

HOMEWORK 4

- 1) Let $m \in \mathbb{Z}$. Prove that $5|m$ if and only if $5|m^2$.
- 2) Let $x, y \in \mathbb{Z}$.
 - a) Prove that if $3 \nmid x$, then $x^2 = 3m + 1$ for some $m \in \mathbb{Z}$.
 - b) Using part a, Prove that if $3 \nmid x$ and $3 \nmid y$, then $3|(x^2 - y^2)$.
- 3) Let $x, y \in \mathbb{Z}$. Prove that $3|xy$ if and only if $3|x$ or $3|y$.
- 4) Let $x, y \in \mathbb{R}$. Prove that if $x^2 - 3x = y^2 - 3y$ and $x \neq y$, then $x + y = 3$.
- 5) Let $x \in \mathbb{R}$. Prove that if $7x^4 + 2x^2 + 1 \leq x^7 + 3x^5 + x^3$, then $x > 0$.
- 6) Let $x, y \in \mathbb{R}$. Prove that $x^2 - xy \geq xy - y^2$.
- 7)
 - a) Disprove: if $a, b \in \mathbb{R}^+$, then $\log\left(\frac{a}{b}\right) = \frac{\log(a)}{\log(b)}$.
 - b) Let $f(x) = 2x^6 + x^2 + 1$. Disprove: there exists a $c \in [-1, 1]$ such that $f(c) = 0$.
 - c) Disprove: there exists $x, y \in \mathbb{R}^+$ such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.
- 8) Prove $\sqrt{3}$ is irrational (you may use Hwk 5.3).
- 9) Prove that the sum of an irrational number and a rational number is irrational.
- 10) Let t be odd and $n = 2t$. Prove there do not exist integers a and b such that $a^2 - b^2 = n$.
- 11) Let $f(x) = x^3 + x + 1$. Prove that there exists a $c \in [-1, 1]$ such that $f(c) = 0$.

Not collected Book problems: 4.3, 4.5, 4.7, 4.19, 4.25, 4.55, 4.66, 4.68,
5.1, 5.11, 5.13, 5.15, 5.32, 5.33, 5.36, 5.43, 5.45, 5.46