# HOMEWORK

1) For each of the following sets find the infimum and supremum. Where necessary you should state one or both do not exist (DNE).
   a) $\{x \in \mathbb{R} : x^2 > 2\}$
   b) $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$
   c) $\{x \in \mathbb{Z} : x + 7 > \pi\}$
   d) Let $a$ be a fixed real number. $\{q \in \mathbb{Q} : q > a^2\}$

2) Determine whether the following are true or false. Justify your answers.
   a) Every nonempty set of real numbers that is bounded below has a least element.
   b) The set of positive rational numbers has a supremum.
   c) If $S$ is a nonempty set of negative real numbers, then $0 \geq \sup S$.
   d) If $S$ is a nonempty set of real numbers that is bounded below and $B$ is a subset of $S$, then $\inf B \geq \inf S$.
   e) $\mathbb{Z}$ is dense in $\mathbb{R}$.
   f) $\mathbb{Q} - \mathbb{Z}$ is dense in $\mathbb{R}$.

3) Let $S$ be a nonempty set of real numbers that is bounded. Prove the following:
   a) $\inf S \leq \sup S$.
   b) If $\inf S \geq \sup S$, then $S$ consists of exactly one number.

4) Fitzpatrick Section 1.1 number 11.

5) Fitzpatrick Section 1.1 number 19.

6) Let $S$ be a nonempty set of real numbers that is bounded below. Prove that $S$ has a minimum if and only if the number $\inf S$ belongs to $S$.

7) Suppose that the number $a$ has the property that for every natural number $n$, $a \leq 1/n$. Prove $a \leq 0$.

8) Let $b$ be a real number and define $S = \{x \in \mathbb{Q} | x < b\}$. Prove $\sup S = b$.

9) Fitzpatrick Section 1.2 number 7.

10) Let $a, b \in \mathbb{R}$ such that $|a - b| \leq 1$. Prove $|a| \leq |b| + 1$.

11)
   a) (Cauchy's inequality) Using the fact that the square of a real number is non-negative, prove for any $a, b \in \mathbb{R}$, $ab < \frac{1}{2}(a^2 + b^2)$.
   b) Using Cauchy's inequality prove that if $x, y$ are non-negative real numbers, then $\sqrt{xy} \leq \frac{1}{2}(x + y)$.
   c) Using Cauchy's inequality prove that if $\ell, m, n$ are non-negative real numbers, then $\ell m + mn + \ell n \leq l^2 + m^2 + n^2$.
   d) Using part $b$ prove that if $\ell, m, n$ are non-negative real numbers, then $8\ell mn \leq (\ell + m)(m + n)(\ell + n)$.