HOMEWORK 15

1) True or false. Justify your answers.
   a) A subsequence of a bounded sequence is bounded.
   b) A subsequence of a monotone sequence is monotone.
   c) A subsequence of a convergent sequence is convergent.
   d) A sequence converges if it has a convergent subsequence.
   e) Every sequence in the interval $(0, 1)$ has a convergent subsequence.
   f) The sum of monotone sequences is monotone.
   g) The product of monotone sequences is monotone.
   h) Every bounded sequence converges.
   i) Every monotone sequence converges.

2) Assume $\{a_n\}$ is monotonically increasing and define $s_n = a_1 + a_2 + \cdots a_n$ for all $n \in \mathbb{N}$. Prove the sequence $\{\frac{s_n}{n}\}$ is also increasing. [Note this can also be done for a decreasing sequence.]

3) Use the Monotone Convergence Theorem to prove each of the following sequences converge.
   a) $y_n = \frac{3n-1}{8n-2}$ for all $n \in \mathbb{N}$.
   b) $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ for all $n \in \mathbb{N}$ (Hint writing $2n$ as $n + n$ might help.)
   c) $\{a_n\}$ by $a_n = \frac{\sqrt{n}}{n+1}$ for all $n \in \mathbb{N}$

4) Let $\{a_n\}$ be a monotone sequence. Prove $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges. [Note: number one g is false] Demonstrate that this is false if you remove the monotonicity assumption.

5) Fitzpatrick section 2.3 number 8.

6) Use monotone convergence to prove the following converge and then find the limit using properties of subsequences.
   a) $s_1 = 1$ and $s_n = \frac{1}{4}(s_{n-1} + 5)$ for $n \geq 2$
   b) $s_1 = 2$ and $s_n = \frac{1}{4}(s_{n-1} + 5)$ for $n \geq 2$
   c) $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1}$ for all $n \in \mathbb{N}$

7) Fitzpatrick Section 2.4 number 7.

8) Fitzpatrick Section 2.4 number 8. Hint: You should use the previous question in your proof.

9) Fitzpatrick Section 2.4 number 9.

10) Let $0 < r < 1$.

   a) Prove $\{r^n\}$ converges using the Monotone Convergence Theorem.

   b) What does $\{r^n\}$ converge to? Prove your answer is correct.