HOMEWORK 3

1) Let $m \in \mathbb{Z}$. Prove that $3|m$ if and only if $3|m^2$.

2) Let $x, y \in \mathbb{Z}$.

   a) Prove that if $3 \nmid x$, then $x^2 = 3m + 1$ for some $m \in \mathbb{Z}$.

   b) Using part a, Prove that if $3 \nmid x$ and $3 \nmid y$, then $3|(x^2 - y^2)$.

3) Let $x, y \in \mathbb{Z}$. Prove that $3|xy$ if and only if $3|x$ or $3|y$.

4) Prove that for every two distinct real numbers a and b, either $\frac{a+b}{2} < a$ or $\frac{a+b}{2} < b$. (Note: for two distinct real numbers a and b, either $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$ is also true.)

5) Let $x, y \in \mathbb{R}$. Prove that if $x^2 - 5x = y^2 - 5y$ and $x \neq y$, then $x + y = 5$.

6) Let $x \in \mathbb{R}$. Prove that if $7x^6 + 2x^4 + x^2 + 1 \leq x^7 + 6x^5 + 3x^3$, then $x > 0$.

7) Let $x, y \in \mathbb{R}$. Prove that $x^2 - xy \geq xy - y^2$.

8) a) Disprove: if $a, b \in \mathbb{R}^+$, then $\log(ab) = \log(a)\log(b)$.

   b) Let $f(x) = 2x^6 + x^2 + 1$. Disprove: there exists a $c \in [-1, 1]$ such that $f(c) = 0$.

   c) Disprove: there exists $x, y \in \mathbb{R}^+$ such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

9) Prove $\sqrt{3}$ is irrational (you may use the above questions as needed).

10) Prove that the product of an irrational number and a non-zero rational number is irrational.

11) Let t be odd and $n = 2t$. Prove there do not exist integers a and b such that $a^2 - b^2 = n$.

12) Let $f(x) = x^3 + x^2 + 1$. Prove that there exists a $c \in [-2, 0]$ such that $f(c) = 0$.

Not collected Book problems: $4.3, 4.5, 4.7, 4.19, 4.25, 4.55, 4.66, 4.68,$
$5.1, 5.11, 5.13, 5.15, 5.32, 5.33, 5.36, 5.43, 5.45, 5.46$