HOMEWORK 10

1) For each of the following sets find the infimum and supremum. Where necessary you should state one or both do not exist (DNE).
   a) $\{x \in \mathbb{R} : x^2 > 5\}$
   b) $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$
   c) $\{x \in \mathbb{Z} : x + 3 > \pi\}$
   d) Let a be a fixed real number. $\{q \in \mathbb{Q} : q > a^2\}$

2) Determine whether the following are true or false. Justify your answers.
   a) Every nonempty set of real numbers that is bounded below has a least element.
   b) The set of positive rational numbers has a supremum.
   c) If S is a nonempty set of positive real numbers, then $0 \leq \inf S$.
   d) $\mathbb{Z}$ is dense in $\mathbb{R}$.
   e) $\mathbb{Q} - \mathbb{Z}$ is dense in $\mathbb{R}$.

3) Let S be a nonempty set of real numbers that is bounded. Prove the following:
   a) $\inf S \leq \sup S$.
   b) If $\inf S \geq \sup S$, then S consists of exactly one number.
   c) If B is a subset of S, then $\inf B \geq \inf S$ and $\sup S \geq \sup B$.

4) Fitzpatrick Section 1.1 number 11.

5) Fitzpatrick Section 1.1 number 19.

6) Let S be a nonempty set of real numbers that is bounded below. Prove that S has a minimum if and only if the number $\inf S$ belongs to S.

7) Suppose that the number a has the property that for every natural number n, $a \leq \frac{1}{n}$. Prove $a \leq 0$.

8) Let b be a real number and define $S = \{x \in \mathbb{Q} | x > b\}$. Prove $\inf S = b$.

9) a) (Cauchy's inequality) Using the fact that the square of a real number is non-negative, prove for any $a, b \in \mathbb{R}$, $ab \leq \frac{1}{2}(a^2 + b^2)$.
   b) Using Cauchy's inequality prove that if x, y are non-negative real numbers, then $\sqrt{xy} \leq \frac{1}{2}(x + y)$.
   c) Using Cauchy's inequality prove that if ℓ, m, n are non-negative real numbers, then $\ell m + mn + \ell n \leq l^2 + m^2 + n^2$.
   d) Using part b prove that if ℓ, m, n are non-negative real numbers, then $8\ell mn \leq (\ell + m)(m + n)(\ell + n)$.