HOMEWORK 3

1) Let $x, y \in \mathbb{Z}$. Prove that $5|xy$ if and only if $5|x$ or $5|y$.

2) Prove that for every two distinct real numbers a and b, either $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$. (Note: for two distinct real numbers a and b, either $\frac{a+b}{2} < a$ or $\frac{a+b}{2} < b$ is also true.)

3) Let $x, y \in \mathbb{R}$. Prove that if $x^2 - 3x = y^2 - 3y$ and $x \neq y$, then $x + y = 3$.

4) Let $x \in \mathbb{R}$. Prove that if $3x^6 + 4x^4 + 5x^2 + 1 \leq x^7 + 4x^5 + 2x^3$, then $x > 0$.

5) Let $x, y \in \mathbb{R}$. Prove that if $x < 0$, then $x^3 - x^2y \leq x^2y - xy^2$.

6) a) Disprove: if $a, b \in \mathbb{R}^+$, then $\log(ab) = \log(a)\log(b)$.

   b) Let $f(x) = 2x^6 + x^4 + 1$. Disprove: there exists a $c \in [-1, 1]$ such that $f(c) = 0$.

   c) Disprove: there exists $x, y \in \mathbb{R}^+$ such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

7) Prove $\sqrt{5}$ is irrational (you may use question 8 from homework 2 as needed).

8) Prove that the quotient of an irrational number and a non-zero rational number is irrational.

9) Let m be odd and $n = 2m$. Prove there do not exist integers x and y such that $x^2 - y^2 = n$.

10) Let $f(x) = x^3 + x^2 + 1$. Prove that there exists a $c \in [-2, -1]$ such that $f(c) = 0$.

Not collected Book problems: $4.3, 4.5, 4.7, 4.19, 4.25, 4.55, 4.66, 4.68,$
$5.1, 5.11, 5.13, 5.15, 5.32, 5.33, 5.36, 5.43, 5.45, 5.46$