S)$. Π_γ induces nontrivial representations of $\mathcal{I}(S)$ on the exterior products

$$V(q,\gamma) = \Lambda^q W_\gamma^-$$

when q is even.

Here is the main result.

Theorem 22. (1) *$\mathcal{I}(S)$ acts trivially on $H^*_{eq.}(\mathfrak{X}(\pi))^{\Gamma_2}$.*

(2) *For $q \in S = \{2j\}_{j=1}^{g-2}$ the action of $\mathcal{I}(S)$ splits as*

$$H^{6g-6-q}_{eq.}(\mathfrak{X}(\pi)) = H^{6g-6-q}_{eq.}(\mathfrak{X}(\pi))^{\Gamma_2} \oplus \bigoplus_{1\neq\gamma\in\Gamma_2} V(q,\gamma)$$

(3) *$\mathcal{I}(S)$ acts trivially on $H^{6g-6-q}_{eq.}(\mathfrak{X}(\pi))$ for $q \notin S$.*

In particular, if we define the *Prym-Torelli group*

$$\mathcal{PI}(S) = \bigcap_{1\neq\gamma\in\Gamma_2} \ker \Pi_\gamma$$

then $\mathcal{PI}(S)$ acts trivially and $\mathcal{I}(S)$ acts nontrivially on $H^*_{eq.}(\mathfrak{X}(\pi))$ for $g > 2$. The splitting of the sum of $V(q,\gamma)$'s is canonically determined by a choice of homology basis of S.

Time does not permit a full description of the proof of Theorem 22. We wish here to give a hint as to the origin of Prym representations. This was already apparent in the work of Hitchin. The key is to look at the higher critical sets.

A split Higgs bundle $E = L_1 \oplus L_2$ has a Higgs field which is diagonal with respect to this splitting. Contained in the "normal" directions to this point in the critical set are off diagonal endomorphisms, for example, parametrized by nonzero elements

$$\varphi \in H^0(L_1^* \otimes L_2 \otimes K)$$

Since we have fixed the determinant, φ is a nonzero section of $(L_2)^2 \otimes K$. While φ is determined by an element of $\mathrm{Sym}^m(S)$ for some m, the line bundle L_2 is only determined up to two torsion. Hence, the parameter space is the 2^{2g} cover $\widetilde{\mathrm{Sym}^m}(S)$. The noninvariant parts of $H^*(\widetilde{\mathrm{Sym}^m}(S))$ give precisely the Prym representations of the Torelli group described above.

Extra work is needed to show these representations split over the long exact sequences that build up the equivariant cohomology of $\mathcal{B}^{ss}$. This is where the Γ_2-action plays an important role. Full details of this result will appear in a forthcoming work [DW[3]].

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