THE GAUGE-UZAWA FINITE ELEMENT METHOD PART I : THE NAVIER-STOKES EQUATIONS

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Abstract. The Gauge-Uzawa FEM is a new first order fully discrete projection method which combines advantages of both the Gauge and Uzawa methods within a variational framework. A time step consists of a sequence of $d+1$ Poisson problems, d being the space dimension, thereby avoiding both the incompressibility constraint as well as dealing with boundary tangential derivatives as in the Gauge Method. This allows for a simple finite element discretization in space of any order in both 2d and 3d. This first part introduces the method for the Navier-Stokes equations of incompressible fluids and shows unconditional stability and error estimates for both velocity and pressure via a variational approach under realistic regularity assumptions. Several numerical experiments document performance of the Gauge-Uzawa FEM and compare it with other projection methods.

Key words. Projection method, Gauge method, Uzawa method, Navier-Stokes equation.

AMS subject classifications. 65M12, 65M15, 65M60.

1. Introduction.

Given an open bounded polygon (or polyhedron) Ω in $\mathbb{R}^d$ with $d = 2$ (or 3), we consider the time dependent Navier-Stokes Equations:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\triangle\mathbf{u} &= \mathbf{f}, && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}^0, && \text{in } \Omega, \end{aligned} \tag{1.1}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. This system models the dynamics of an incompressible viscous Newtonian fluid. The viscosity $\mu = Re^{-1}$ is the reciprocal of the Reynolds number. The unknowns are vector function $\mathbf{u}$ (velocity) and scalar function p (pressure).

The incompressibility condition div $\mathbf{u} = 0$ in (1.1) leads to a saddle point structure, which requires compatibility between the discrete spaces for $\mathbf{u}$ and p [1, 2, 10] (*inf-sup condition*). To circumvent this difficulty, projection methods have been studied since the late 60's, which exploit the time dependence in (1.1) [4, 9, 11, 18, 21, 24, 25]. However, such methods

- yield momentum equations inconsistent with the first equation in (1.1) ;
- impose artificial boundary conditions on pressure (or related variables), which are responsible for boundary layers and reduced accuracy [4, 9];
- require sometimes knowing a suitable initial pressure which is incompatible with the elliptic nature of the Lagrange multiplier p and equation div $\mathbf{u} = 0$ [11, 18];
- are often studied without space discretization [3, 4, 18, 20, 21, 25], and the ensuing analysis may not apply to full discretizations;
- often require unrealistic regularity assumptions in their analysis, particularly so for fully discrete schemes; for instance $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^2)$, $\mathbf{u}_{ttt} \in L^\infty(\mathbf{H}^1)$, $p_{tt} \in L^\infty(H^2)$, $p_{ttt} \in L^\infty(L^2)$ is required in [11] for a Chorin finite element method, and similar strong assumptions are made in [27] for a Gauge finite difference method.

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The Gauge Method is a projection method, due to Osedelets [17] and E and Liu [7], meant to circumvent these difficulties. It introduces new variables $\mathbf{a}$ and ϕ (gauge) such that $\mathbf{u} = \mathbf{a} + \nabla\phi$ and couple them via the boundary condition $\mathbf{u} = \mathbf{0}$. The method has been studied in [27] using asymptotic methods and in [16] employing variational techniques. The boundary coupling is responsible for accuracy degradation in problems with singular solutions (due to reentrant corners), as will be illustrated below. It also makes the use of finite element methods (FEM) problematic for space discretization. In this paper, we construct a Gauge-Uzawa FEM (GU-FEM) which inherits some beneficial properties of both the Gauge Method and the Uzawa Method and avoids dealing with boundary derivatives. We also prove that the fully discrete method is unconditionally stable and derive error estimates for both velocity and pressure under realistic regularity requirements.

1.1. The Gauge-Uzawa Finite Element Method. To motivate the new method we start from the Gauge Method of Oseledets [17] and E and Liu [7]; see also [16, 19]. Let ϕ be an auxiliary scalar variable, the so-called *gauge* variable, and $\mathbf{a}$ be a vector unknown such that $\mathbf{u} = \mathbf{a} + \nabla\phi$. If ϕ and p satisfy the heat equation $\partial_t\phi - \mu\Delta\phi = -p$, then the momentum and incompressibility equations become

$$\begin{aligned} \partial_t\mathbf{a} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\triangle\mathbf{a} = \mathbf{f}, \quad &\text{in } \Omega, \\ -\triangle\phi = \operatorname{div}\mathbf{a}, \quad &\text{in } \Omega. \end{aligned}$$

This formulation is equivalent to (1.1) at the PDE level. We are now free to choose boundary conditions for the non-physical variables $\mathbf{a}$ and ϕ for as long as $\mathbf{u} = \mathbf{0}$ is enforced. Hereafter, we employ a Neumann condition on ϕ which, according to [7, 16, 19, 27], is the most advantageous:

$$\partial_{\boldsymbol{\nu}}\phi = 0, \quad \mathbf{a}\cdot\boldsymbol{\nu} = 0, \quad \mathbf{a}\cdot\boldsymbol{\tau} = -\partial_{\boldsymbol{\tau}}\phi;$$

$\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the unit vectors in the normal and tangential directions, respectively. Upon discretizing in time via the backward Euler method [7, 27], and a semi-implicit treatment of the convection term, we end up with the following unconditionally stable method [16, 19]:

ALGORITHM 1 (Gauge Method). *Start with* $\phi^0 = 0$ *and* $\mathbf{a}^0 = \mathbf{u}^0$. *Repeat the steps*

Step 1: *Find* $\mathbf{a}^{n+1}$ *as the solution of*

$$\begin{aligned} \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\tau} + (\mathbf{u}^n\cdot\nabla)(\mathbf{a}^{n+1} + \nabla\phi^n) - \mu\triangle\mathbf{a}^{n+1} = \mathbf{f}(t^{n+1}), \quad &\text{in } \Omega, \\ \mathbf{a}^{n+1}\cdot\boldsymbol{\nu} = 0, \qquad \mathbf{a}^{n+1}\cdot\boldsymbol{\tau} = -\partial_{\boldsymbol{\tau}}\phi^n, \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Step 2: *Find* ϕ^{n+1} *as the solution of*

$$\begin{aligned} -\triangle\phi^{n+1} = \operatorname{div}\mathbf{a}^{n+1}, \quad &\text{in } \Omega, \\ \partial_{\boldsymbol{\nu}}\phi^{n+1} = 0, \quad &\text{on } \partial\Omega. \end{aligned}$$

Step 3: *Update* $\mathbf{u}^{n+1}$ *according to*

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1}. \tag{1.3}$$

We point out that the momentum equation is linear in $\mathbf{a}^{n+1}$, and that the explicit boundary condition $\mathbf{a}^{n+1}\cdot\boldsymbol{\tau} = -\partial_{\boldsymbol{\tau}}\phi^n$ is crucial to decouple the equations for $\mathbf{a}^{n+1}$ and ϕ^{n+1}. Since this formulation is consistent with (1.1), except for $\mathbf{u}^{n+1}\cdot\boldsymbol{\tau} = \partial_{\boldsymbol{\tau}}(\phi^{n+1} - \phi^n)$, normal mode analysis can be used to show full accuracy for smooth solutions [3, 20]. However, several deficiencies of this algorithm are now apparent:

- The boundary term $\partial_{\boldsymbol\tau}\phi^n$ is non-variational and thus difficult to implement within a finite element context, especially in 3d.
- The computation of $\partial_{\boldsymbol\tau}\phi^n$, which involves numerical differentiation, yields loss of accuracy and is problematic at corners of $\partial\Omega$ where $\boldsymbol\tau$ is not well defined. This is remarkably important for reentrant corners as illustrated in the comparisons below.
- The computation of $p^{n+1} = \mu\Delta\phi^{n+1} - \tau^{-1}(\phi^{n+1} - \phi^n)$ is also unstable. This yields a reduced rate of convergence or lack of convergence altogether [16, 19, 27].
- Numerical experiments indicate that the polynomial degree for ϕ must be of higher order than that for p [19]. A suitable combination of finite element spaces for $(\mathbf{a}, \mathbf{u}, \phi, p)$ is continuous piecewise polynomials $(\mathcal{P}^2, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^1)$, which is consistent with (1.3) and the previous expression for p^{n+1}. This is however rather costly computationally since ϕ is just an auxiliary variable without intrinsic interest [19].

The purpose of this paper is to construct and study the Gauge-Uzawa FEM, which overcomes these shortcomings without losing advantages of the Gauge Method. We start by introducing a new vector variable $\widehat{\mathbf{u}}^{n+1}$ having zero boundary values

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^n.$$

Inserting this into (1.2), we readily get

$$\frac{\widehat{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n\cdot\nabla)\widehat{\mathbf{u}}^{n+1} - \mu\triangle\widehat{\mathbf{u}}^{n+1} + \mu\nabla\triangle\phi^n = \mathbf{f}(t^{n+1}), \quad \text{in } \Omega. \tag{1.4}$$

To deal with the third order term $\nabla\triangle\phi^n$, which is a source of trouble due to lack of commutativity of the differential operators at the discrete level, we introduce the variable $s^{n+1} = \triangle\phi^{n+1}$ and note the connection with the *Uzawa iteration*:

$$s^{n+1} = \triangle\phi^{n+1} = -\operatorname{div}\mathbf{a}^{n+1} = \triangle\phi^n - \operatorname{div}\widehat{\mathbf{u}}^{n+1} = s^n - \operatorname{div}\widehat{\mathbf{u}}^{n+1}. \tag{1.5}$$

If we also set $\rho^{n+1} = \phi^{n+1} - \phi^n$, then

$$-\triangle\rho^{n+1} = -\triangle(\phi^{n+1} - \phi^n) = \operatorname{div}\widehat{\mathbf{u}}^{n+1}. \tag{1.6}$$

Combining (1.4), (1.5) and (1.6) we arrive at the discrete-time Gauge-Uzawa method.

In order to introduce the finite element discretization we need further notation. Let $H^s(\Omega)$ be the Sobolev space with s derivatives in $L^2(\Omega)$, set $\mathbf{L}^2(\Omega) = \big(L^2(\Omega)\big)^d$ and $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, where $d = 2$ or 3, and denote by $L^2_0(\Omega)$ the subspace of $L^2(\Omega)$ of functions with vanishing meanvalue. We indicate with $\|\cdot\|_s$ the norm in $H^s(\Omega)$, and with $\langle\cdot\,,\,\cdot\rangle$ the inner product in $L^2(\Omega)$. Let $\mathfrak{T} = \{K\}$ be a shape-regular quasi-uniform partition of Ω of meshsize h into closed elements K [1, 2, 10]. The vector and scalar finite element spaces are:

$$\begin{aligned}
&\mathbb{W}_h := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) : \mathbf{v}_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathfrak{T}\}, \quad \mathbb{V}_h := \mathbb{W}_h \cap \mathbf{H}^1_0(\Omega),\\
&\mathbb{P}_h := \{q_h \in L^2_0(\Omega) \cap C^0(\Omega) : q_h|_K \in \mathcal{Q}(K) \quad \forall K \in \mathfrak{T}\},
\end{aligned}$$

where $\mathcal{P}(K)$ and $\mathcal{Q}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathfrak{T}$ [2, 10]. We stress that the space $\mathbb{P}_h$ is composed of continuous functions for (1.6) to make sense. This implies the crucial equality

$$\langle \operatorname{div}\mathbf{v}_h\,,\, q_h\rangle = -\langle \mathbf{v}_h\,,\,\nabla q_h\rangle\,, \quad \forall \mathbf{v}_h \in \mathbb{V}_h, q_h \in \mathbb{P}_h.$$

Using the following discrete counterpart of the form $\mathfrak{N}(\mathbf{u},\mathbf{v},\mathbf{w}) = \langle(\mathbf{u}\cdot\nabla)\mathbf{v}\,,\,\mathbf{w}\rangle$

$$\mathfrak{N}_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = \frac{1}{2}\langle(\mathbf{u}_h\cdot\nabla)\mathbf{v}_h\,,\,\mathbf{w}_h\rangle - \frac{1}{2}\langle(\mathbf{u}_h\cdot\nabla)\mathbf{w}_h\,,\,\mathbf{v}_h\rangle\,, \tag{1.7}$$

we are ready to write the Gauge-Uzawa finite element method:

Algorithm 2 (Gauge-Uzawa FEM). *Start with* $s_h^0 = 0$ *and* $\mathbf{u}_h^0$ *as a solution of* $\langle \mathbf{u}_h^0 , \mathbf{w}_h \rangle = \langle \mathbf{u}^0 , \mathbf{w}_h \rangle$ *for all* $\mathbf{w}_h \in \mathbb{V}_h$.

Step 1: *Find* $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h$ *as the solution of*

$$\begin{aligned} \tau^{-1} \left\langle \widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n , \mathbf{w}_h \right\rangle + \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \mu \left\langle \nabla \widehat{\mathbf{u}}_h^{n+1} , \nabla \mathbf{w}_h \right\rangle \\ - \mu \left\langle s_h^n , \operatorname{div} \mathbf{w}_h \right\rangle = \left\langle \mathbf{f}(t^{n+1}) , \mathbf{w}_h \right\rangle , \quad \forall \mathbf{w}_h \in \mathbb{V}_h. \end{aligned} \tag{1.8}$$

Step 2: *Find* $\rho_h^{n+1} \in \mathbb{P}_h$ *as the solution of*

$$\left\langle \nabla \rho_h^{n+1} , \nabla \psi_h \right\rangle = \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} , \psi_h \right\rangle , \qquad \forall \psi_h \in \mathbb{P}_h. \tag{1.9}$$

Step 3: *Update* $s_h^{n+1} \in \mathbb{P}_h$ *according to*

$$\left\langle s_h^{n+1} , q_h \right\rangle = \left\langle s_h^n , q_h \right\rangle - \left\langle \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} , q_h \right\rangle , \qquad \forall q_h \in \mathbb{P}_h. \tag{1.10}$$

Step 4: *Update* $\mathbf{u}_h^{n+1} \in \mathbb{W}_h$ *according to*

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} + \nabla \rho_h^{n+1}. \tag{1.11}$$

We note that $\mathbf{u}_h^{n+1}$ is a discontinuous function across inter-element boundaries and that, in light of (1.9), $\mathbf{u}_h^{n+1}$ is discrete divergence free in the sense that

$$\left\langle \mathbf{u}_h^{n+1} , \nabla \psi_h \right\rangle = 0, \quad \forall \psi_h \in \mathbb{P}_h. \tag{1.12}$$

In addition, the discrete pressure $p_h^{n+1} \in \mathbb{P}_h$ can be computed via

$$p_h^{n+1} = \mu s_h^{n+1} - \tau^{-1} \rho_h^{n+1}. \tag{1.13}$$

Consequently, the ensuing momentum equations for either $(\widehat{\mathbf{u}}^{n+1}, p^n)$ or $(\mathbf{u}^{n+1}, p^{n+1})$ are fully consistent with (1.1), a distinctive feature of this new formulation:

$$\begin{aligned} \tau^{-1} \left\langle \widehat{\mathbf{u}}_h^{n+1} - \widehat{\mathbf{u}}_h^n , \mathbf{w}_h \right\rangle + \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \mu \left\langle \nabla \widehat{\mathbf{u}}_h^{n+1} , \nabla \mathbf{w}_h \right\rangle \\ - \left\langle p_h^n , \operatorname{div} \mathbf{w}_h \right\rangle = \left\langle \mathbf{f}(t^{n+1}) , \mathbf{w}_h \right\rangle , \quad \forall \mathbf{w}_h \in \mathbb{V}_h. \end{aligned} \tag{1.14}$$

1.2. Comparison with Other Projection Methods. We now compare the Gauge-Uzawa FEM of Algorithm 2 with the original Chorin Method [4, 25], the Chorin-Uzawa Method [18], and the Gauge Method of Algorithm 1 [7, 16, 19] using finite elements of degree 2 for $\mathbf{u}, \widehat{\mathbf{u}}, \mathbf{a}$, of degree 1 for p, s, ρ, and of degree 3 for ϕ. We consider the L-shaped domain $\Omega = ((-1,1) \times (-1,1)) - ([0,1) \times (-1,0])$ and the corresponding time-dependent singular solution of the Stokes equation ($\mathfrak{N}_h = 0$) [26]

$$\mathbf{u}(r,\theta) = \frac{3-\cos(5t)}{4} r^\alpha \begin{bmatrix} \cos(\theta)\psi'(\theta) + (1+\alpha)\sin(\theta)\psi(\theta) \\ \sin(\theta)\psi'(\theta) - (1+\alpha)\cos(\theta)\psi(\theta) \end{bmatrix},$$

$$p(r,\theta) = -\frac{3-\cos(5t)}{4} r^{\alpha-1} \frac{(1+\alpha)^2\psi'(\theta) + \psi'''(\theta)}{1-\alpha},$$

where $\omega = \frac{3\pi}{2}, \alpha = 0.544$,

$$\psi(\theta) = \frac{\sin((1+\alpha)\theta)\cos(\alpha\omega)}{1+\alpha} - \cos((1+\alpha)\theta) + \frac{\sin((\alpha-1)\theta)\cos(\alpha\omega)}{1-\alpha} + \cos((\alpha-1)\theta),$$

and $T = 5$. Since $\alpha < 1$, the pressure p is unbounded at the origin. The initial mesh and time steps are $\tau = h = 1/8$ and are subsequently halved for every experiment.

Figure 1.1 clearly shows the superior performance of the Gauge-Uzawa FEM, particularly so in regard to pressure approximation for which the Gauge Method fails to converge. These experiments, as well as those in §7, were carried out within the software platform ALBERT of Schmidt and Siebert [22].

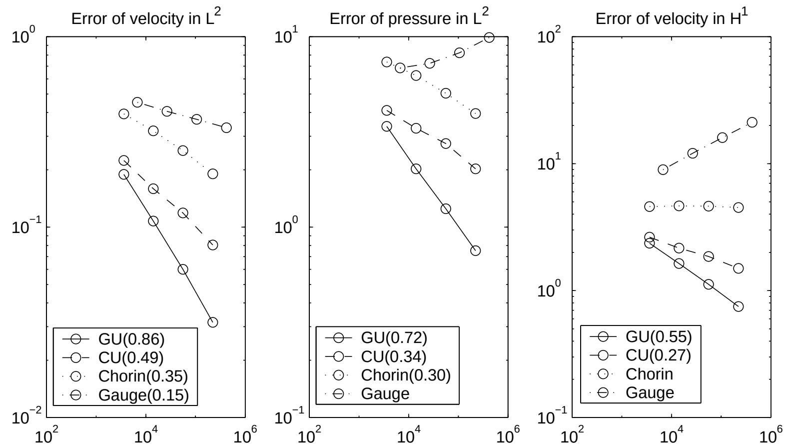

Fig. 1.1. *Error decay vs number of degrees of freedom for four projection methods; the errors are measured in $L^2(\mathbf{L}^2)$ and $L^2(\mathbf{H}^1)$ for velocity and $L^2(L^2)$ for pressure. Velocity and pressure do not always converge for the Gauge Method, even though we use the best finite element combination $(\mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^3)$ for $(\mathbf{u}, p, \phi)$. The Gauge-Uzawa FEM exhibits a superior performance overall. The numbers in parenthesis are the experimental orders of convergence.*

1.3. The Main Results. We now summarize our theoretical results of the rest of this paper for the Gauge-Uzawa FEM. In §3 we prove stability.

Theorem 1.1 (Stability). *The Gauge-Uzawa FEM is unconditionally stable in the sense that, for all $\tau > 0$, the following a priori bound holds:*

$$\begin{aligned}\left\|\mathbf{u}_h^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_0^2 + \frac{\mu\tau}{2}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\ + 2\sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2 + \mu\tau\left\|s_h^{N+1}\right\|_0^2 \le \left\|\mathbf{u}_h^0\right\|_0^2 + C\tau\sum_{n=0}^{N}\left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2.\end{aligned} \tag{1.15}$$

We then study the rate of convergence of the various unknowns under appropriate assumptions $A1-6$ described in §2. In §4 we prove error estimates for velocity.

Theorem 1.2 (Error Estimates for Velocity). *If A1-6 hold and $h^2 \le C\tau$, with $C > 0$ arbitrary, then we have the error estimates*

$$\tau\sum_{n=0}^{N}\left\|\nabla\left(\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right)\right\|_0^2 \le C(\tau + h^2),$$

$$\tau\sum_{n=0}^{N}\left(\left\|\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1}\right\|_0^2 + \left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_0^2\right) \le C(\tau + h^2)^2.$$

Given a sequence $\{W^n\}_{n=0}^N$, we define its discrete time derivative to be

$$\delta W^{n+1} := \frac{W^{n+1} - W^n}{\tau}.$$

We also define the discrete weight $\sigma^n := \min(t^n, 1)$ for $1 \le n \le N$. In §5 we derive an error estimate for time derivative of velocity and utilize it in §6 to prove and error estimate for pressure.

THEOREM 1.3 (Error estimates for Time Derivative of Velocity and Pressure). *Let A1-6 hold and* $C_1 h^2 \le \tau \le C_2 h^{\frac{d}{3}(1+\varepsilon)}$ *be valid with arbitrary constants* $C_1 > 0$ *and* $C_2 > 0$*, where* d *is the space dimension. Then the following weighted estimates hold*

$$\tau \sum_{n=0}^{N} \sigma^{n+1} \left(\left\| \delta(\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1}) \right\|_0^2 + \left\| p(t^{n+1}) - p_h^{n+1} \right\|_0^2 \right) \le C(\tau + h^2).$$

If NLC of §2 is also satisfied, then the following uniform error estimates are valid

$$\tau \sum_{n=0}^{N} \left(\left\| \delta(\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1}) \right\|_0^2 + \left\| p(t^{n+1}) - p_h^{n+1} \right\|_0^2 \right) \le C(\tau + h^2).$$

The proofs of Theorems 1.1-1.3 follow the variational approach of [16, 19]. We finally conclude in §7 with numerical experiments which document both accuracy and performance of the Gauge-Uzawa FEM.

2. Basic Assumptions and Regularity. This section is mainly devoted to stating assumptions and basic regularity results. We refer to Constantin and Foias [5], Heywood and Rannacher [12], Prohl [18] for details.

2.1. Regularity. We start with three basic assumptions about data Ω, $\mathbf{u}^0$, $\mathbf{f}$, and $\mathbf{u}$. We consider first the stationary Stokes equations, which will be used in a duality argument:

$$\begin{aligned} -\triangle \mathbf{v} + \nabla q &= \mathbf{g}, \quad \text{in } \Omega, \\ \text{div } \mathbf{v} &= 0, \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0}, \quad \text{on } \Omega. \end{aligned} \tag{2.1}$$

ASSUMPTION A 1 (Regularity of $(\mathbf{v}, q)$). *The unique solution* $(\mathbf{v}, q) \in H_0^1(\Omega) \times L_0^2(\Omega)$ *of the stationary Stokes equations (2.1) satisfies*

$$\|\mathbf{v}\|_2 + \|q\|_1 \le C \|\mathbf{g}\|_0.$$

We remark that A1 is valid provided $\partial\Omega$ is of class C^2 [5], or if Ω is a *convex* two-dimensional polygon [13] or three-dimensional polyhedron [6].

ASSUMPTION A 2 (Data Regularity). *The initial velocity* $\mathbf{u}^0$ *and the forcing term* $\mathbf{f}$ *in (1.1) satisfy*

$$\mathbf{u}^0 \in \mathbf{H}^2(\Omega) \cap \mathbf{Z}(\Omega) \quad \text{and} \quad \mathbf{f},\, \mathbf{f}_t \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)),$$

where $\mathbf{Z}(\Omega) := \{\mathbf{z} \in \mathbf{H}_0^1(\Omega) : \text{div } \mathbf{z} = 0\}$.

ASSUMPTION A 3 (Regularity of the Solution $\mathbf{u}$). *There exists* $M > 0$ *such that*

$$\sup_{t \in [0,T]} \|\nabla \mathbf{u}(t)\|_0 \le M.$$

We note that A3 is always satisfied in 2d, whereas it is valid in 3d provided $\left\|\mathbf{u}^0\right\|_1$ and $\|\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$ are sufficiently small [12].

LEMMA 2.1 (Uniform and Weighted A Priori Estimates [12]). *Let* $\sigma(t) = \min\{t, 1\}$ *be a weight function. Let A1-3 hold and* $0 < T \le \infty$*. Then the solution* $(\mathbf{u}, p)$ *of* (1.1) *satisfies*

$$\sup_{0<t<T} \left(\|\mathbf{u}\|_2 + \|\mathbf{u}_t\|_0 + \|p\|_1 \right) \le M, \qquad \int_0^T \|\mathbf{u}_t\|_1^2 dt \le M, \tag{2.2}$$

and

$$\sup_{0<t<T}\Big(\sigma(t)\|\mathbf{u}_t\|_1^2\Big) \le M, \qquad \int_0^T \sigma(t)\Big(\|\mathbf{u}_t\|_2^2 + \|\mathbf{u}_{tt}\|_0^2 + \|p_t\|_1^2\Big)dt \le M. \tag{2.3}$$

Consequently, $(\mathbf{u}, p) \in L^\infty(0, T; \mathbf{H}^2(\Omega) \times H^1(\Omega))$ provided A1-3 are valid.

The following *nonlocal* assumption is used to remove the weight $\sigma(t)$ for the error estimates for $\mathbf{u}_t$ in §5 and pressure in §6.

ASSUMPTION NLC (Nonlocal Compatibility). *The data $\mathbf{u}^0$ and $\mathbf{f}^0 = \mathbf{f}(0,\cdot)$ are such that $\|\nabla\mathbf{u}_t(0)\|_0 \le M$.*

In view of [12, Corollary 2.1], we realize that NLC is equivalent to the initial data $\mathbf{u}^0, p^0 = p(0,\cdot), \mathbf{f}^0$ satisfying the overdetermined system

$$\Delta p^0 = \mathrm{div}\,\big(\mathbf{f}^0 - (\mathbf{u}^0\cdot\nabla)\mathbf{u}^0\big) \quad \text{in } \Omega, \qquad \nabla p^0 = \Delta\mathbf{u}^0 + \mathbf{f}^0 - (\mathbf{u}^0\cdot\nabla)\mathbf{u}^0 \quad \text{on } \partial\Omega.$$

This is true if $\mathbf{u}^0 = \mathbf{f}^0 = \mathbf{0}$, in which case also $p^0 = 0$ and $\|\nabla\mathbf{u}_t(0)\|_0 = 0$. However, $\|\nabla\mathbf{u}_t(t)\|_0$ blows-up in general as $t \downarrow 0$, thereby uncovering the practical limitations of results based on higher regularity than (2.2) and (2.3) uniformly for $t \downarrow 0$ [11, 27].

LEMMA 2.2 (Uniform A Priori Estimates [12, Corollary 2.1]). *Suppose A1-3 hold and let $0 < T \le \infty$. Then NLC is valid if and only if*

$$\int_0^T \|\mathbf{u}_{tt}(t)\|_0^2 dt + \sup_{0<t<T}\|\nabla\mathbf{u}_t(t)\|_0^2 \le M. \tag{2.4}$$

Furthermore, if NLC holds, then

$$\int_0^T \Big(\|p_t(t)\|_1^2 + \|\mathbf{u}_t(t)\|_2^2\Big)\,dt \le M.$$

LEMMA 2.3 (A Priori Estimates on $\mathbf{Z}(\Omega)^*$ [16, 19]). *If A1-3 hold, then we have*

$$\int_0^T \|\mathbf{u}_{tt}(t)\|_*^2 dt \le M, \tag{2.5}$$

where $\mathbf{Z}(\Omega)^*$ is a dual space of $\mathbf{Z}(\Omega)$. Furthermore, if NLC also hold, then

$$\sup_{0<t<T}\|\mathbf{u}_{tt}(t)\|_*^2 \le M.$$

LEMMA 2.4 (Div-Grad Relation [15, 16, 19, 24]). *If $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, then*

$$\|\mathrm{div}\ \mathbf{v}\|_0 \le \|\nabla\mathbf{v}\|_0.$$

2.2. Properties of FEM. We impose the following properties on $\mathbb{V}_h, \mathbb{P}_h$.

ASSUMPTION A4 (Discrete Inf-Sup). *There exists a constant $\beta > 0$ such that*

$$\inf_{q_h\in\mathbb{P}_h}\sup_{\mathbf{v}_h\in\mathbb{V}_h}\frac{\langle \mathrm{div}\ \mathbf{v}_h\,,\, q_h\rangle}{\|\mathbf{v}_h\|_1\|q_h\|_0} \ge \beta.$$

ASSUMPTION A5 (Shape Regularity and Quasiuniformity [1, 2, 10]). *There exists a constant $C > 0$ such that the ratio between the diameter h_K of an element $K \in \mathfrak{T}$*

and the diameter of the largest ball contained in K is bounded uniformly by C, and h_K is comparable with the meshsize h for all $K \in \mathfrak{T}$.

ASSUMPTION A6 (Approximability [1, 2, 10]). *For each $(\mathbf{v}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$, there exist approximations $(\mathbf{v}_h, q_h) \in \mathbb{V}_h \times \mathbb{P}_h$ such that*

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 \le Ch^2\|\mathbf{v}\|_2 \quad \text{and} \quad \|q - q_h\|_0 \le Ch\|q\|_1.$$

Let now $(\mathbf{v}_h, q_h) \in \mathbb{V}_h \times \mathbb{P}_h$ indicate the finite element solution of (2.1), namely,

$$\begin{aligned} \langle \nabla \mathbf{v}_h \,,\, \nabla \mathbf{w}_h \rangle - \langle q_h \,,\, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{g} \,,\, \mathbf{w}_h \rangle \,, \quad &\forall \mathbf{w}_h \in \mathbb{V}_h, \\ \langle r_h \,,\, \operatorname{div} \mathbf{v}_h \rangle = 0, \qquad\qquad &\forall r_h \in \mathbb{P}_h. \end{aligned} \tag{2.6}$$

LEMMA 2.5 (Error Estimates for Mixed FEM [1, 2, 10]). *Let $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ be the solutions of (2.1) and $(\mathbf{v}_h, q_h) = \mathfrak{S}_h(\mathbf{v}, q) \in \mathbb{V}_h \times \mathbb{P}_h$ be the Stokes projections defined by (2.6), respectively. If A4-6 hold, then*

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 + h\|q - q_h\|_0 \le Ch^2 \left(\|\mathbf{v}\|_2 + \|q\|_1\right). \tag{2.7}$$

If also A1 holds, then the right-hand side is bounded by $Ch^2\|\mathbf{g}\|_0$ and if $d \le 4$

$$\|\mathbf{g}\|_* \le C\|\nabla \mathbf{v}\|_0 \le Ch\|\mathbf{g}\|_0 + C\|\nabla \mathbf{v}_h\|_0, \tag{2.8}$$

$$|\!|\!|\mathbf{v} - \mathbf{v}_h|\!|\!| := \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{\mathbf{L}^3(\Omega)} \le C\|\mathbf{g}\|_0. \tag{2.9}$$

Proof. Inequality (2.7) is standard [1, 2, 10]. To prove (2.8) we simply test (2.1) with an arbitrary $\mathbf{z} \in \mathbf{Z}(\Omega)$ for the first inequality, and next use (2.7) for the second one. To establish (2.9) we just deal with the L^∞-norm since the other can be treated similarly. If I_h denotes the Clement interpolant, then $\|\mathbf{v} - I_h\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \le C\|\mathbf{v}\|_2$ and

$$\|I_h\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^\infty(\Omega)} \le Ch^{-d/2}\|I_h\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C\|\mathbf{v}\|_2$$

as a consequence of an inverse estimate and (2.7). This completes the proof. ∎

REMARK 2.6 (H^1 Stability of q_h). The bound $\|\nabla q_h\|_0 \le C\left(\|\mathbf{v}\|_2 + \|q\|_1\right)$ is a simple by-product of (2.7). To see this, we add and subtract I_hq, use the stability of I_h in H^1, and observe that (2.7) implies $\|\nabla(q_h - I_hq)\|_0 \le Ch^{-1}\|q_h - I_hq\| \le C$.

We finally state without proof several properties of the nonlinear form $\mathfrak{N}_h$. In view of (1.7), we have a following properties of $\mathfrak{N}_h$ for all $\mathbf{u}_h, \mathbf{v}_h\mathbf{w}_h \in \mathbb{V}_h$:

$$\mathfrak{N}_h(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) = -\mathfrak{N}_h(\mathbf{u}_h, \mathbf{w}_h, \mathbf{v}_h), \qquad \mathfrak{N}_h(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) = 0, \tag{2.10}$$

and

$$\operatorname{div} \mathbf{u} = 0 \quad \Rightarrow \quad \mathfrak{N}_h(\mathbf{u}, \mathbf{v}_h, \mathbf{w}_h) = \mathfrak{N}(\mathbf{u}, \mathbf{v}_h, \mathbf{w}_h) = -\mathfrak{N}(\mathbf{u}, \mathbf{w}_h, \mathbf{v}_h).$$

Applying Sobolev imbedding Lemma yields the following useful results.

LEMMA 2.7 (Bounds on Nonlinear Convection [11, 12]). *Let $\mathbf{u}, \mathbf{v} \in \mathbf{H}^2(\Omega)$ with $\operatorname{div} \mathbf{u} = 0$, and let $\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbb{V}_h$. Then*

$$\mathfrak{N}_h(\mathbf{u}, \mathbf{v}_h, \mathbf{w}_h) \le C \begin{cases} \|\mathbf{u}\|_1\|\mathbf{v}_h\|_1\|\mathbf{w}_h\|_1 \\ \|\mathbf{u}\|_2\|\nabla \mathbf{v}_h\|_0\|\mathbf{w}_h\|_0 \\ \|\mathbf{u}\|_2\|\mathbf{v}_h\|_0\|\nabla \mathbf{w}_h\|_0, \end{cases} \tag{2.11}$$

$$\mathfrak{N}_h(\mathbf{u}_h, \mathbf{v}, \mathbf{w}_h) \le \|\mathbf{u}_h\|_0\|\mathbf{v}\|_2\|\nabla \mathbf{w}_h\|_0. \tag{2.12}$$

If in addition $d \le 3$, then

$$\mathfrak{N}_h(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) \le C \begin{cases} C\|\mathbf{u}_h\|_0 \|\!|\mathbf{v}_h|\!\| \|\nabla \mathbf{w}_h\|_0 \\ C\|\mathbf{u}_h\|_{\mathbf{L}^3(\Omega)} \|\mathbf{v}_h\|_1 \|\nabla \mathbf{w}_h\|_0. \end{cases} \tag{2.13}$$

3. Theorem 1.1: Stability. In this section, we show that the Gauge-Uzawa FEM is unconditionally stable via a standard energy method. We choose $\mathbf{w}_h = 2\tau \widehat{\mathbf{u}}_h^{n+1}$ in (1.8) and observe the following relation for the first term in (1.8)

$$\left\langle \widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n \,,\, \widehat{\mathbf{u}}_h^{n+1} \right\rangle = \left\langle \mathbf{u}_h^{n+1} - \mathbf{u}_h^n \,,\, \mathbf{u}_h^{n+1} \right\rangle + \left\langle \nabla \rho_h^{n+1} \,,\, \nabla \rho_h^{n+1} \right\rangle,$$

because of (1.11). Since the convection term vanishes from (2.10), we then obtain

$$\begin{aligned} \left\|\mathbf{u}_h^{n+1}\right\|_0^2 - \|\mathbf{u}_h^n\|_0^2 + \left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_0^2 + 2\left\|\nabla \rho_h^{n+1}\right\|_0^2 + 2\mu\tau \left\|\nabla \widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\ = 2\mu\tau \left\langle s_h^n \,,\, \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \right\rangle + 2\tau \left\langle \mathbf{f}(t^{n+1}) \,,\, \widehat{\mathbf{u}}_h^{n+1} \right\rangle. \end{aligned}$$

According to (1.10), we can write

$$2\left\langle s_h^n \,,\, \operatorname{div} \widehat{\mathbf{u}}_h^{n+1} \right\rangle = 2\left\langle s_h^n \,,\, s_h^n - s_h^{n+1} \right\rangle = \|s_h^n\|_0^2 - \left\|s_h^{n+1}\right\|_0^2 + \left\|s_h^n - s_h^{n+1}\right\|_0^2.$$

Combining now (1.10) with Lemma 2.4, we infer that $\left\|s_h^{n+1} - s_h^n\right\|_0 \le \left\|\operatorname{div} \widehat{\mathbf{u}}_h^{n+1}\right\|_0 \le \left\|\nabla \widehat{\mathbf{u}}_h^{n+1}\right\|$, whence

$$\begin{aligned} \left\|\mathbf{u}_h^{n+1}\right\|_0^2 - \|\mathbf{u}_h^n\|_0^2 + \left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_0^2 + 2\left\|\nabla \rho_h^{n+1}\right\|_0^2 + 2\mu\tau \left\|\nabla \widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\ + \mu\tau \left\|s_h^{n+1}\right\|_0^2 - \mu\tau \|s_h^n\|_0^2 \le \frac{\tau}{2} \left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2 + \frac{3\mu\tau}{2} \left\|\nabla \widehat{\mathbf{u}}_h^{n+1}\right\|_0^2. \end{aligned}$$

Adding over n from 0 to N, we obtain (1.15) and complete the proof of Theorem 1.1.

4. Theorem 1.2: Error Analysis for Velocity. In this section, we prove weak and strong error estimates for velocity for the Gauge-Uzawa FEM of Algorithm 2. The proof is rather intricate because of the limited regularity of §2.1, particularly that $\mathbf{u}_{tt} \notin L^2(0, T; \mathbf{L}^2(\Omega))$, and consists of 3 steps as follows:

- *Time-Discrete Stokes*: We first consider a sequence of Stokes equations with exact forcing and convection, namely $\mathbf{U}^{n+1} \in \mathbf{H}_0^1(\Omega), P^{n+1} \in L_0^2(\Omega)$ satisfy $\mathbf{U}^0 = \mathbf{u}^0$ and

$$\delta \mathbf{U}^{n+1} - \mu \Delta \mathbf{U}^{n+1} + \nabla P^{n+1} = \mathbf{f}(t^{n+1}) - \big((\mathbf{u} \cdot \nabla)\mathbf{u}\big)(t^{n+1}), \quad \operatorname{div} \mathbf{U}^{n+1} = 0. \tag{4.1}$$

  In Lemma 4.1 we derive estimates for the errors

$$\mathbf{G}^{n+1} := \mathbf{u}(t^{n+1}) - \mathbf{U}^{n+1}, \qquad g^{n+1} := p(t^{n+1}) - P^{n+1},$$

  which rely solely on the regularity $\mathbf{u}_{tt} \in L^2\left([0:T] : \mathbf{Z}(\Omega)^*\right)$ of Lemma 2.3. This is possible because the test function $\mathbf{w} = \mathbf{u}(t^{n+1}) - \mathbf{U}^{n+1}$ is divergence free and thus allows us to work on the spaces $\mathbf{Z}(\Omega)$ and $\mathbf{Z}(\Omega)^*$.
- *Stokes Projection*: We define $(\mathbf{U}_h^{n+1}, P_h^{n+1}) := \mathfrak{S}_h(\mathbf{u}(t^{n+1}), p(t^{n+1})) \in \mathbb{V}_h \times \mathbb{P}_h$ to be the Stokes projection of the true solution at time t^{n+1}, and derive error estimates in Lemma 4.3 for the errors

$$\mathbf{G}_h^{n+1} := \mathbf{u}(t^{n+1}) - \mathbf{U}_h^{n+1}, \qquad g_h^{n+1} := p(t^{n+1}) - P_h^{n+1}.$$

  We point out that this choice of space discretization is more handy than discretizing (4.1) by finite elements, and still gives estimates for the errors $\mathbf{F}^{n+1} := \mathbf{U}^{n+1} - \mathbf{U}_h^{n+1}$ and $f^{n+1} := P^{n+1} - P_h^{n+1}$ by combining the first two steps.

- *Comparing* (4.1) *with* (1.8)-(1.11): We derive strong estimates of order 1/2 and use then to prove weak estimates of order 1 for the errors

$$\mathbf{E}^{n+1} := \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}^{n+1} := \mathbf{U}^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e^{n+1} := P^{n+1} - p_h^{n+1}. \tag{4.2}$$

This is the most technical step since we now must deal with the fact that $\widehat{\mathbf{u}}_h^{n+1}$ is not divergence free whereas $\mathbf{u}_h^{n+1}$ does not vanish on $\partial\Omega$; this is carried out in §4.3. Upon combining the estimates of these 3 steps, we readily obtain Theorem 1.2.

4.1. Time-Discrete Stokes Problem. We now show error bounds for (4.1).

LEMMA 4.1 (Uniform estimates). *Let A1-3 hold. Then*

$$\big\|\mathbf{G}^{N+1}\big\|_0^2 + \sum_{n=0}^{N} \big\|\mathbf{G}^{n+1} - \mathbf{G}^n\big\|_0^2 + \mu\tau \sum_{n=0}^{N} \big\|\nabla \mathbf{G}^{n+1}\big\|_0^2 \le C\tau^2, \tag{4.3}$$

$$\tau \sum_{n=0}^{N} \big\|g^{n+1}\big\|_0^2 \le C\tau. \tag{4.4}$$

Proof. We subtract (4.1) from (1.1) at $t = t^{n+1}$ and thereby write

$$\delta\mathbf{G}^{n+1} - \mu\Delta\mathbf{G}^{n+1} + \nabla g^{n+1} = \mathbf{R}^{n+1} := \frac{1}{\tau}\int_{t^n}^{t^{n+1}} (t - t^n)\mathbf{u}_{tt}(\cdot, t)dt, \tag{4.5}$$

where $\mathbf{R}^{n+1}$ is the truncation error. We multiply this elliptic PDE by the admissible test function $2\tau\mathbf{G}^{n+1} \in \mathbf{Z}(\Omega)$ to arrive at

$$\big\|\mathbf{G}^{n+1}\big\|_0^2 - \|\mathbf{G}^n\|_0^2 + \big\|\mathbf{G}^{n+1} - \mathbf{G}^n\big\|_0^2 + 2\mu\tau\big\|\nabla\mathbf{G}^{n+1}\big\|_0^2 \le 2\tau\big\|\mathbf{R}^{n+1}\big\|_*\big\|\nabla\mathbf{G}^{n+1}\big\|_0.$$

Adding over n and using (2.5) yield (4.3). To prove (4.4) we use the error equation (4.5) to obtain for any $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$

$$\big\langle g^{n+1}, \operatorname{div}\mathbf{w}\big\rangle \le \frac{1}{\tau}\big\|\mathbf{G}^{n+1} - \mathbf{G}^n\big\|_0\|\mathbf{w}\|_0 + \mu\big\|\nabla\mathbf{G}^{n+1}\big\|_0\|\nabla\mathbf{w}\|_0 + \big\|\mathbf{R}^{n+1}\big\|_0\|\mathbf{w}\|_0.$$

Since $\big\|\mathbf{R}^{n+1}\big\|_0^2 \le \frac{1}{2}\int_{t^n}^{t^{n+1}} \sigma\|\mathbf{u}_{tt}\|_0^2$, then (2.3) and (4.3) together with the continuous inf-sup condition imply (4.4). ∎

LEMMA 4.2 (Weighted Estimates). *Let A1-3 hold. Then*

$$\sigma^{N+1}\big\|\delta\mathbf{G}^{N+1}\big\|_0^2 + \sum_{n=1}^{N}\sigma^{n+1}\big\|\delta\mathbf{G}^{n+1} - \delta\mathbf{G}^n\big\|_0^2 + \frac{\mu\tau}{2}\sum_{n=1}^{N}\sigma^{n+1}\big\|\nabla\delta\mathbf{G}^{n+1}\big\|_0^2 \le C\tau, \tag{4.6}$$

$$\sup_{0\le n\le N+1}\sigma^n\|g^n\|_0^2 + \sum_{n=0}^{N}\sigma^{n+1}\left(\big\|g^{n+1}\big\|_0^2 + \big\|\delta g^{n+1}\big\|_0^2\right) \le C\tau. \tag{4.7}$$

If NLC is also valid, then (4.6) *and* (4.7) *become uniform, namely without weights.*

Proof. To prove (4.6) we subtract two consecutive equations (4.5) and thus derive an equation for $\delta\mathbf{G}^{n+1}$. We next multiply this equation by $2\sigma^{n+1}\delta\mathbf{G}^{n+1}$ and proceed as in Lemma 4.1 to discover that $I^{n+1} := 2\sigma^{n+1}\big\langle\delta(\mathbf{G}^{n+1} - \mathbf{G}^n), \delta\mathbf{G}^{n+1}\big\rangle$ and $II^{n+1} := 2\tau\sigma^{n+1}\big\langle\delta\mathbf{R}^{n+1}, \delta\mathbf{G}^{n+1}\big\rangle$ must be estimated. We see that

$$I^{n+1} = \sigma^{n+1}\big\|\delta\mathbf{G}^{n+1}\big\|_0^2 - \sigma^n\|\delta\mathbf{G}^n\|_0^2 + \sigma^{n+1}\big\|\delta\mathbf{G}^{n+1} - \delta\mathbf{G}^n\big\|_0^2 - (\sigma^{n+1} - \sigma^n)\|\delta\mathbf{G}^n\|_0^2,$$

and realize that, upon summation over n, the first two terms on the right-hand side telescope whereas the last one leads to $\frac{1}{\tau}\sum_{n=1}^{N}\left\|\mathbf{G}^{n+1}-\mathbf{G}^{n}\right\|_0^2 \le C\tau$ in view of (4.3). On the other hand, II^{n+1} can be written equivalently as follows:

$$\begin{aligned} II^{n+1} &= 2\sigma^{n+1}\left\langle \mathbf{R}^{n+1}, \delta\mathbf{G}^{n+1}\right\rangle - 2\sigma^{n}\left\langle \mathbf{R}^{n}, \delta\mathbf{G}^{n}\right\rangle \\ &\quad + 2\sigma^{n}\left\langle \mathbf{R}^{n}, \delta\mathbf{G}^{n}-\delta\mathbf{G}^{n+1}\right\rangle + 2(\sigma^{n}-\sigma^{n+1})\left\langle \mathbf{R}^{n}, \delta\mathbf{G}^{n+1}\right\rangle. \end{aligned}$$

We now add on n and observe that the first two terms telescope. The third term can be handled via the estimate $\sum_{n=1}^{N}\sigma^n\|\mathbf{R}^n\|_0^2 \le C\tau\int_0^T \sigma\|\mathbf{u}_{tt}\|_0^2 \le C\tau$, which results from (2.3), together with the bound for $\sum_{n=1}^{N} I^{n+1}$. Using again $\sum_{n=1}^{N}\sigma^n\|\mathbf{R}^n\|_0^2 \le C\tau$, now coupled with $\sum_{n=1}^{N}\|\delta\mathbf{G}^n\|_0^2 \le C$ from (4.3), takes care of the last term in II^{n+1}.

We finally observe that the presence of weights allows us to employ regularity (2.3) for $\mathbf{u}_{tt}$. If we further assume NLC, then we could omit weights and instead resort to regularity (2.4) to establish uniform bounds. This completes the proof. □

4.2. Stokes Projection. We now establish simple estimates for $(\mathbf{G}_h^{n+1}, g_h^{n+1})$.

LEMMA 4.3 (Stokes Projection). *Let A1-6 hold. Then*

$$\left\|\mathbf{G}_h^{n+1}\right\|_0 + h\left\|\mathbf{G}_h^{n+1}\right\|_1 + h\left\|g_h^{n+1}\right\|_0 \le Ch^2, \tag{4.8}$$

$$\tau\sum_{n=0}^{N}\sigma^{n+1}\left(\left\|\delta\mathbf{G}_h^{n+1}\right\|_0^2 + h^2\left\|\delta\mathbf{G}_h^{n+1}\right\|_1^2 + h^2\left\|\delta g_h^{n+1}\right\|_0^2\right) \le Ch^4. \tag{4.9}$$

If NLC also holds, then (4.9) *becomes* uniform*, namely without weights.*

Proof. Estimate (4.8) is a direct consequence of Lemma 2.5 and (2.2). Since the Stokes operator $\mathfrak{S}_h$ is linear, we readily have $(\delta\mathbf{U}_h^n, \delta P_h^n) = \mathfrak{S}_h(\delta\mathbf{u}(t^n), \delta p(t^n))$, and Lemma 2.5 applies again. Upon multiplying by $\tau\sigma^{n+1}$, the square of the right-hand side of (2.7) can be bounded by

$$h^4\tau^{-1}\sum_{n=0}^{N}\sigma^{n+1}\Big(\left\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\right\|_2^2 + \left\|p(t^{n+1})-p(t^n)\right\|_1^2\Big).$$

We examine the velocity term only since the other one is similar. For $n=0$ we recall (2.2), along with $\sigma^1=\tau$, to write $\sigma^1\left\|\mathbf{u}(t^1)-\mathbf{u}(t^0)\right\|_2^2 \le C\tau$. For $n \ge 1$, instead, we use that $\sigma^{n+1} \le 2\sigma(t)$ for $t^n \le t \le t^{n+1}$, whence

$$\sum_{n=1}^{N}\sigma^{n+1}\left\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\right\|_2^2 \le C\tau\int_0^T \sigma\|\mathbf{u}_t\|_2^2 \le C\tau,$$

because of (2.3). This completes the proof. □

4.3. Comparing (4.1) **with** (1.8)-(1.11)**.** We derive strong estimates of order 1/2 and use then to prove weak estimates of order 1 for the errors in (4.2), namely,

$$\mathbf{E}^{n+1} = \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}^{n+1} = \mathbf{U}^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e^{n+1} = P^{n+1} - p_h^{n+1}.$$

Before embarking on this discussion, we mention several useful properties of the error functions. If $\mathbf{E}_h^{n+1} := \mathbf{U}_h^{n+1} - \mathbf{u}_h^{n+1}$, $\widehat{\mathbf{E}}_h^{n+1} := \mathbf{U}_h^{n+1} - \widehat{\mathbf{u}}_h^{n+1}$, and $\mathbf{F}^{n+1} = \mathbf{U}^{n+1} - \mathbf{U}_h^{n+1}$, then

$$\begin{aligned} \widehat{\mathbf{E}}^{n+1} &= \mathbf{E}^{n+1} + \nabla\rho_h^{n+1}, \qquad & \widehat{\mathbf{E}}_h^{n+1} &= \mathbf{E}_h^{n+1} + \nabla\rho_h^{n+1}, \\ \widehat{\mathbf{E}}^{n+1} &= \mathbf{F}^{n+1} + \widehat{\mathbf{E}}_h^{n+1}, \qquad & \mathbf{E}^{n+1} &= \mathbf{F}^{n+1} + \mathbf{E}_h^{n+1}, \end{aligned}$$

as well as

$$\left\langle \mathbf{E}^{n+1}, \nabla q_h \right\rangle = \left\langle \mathbf{E}_h^{n+1}, \nabla q_h \right\rangle = \left\langle \mathbf{F}^{n+1}, \nabla q_h \right\rangle = 0, \quad \forall q_h \in \mathbb{P}_h, \tag{4.10}$$

whence we deduce crucial orthogonality properties:

$$\|\widehat{\mathbf{E}}^{n+1}\|_0^2 = \|\mathbf{E}^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2, \qquad \|\widehat{\mathbf{E}}_h^{n+1}\|_0^2 = \|\mathbf{E}_h^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2. \tag{4.11}$$

Since $\mathbf{F}^{n+1} = \mathbf{G}_h^{n+1} - \mathbf{G}^{n+1}, f^{n+1} = g_h^{n+1} - g^{n+1}$, Lemmas 4.1 and 4.3 give rise to the following estimates provided A1-6 hold

$$\begin{gathered} \|\mathbf{F}^{n+1}\|_0^2 \le C(\tau^2 + h^4), \quad \mu\tau \sum_{n=1}^N \|\nabla \mathbf{F}^{n+1}\|_0^2 \le C(\tau^2 + h^2), \\ \tau \sum_{n=1}^N \|f^{n+1}\|_0^2 \le C(\tau + h^2). \end{gathered} \tag{4.12}$$

We also point out that, owing to Lemma 2.4, $s_h^{n+1} \in \mathbb{P}_h$ defined in (1.10) satisfies

$$\left\| s_h^{n+1} - s_h^n \right\|_0 \le \|\nabla \widehat{\mathbf{E}}^{n+1}\|_0. \tag{4.13}$$

LEMMA 4.4 (Reduced Rate of Convergence for Velocity). *Let A1-6 and $h^2 \le C\tau$ be valid with arbitrary constant $C > 0$. Then the velocity error functions satisfy*

$$\begin{aligned} &\left\|\mathbf{E}^{N+1}\right\|_0^2 + \left\|\widehat{\mathbf{E}}^{N+1}\right\|_0^2 + \mu\tau\left\|s_h^{N+1}\right\|_0^2 + \frac{1}{2}\sum_{n=0}^N \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 \\ &\qquad + \sum_{n=0}^N \left\|\nabla \rho_h^{n+1}\right\|_0^2 + \frac{\mu\tau}{2}\sum_{n=0}^N \left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2 \le C(\tau + h^2). \end{aligned} \tag{4.14}$$

Proof. Subtracting (1.8) from (4.1) yields, for all $\mathbf{w}_h \in \mathbb{V}_h$,

$$\begin{aligned} \tau^{-1}\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \mathbf{w}_h \right\rangle + \mu \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{w}_h \right\rangle = \left\langle P^{n+1}, \operatorname{div} \mathbf{w}_h \right\rangle \\ - \mu \left\langle s_h^n, \operatorname{div} \mathbf{w}_h \right\rangle - \mathfrak{N}_h(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \mathbf{w}_h) + \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h). \end{aligned} \tag{4.15}$$

Choosing $\mathbf{w}_h = 2\tau\widehat{\mathbf{E}}_h^{n+1} = 2\tau(\widehat{\mathbf{E}}^{n+1} - \mathbf{F}^{n+1})$ in (4.15), and using (4.10), we easily get

$$\left\|\mathbf{E}^{n+1}\right\|_0^2 - \|\mathbf{E}^n\|_0^2 + \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + 2\mu\tau\left\|\nabla \widehat{\mathbf{E}}^{n+1}\right\|_0^2 + 2\left\|\nabla \rho_h^{n+1}\right\|_0^2 = \sum_{i=1}^4 A_i, \tag{4.16}$$

with

$$\begin{aligned} A_1 &:= 2\left\langle \mathbf{E}^{n+1} - \mathbf{E}^n, \mathbf{F}^{n+1} \right\rangle + 2\mu\tau \left\langle \nabla \widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{F}^{n+1} \right\rangle, \\ A_2 &:= 2\tau \left\langle P^{n+1}, \operatorname{div} \widehat{\mathbf{E}}_h^{n+1} \right\rangle, \\ A_3 &:= -2\tau \left(\mathfrak{N}_h(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \widehat{\mathbf{E}}_h^{n+1}) - \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}) \right), \\ A_4 &:= -2\mu\tau \left\langle s_h^n, \operatorname{div} \widehat{\mathbf{E}}_h^{n+1} \right\rangle. \end{aligned}$$

We now estimate each term A_i separately. Applying Hölder inequality, we find a bound of the first term

$$A_1 \le \frac{1}{2}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + C\left\|\mathbf{F}^{n+1}\right\|_0^2 + \frac{\mu\tau}{4}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + C\mu\tau\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2. \tag{4.17}$$

Since $\mathbf{U}_h^{n+1}$ is discrete divergence free, but not so $\widehat{\mathbf{u}}_h^{n+1}$, we add and subtract P_h^{n+1} and $p(t^{n+1})$, and recall (1.9) and Remark 2.6 to derive

$$\begin{aligned} A_2 &= 2\tau\left\langle f^{n+1}\,,\,\mathrm{div}\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle + 2\tau\left\langle\nabla g_h^{n+1}\,,\,\nabla\rho_h^{n+1}\right\rangle - 2\tau\left\langle\nabla p(t^{n+1})\,,\,\nabla\rho_h^{n+1}\right\rangle \\ &\le C\tau^2 B^{n+1} + \frac{C\tau}{\mu}\left\|f^{n+1}\right\|_0^2 + \frac{\mu\tau}{8}\left(\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\nabla\mathbf{F}^{n+1}\right\|_0^2\right) + \left\|\nabla\rho_h^{n+1}\right\|_0^2, \end{aligned} \tag{4.18}$$

where $B^{n+1} := \left\|\mathbf{u}(t^{n+1})\right\|_2^2 + \left\|\nabla p(t^{n+1})\right\|_0^2$. To tackle A_3 we first add and subtract $\mathbf{u}(t^{n+1}), \mathbf{u}_h^n$, and realize that $\mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}) = 0$ according to (2.10). This yields

$$\begin{aligned} A_3 = &- 2\tau\mathfrak{N}_h((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}), \widehat{\mathbf{E}}_h^{n+1}) \\ &- 2\tau\mathfrak{N}_h((\mathbf{u}(t^n) - \mathbf{u}_h^n), \mathbf{u}(t^{n+1}), \widehat{\mathbf{E}}_h^{n+1}) - 2\tau\mathfrak{N}_h(\mathbf{u}_h^n, \mathbf{G}_h^{n+1}, \widehat{\mathbf{E}}_h^{n+1}). \end{aligned}$$

Since $\left\|\mathbf{u}(t^{n+1})\right\|_2 + \left|\!\left|\!\left|\mathbf{G}_h^{n+1}\right|\!\right|\!\right| \le C$ in view of (2.2) and (2.9), and $\widehat{\mathbf{E}}_h^{n+1} = \widehat{\mathbf{E}}^{n+1} - \mathbf{F}^{n+1}$, (2.11) and (2.13) give

$$A_3 \le \frac{C\tau^2}{\mu}D^{n+1} + \frac{C\tau}{\mu}\left(\left\|\mathbf{E}^n\right\|_0^2 + \left\|\mathbf{G}^n\right\|_0^2 + \left\|\mathbf{G}_h^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{8}\left\|\nabla\widehat{\mathbf{F}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{8}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2,$$

with $D^{n+1} := \int_{t^n}^{t^{n+1}} \left\|\mathbf{u}_t(t)\right\|_0^2 dt$. Next, making use of (1.10) and (4.13), we arrive at

$$\begin{aligned} A_4 &= 2\mu\tau\left\langle s_h^n\,,\,\mathrm{div}\,\widehat{\mathbf{u}}_h^{n+1}\right\rangle = 2\mu\tau\left\langle s_h^n - s_h^{n+1}\,,\,s_h^n\right\rangle \\ &\le \mu\tau\left(\left\|s_h^n\right\|_0^2 - \left\|s_h^{n+1}\right\|_0^2\right) + \mu\tau\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2. \end{aligned}$$

Inserting the above estimates into (4.16), summing over n from 0 to N gives

$$\begin{aligned} &\left\|\mathbf{E}^{N+1}\right\|_0^2 + \frac{1}{2}\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_0^2 + \frac{\mu\tau}{2}\sum_{n=0}^{N}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ &\qquad + \mu\tau\left\|s_h^{N+1}\right\|_0^2 + \sum_{n=0}^{N}\left\|\nabla\rho_h^{n+1}\right\|_0^2 \le C(\tau + h^2) + \frac{C\tau}{\mu}\sum_{n=0}^{N}\left\|\mathbf{E}^n\right\|_0^2, \end{aligned} \tag{4.19}$$

where we have used (2.2) to bound B^{n+1}, D^{n+1}, together with (4.3) and (4.8) to estimate $\|\mathbf{G}^n\|_0$ and $\|\mathbf{G}_h^{n+1}\|_0$, respectively, and (4.12) as well as $h^2 \le C\tau$ to bound $\|\mathbf{F}^{n+1}\|_0, \|\mathbf{F}^{n+1}\|_0$ and $\|f^{n+1}\|_0$. The discrete Gronwall lemma finally yields (4.14) except for $\|\widehat{\mathbf{E}}^{n+1}\|_0^2$. The latter results from (4.11) and completes the proof. □

Remark 4.5 (Initial Errors). If $N = 0$ in (4.19), then Lemmas 4.1 and 4.3 give

$$\begin{aligned} &\left\|\mathbf{E}^1\right\|_0^2 + \frac{1}{2}\left\|\mathbf{E}^1 - \mathbf{E}^0\right\|_0^2 + \frac{\mu\tau}{2}\left\|\nabla\widehat{\mathbf{E}}^1\right\|_0^2 + \mu\tau\left\|s_h^1\right\|_0^2 + \left\|\nabla\rho_h^1\right\|_0^2 \\ &\qquad \le C(\tau^2 + \tau h^2 + h^4) + \frac{C\tau}{\mu}\left\|f^1\right\|_0^2 \le C(\tau + \tau h^2 + h^4), \end{aligned}$$

or alternatively $\leq C(\tau^2 + \tau h^2 + h^4)$ provided NLC holds in conjunction with (4.7).

REMARK 4.6 (Suboptimal Order). The suboptimal order $\mathcal{O}(\tau + h^2)$ of Lemma 4.4 is due to terms $\|\mathbf{F}^{n+1}\|_0^2 + \tau\|\nabla\mathbf{F}^{n+1}\|_0^2$ in (4.17) and the fact that $\widehat{\mathbf{E}}_h^{n+1}$ in (4.18) is not discrete divergence free. To improve upon this we must get rid of both terms.

LEMMA 4.7 (Full Rate of Convergence for Velocity). *Let A1-6 hold and $h^2 \leq C\tau$ be valid with arbitrary constant $C > 0$. Then we have*

$$\left\|\mathbf{E}^{N+1}\right\|_*^2 + \sum_{n=0}^{N} \left\|\mathbf{E}^{n+1} - \mathbf{E}^n\right\|_*^2 + \left(\mu\tau\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \leq C\left(\tau^2 + h^4\right). \quad (4.20)$$

Proof. Let $(\mathbf{v}^n, q^n)$ and $(\mathbf{v}_h^n, q_h^n)$ be solutions of the Stokes equations (2.1) and (2.6) with $\mathbf{g} = \mathbf{E}^n$. Then Lemma 2.5 and A1 yield a crucial inequality

$$\|\mathbf{v}^n - \mathbf{v}_h^n\|_0 + h\|\mathbf{v}^n - \mathbf{v}_h^n\|_1 + h\|q^n - q_h^n\|_0 \leq Ch^2\|\mathbf{E}^n\|_0. \quad (4.21)$$

Since $\mathbf{v}_h^{n+1}$ is discrete divergence free, then $\left\langle \nabla\rho_h^{n+1}, \mathbf{v}_h^{n+1}\right\rangle = 0$ and

$$\left\langle \widehat{\mathbf{E}}^{n+1} - \mathbf{E}^n, \mathbf{v}_h^{n+1}\right\rangle = \left\langle \mathbf{E}^{n+1} - \mathbf{E}^n, \mathbf{v}_h^{n+1}\right\rangle = \left\langle \nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n), \nabla\mathbf{v}_h^{n+1}\right\rangle.$$

Choosing $\mathbf{w}_h = 2\tau\mathbf{v}_h^{n+1}$ in (4.15), thus yields

$$\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 - \|\nabla\mathbf{v}_h^n\|_0^2 + \left\|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\right\|_0^2 + 2\mu\tau\left\|\mathbf{E}^{n+1}\right\|_0^2 = \sum_{i=1}^{4} A_i, \quad (4.22)$$

with

$$\begin{aligned}
A_1 &:= -2\mu\tau\left\langle \nabla\mathbf{F}^{n+1}, \nabla\mathbf{v}_h^{n+1}\right\rangle, \\
A_2 &:= 2\mu\tau\left(\left\langle \mathbf{F}^{n+1}, \mathbf{E}^{n+1}\right\rangle + \left\langle \nabla\rho_h^{n+1}, \nabla q_h^{n+1}\right\rangle\right), \\
A_3 &:= 2\tau\left\langle P^{n+1}, \operatorname{div}\mathbf{v}_h^{n+1}\right\rangle, \\
A_4 &:= -2\tau\left(\mathfrak{N}_h(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \mathbf{v}_h^{n+1}) - \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1})\right).
\end{aligned}$$

We now estimate A_1 to A_4 separately. We use the inequality (4.21) to get

$$\begin{aligned}
A_1 &= 2\mu\tau\left\langle \nabla\mathbf{F}^{n+1}, \nabla\left(\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\right) - \nabla\mathbf{v}^{n+1}\right\rangle \\
&\leq C\mu\tau\left(h^2\left\|\nabla\mathbf{F}^{n+1}\right\|_0^2 + \left\|\mathbf{F}^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{6}\left\|\mathbf{E}^{n+1}\right\|_0^2
\end{aligned}$$

as well as

$$A_2 \leq C\mu\tau\left(\left\|\mathbf{F}^{n+1}\right\|_0^2 + \left\|\nabla\rho_h^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{6}\left\|\mathbf{E}^{n+1}\right\|_0^2.$$

We next use that $\mathbf{v}_h^{n+1}$ is discrete divergence free and $\mathbf{v}^{n+1}$ is divergence free. Hence

$$\begin{aligned}
A_3 &= 2\tau\left\langle P^{n+1} - P_h^{n+1}, \operatorname{div}(\mathbf{v}_h^{n+1} - \mathbf{v}^{n+1})\right\rangle \\
&\leq C\tau h\left\|f^{n+1}\right\|_0\left\|\mathbf{v}^{n+1}\right\|_2 \leq \frac{C\tau h^2}{\mu}\left\|f^{n+1}\right\|_0^2 + \frac{\mu\tau}{6}\left\|\mathbf{E}^{n+1}\right\|_0^2.
\end{aligned}$$

At the same time, the convection term A_4 can be rewritten as $A_4 = \sum_{i=1}^{3} A_{4,i}$ with

$$\begin{aligned}
A_{4,1} &:= -2\tau\mathfrak{N}_h((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) + (\mathbf{u}(t^n) - \mathbf{u}_h^n), \mathbf{u}(t^{n+1}), \mathbf{v}_h^{n+1}), \\
A_{4,2} &:= 2\tau\mathfrak{N}_h\left(\mathbf{u}(t^n) - \mathbf{u}_h^n, \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, (\mathbf{v}_h^{n+1} - \mathbf{v}^{n+1}) + \mathbf{v}^{n+1}\right), \\
A_{4,3} &:= -2\tau\mathfrak{N}_h\left(\mathbf{u}(t^n), \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1}\right).
\end{aligned}$$

Since $\mathbf{u}(t^n)-\mathbf{u}_h^n=\mathbf{E}^n+\mathbf{G}^n$, (2.2) in conjunction with (2.12) yields

$$A_{4,1}\le C\tau^2\mu\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t(t)\|_0^2dt+\frac{\mu\tau}{6}\left(\|\mathbf{E}^n\|_0^2+\|\mathbf{G}^n\|_0^2\right)+\frac{C\tau}{\mu}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2.$$

Before tacking $A_{4,2}$ we observe that (4.3) and (4.14) imply $\|\mathbf{u}(t^n)-\mathbf{u}_h^n\|_0\le C(h+\tau^{1/2})$, and that (2.9) and (4.21) yield

$$\left|\!\left|\!\left|\mathbf{v}_h^{n+1}-\mathbf{v}^{n+1}\right|\!\right|\!\right|\le C\|\mathbf{v}^{n+1}\|_2\le C\|\mathbf{E}^{n+1}\|_0.$$

Therefore, (2.12) and (2.13) lead to

$$A_{4,2}\le\frac{C\tau}{\mu}\left(\tau+h^2\right)\left(\left\|\nabla\mathbf{G}^{n+1}\right\|_0^2+\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right)+\frac{\mu\tau}{6}\left\|\mathbf{E}^{n+1}\right\|_0^2.$$

Since $\mathbf{u}(t^n)$ is divergence free, we can resort to (2.11) and (4.11) to obtain

$$A_{4,3}\le\frac{\mu\tau}{6}\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla\rho_h^{n+1}\right\|_0^2+\left\|\mathbf{G}^{n+1}\right\|_0^2\right)+\frac{C\tau}{\mu}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2.$$

Inserting the above estimates into (4.22) and summing over n from 0 to N, we deduce

$$\begin{aligned}\left\|\nabla\mathbf{v}_h^{N+1}\right\|_0^2+\sum_{n=0}^{N}\left\|\nabla(\mathbf{v}_h^{n+1}-\mathbf{v}_h^n)\right\|_0^2+\mu\tau\sum_{n=0}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2&\\ \le C\left(\tau^2+h^4\right)+\frac{C\tau}{\mu}\sum_{n=0}^{N}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2&\end{aligned}\tag{4.23}$$

because of (4.3), (4.12), and (4.14) bound the remaining terms. The discrete Gronwall lemma and (2.8) allows us to remove the rightmost term in (4.23), and thereby arrive at (4.20) upon invoking (2.8). However, this does not give a bound for $\|\widehat{\mathbf{E}}^{n+1}\|_0$, which comes from (4.11) and (4.14) instead. The proof is thus finished. □

Proof of Theorem 1.2. This is a consequence of Lemmas 4.1, 4.4, and 4.7. ■

Remark 4.8 (Estimates for $\left\|\nabla\mathbf{v}_h^1\right\|_0$). These estimates will be crucial in §5 and can be extracted from (4.23) upon invoking Remark 4.5 and choosing $N=0$. Since $\mathbf{v}_h^0=0$ because $\mathbf{E}^0$ is orthogonal to $\mathbb{V}_h$, (4.23) reduces to

$$\left\|\nabla\mathbf{v}_h^1\right\|_0^2\le C\tau(\tau^2+h^4)+\frac{C\tau h^2}{\mu}\left\|f^1\right\|_0^2\le C\tau(\tau^2+h^2).$$

On the other hand, if NLC is also valid then $\|f^1\|_0^2\le C\tau$ and $\left\|\nabla\mathbf{v}_h^1\right\|_0^2\le C\tau(\tau^2+h^4)$.

5. Theorem 1.3: Error Analysis for Time Derivative of Velocity. In this section we embark on an error analysis for the time derivative of velocity.

Lemma 5.1 (Stability of Time-Derivative of Velocity). *Let A1-6 hold and* $h^2\le C_1h^2\le\tau\le C_2h^{\frac{d}{3}(1+\varepsilon)}$ *be valid with arbitrary constants* $C_1,C_2>0$. *Then the error functions satisfy the weighted estimates*

$$\begin{aligned}\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2+\sum_{n=1}^{N}\sigma^{n+1}\left\|\delta\mathbf{E}^{n+1}-\delta\mathbf{E}^n\right\|_0^2+\sum_{n=1}^{N}\sigma^{n+1}\left\|\nabla\delta\rho_h^{n+1}\right\|_0^2&\\ +\mu\tau\sigma^{N+1}\left\|\delta s_h^{N+1}\right\|_0^2+\mu\tau\sum_{n=1}^{N}\sigma^{n+1}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2\le C.&\end{aligned}\tag{5.1}$$

If NLC also valid, then (5.1) *become uniform, namely without weights.*

Proof. Subtracting two consecutive expressions (4.15) yields

$$\begin{aligned}\Big\langle \delta\widehat{\mathbf{E}}^{n+1} - \delta\mathbf{E}^{n}\,,\,\mathbf{w}_h\Big\rangle + \mu\tau\Big\langle \nabla\delta\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\mathbf{w}_h\Big\rangle \\ = \tau\left\langle \delta P^{n+1}\,,\,\mathrm{div}\,\mathbf{w}_h\right\rangle - \mu\tau\left\langle \delta s_h^{n}\,,\,\mathrm{div}\,\mathbf{w}_h\right\rangle \\ - \mathfrak{N}_h(\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1}),\mathbf{w}_h) + \mathfrak{N}_h(\mathbf{u}_h^{n},\widehat{\mathbf{u}}_h^{n+1},\mathbf{w}_h) \\ + \mathfrak{N}_h(\mathbf{u}(t^{n}),\mathbf{u}(t^{n}),\mathbf{w}_h) - \mathfrak{N}_h(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n},\mathbf{w}_h).\end{aligned} \tag{5.2}$$

Choosing $\mathbf{w}_h = 2\delta\widehat{\mathbf{E}}_h^{n+1} = 2\delta(\widehat{\mathbf{E}}^{n+1} - \mathbf{F}^{n+1})$ in (5.2), and using (4.10), implies

$$\begin{aligned}\left\|\delta\mathbf{E}^{n+1}\right\|_0^2 - \left\|\delta\mathbf{E}^{n}\right\|_0^2 + \left\|\delta\mathbf{E}^{n+1} - \delta\mathbf{E}^{n}\right\|_0^2 \\ + 2\left\|\nabla\delta\rho_h^{n+1}\right\|_0^2 + 2\mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 = \sum_{i=1}^{4} A_i,\end{aligned} \tag{5.3}$$

with

$$\begin{aligned}A_1 &:= 2\left\langle \delta\mathbf{E}^{n+1} - \delta\mathbf{E}^{n}\,,\,\delta\mathbf{F}^{n+1}\right\rangle + 2\mu\tau\Big\langle \nabla\delta\widehat{\mathbf{E}}^{n+1}\,,\,\nabla\delta\mathbf{F}^{n+1}\Big\rangle,\\ A_2 &:= 2\tau\Big\langle \delta P^{n+1}\,,\,\mathrm{div}\,\delta\widehat{\mathbf{E}}_h^{n+1}\Big\rangle,\\ A_3 &:= -2\mu\tau\Big\langle \delta s_h^{n}\,,\,\mathrm{div}\,\delta\widehat{\mathbf{E}}_h^{n+1}\Big\rangle,\\ A_4 &:= -2\mathfrak{N}_h(\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}_h^{n+1}) + 2\mathfrak{N}_h(\mathbf{u}_h^{n},\widehat{\mathbf{u}}_h^{n+1},\delta\widehat{\mathbf{E}}_h^{n+1}),\\ &\quad + 2\mathfrak{N}_h(\mathbf{u}(t^{n}),\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}_h^{n+1}) - 2\mathfrak{N}_h(\mathbf{u}_h^{n-1},\widehat{\mathbf{u}}_h^{n},\delta\widehat{\mathbf{E}}_h^{n+1}).\end{aligned}$$

We now estimate each term A_i separately. First, we easily find out that

$$\begin{aligned}A_1 &\le \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + C\mu\tau\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2 + \frac{1}{2}\left\|\delta\mathbf{E}^{n+1} - \delta\mathbf{E}^{n}\right\|_0^2 + C\left\|\delta\mathbf{F}^{n+1}\right\|_0^2,\\ A_2 &= 2\tau\Big\langle \delta p(t^{n+1}) - \delta g^{n+1}\,,\,\mathrm{div}\,\delta\widehat{\mathbf{E}}_h^{n+1}\Big\rangle\\ &\le \frac{C}{\mu}\int_{t^n}^{t^{n+1}}\|p_t(t)\|_0^2 dt + \frac{C\tau}{\mu}\left\|\delta g^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2.\end{aligned}$$

Since $\mathbf{U}_h^{n+1}$ is discrete divergence free, then $A_3 = 2\mu\tau\left\langle \delta s_h^{n}\,,\,\mathrm{div}\,\delta\widehat{\mathbf{u}}_h^{n+1}\right\rangle$. Consequently, making use of (1.10) and (4.13), we arrive at

$$A_3 = 2\mu\tau\left\langle \delta s_h^{n}\,,\,\delta s_h^{n} - \delta s_h^{n+1}\right\rangle \le \mu\tau\left(\left\|\delta s_h^{n}\right\|_0^2 - \left\|\delta s_h^{n+1}\right\|_0^2\right) + \mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2.$$

At the same time, we further split A_4 to read $A_4 = A_{4,1} + A_{4,2}$ with

$$\begin{aligned}A_{4,1} :=& -2\tau\big(\mathfrak{N}_h(\delta\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}_h^{n+1}) - \mathfrak{N}_h(\delta\mathbf{u}(t^{n}),\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}_h^{n+1})\\ &- \mathfrak{N}_h(\mathbf{u}(t^{n}) - \mathbf{u}_h^{n},\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}_h^{n+1}) + \mathfrak{N}_h(\mathbf{u}(t^{n-1}) - \mathbf{u}_h^{n-1},\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}_h^{n+1})\big),\\ A_{4,2} :=& -2\left(\mathfrak{N}_h(\mathbf{u}_h^{n},\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1},\delta\widehat{\mathbf{E}}_h^{n+1}) - \mathfrak{N}_h(\mathbf{u}_h^{n-1},\mathbf{u}(t^{n}) - \widehat{\mathbf{u}}_h^{n},\delta\widehat{\mathbf{E}}_h^{n+1})\right).\end{aligned}$$

In light of (2.2) and definitions of $\mathbf{G}^i$ and $\mathbf{E}^i$, (2.12) produces

$$A_{4,1} \le \frac{C}{\mu}D^{n+1} + \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2 + \frac{C}{\mu\tau}\sum_{i=n-1}^{n}\left(\left\|\mathbf{G}^i\right\|_0^2 + \left\|\mathbf{E}^i\right\|_0^2\right),$$

with $D^{n+1} := \int_{t^{n-1}}^{t^{n+1}} \|\mathbf{u}_t(t)\|_0^2 dt$. To bound $A_{4,2}$ we rewrite as $A_{4,2} = \sum_{i=1}^3 B_i$ with

$$\begin{aligned} B_1 &:= -2\mathfrak{N}_h(\mathbf{u}_h^n, \mathbf{G}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}),\\ B_2 &:= 2\mathfrak{N}_h(\mathbf{u}_h^{n-1}, \mathbf{G}_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}),\\ B_3 &:= -2\mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{E}}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}) + 2\mathfrak{N}_h(\mathbf{u}_h^{n-1}, \widehat{\mathbf{E}}_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}). \end{aligned}$$

Since $\left\|\!\left|\mathbf{G}_h^{n+1}\right|\!\right\| \le C\big(\|\mathbf{u}(t^{n+1})\|_2 + \|p(t^{n+1})\|_1\big) \le C$, (2.11) and (2.13) give

$$\begin{aligned} B_1 &= 2\mathfrak{N}_h\left((\mathbf{u}(t^n) - \mathbf{u}_h^n) - \mathbf{u}(t^n), \mathbf{G}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right)\\ &\le \frac{C}{\mu\tau}\left(\|\mathbf{E}^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \left\|\mathbf{G}_h^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2, \end{aligned}$$

as well as

$$B_2 \le \frac{C}{\mu\tau}\left(\left\|\mathbf{E}^{n-1}\right\|_0^2 + \left\|\mathbf{G}^{n-1}\right\|_0^2 + \|\mathbf{G}_h^n\|_0^2\right) + \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2.$$

Invoking crucial properties of $\mathfrak{N}_h$, written in (2.10), we infer that

$$B_3 = \frac{2}{\tau}\left(\mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^n) + \mathfrak{N}_h(\mathbf{u}_h^{n-1}, \widehat{\mathbf{E}}_h^n, \widehat{\mathbf{E}}_h^{n+1})\right) = 2\tau\mathfrak{N}_h(\delta\mathbf{u}_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^n).$$

Hence

$$B_3 = -2\tau\mathfrak{N}_h(\delta\mathbf{G}_h^n - \delta\mathbf{u}(t^n), \delta\widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^n) - 2\tau\mathfrak{N}_h(\delta\mathbf{E}_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}, \widehat{\mathbf{E}}_h^n) = B_4 + B_5.$$

Since $\|\widehat{\mathbf{E}}_h^n\|_1 \le C$ according to (4.12) and (4.14), then (2.11) yields

$$\begin{aligned} B_4 &\le C\tau\left(\|\delta\mathbf{G}_h^n\|_1 + \|\delta\mathbf{u}(t^n)\|_1\right)\left\|\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_1\left\|\widehat{\mathbf{E}}_h^n\right\|_1\\ &\le \frac{C\tau}{\mu}\|\nabla\delta\mathbf{G}_h^n\|_0^2 + \frac{C}{\mu}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t(t)\|_1^2 dt + \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2. \end{aligned}$$

We now deal with B_5 via (2.13), namely $B_5 \le C\tau\|\delta\mathbf{E}_h^n\|_{\mathbf{L}^3(\Omega)}\|\delta\widehat{\mathbf{E}}_h^{n+1}\|_1\|\widehat{\mathbf{E}}_h^n\|_1$. In contrast to [16], here we no longer have $\mathbf{E}_h^{n+1} \in \mathbf{H}_0^1$ and we have to resort to the inverse inequality $\|\delta\mathbf{E}_h^n\|_{\mathbf{L}^3(\Omega)} \le Ch^{-\frac{d}{6}}\|\delta\mathbf{E}_h^n\|_0$, whence

$$B_5 \le \underbrace{\frac{C\tau h^{-\frac{d}{3}}}{\mu}\|\delta\mathbf{E}_h^n\|_0^2\left\|\nabla\widehat{\mathbf{E}}_h^n\right\|_0^2}_{=:\Lambda^n} + \frac{\mu\tau}{14}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{\mu\tau}{14}\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2.$$

We postpone the discussion of Λ^n until the end since it is rather delicate. We now insert the above estimates into (5.3), multiply by the weight σ^{n+1}, and add over n from 1 to N. Arguing as in Lemma 4.4, we see that the first two terms in (5.3) become

$$\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 - \sigma^1\left\|\delta\mathbf{E}^1\right\|_0^2 - \tau\sum_{n=1}^N\|\delta\mathbf{E}^n\|_0^2 \ge -C + \sigma^{N+1}\left\|\delta\mathbf{E}^{n+1}\right\|_0^2. \qquad (5.4)$$

On the other hand, we resort to property $\frac{\sigma^{n+1}}{\sigma^n} \le 2$ for $n \ge 1$ to write

$$\begin{aligned}\sum_{n=1}^{N} \sigma^{n+1} A_2 \le & \frac{C\tau}{\mu} \sum_{n=1}^{N} \sigma^{n+1} \left\|\delta g^{n+1}\right\|_0^2 + \frac{C}{\mu} \int_{t^1}^{t^{N+1}} \sigma(t) \|p_t(t)\|_0^2 dt \\ & + \frac{\mu\tau}{14} \sum_{n=1}^{N} \sigma^{n+1} \left(\left\|\nabla \delta \widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \left\|\nabla \delta \mathbf{F}^{n+1}\right\|_0^2 \right).\end{aligned} \tag{5.5}$$

Collecting these estimates, and using Lemmas 4.1-4.4 and 4.7, we get for $D_1, D_2 > 0$

$$\begin{aligned}\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 + \frac{1}{2}\sum_{n=1}^{N}\sigma^{n+1}\left\|\delta\mathbf{E}^{n+1} - \delta\mathbf{E}^n\right\|_0^2 + \frac{\mu\tau}{2}\sum_{n=1}^{N}\sigma^{n+1}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 \\ + \sum_{n=1}^{N}\sigma^{n+1}\left\|\nabla\delta\rho_h^{n+1}\right\|_0^2 + \mu\tau\sigma^{N+1}\left\|\delta s_h^{N+1}\right\|_0^2 \le D_1 + D_2\sum_{n=1}^{N}\sigma^{n+1}\Lambda^n.\end{aligned}$$

To complete this proof, it suffices to show $\sum_{n=1}^{N} \sigma^{n+1} B_6^n \le C$. To do so, we start with a simpler form of the above estimate, namely,

$$\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 \le D_1 + D_2 \tau h^{-\frac{d}{3}} \sum_{n=1}^{N} \sigma^{n+1} \|\delta\mathbf{E}_h^n\|_0^2 \left\|\nabla\widehat{\mathbf{E}}_h^n\right\|_0^2. \tag{5.6}$$

Since $\tau^2 \sum_{n=1}^{N} \|\delta\mathbf{E}^n\|_0^2 = \sum_{n=1}^{N} \left\|\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0^2 \le C\tau$ and $\left\|\nabla\widehat{\mathbf{E}}_h^n\right\|_0^2 \le C$ according to Lemma 4.4, we readily obtain the rough estimate

$$\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 \le C h^{-\frac{d}{3}}.$$

To improve upon this, we utilize $\sum_{n=1}^{N} \|\nabla\widehat{\mathbf{E}}_h^n\|_0^2 \le C$, a by-product of (4.12) and (4.14). Hence

$$\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 \le D_1 + D_2 \tau h^{-\frac{2d}{3}} \sum_{n=1}^{N} \left\|\nabla\widehat{\mathbf{E}}_h^n\right\|_0^2 \le C\tau h^{-\frac{2d}{3}}.$$

We realize that the net effect is a an additional factor $C\tau h^{-d/3}$ in (5.6). After m iterations, we obtain

$$\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 \le M(m)\left(\tau h^{-\frac{d}{3}}\right)^m h^{-\frac{d}{3}},$$

where $M(m) > 0$ possibly grows with m. Since $\tau h^{-\frac{d}{3}} \le C_2 h^{\frac{d\varepsilon}{3}}$, for $m > \varepsilon^{-1}$ we obtain $\sum_{n=1}^{N} \sigma^{n+1}\Lambda^n \le C$. This shows our assertion (5.1).

If NLC is valid, then so is Lemma 2.2, thereby making unnecessary the use of weight σ^{n+1} in (5.4) and (5.5). This yields an inequality similar to (5.1) without weights, and implies the asserted uniform estimate. ∎

LEMMA 5.2 (Rate of Convergence for Time-Derivative of Velocity). *Let A1-6 hold and $C_1 h^2 \le \tau \le C_2 h^{\frac{d}{3}(1+\varepsilon)}$ be valid with arbitrary constants $C_1, C_2 > 0$. Then the error function $\mathbf{E}^n$ satisfies the weighted estimates*

$$\sigma^{N+1}\left\|\delta\mathbf{E}^{N+1}\right\|_*^2 + \sum_{n=1}^{N} \sigma^{n+1} \left(\left\|\delta\mathbf{E}^{n+1} - \delta\mathbf{E}^n\right\|_*^2 + \mu\tau\left\|\delta\mathbf{E}^{n+1}\right\|_0^2 \right) \le C\left(\tau + h^2\right). \tag{5.7}$$

If NLC also valid, then the following uniform error estimates hold

$$\left\|\delta \mathbf{E}^{N+1}\right\|_*^2 + \frac{1}{2}\sum_{n=1}^{N}\left\|\delta \mathbf{E}^{n+1} - \delta \mathbf{E}^{n}\right\|_*^2 + \mu\tau \sum_{n=1}^{N}\left\|\delta \mathbf{E}^{n+1}\right\|_0^2 \le C\left(\tau + h^2\right). \tag{5.8}$$

Proof. Let $(\mathbf{v}^n, q^n)$ and $(\mathbf{v}_h^n, q_h^n)$ be solutions of the Stokes equations (2.1) and (2.6) with $\mathbf{g} = \mathbf{E}^{n+1}$. Choosing $\mathbf{w}_h = 2\delta\mathbf{v}_h^{n+1}$ in (5.2), we arrive at

$$\left\|\nabla\delta\mathbf{v}_h^{n+1}\right\|_0^2 - \|\nabla\delta\mathbf{v}_h^{n}\|_0^2 + \left\|\nabla(\delta\mathbf{v}_h^{n+1} - \delta\mathbf{v}_h^{n})\right\|_0^2 + 2\mu\tau\left\|\delta\mathbf{E}^{n+1}\right\|_0^2 = \sum_{i=1}^{4} A_i, \tag{5.9}$$

with

$$\begin{aligned}
A_1 &:= -2\mu\tau\left\langle \nabla\delta\mathbf{F}^{n+1}, \nabla\delta\mathbf{v}_h^{n+1}\right\rangle, \\
A_2 &:= 2\mu\tau\left(\left\langle \delta\mathbf{E}^{n+1}, \delta\mathbf{F}^{n+1}\right\rangle + \left\langle \nabla\delta q_h^{n+1}, \nabla\delta\rho_h^{n+1}\right\rangle\right), \\
A_3 &:= 2\tau\left\langle \delta P^{n+1}, \operatorname{div}\delta\mathbf{v}_h^{n+1}\right\rangle, \\
A_4 &:= 2\mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \delta\mathbf{v}_h^{n+1}) - 2\mathfrak{N}_h(\mathbf{u}_h^{n-1}, \widehat{\mathbf{u}}_h^{n}, \delta\mathbf{v}_h^{n+1}) \\
&\quad - 2\mathfrak{N}_h(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \delta\mathbf{v}_h^{n+1}) + 2\mathfrak{N}_h(\mathbf{u}(t^{n}), \mathbf{u}(t^{n}), \delta\mathbf{v}_h^{n+1}).
\end{aligned}$$

Except for A_4, we can proceed as in Lemma 4.7 to estimate $A_1 - A_3$, whence

$$\begin{aligned}
A_1 &\le C\mu\tau\left(h^2\left\|\nabla\delta\mathbf{F}^{n+1}\right\|_0^2 + \left\|\delta\mathbf{F}^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{6}\left\|\delta\mathbf{E}^{n+1}\right\|_0^2, \\
A_2 &\le C\mu\tau\left(\left\|\delta\mathbf{F}^{n+1}\right\|_0^2 + \left\|\nabla\delta\rho_h^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{6}\left\|\delta\mathbf{E}^{n+1}\right\|_0^2, \\
A_3 &\le \frac{C\tau h^2}{\mu}\left\|\delta f^{n+1}\right\|_0^2 + \frac{\mu\tau}{6}\left\|\delta\mathbf{E}^{n+1}\right\|_0^2.
\end{aligned}$$

The remaining term A_4 gives rise to rather technical calculations. A tedious but simple rearrangement yields $A_4 = \sum_{i=1}^{6} A_{4,i}$ with each term A_i to be examined separately

$$\begin{aligned}
A_{4,1} &:= -2\mathfrak{N}_h(\mathbf{u}(t^{n+1}) - 2\mathbf{u}(t^n) + \mathbf{u}(t^{n-1}), \mathbf{u}(t^{n+1}), \delta\mathbf{v}_h^{n+1}), \\
A_{4,2} &:= -2\mathfrak{N}_h((\mathbf{u}(t^n) - \mathbf{u}_h^n) - (\mathbf{u}(t^{n-1}) - \mathbf{u}_h^{n-1}), \mathbf{u}(t^{n+1}), \delta\mathbf{v}_h^{n+1}), \\
A_{4,3} &:= -2\mathfrak{N}_h(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \delta\mathbf{v}_h^{n+1}), \\
A_{4,4} &:= -2\mathfrak{N}_h(\mathbf{u}(t^n) - \mathbf{u}(t^{n-1}), \mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \delta\mathbf{v}_h^{n+1}), \\
A_{4,5} &:= -2\mathfrak{N}_h(\mathbf{u}(t^{n-1}) - \mathbf{u}_h^{n-1}, \mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \delta\mathbf{v}_h^{n+1}), \\
A_{4,6} &:= -2\mathfrak{N}_h(\mathbf{u}_h^{n-1}, (\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) - (\widehat{\mathbf{u}}_h^{n+1} - \widehat{\mathbf{u}}_h^{n}), \delta\mathbf{v}_h^{n+1}).
\end{aligned}$$

Since $\left\|\mathbf{u}(t^{n+1}) - 2\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})\right\|_0^2 \le C\tau^2 \int_{t^{n-1}}^{t^{n+1}} \sigma\|\mathbf{u}_{tt}\|_0^2\,dt$, (2.2) and (2.12) yield

$$A_{4,1} \le C\tau\int_{t^{n-1}}^{t^{n+1}} \sigma(t)\|\mathbf{u}_{tt}(t)\|_0^2 dt + C\tau\left\|\nabla\delta\mathbf{v}_h^{n+1}\right\|_0^2,$$

as well as

$$A_{4,2} \le \frac{\mu\tau}{8}(\|\delta\mathbf{G}^n\|_0^2 + \|\delta\mathbf{E}^n\|) + \frac{C\tau}{\mu}\left\|\nabla\delta\mathbf{v}_h^{n+1}\right\|_0^2.$$

Dealing with $A_{4,3}$ entails further rearrangement as follows:

$$\begin{aligned}A_{4,3} &= 2\tau\mathfrak{N}_h(\delta\mathbf{u}(t^n)-\delta\mathbf{u}_h^n,\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}_h^{n+1}-\delta\mathbf{v}^{n+1})\\ &\quad+2\tau\mathfrak{N}_h(\delta\mathbf{u}(t^n)-\delta\mathbf{u}_h^n,\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}^{n+1})\\ &\quad-2\tau\mathfrak{N}_h(\delta\mathbf{u}(t^n),\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}_h^{n+1}-\delta\mathbf{v}^{n+1})\\ &\quad-2\tau\mathfrak{N}_h(\delta\mathbf{u}(t^n),\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}^{n+1}).\end{aligned}$$

In view of (2.12) and (2.13), we can thus write

$$\begin{aligned}A_{4,3} &\le C\tau\|\delta\mathbf{u}(t^n)-\delta\mathbf{u}_h^n\|_{\mathbf{L}^3(\Omega)}\big\|\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1}\big\|_1\big\|\delta\mathbf{v}^{n+1}-\delta\mathbf{v}_h^{n+1}\big\|_1\\ &\quad+C\tau\|\delta\mathbf{u}(t^n)-\delta\mathbf{u}_h^n\|_0\big\|\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1}\big\|_1\big\|\delta\mathbf{v}^{n+1}\big\|_2\\ &\quad+C\tau\|\delta\mathbf{u}(t^n)\|_1\big\|\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1}\big\|_1\big\|\delta\mathbf{v}^{n+1}-\delta\mathbf{v}_h^{n+1}\big\|_1\\ &\quad+C\tau\|\delta\mathbf{u}(t^n)\|_1\big\|\mathbf{u}(t^{n+1})-\widehat{\mathbf{u}}_h^{n+1}\big\|_0\big\|\delta\mathbf{v}^{n+1}\big\|_2.\end{aligned}$$

Since $\|\delta(\mathbf{v}^{n+1}-\mathbf{v}_h^{n+1})\|_1\le Ch\|\mathbf{E}^{n+1}\|_0$ because of (2.7), we see that the problematic term with $\mathbf{L}^3$ norm can be easily handled. In fact, invoking Lemma 5.1 together with an inverse inequality from L^3 to L^2 gives

$$\sigma^n\|\delta\mathbf{u}(t^n)-\delta\mathbf{u}_h^n\|_0^2+\sigma^n h^2\|\delta\mathbf{u}(t^n)-\delta\mathbf{u}_h^n\|_{\mathbf{L}^3(\Omega)}^2\le C.$$

We note that this inequality also holds without weight σ^n if NLC is valid. Since, according with (2.2), we have $\|\delta\mathbf{u}(t^n)\|_0^2\le M$ and $\|\delta\mathbf{u}(t^n)\|_1^2\le\tau^{-1}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t(t)\|_1^2dt\le M\tau^{-1}$, after a simple calculation we get

$$A_{4,3}\le\frac{C}{\mu}(\tau+h^2)D^n+\frac{C\tau}{\sigma^n\mu}\left(\Big\|\nabla\widehat{\mathbf{E}}^{n+1}\Big\|_0^2+\big\|\nabla\mathbf{G}^{n+1}\big\|_0^2\right)+\frac{\mu\tau}{8}\big\|\delta\mathbf{E}^{n+1}\big\|_0^2,$$

where $D^n:=\int_{t^{n-1}}^{t^n}\|\nabla\mathbf{u}_t(t)\|_0^2dt$. We use again the bound for $\|\delta\mathbf{u}(t^n)\|_1$ to get

$$A_{4,4}\le C\tau^2\|\delta\mathbf{u}(t^n)\|_1\big\|\delta\mathbf{u}(t^{n+1})\big\|_1\big\|\delta\mathbf{v}_h^{n+1}\big\|_1\le C\tau D^n+C\tau\big\|\nabla\delta\mathbf{v}_h^{n+1}\big\|_0^2.$$

To estimate $A_{4,5},A_{4,6}$ we again have to handle an L^3 norm, this time for $\mathbf{u}(t^n)-\mathbf{u}_h^n$. Combining once more Lemma 5.1 with an inverse estimate, yields $h\|\mathbf{u}(t^n)-\mathbf{u}_h^n\|_{\mathbf{L}^3(\Omega)}\le C\|\mathbf{u}(t^n)-\mathbf{u}_h^n\|_0\le C(\tau+h^2)^{\frac{1}{2}}$. Consequently,

$$\begin{aligned}A_{4,5}\le&C\tau\big\|\mathbf{u}(t^{n-1})-\mathbf{u}_h^{n-1}\big\|_{\mathbf{L}^3(\Omega)}\big\|\delta\mathbf{u}(t^{n+1})\big\|_1\big\|\delta\mathbf{v}^{n+1}-\delta\mathbf{v}_h^{n+1}\big\|_1\\ &+C\tau\big\|\mathbf{u}(t^{n-1})-\mathbf{u}_h^{n-1}\big\|_0\big\|\delta\mathbf{u}(t^{n+1})\big\|_1\big\|\delta\mathbf{v}^{n+1}\big\|_2\\ \le&\frac{C}{\mu}(\tau+h^2)D^{n+1}+\frac{\mu\tau}{8}\big\|\delta\mathbf{E}^{n+1}\big\|_0^2.\end{aligned}$$

In addition, since

$$\begin{aligned}A_{4,6}=&2\tau\mathfrak{N}_h(\mathbf{u}(t^{n-1})-\mathbf{u}_h^{n-1},\delta\mathbf{u}(t^{n+1})-\delta\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}_h^{n+1}-\delta\mathbf{v}^{n+1})\\ &+2\tau\mathfrak{N}_h(\mathbf{u}(t^{n-1})-\mathbf{u}_h^{n-1},\delta\mathbf{u}(t^{n+1})-\delta\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}^{n+1})\\ &-2\tau\mathfrak{N}_h(\mathbf{u}(t^{n-1}),\delta\mathbf{u}(t^{n+1})-\delta\widehat{\mathbf{u}}_h^{n+1},\delta\mathbf{v}_h^{n+1}),\end{aligned}$$

a similar argument leads to

$$A_{4,6}\le\frac{C\tau}{\mu}\left(\Big\|\delta\widehat{\mathbf{E}}^{n+1}+\delta\mathbf{G}^{n+1}\Big\|_0^2+(\tau+h^2)\Big\|\delta\widehat{\mathbf{E}}^{n+1}+\delta\mathbf{G}^{n+1}\Big\|_1^2\right)+\frac{\mu\tau}{8}\big\|\delta\mathbf{E}^{n+1}\big\|_0^2.$$

We now multiply both sides of (5.9) by the weight σ^{n+1} and sum over n for $1 \le n \le N$. We first examine the ensuing first two terms on the left-hand side of (5.9). In light of $\sigma^1 = \tau$, $h^2 \le C\tau$, $\sum_{n=1}^N \|\nabla\delta\mathbf{v}_h^n\|_0^2 \le C$ (see Lemma 4.7) and $\sigma^1\|\nabla\delta\mathbf{v}_h^1\|_0^2 \le C\tau$ (see Remark 4.8), we deduce

$$\sum_{n=1}^N \Big(\sigma^{n+1}\big\|\nabla\delta\mathbf{v}^{n+1}\big\|_0^2 - \sigma^n\|\nabla\delta\mathbf{v}_h^n\|_0^2 - \big(\sigma^{n+1}-\sigma^n\big)\|\nabla\delta\mathbf{v}_h^n\|_0^2\Big)$$
$$\ge \sigma^{N+1}\big\|\nabla\delta\mathbf{v}_h^{N+1}\big\|_0^2 - \sigma^1\big\|\nabla\delta\mathbf{v}_h^1\big\|_0^2 - \tau\sum_{n=1}^N \|\nabla\delta\mathbf{v}_h^n\|_0^2 \ge \sigma^{N+1}\big\|\nabla\delta\mathbf{v}_h^{N+1}\big\|_0^2 - C\tau.$$

Since $\frac{\sigma^{n+1}}{\sigma^n} \le 2$ for $n \ge 1$, we can replace σ/σ^n in $A_{4,3}$ by a constant. Therefore, we can achieve an estimate for $\sigma^{n+1}\|\nabla\delta\mathbf{v}_h^{n+1}\|_0^2$ with the aid of Lemmas 4.1, 4.7, and 5.1, as well as the discrete Gronwall lemma. The asserted *weighted* error estimate follows from (2.8).

If NLC is valid, we do not need to multiply (5.9) by σ^{n+1} to derive the *uniform* error estimate (5.8). In this case we have, instead, $\|\delta\mathbf{G}^n\|_0 + \|\delta\mathbf{E}^n\|_0 \le C$ (see Lemmas 4.2 and 5.1). We finally proceed as before to obtain (5.8). ∎

6. Theorem 1.3: Error Analysis for Pressure. We derive here the error of pressure of Theorem 1.3 by exploiting all previous results.

LEMMA 6.1 (Rate of Convergence for Pressure). *Let A1-6 hold and $C_1h^2 \le \tau \le C_2h^{\frac{d}{3}(1+\varepsilon)}$ be valid with arbitrary constants $C_1, C_2 > 0$. Then the pressure error function satisfies the weighted estimates*

$$\tau\sum_{n=0}^N \sigma^{n+1}\big\|e_h^{n+1}\big\|_0^2 \le C\left(\tau + h^2\right). \tag{6.1}$$

If NLC is also valid, then the following uniform error estimate holds

$$\tau\sum_{n=0}^N \big\|e_h^{n+1}\big\|_0^2 \le C\left(\tau + h^2\right). \tag{6.2}$$

Proof. Since $p_h^{n+1} = \mu s_h^{n+1} - \tau^{-1}\rho_h^{n+1}$ and $\widehat{\mathbf{E}}_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla\rho_h^{n+1}$ according to (1.11) and (1.13), we can rearrange (4.15) to read $\left\langle e_h^{n+1}, \operatorname{div}\mathbf{w}_h\right\rangle = A_1 + A_2$ with

$$A_1 := \left\langle\delta\mathbf{E}^{n+1}, \mathbf{w}_h\right\rangle + \mu\left\langle\nabla\widehat{\mathbf{E}}^{n+1}, \nabla\mathbf{w}_h\right\rangle - \left\langle\mu(s_h^{n+1} - s_h^n) + f^{n+1}, \operatorname{div}\mathbf{w}_h\right\rangle,$$
$$A_2 := \mathfrak{N}_h\left(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \mathbf{w}_h\right) - \mathfrak{N}_h\left(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h\right).$$

In view of (4.13), A_1 can be bounded as follows:

$$\sup_{\mathbf{w}_h\in\mathbb{V}_h} \frac{|A_1|}{\|\nabla\mathbf{w}_h\|_0} \le C\big\|\delta\mathbf{E}^{n+1}\big\|_0 + C\mu\Big\|\nabla\widehat{\mathbf{E}}^{n+1}\Big\|_0 + C\big\|f^{n+1}\big\|_0.$$

The remaining term A_2 can be further split as follows:

$$\begin{aligned} A_2 = &- \mathfrak{N}_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}), \mathbf{z}_h^{n+1}) \\ &- \mathfrak{N}_h(\mathbf{u}(t^n) - \mathbf{u}_h^n, \mathbf{u}(t^{n+1}), \mathbf{z}_h^{n+1}) \\ &- \mathfrak{N}_h(\mathbf{u}(t^n), \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \mathbf{z}_h^{n+1}) \\ &- \mathfrak{N}_h(\mathbf{u}_h^n - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, \mathbf{z}_h^{n+1}). \end{aligned}$$

The only problematic term is the last one because it requires use of (2.13). To this end, note that $\|\mathbf{u}(t^n) - \mathbf{u}_h^n\|_{\mathbf{L}^3(\Omega)} \le C + Ch^{-\frac{d}{6}}(\tau^{\frac{1}{2}} + h) \le C$, as results from adding and subtracting $I_h \mathbf{u}(t^n)$, and employing an inverse inequality together with (4.14). Therefore, since (2.2) implies $\int_{t^n}^{t^{n+1}} \|\mathbf{u}_t(t)\|_0 dt \le C\tau$,

$$\sup_{\mathbf{w}_h \in \mathbb{V}_h} \frac{|A_2|}{\|\nabla \mathbf{w}_h\|_0} \le C\tau + C\left(\|\mathbf{E}^n\|_0 + \|\mathbf{G}^n\|_0 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_1 + \left\|\mathbf{G}^{n+1}\right\|_1\right).$$

Altogether, invoking the inf-sup condition $A4$ in conjunction with (4.3) and (4.14), we thus obtain

$$\begin{aligned} \beta\|e_h^{n+1}\|_0 &\le \sup_{\mathbf{w}_h \in \mathbb{V}_h} \frac{\langle e_h^{n+1}, \operatorname{div} \mathbf{w}_h\rangle}{\|\nabla \mathbf{w}_h\|_0} \\ &\le C\big(\tau^{\frac{1}{2}} + h\big) + C\left(\left\|\delta \mathbf{E}^{n+1}\right\|_0 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_1 + \left\|\mathbf{G}^{n+1}\right\|_1 + \left\|f^{n+1}\right\|_0\right). \end{aligned}$$

What remains now is to square, multiply by $\tau\sigma^{n+1}$ (resp. τ in case NLC is valid)), and sum over n from 0 to N. Recalling (4.3), (4.12), (4.14) and (5.7), assertion (6.1) (resp. (6.2)) follows immediately. This concludes the proof. □

7. Numerical Experiments. In this section, we document the computational performance of the Gauge-Uzawa FEM with two relevant examples. They were both computed within the finite element toolbox ALBERT of Schmidt and Siebert [22].

7.1. Example 1: Smooth Solution. This example is meant to confirm our main theorems numerically. The domain is the unit square $\Omega = [0, 1] \times [0, 1]$ and the (smooth) solution is given by

$$\begin{cases} u(x, y, t) = \cos(t)(x^2 - 2x^3 + x^4)(2y - 6y^2 + 4y^3) \\ v(x, y, t) = -\cos(t)(y^2 - 2y^3 + y^4)(2x - 6x^2 + 4x^3) \\ p(x, y, t) = \cos(t)(x^2 + y^2 - \frac{2}{3}). \end{cases}$$

The forcing term $\mathbf{f}(t)$ is determined accordingly for any μ; here $\mu = 1$. Computations are performed with the Taylor-Hood $(\mathcal{P}^2, \mathcal{P}^1)$ finite element pair on quasi-uniform meshes of size h. However, the coarsest mesh is quite distorted to avoid superconvergence effects. Since we expect a rate of convergence in $L^2(\mathbf{H}^1 \times L^2)$ of order $\mathcal{O}(\tau + h^2)$, we impose the relation $\tau = h^2$ to avoid dominance of either space or time error over the other. Table 7.1 shows second order accuracy for both velocity and pressure. This computational result is consistent with our theory for velocity in $L^2(\mathbf{L}^2)$ but is better than we predict for pressure as well as several stronger norms for both velocity and pressure.

7.2. Example 2: Backward Step and Do-nothing Boundary Condition. In order to explore the applicability of the Gauge-Uzawa method beyond the theory, we consider the backward step flow problem with *do-nothing boundary condition*; this is a natural boundary condition for the stress, namely

$$(-\nabla \mathbf{u} + \mathbf{I}p) \cdot \boldsymbol{\nu} = 0, \quad \text{on } \Gamma_{out}, \tag{7.1}$$

where $\Gamma_{out} \subset \partial\Omega$. This condition can be imposed for fluid problems with an open outlet without forcing. Conditions involving the stress and geometric quantities such as mean curvature are ubiquitous in dealing with free boundary problems for fluids.

h		1/8	1/16	1/32	1/64	1/128
$\\|\mathbf{E}\\|_{L^\infty(\mathbf{L}^2)}$		0.000620853	0.00015719	3.93629e-05	9.84413e-06	2.46124e-06
	Order		1.981742	1.997601	1.999501	1.999878
$\\|\mathbf{E}\\|_{L^\infty(\mathbf{L}^\infty)}$		0.00161487	0.000405717	9.99044e-05	2.47218e-05	6.14264e-06
	Order		1.992872	2.021854	2.014764	2.008853
$\\|\mathbf{E}\\|_{L^2(\mathbf{L}^2)}$		0.00156787	0.000430621	0.000111099	2.80291e-05	7.02442e-06
	Order		1.864315	1.954573	1.986848	1.996474
$\\|\mathbf{E}\\|_{L^\infty(\mathbf{H}^1)}$		0.00823813	0.0021339	0.00053749	0.000134617	3.36693e-05
	Order		1.948824	1.989183	1.997377	1.999355
$\\|\mathbf{E}\\|_{L^2(\mathbf{H}^1)}$		0.0220663	0.00643655	0.00171973	0.000442083	0.000111798
	Order		1.777485	1.904106	1.959793	1.983423
$\\|e\\|_{L^\infty(L^2)}$		0.0105357	0.0027511	0.000694088	0.000173903	4.34992e-05
	Order		1.937206	1.986818	1.996836	1.999222
$\\|e\\|_{L^\infty(\mathbf{L}^\infty)}$		0.0894505	0.0293408	0.00887096	0.00258632	0.000737458
	Order		1.608181	1.725746	1.778189	1.810268
$\\|e\\|_{L^2(\mathbf{L}^2)}$		0.0894505	0.00836859	0.0023322	0.000620179	0.0001615
	Order		3.418033	1.843293	1.910935	1.941151

TABLE 7.1

Example 7.1: The Error Decay of Gauge-Uzawa FEM for a smooth solution and several norms for velocity and pressure. The computations are performed with the Taylor-Hood $(\mathcal{P}^2, \mathcal{P}^1)$ finite element pair on quasi-uniform meshes. The meshes are distorted though to prevent superconvergence effects. The table shows second order accuracy for both velocity and pressure for the relation $\tau = h^2$.

The mere fact that projection methods decouple velocity and pressure computations, and that both $\mathbf{u}$ and p appear together in (7.1) makes its implementation a challenge. This is the case for several projections methods such as the Chorin's method [4, 21, 18] and the Gauge method [8, 7, 27].

Since the momentum equation (1.8) is consistent for the pair $(\widehat{\mathbf{u}}_h^{n+1}, p_h^n)$, as written in (1.14), to impose (7.1) on the Gauge-Uzawa method, we use the modified form $(-\nabla \mathbf{u}^{n+1} + \mathbf{I}p^n) \cdot \boldsymbol{\nu} = 0$. This amounts to solving (1.14), namely,

$$\tau^{-1} \left\langle \widehat{\mathbf{u}}_h^{n+1} - \widehat{\mathbf{u}}_h^n \,,\, \mathbf{w}_h \right\rangle + \mathfrak{N}_h(\mathbf{u}_h^n, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \mu \left\langle \nabla \widehat{\mathbf{u}}_h^{n+1} \,,\, \nabla \mathbf{w}_h \right\rangle \\ - \mu \left\langle p_h^n \,,\, \mathrm{div}\, \mathbf{w}_h \right\rangle = \left\langle \mathbf{f}(t^{n+1}) \,,\, \mathbf{w}_h \right\rangle,$$

but with test function $\mathbf{w}_h$ free on Γ_{out}. This leads, however, to an incompatible Poisson problem (1.9) if we insist on a homogeneous Neumann condition; note that now it is plausible that $\int_{\partial\Omega} \widehat{\mathbf{u}}_h^{n+1} \cdot \boldsymbol{\nu} \neq 0$.

To circumvent this issue, we consider a space-continuous Gauge-Uzawa formulation. In view of (1.9) and (1.12), we can write

$$\left\langle \nabla \rho^{n+1} \,,\, \nabla \psi \right\rangle = - \left\langle \widehat{\mathbf{u}}^{n+1} \,,\, \nabla \psi \right\rangle = \left\langle \mathbf{u}^{n+1} - \widehat{\mathbf{u}}^{n+1} \,,\, \nabla \psi \right\rangle, \qquad \forall \psi \in \mathbb{P}.$$

This amounts to the natural boundary condition $\partial_{\boldsymbol{\nu}} \rho^{n+1} = (\mathbf{u}^{n+1} - \widehat{\mathbf{u}}^{n+1}) \cdot \boldsymbol{\nu}$, which is not computable since we do not yet know $\mathbf{u}^{n+1}$. We now decompose $\partial\Omega$ into an inflow part Γ_{in}, where we prescribe velocity, an outflow part Γ_{out}, where we impose (7.1), and the rest where $\widehat{\mathbf{u}}^{n+1} \cdot \boldsymbol{\nu} = \mathbf{u}^{n+1} \cdot \boldsymbol{\nu} = 0$. Since $\int_{\partial\Omega} \mathbf{u}^{n+1} \cdot \boldsymbol{\nu} = \int_\Omega \mathrm{div}\, \mathbf{u}^{n+1} = 0$

$$\int_{\Gamma_{out}} \mathbf{u}^{n+1} \cdot \boldsymbol{\nu} = - \int_{\Gamma_{in}} \mathbf{u}^{n+1} \cdot \boldsymbol{\nu} = - \int_{\Gamma_{in}} \widehat{\mathbf{u}}^{n+1} \cdot \boldsymbol{\nu},$$

whence

$$\int_{\Gamma_{out}} (\mathbf{u}^{n+1} - \widehat{\mathbf{u}}^{n+1}) \cdot \boldsymbol{\nu} = \int_{\partial\Omega} \widehat{\mathbf{u}}^{n+1} \cdot \boldsymbol{\nu}.$$

We thus solve (1.9) with a constant flux condition, namely,

$$\partial_{\boldsymbol{\nu}} \rho^{n+1} = |\Gamma_{out}|^{-1} \int_{\partial\Omega} \widehat{\mathbf{u}}^{n+1} \cdot \boldsymbol{\nu} \quad \text{on } \Gamma_{out}.$$

We consider a simple geometry consisting of a backward step flow with *do-nothing boundary condition*. This example has been studied extensively and our results are consistent with those in the literature [14, 23]. The computational domain Ω is $[0,6] \times [0,1]$ with an obstacle $[1.2, 1.6] \times [0, 0.4]$ (see Figure 7.1). No slip boundary condition is imposed except on the inflow boundary Γ_{in} and on the outflow boundary Γ_{out}. We assign $\mathbf{u} = (1,0)$ on Γ_{in} and (7.1) on Γ_{out} for all time t. The viscosity is $\mu = 0.005$ and the discretization parameters are $\tau = 0.05$ and $h = 1/32$.

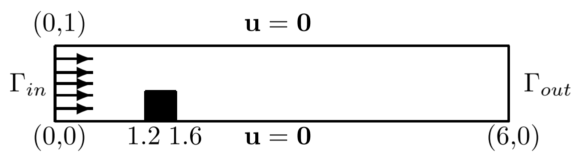

FIG. 7.1. *Example 7.2: The computational domain and boundary values. The viscosity is* $\mu = 0.005$ *and the discretization parameters are* $\tau = 0.05$ *and* $h = 1/32$.

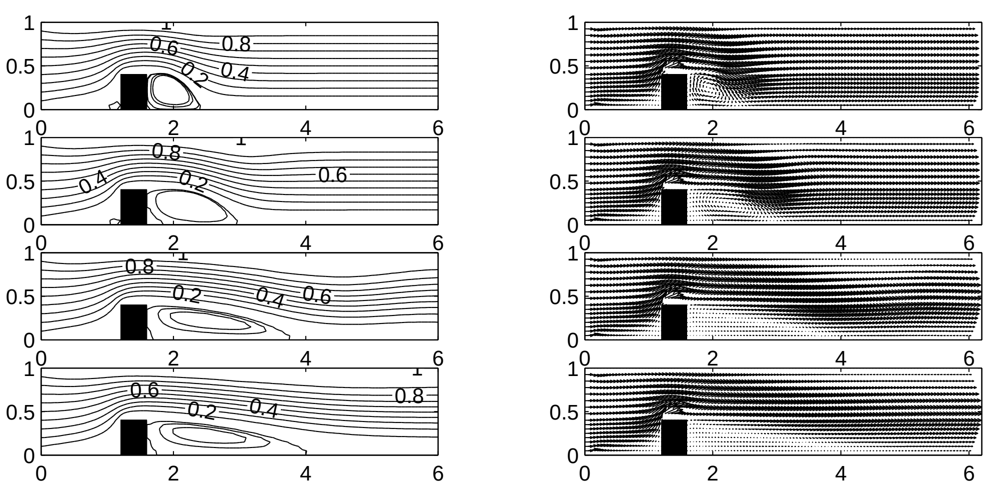

FIG. 7.2. *Example 7.2: The streamlines and velocity vector fields at times* $t = 1, 2, 5,$ *and* 50.

Figure 7.2 is a time sequence of streamlines and velocity vector fields for $t = 1, 2, 5, 50$. For $t = 50$ the evolution already became stationary. Figure 7.3 displays zooms of the recirculation zone behind the step.

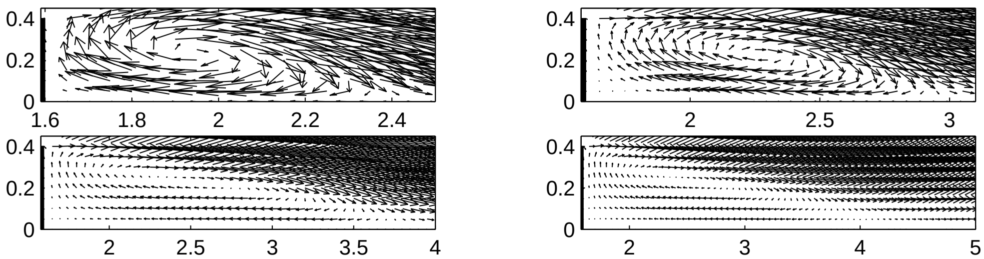

FIG. 7.3. *Example 7.2: Zooms of velocity vector field in the recirculation zone behind the step at times* $t = 1,2,\ 5,$ *and* 50

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