HOMEWORK 12

1) Determine whether the following are true or false. Justify your answers.
   a) If the sequence $\{a_n^2\}$ converges, then $\{a_n\}$ converges.
   b) If the sequence $\{a_n + b_n\}$ converges, then $\{a_n\}$ and $\{b_n\}$ converge.
   c) If the sequence $\{|a_n|\}$ converges, then $\{a_n\}$ converges.
   d) Every bounded sequence converges.
   e) A convergent sequence of positive numbers has a positive limit.
   f) The limit of a convergent sequence in (a, b) also belongs to (a, b).

2) Prove that if the sequences $\{a_n\}$ and $\{b_n + a_n\}$ converge, then the sequence $\{b_n\}$ converges also.

3) Suppose the sequence $\{a_n\}$ converges to $a < 1$. Prove there exists an index N such that for all $n \geq N$, $a_n < 1$.

4) Prove that $\{c_n\} \to c$ if and only if $\{c_n - c\} \to 0$.

5) Fitzpatrick section 2.1 number 11. We have proven that the sequence $\{\frac{1}{n}\}$ converges to zero and that it does not converge to any other values. Use this to help you prove that none of the following statements are equivalent to the definition of convergence of a sequence $\{a_n\}$ to the number a.
   a) For some $\epsilon > 0$ there is an index N such that for all indices $n \geq N$, $|a_n - a| < \epsilon$.
   b) For each $\epsilon > 0$ and each index N, for all indices $n \geq N$, $|a_n - a| < \epsilon$.
   c) There exists an index N such that for every number $\epsilon > 0$, for all indices $n \geq N$, $|a_n - a| < \epsilon$.

6) Prove the set $[2, 5) \cup (5, 8]$ is not closed.

7) Prove the set $[0, \pi]$ is closed.