EXAM 2: **NAME:**________________

1) (24 pts) Prove the following by induction:

(a) Prove $3^n \geq 2^{n+1}$ for $n \geq 2$.

(b) Prove $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$ where F_i represents the Fibonacci numbers.

(c) Let $\langle a \rangle$ be a sequence satisfying $a_1 = 1, a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for $n \in \mathbb{N}$.

2) (10 pts) Are the following one-to-one and/or onto? Justify with a counterexample or proof.

(a) $f(x) = 3x + 1$ on $\mathbb{R}$.

(b) $g(x) = x^2$ on $\mathbb{R}$.

3) (12 pts) Are the following True or False? Remember to consider non-continuous functions. Give a counterexample or sentence as justification.

(a) Every decreasing function from $\mathbb{R}$ to $\mathbb{R}$ is surjective.

(b) Every surjective function from $\mathbb{R}$ to $\mathbb{R}$ is unbounded.

(c) Let $f : A \to B$ and $g : B \to A$. If $f(g(y)) = y$ for all $y \in B$, then f is a bijection.

4) (16 pts) Prove the following:

(a) The composition of two surjections is a surjection.

(b) If $f : A \to B$ and $g : B \to C$, let $h = g \circ f$. Prove if h is injective, then f is injective.

5) (16 pts) (a) Let B be a proper subset of a set A, and let f be a bijection from A to B. Prove that A is an infinite set.

(b) Give a bijection from the even natural numbers to the natural numbers.

6) (10 pts) Let S be a subset of $\{1, 2, \ldots, 3n\}$ having size $2n + 1$. Prove that S must contain 3 consecutive numbers. Show that this is the best possible by exhibiting a set of size $2n$ for which the conclusion is false.

7) (12 pts) Let A be a subset of $\mathbb{R}$. Let S be the set of all functions $\mathbb{R} \to \mathbb{R}$. Let R be the relation on S defined by $(f, g) \in R$ iff $f(x) = g(x)$ for all $x \in A$. Prove R is an equivalence relation.