MATH 406 SUMMER SESSION II 2005 FINAL

1) (8 pts) Show by induction that $2^n > n$ whenever n is an integer greater than 1.

2) (12 pts) Show that if $a^3 | b^2$ then $a|b$.

3) a) (6 pts) Let $a, b, m, n \in \mathbb{Z}$ with m and n positive such that $n|m$. Show that if $a \equiv b \pmod{m}$ then $a \equiv b \pmod{n}$.

b) (6 pts) Let $a, b, c, m \in \mathbb{Z}$ with m and c positive. Show that if $a \equiv b \pmod{m}$ then $ac \equiv bc \pmod{mc}$.

4) a) (6 pts) Show that if p is an odd prime then $(p-1)! \cdot a^p \equiv -a \pmod{p}$ for any integer a.

b) (6 pts) State Euler's Theorem. (be careful to state all assumptions)

5) a) (8 pts) Show there are no n such that $\phi(n) = 14$.

b) (3 pts) Find $\sigma(60)$.

c) (3 pts) Find $\tau(60)$.

6) a) (4 pts) Find the order of 2 modulo 31.

b) (10 pts) Find a complete set of incongruent primitive roots modulo 13. Reduce them into the interval $[0, 13)$.

c) (6 pts) Find a complete set of incongruent primitive roots modulo $2 \cdot 13$.

d) (4 pts) Given that 2 is a primitive root modulo 5 and that 7 is not a primitive root modulo 25 find a primitive root modulo 25.

e) (4 pts) Using part (d) find a primitive root modulo 5^k for all positive integers k.

7) (12 pts) Let $-a$ be a quadratic residue modulo p. For what p is $+a$ also a quadratic residue?

8) a) (12 pts) Find a congruence describing all odd primes for which 3 is a quadratic residue.

b) (6 pts) How does part (a) change if we instead ask for a congruence describing when $\left(\frac{3}{n}\right) = 1$ for n an odd positive integer? (Explain your answer.)

9) (16 pts) Find formulas for the integers of all Primitive Pythagorean triples x, y, z with $z = y+2$ and y odd.

10) a) (8 pts) Show that 5 is not a Gaussian prime.

b) (10 pts) Show that if $\alpha, \beta, \gamma, \nu, \mu \in \mathbb{Z}[i]$ with $\gamma|\alpha$ and $\gamma|\beta$ then $\gamma|(\alpha\nu + \beta\mu)$.